ISSN 1974-4110

Università degli Studi di Urbino "Carlo Bo"
Facoltà di Economia
www.econ.uniurb.it

WP-EMS
Working Papers Series in
Economics, Mathematics and Statistics

# "GENERALIZED DIFFERENTIABILITY OF FUZZY-VALUED FUNCTIONS"

• Barnabás Bede, (DigiPen Inst. of Technology, Redmond, Washington, USA)
• Luciano Stefanini, (U. Urbino)

WP-EMS # 2012/9

# Generalized Differentiability of Fuzzy-valued Functions

Barnabás Bede [a,*]

[a] *Department of Mathematics, DigiPen Institute of Technology, 9931 Willows Rd NE, Redmond, Washington 98052, USA*

Luciano Stefanini [b]

[b] *DESP - Department of Economics, Society and Political Sciences, University of Urbino "Carlo Bo", Italy*

**Abstract**

In the present paper, using novel generalizations of the Hukuhara difference for fuzzy sets, we introduce and study new generalized differentiability concepts for fuzzy valued functions. Several properties of the new concepts are investigated and they are compared to similar fuzzy differentiabilities finding connections between them. Characterization and relatively simple expressions are provided for the new derivatives.

*Key words:* Fuzzy-valued function, Strongly generalized differentiability, Generalized Hukuhara differentiability, Generalized fuzzy derivative

## 1 Introduction

The purpose of the present paper is to use the fuzzy gH-difference introduced in [37], [38] to define and study new generalizations of the differentiability for fuzzy-number-valued functions. Several generalized fuzzy derivative concepts are studied in relation with the similar notions in [2], [39]. We also show connections to the ideas of [22], [23], [26]. As a consequence, the paper presents several new results and discusses old ones in the light of the new concepts introduced recently and studied here.

* Corresponding author

*Email addresses:* `bbede@digipen.edu` ( Barnabás Bede), `lucste@uniurb.it` (Luciano Stefanini).

These new generalized derivatives are motivated by their usefulness in a very quickly developing area at the intersection of set-valued analysis and fuzzy sets, namely, the area of fuzzy analysis and fuzzy differential equations [1], [5], [6], [7], [9], [15], [18], [19], [20], [21], [27], [29], [30], [33], [34], [35], [43] etc.

As we can see, a key point in our investigation is played by the difference concepts for fuzzy numbers. A recent very promising concept, the g-difference proposed by [37], [38] is studied here in detail. We observe that this difference has a great advantage over peer concepts, namely that it always exists. We obtain relatively simple expressions, a minimality property and a characterization for the g-difference.

It is well-known that the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [10], [11], [18]. The gH-difference of two fuzzy numbers exists under much less restrictive conditions, however it does not always exist [36], [37]. The g-difference proposed in [38] overcomes these shortcomings of the above discussed concepts and the g-difference of two fuzzy numbers always exists. The same remark is valid if we regard differentiability concepts in fuzzy setting.

Based on the gH-difference coming from [38], [39], [40], new gH-derivative concepts that generalize those in [2] are investigated, mainly in view of their characterization. Based on the g-difference a new, very general fuzzy differentiability concept is defined and studied, the so-called g-derivative. It is carefully compared with the generalized fuzzy differentiabilities in [2], [39], [40], and it is shown that the g-difference is the most general among all similar definitions. The properties we obtain show characterization of the new g-differentiability, an interesting minimality property and some computational results.

The relation between the new fuzzy derivatives and the fuzzy integral is studied, and Newton-Leibniz type formulas are non-trivially extended to the fuzzy case.

The paper is organized as follows; section 2 introduces the generalized fuzzy difference and presents some new results; next, we show some new properties of the generalized Hukuhara derivative of a fuzzy valued function (section 3) and we introduce the new concept of generalized derivative (section 4); the paper concludes with section 5, where the basic relations between gH-differentiability and the integral are examined.

## 2 Generalized fuzzy difference

One of the first definitions of difference and derivative for set-valued functions was given by Hukuhara [16] (H-difference and H-derivative); it has been extended to the fuzzy case in [32] and applied to fuzzy differential equations (FDE) by many authors in several papers (see [12], [17], [18], [19], [20], [29]). But the H-derivative in FDE suffers certain disadvantages (see [2], [3], [6], [7], [10], [13], [36]) related to the properties of the space $\mathcal{K}^n$ of all nonempty compact sets of $\mathbb{R}^n$ and in particular to the fact that Minkowski addition does not possess an inverse subtraction. On the other hand, a more general definition of subtraction for compact convex sets, and in particular for compact intervals, has been introduced by several authors. Markov [22], [23], [26] defined a non-standard difference, also called *inner*-difference, and extended its use to interval arithmetic and to interval calculus, including interval differential equations (see [24], [25]). In the setting of Hukuhara difference, the interval and fuzzy generalized Hukuhara differences have been recently examined in [37], [38].

We start with a brief account of these concepts.

Let $\mathcal{K}_C^n$ be the space of nonempty compact and convex sets of $\mathbb{R}^n$. The Hukuhara H-difference has been introduced as a set $C$ for which $A \ominus_H B = C \iff A = B + C$ and an important property of $\ominus_H$ is that $A \ominus_H A = \{0\}$ $\forall A \in \mathcal{K}_C^n$ and $(A + B) \ominus_H B = A$, $\forall A, B \in \mathcal{K}_C^n$. The H-difference is unique, but it does not always exist (a necessary condition for $A \ominus_H B$ to exist is that $A$ contains a translate $\{c\} + B$ of $B$). A generalization of the Hukuhara difference aims to overcome this situation. The generalized Hukuhara difference of two sets $A, B \in \mathcal{K}_C^n$ (gH-difference for short) is defined as follows

$$A \ominus_{gH} B = C \iff \begin{cases} \quad (a)\ A = B + C \\ \text{or } (b)\ \ B = A + (-1)C \end{cases} \tag{1}$$

The *inner*-difference in [26], denoted with the symbol "$-^-$", is defined by first introducing the *inner*-sum of $A$ and $B$ by

$$A +^- B = \begin{cases} X \text{ if } X \text{ solves } (-A) + X = B \\ Y \text{ if } Y \text{ solves } (-B) + Y = A \end{cases} \tag{2}$$

and then

$$A -^- B = A +^- (-B). \tag{3}$$

It is not difficult to see that $A \ominus_{gH} B = A -^- B$; in fact, $A +^- (-B) = C$ means $(-A) + C = (-B)$ i.e. case $(b)$ of (1), or $(-(-B)) + C = A$ i.e. case

$(a)$ of (1).

In case $(a)$ of (1) the gH-difference is coincident with the H-difference. Thus the gH-difference, or the *inner*-difference, is a generalization of the H-difference.

The $gH$-difference (1) or, equivalently, the *inner*-difference (3) for intervals or for compact convex sets is the basis for the definition of a new difference in the fuzzy context.

We will denote $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers, i.e. normal, fuzzy convex, upper semi continuous and compactly supported fuzzy sets defined over the real line. Fundamental concepts in fuzzy sets theory are the *support*, the *level-sets* (or *level-cuts*) and the *core* of a fuzzy number.

Here, $cl(X)$ denotes the closure of set $X$.

**Definition 1** *Let $u \in \mathbb{R}_{\mathcal{F}}$ be a fuzzy number. For $\alpha \in ]0,1]$, the $\alpha$-level set of $u$ (or simply the $\alpha-cut$) is defined by $[u]_\alpha = \{x | x \in \mathbb{R}, u(x) \geq \alpha\}$ and for $\alpha = 0$ by the closure of the support $[u]_0 = cl\{x | x \in \mathbb{R}, u(x) > 0\}$. The core of $u$ is the set of elements of $\mathbb{R}$ having membership grade 1, i.e., $[u]_1 = \{x | x \in \mathbb{R}, u(x) = 1\}$.*

It is well-known that the $level-cuts$ are "nested", i.e. $[u]_\alpha \subseteq [u]_\beta$ for $\alpha > \beta$. A fuzzy set $u$ is a fuzzy number if and only if the $\alpha - cuts$ are nonempty compact intervals of the form $[u]_\alpha = [u_\alpha^-, u_\alpha^+] \subset \mathbb{R}$. The "nested" property is the basis for the LU representation (L for lower, U for upper) (see [14], [42]).

**Proposition 2** *A fuzzy number $u$ is completely determined by any pair $u = (u^-, u^+)$ of functions $u^-, u^+ : [0,1] \longrightarrow \mathbb{R}$, defining the end-points of the $\alpha - cuts$, satisfying the three conditions:*
*(i) $u^- : \alpha \longrightarrow u_\alpha^- \in \mathbb{R}$ is a bounded monotonic non decreasing left-continuous function $\forall \alpha \in ]0,1]$ and right-continuous for $\alpha = 0$;*
*(ii) $u^+ : \alpha \longrightarrow u_\alpha^+ \in \mathbb{R}$ is a bounded monotonic non increasing left-continuous function $\forall \alpha \in ]0,1]$ and right-continuous for $\alpha = 0$;*
*(iii) $u_1^- \leq u_1^+$ for $\alpha = 1$, which implies $u_\alpha^- \leq u_\alpha^+$ $\forall \alpha \in [0,1]$.*

The following result is well known [28]:

**Proposition 3** *Let $\{U_\alpha | \alpha \in ]0,1]\}$ be a family of real intervals such that the following three conditions are satisfied:*
*1. $U_\alpha$ is a nonempty compact interval for all $\alpha \in ]0,1]$;*
*2. if $0 < \alpha < \beta \leq 1$ then $U_\beta \subseteq U_\alpha$;*
*3. given any non decreasing sequence $\alpha_n \in ]0,1]$ with $\lim\limits_{n \longrightarrow \infty} \alpha_n = \alpha > 0$ it is $U_\alpha = \bigcap\limits_{n=1}^{\infty} U_{\alpha_n}$.*
*Then there exists a unique LU-fuzzy quantity $u$ such that $[u]_\alpha = U_\alpha$ for all*

$\alpha \in ]0,1]$ *and* $[u]_0 = cl\left(\bigcup_{\alpha\in]0,1]} U_\alpha\right)$.

We refer to the functions $u^-_{(.)}$ and $u^+_{(.)}$ as the lower and upper branches of $u$, respectively. A trapezoidal fuzzy number, denoted by $u = \langle a,b,c,d\rangle$, where $a \leq b \leq c \leq d$, has $\alpha - cuts$ $[u]_\alpha = [a + \alpha(b-a), d - \alpha(d-c)]$, $\alpha \in [0,1]$, obtaining a triangular fuzzy number if $b = c$.

The addition $u + v$ and the scalar multiplication $ku$ are defined as having the level cuts

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = \{x+y | x \in [u]_\alpha,\ y \in [v]_\alpha\}$$
$$[ku]_\alpha = k[u]_\alpha = \{kx | x \in [u]_\alpha\},\ [0]_\alpha = \{0\}\ \forall \alpha \in [0,1]$$

The subtraction of fuzzy numbers $u - v$ is defined as the addition $u + (-v)$ where $-v = (-1)v$.

The standard Hukuhara difference (H-difference $\ominus_H$) is defined by $u \ominus_H v = w \iff u = v + w$, being $+$ the standard fuzzy addition; if $u \ominus_H v$ exists, its $\alpha - cuts$ are $[u \ominus_H v]_\alpha = [u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha]$. It is well known that $u \ominus_H u = 0$ (here $0$ stands for the singleton $\{0\}$) for all fuzzy numbers $u$, but $u - u \neq 0$.

The Hausdorff distance on $\mathbb{R}_\mathcal{F}$ is defined by

$$D(u,v) = \sup_{\alpha\in[0,1]} \left\{ \|[u]_\alpha \ominus_{gH} [v]_\alpha\|_* \right\},$$

where, for an interval $[a,b]$, the norm is

$$\|[a,b]\|_* = \max\{|a|,|b|\}.$$

The metric $D$ is well defined since the gH-difference of intervals, $[u]_\alpha \ominus_{gH} [v]_\alpha$ always exists. Also, this allows us to deduce that $(\mathbb{R}_\mathcal{F}, D)$ is a complete metric space. This definition is equivalent to the usual definitions for metric spaces of fuzzy numbers in e.g., [12], [18], [14].

The next lemma will be used throughout the paper.

**Lemma 4** *Let $f : \mathbb{R} \to \mathbb{R}_\mathcal{F}$ be a fuzzy-number-valued function. Let $x_0 \in \mathbb{R}$. Then if*
*(i)* $\lim_{x\to x_0}[f(x)]_\alpha = U_\alpha = [u^-_\alpha, u^+_\alpha]$ *uniformly with respect to* $\alpha \in [0,1]$,
*(ii)* $u^-_\alpha, u^+_\alpha$ *fulfill the conditions in Proposition 2 or equivalently $U_\alpha$ fulfill the conditions in Proposition 3,*
*then* $\lim_{x\to x_0} f(x) = u$, *with* $[u]_\alpha = U_\alpha = [u^-_\alpha, u^+_\alpha]$.

**PROOF.** By condition (ii) the intervals $U_\alpha$ define a fuzzy number, denoted

$u$. Then, by condition (i), we have

$$\lim_{x\to x_0} D(f(x),u) = \lim_{x\to x_0} \sup_{\alpha\in[0,1]} \left\{ \|[f(x)]_\alpha \ominus_{gH} [u]_\alpha\|_* \right\} = 0,$$

i.e., $\lim_{x\to x_0} f(x) = u$.□

**Definition 5** *Given two fuzzy numbers $u, v \in \mathbb{R}_\mathcal{F}$, the generalized Hukuhara difference (gH-difference for short) is the fuzzy number $w$, if it exists, such that*

$$u \ominus_{gH} v = w \iff \begin{cases} \quad (i) \quad u = v + w \\ or \quad (ii) \quad v = u - w \end{cases}. \tag{4}$$

*It is easy to show that (i) and (ii) are both valid if and only if $w$ is a crisp number.*

In terms of $\alpha - cuts$ we have $[u \ominus_{gH} v]_\alpha = [\min\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}, \max\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}]$ and if the H-difference exists, then $u \ominus_H v = u \ominus_{gH} v$; the conditions for the existence of $w = u \ominus_{gH} v \in \mathbb{R}_\mathcal{F}$ are

$$\begin{array}{ll} \text{case } (i) \begin{cases} w_\alpha^- = u_\alpha^- - v_\alpha^- \text{ and } w_\alpha^+ = u_\alpha^+ - v_\alpha^+ \\ \text{with } w_\alpha^- \text{ increasing , } w_\alpha^+ \text{ decreasing , } w_\alpha^- \le w_\alpha^+ \end{cases} & , \forall\alpha \in [0,1] \\ \text{case } (ii) \begin{cases} w_\alpha^- = u_\alpha^+ - v_\alpha^+ \text{ and } w_\alpha^+ = u_\alpha^- - v_\alpha^- \\ \text{with } w_\alpha^- \text{ increasing, } w_\alpha^+ \text{ decreasing, } w_\alpha^- \le w_\alpha^+ \ . \end{cases} & , \forall\alpha \in [0,1] \end{array} \tag{5}$$

The following properties were obtained in [38].

**Proposition 6** *([38]) Let $u, v \in \mathbb{R}_\mathcal{F}$ be two fuzzy numbers; then*
*i) if the gH-difference exists, it is unique;*
*ii) $u \ominus_{gH} v = u \ominus_H v$ or $u \ominus_{gH} v = -(v \ominus_H u)$ whenever the expressions on the right exist; in particular, $u \ominus_{gH} u = u \ominus_H u = 0$,*
*iii) if $u \ominus_{gH} v$ exists in the sense $(i)$, then $v \ominus_{gH} u$ exists in the sense $(ii)$ and vice versa,*
*iv) $(u + v) \ominus_{gH} v = u$,*
*v) $0 \ominus_{gH} (u \ominus_{gH} v) = v \ominus_{gH} u$,*
*vi) $u \ominus_{gH} v = v \ominus_{gH} u = w$ if and only if $w = -w$; furthermore, $w = 0$ if and only if $u = v$.*

In the fuzzy case, it is possible that the gH-difference of two fuzzy numbers does not exist. For example we can consider a triangular and a trapezoidal fuzzy number $u = \langle 0, 2, 2, 4\rangle$ and $v = \langle 0, 1, 2, 3\rangle$; level-wise, the gH-differences exist and they are e.g. for both the 0 and 1 level sets the same $[0, 1]$, but the gH-difference $u \ominus_{gH} v$ does not exist. Indeed, if we suppose that it exists then either case (i) or (ii) of (5) should hold for any $\alpha \in [0, 1]$. But $w_0^- = u_0^- - v_0^- =$

$0 < w_0^+ = u_0^+ - v_0^+ = 1$ while $w_1^- = 1 > w_1^+ = 0$, so neither case (i) or (ii) is true from (5). To solve this shortcoming, in [37], [38] a new difference between fuzzy numbers was proposed, a difference that always exists.

**Definition 7** *The generalized difference (g-difference for short) of two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ is given by its level sets as*

$$[u \ominus_g v]_\alpha = cl \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_{gH} [v]_\beta), \forall \alpha \in [0,1], \tag{6}$$

*where the gH-difference $\ominus_{gH}$ is with interval operands $[u]_\beta$ and $[v]_\beta$.*

**Proposition 8** *The g-difference (6) is given by the expression*

$$[u \ominus_g v]_\alpha = \left[ \inf_{\beta \geq \alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta \geq \alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\} \right]$$

**PROOF.** Let $\alpha \in [0,1]$ be fixed. We observe that for any $\beta \geq \alpha$ we have

$$\begin{aligned} [u]_\beta \ominus_{gH} [v]_\beta &= \left[ \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\} \right] \\ &\subseteq \left[ \inf_{\lambda \geq \beta} \min\{u_\lambda^- - v_\lambda^-, u_\lambda^+ - v_\lambda^+\}, \sup_{\lambda \geq \beta} \max\{u_\lambda^- - v_\lambda^-, u_\lambda^+ - v_\lambda^+\} \right] \end{aligned}$$

and it follows that

$$cl \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_{gH} [v]_\beta) \subseteq \left[ \inf_{\beta \geq \alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta \geq \alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\} \right]$$

Let us consider now

$$cl \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_{gH} [v]_\beta) = cl \bigcup_{\beta \geq \alpha} \left[ \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\} \right].$$

For any $n \geq 1$, there exist $a_n \in \{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+ : \beta \geq \alpha\}$ such that $\inf_{\beta \geq \alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\} > a_n - \frac{1}{n}$. Also there exist $b_n \in \{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+ : \beta \geq \alpha\}$ such that $\sup_{\beta \geq \alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\} < b_n + \frac{1}{n}$. We have $cl \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_{gH} [v]_\beta) \supseteq [a_n, b_n], \forall n \geq 1$ and we obtain

$$cl \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_{gH} [v]_\beta) \supseteq \bigcup_{n \geq 1} [a_n, b_n] \supseteq \left( \lim_{n \to \infty} a_n, \lim_{n \to \infty} b_n \right)$$

and finally

$$cl \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_{gH} [v]_\beta) \supseteq \left[ \inf_{\beta \geq \alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta \geq \alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\} \right].$$

The conclusion

$$\left[\inf_{\beta\geq\alpha}\min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta\geq\alpha}\max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}\right] = cl\bigcup_{\beta\geq\alpha}([u]_\beta \ominus_{gH} [v]_\beta)$$

of the proposition follows.□

**Remark 9** *The property in the previous proposition 8 holds in particular for* $\alpha = 0$, *case which is covered because of the right continuity of the functions* $u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+$.

The following proposition gives a simplified notation for $u \ominus_g v$ and $v \ominus_g u$.

**Proposition 10** *For any two fuzzy numbers* $u, v \in \mathbb{R}_\mathcal{F}$ *the two g-differences* $u \ominus_g v$ *and* $v \ominus_g u$ *exist and, for any* $\alpha \in [0,1]$, *we have* $u \ominus_g v = -(v \ominus_g u)$ *with*

$$[u \ominus_g v]_\alpha = [d_\alpha^-, d_\alpha^+] \text{ and } [v \ominus_g u]_\alpha = [-d_\alpha^+, -d_\alpha^-] \tag{7}$$

*where*

$$d_\alpha^- = \inf(D_\alpha), \quad d_\alpha^+ = \sup(D_\alpha)$$

*and the sets* $D_\alpha$ *are*

$$D_\alpha = \{u_\beta^- - v_\beta^- | \beta \geq \alpha\} \cup \{u_\beta^+ - v_\beta^+ | \beta \geq \alpha\}.$$

**PROOF.** Consider a fixed $\alpha \in [0,1]$. Clearly, using Proposition 8,

$$\begin{aligned}[u \ominus_g v]_\alpha &= \left[\inf_{\beta\geq\alpha}\min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta\geq\alpha}\max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}\right] \\ &\subseteq [\inf(D_\alpha), \sup(D_\alpha)] = [d_\alpha^-, d_\alpha^+].\end{aligned}$$

Vice versa, for all $n \geq 1$ and from the definition of $d_\alpha^-$ and $d_\alpha^+$, there exist $a_n, b_n \in D_\alpha$ such that

$$d_\alpha^- \leq a_n < d_\alpha^- + \frac{1}{n}$$
$$d_\alpha^+ - \frac{1}{n} < b_n \leq d_\alpha^+$$

and the following limits exist

$$\lim a_n = d_\alpha^- \ , \quad \lim b_n = d_\alpha^+;$$

on the other hand, $[a_n, b_n] \subseteq cl\bigcup_{\beta\geq\alpha}([u]_\alpha \ominus_{gH} [v]_\alpha)$ for all $n \geq 1$ and then

$$\bigcup_{n\geq 1}[a_n, b_n] \subseteq cl\bigcup_{\beta\geq\alpha}([u]_\alpha \ominus_{gH} [v]_\alpha).$$

It follows that

$$[d_\alpha^-, d_\alpha^+] = [\lim a_n,\ \lim b_n] \subseteq cl \bigcup_{n\geq 1} [a_n, b_n] \subseteq cl \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_{gH} [v]_\beta)$$

and the proof is complete.

**Remark 11** *We observe that there are other possible different expressions for the g-difference as e.g.,*

$$[u \ominus_g v]_\alpha = \left[ \min\{\inf_{\beta\geq\alpha}(u_\beta^- - v_\beta^-), \inf_{\beta\geq\alpha}(u_\beta^+ - v_\beta^+)\}, \max\{\sup_{\beta\geq\alpha}(u_\beta^- - v_\beta^-), \sup_{\beta\geq\alpha}(u_\beta^+ - v_\beta^+)\} \right].$$

The next proposition shows that the g-difference is well defined for any two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$.

**Proposition 12** *([38]) For any fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ the g-difference $u \ominus_g v$ exists and it is a fuzzy number.*

**PROOF.** We regard the LU-fuzzy quantity $u \ominus_g v$. Then according to the previous result, if we denote $w^- = (u \ominus_g v)^-$ and $w^+ = (u \ominus_g v)^+$ we have

$$w^-(\alpha) = \inf_{\beta\geq\alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\} \leq w^+(\alpha) = \sup_{\beta\geq\alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}.$$

Obviously $w^-$ is bounded and non decreasing while $w^+$ is bounded non increasing. Also, $w^-, w^+$ are left continuous on $(0, 1]$, since $u^- - v^-, u^+ - v^+$ are left continuous on $(0, 1]$ and they are right continuous at 0 since so are the functions $u^- - v^-, u^+ - v^+$.$\square$

Let us consider the fuzzy inclusion defined as $u \subseteq v \iff u(x) \leq v(x)\,, \forall x \in \mathbb{R}$ $\iff [u]_\alpha \subseteq [v]_\alpha, \forall \alpha \in [0, 1]$. The following proposition provides a minimality property for the g-difference.

**Proposition 13** *The g-difference $u \ominus_g v$ is the smallest fuzzy number $w$ in the sense of fuzzy inclusion such that*

$$[u]_\alpha \ominus_{gH} [v]_\alpha \subseteq [w]_\alpha\,, \forall \alpha \in [0, 1]$$

*and*

$$\begin{cases} u \subseteq v + w \\ v \subseteq u - w \end{cases}.$$

**PROOF.** For the proof, first we observe that

$$[u]_\alpha \ominus_{gH} [v]_\alpha \subseteq [u \ominus_g v]_\alpha, \forall \alpha \in [0,1].$$

Let $w \in \mathbb{R}_\mathcal{F}$ fulfill

$$[u]_\alpha \ominus_{gH} [v]_\alpha \subseteq [w]_\alpha, \forall \alpha \in [0,1].$$

Then for any $\alpha, \beta \in [0,1], \alpha \leq \beta$ we have

$$[u]_\beta \ominus_{gH} [v]_\beta \subseteq [w]_\beta \subseteq [w]_\alpha.$$

and so

$$\bigcup_{\beta \geq \alpha} [u]_\beta \ominus_{gH} [v]_\beta \subseteq [w]_\alpha,$$

and since $[w]_\alpha$ is closed we obtain

$$[u \ominus_g v]_\alpha = cl \bigcup_{\beta \geq \alpha} [u]_\beta \ominus_{gH} [v]_\beta \subseteq [w]_\alpha, \forall \alpha \in [0,1].$$

As a conclusion $u \ominus_g v \subseteq w$. The inclusions $u \subseteq v + w$ and $v \subseteq u - w$ follow from the definition of $\ominus_{gH}$.□

The following properties turn out to be true for the g-difference.

**Proposition 14** *Let $u, v \in \mathbb{R}_\mathcal{F}$ be two fuzzy numbers; then*
*i) $u \ominus_g v = u \ominus_{gH} v$, whenever the expression on the right exists; in particular $u \ominus_g u = 0$,*
*ii) $(u + v) \ominus_g v = u$,*
*iii) $0 \ominus_g (u \ominus_g v) = v \ominus_g u$,*
*iv) $u \ominus_g v = v \ominus_g u = w$ if and only if $w = -w$; furthermore, $w = 0$ if and only if $u = v$.*

**PROOF.** The proof of i) is immediate.
For ii) we can use i). Indeed, in this case the classical Hukuhara difference $(u + v) \ominus v$ exists (and so the gH-difference $(u + v) \ominus_{gH} v$ also exists) and we have $(u + v) \ominus_g v = (u + v) \ominus_{gH} v = u$.
The proof of iii) follows from (7) for all $\alpha \in [0,1]$:

$$\begin{aligned}
[0 \ominus_g (u \ominus_g v)]_\alpha &= [0,0] \ominus_g [d_\alpha^-, d_\alpha^+] \\
&= \left[\min\{0 - d_\alpha^-, 0 - d_\alpha^+\}, \max\{0 - d_\alpha^-, 0 - d_\alpha^+\}\right] \\
&= [-d_\alpha^+, -d_\alpha^-] = [v \ominus_g u]_\alpha.
\end{aligned}$$

To prove iv), consider again (7); for all $\alpha \in [0,1]$ we have $[w]_\alpha = [u \ominus_g v]_\alpha = [d_\alpha^-, d_\alpha^+]$ and $[w]_\alpha = [v \ominus_g u]_\alpha = [-d_\alpha^+, -d_\alpha^-]$ so that $w = -w$ and vice versa; the last part of iv) follows from the last part of i) and the fact that $w = -w = 0$

if and only if $d_\alpha^- = d_\alpha^+$ for all $\alpha \in [0,1]$ and from the definition of $d_\alpha^-, d_\alpha^+$ this is true if and only if $u_\alpha^- - v_\alpha^- = 0$, $u_\alpha^+ - v_\alpha^+ = 0$ i.e. $u_\alpha^- = v_\alpha^-$, $u_\alpha^+ = v_\alpha^+$ for all $\alpha \in [0,1]$.□

The connection between the gH-difference, the g-difference and the Hausdorff distance adds a geometric interpretation for the differences constructed.

**Proposition 15** *For all $u, v \in \mathbb{R}_\mathcal{F}$ we have*

$$D(u,v) = \sup_{\alpha\in[0,1]} \|[u]_\alpha \ominus_{gH} [v]_\alpha\|_* = \|u \ominus_g v\|$$

*where $\|\cdot\| = D(\cdot, 0)$.*

**PROOF.** We have that $w = u \ominus_g v$ is a fuzzy number, then $\|w\| = \sup_{\alpha\in[0,1]} \max\{|w_\alpha^-|, |w_\alpha^+|\} = \max\{|w_0^-|, |w_0^+|\}$ and

$$\begin{aligned}
&\sup_{\alpha\in[0,1]} \|[u]_\alpha \ominus_{gH} [v]_\alpha\|_* = \\
&= \sup_{\alpha\in[0,1]} \left\|[\min\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}, \max\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}]\right\|_* \\
&= \sup_{\alpha\in[0,1]} \max\{\left|\min\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}\right|, \left|\max\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}\right|\} \\
&= \sup_{\alpha\in[0,1]} \max\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\} = D(u,v).
\end{aligned}$$

Now, since max and sup are idempotent operators, we obtain

$$\begin{aligned}
\|u \ominus_g v\| &= \sup_{\alpha\in[0,1]} \|[u \ominus_g v]_\alpha\|_* \\
&= \sup_{\alpha\in[0,1]} \left\|\left[\inf_{\beta\geq\alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta\geq\alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}\right]\right\|_* \\
&= \sup_{\alpha\in[0,1]} \left\{\sup_{\beta\geq\alpha} \max\{\left|u_\beta^- - v_\beta^-\right|, \left|u_\beta^+ - v_\beta^+\right|\}\right\} \\
&= \sup_{\alpha\in[0,1]} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\} = D(u,v).\square
\end{aligned}$$

**Example 16** *Let us consider some examples when the gH-difference does not exist, while the g-difference exists. At the beginning of this section we have considered two trapezoidal fuzzy numbers $u = \langle 0,2,2,4\rangle$ and $v = \langle 0,1,2,3\rangle$. Their g-difference is the $[0,1]$ interval (interpreted as the trapezoidal fuzzy number $\langle 0,0,1,1\rangle$. If we consider the trapezoidal number $u = \langle 2,3,5,6\rangle$ and the triangular number $v = \langle 0,4,4,8\rangle$ we can see that their gH-difference does not exist. Their g-difference however, exists and it is given as in Fig. 1.*

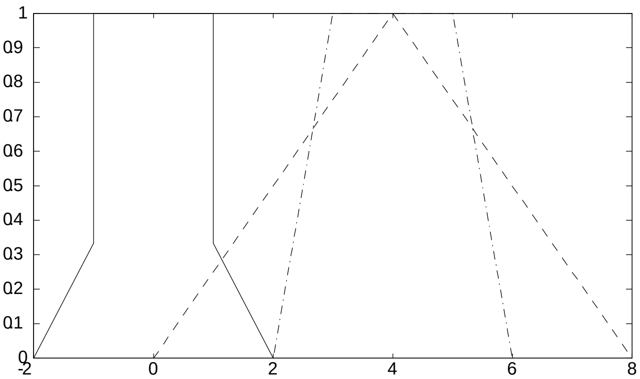


Figure 1. The g-difference $u \ominus_g v$ (solid line) of a trapezoidal $u = \langle 2, 3, 5, 6 \rangle$ (dash-dot line) and a triangular $v = \langle 0, 4, 4, 8 \rangle$ (dashed line) fuzzy number.

**Remark 17** *We observe that since $u \ominus_g v = u \ominus_{gH} v$ whenever the right side exists we can also conclude*

$$D(u, v) = \|u \ominus_g v\| = \|u \ominus_{gH} v\|,$$

*whenever $u \ominus_{gH} v$ exists.*

## 3 Generalized Hukuhara differentiability (gH-differentiability)

Generalized differentiability concepts were first considered for interval-valued functions in the works of Markov ([24], [25]). This line of research is continued by several papers [2], [8], [31], [39] etc. dealing with interval and fuzzy-valued functions. In this section we focus on the fuzzy case and we present and compare alternative definitions for the derivative of a fuzzy-valued function.

The first two concepts were presented in [2] for the fuzzy case and in [39], [40]. These are using the usual Hukuhara difference $\ominus_H$.

**Definition 18** *([2]) Let $f : ]a, b[ \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in ]a, b[$. We say that $f$ is strongly generalized Hukuhara differentiable at $x_0$ (GH-differentiable for short) if there exists an element $f'_G(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that, for all $h > 0$ sufficiently small,*

*(i) $\exists f(x_0 + h) \ominus_H f(x_0)$, $f(x_0) \ominus_H f(x_0 - h)$ and*

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_G(x_0),$$

*or (ii) $\exists f(x_0) \ominus_H f(x_0 + h)$, $f(x_0 - h) \ominus_H f(x_0)$ and*

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus_H f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus_H f(x_0)}{(-h)} = f'_G(x_0),$$

*or (iii)* $\exists f(x_0+h)\ominus_H f(x_0),\ f(x_0-h)\ominus_H f(x_0)$ *and*

$$\lim_{h\searrow 0}\frac{f(x_0+h)\ominus_H f(x_0)}{h}=\lim_{h\searrow 0}\frac{f(x_0-h)\ominus_H f(x_0)}{(-h)}=f'_G(x_0),$$

*or (iv)* $\exists f(x_0)\ominus_H f(x_0+h),\ f(x_0)\ominus_H f(x_0-h)$ *and*

$$\lim_{h\searrow 0}\frac{f(x_0)\ominus_H f(x_0+h)}{(-h)}=\lim_{h\searrow 0}\frac{f(x_0)\ominus_H f(x_0-h)}{h}=f'_G(x_0).$$

**Definition 19** *([2]) Let* $f:]a,b[\to\mathbb{R}_{\mathcal{F}}$ *and* $x_0\in]a,b[$*. For a sequence* $h_n\searrow 0$ *and* $n_0\in N$*, let us denote*

$$A_{n_0}^{(1)}=\left\{n\geq n_0;\exists E_n^{(1)}:=f(x_0+h_n)\ominus_H f(x_0)\right\},$$

$$A_{n_0}^{(2)}=\left\{n\geq n_0;\exists E_n^{(2)}:=f(x_0)\ominus_H f(x_0+h_n)\right\},$$

$$A_{n_0}^{(3)}=\left\{n\geq n_0;\exists E_n^{(3)}:=f(x_0)\ominus_H f(x_0-h_n)\right\},$$

$$A_{n_0}^{(4)}=\left\{n\geq n_0;\exists E_n^{(4)}:=f(x_0-h_n)\ominus_H f(x_0)\right\}.$$

*We say that* $f$ *is weakly generalized (Hukuhara) differentiable on* $x_0$*, if for any sequence* $h_n\searrow 0$*, there exists* $n_0\in N$*, such that* $A_{n_0}^{(1)}\cup A_{n_0}^{(2)}\cup A_{n_0}^{(3)}\cup A_{n_0}^{(4)}=\{n\in N;n\geq n_0\}$ *and moreover, there exists an element in* $\mathbb{R}_{\mathcal{F}}$ *denoted by* $f'_w(x_0)$*, such that if for some* $j\in\{1,2,3,4\}$ *we have* $card(A_{n_0}^{(j)})=+\infty$*, then*

$$\lim_{\substack{n\to\infty\\ n\in A_{n_0}^{(j)}}} D\left(\frac{E_n^{(j)}}{(-1)^{j+1}h_n},f'_w(x_0)\right)=0.$$

Based on the gH-difference we obtain the following definition (for interval-valued functions, the same definition was suggested in [25] using *inner*-difference):

**Definition 20** *Let* $x_0\in]a,b[$ *and* $h$ *be such that* $x_0+h\in]a,b[$*, then the gH-derivative of a function* $f:]a,b[\to\mathbb{R}_{\mathcal{F}}$ *at* $x_0$ *is defined as*

$$f'_{gH}(x_0)=\lim_{h\to 0}\frac{1}{h}[f(x_0+h)\ominus_{gH} f(x_0)]. \tag{8}$$

*If* $f'_{gH}(x_0)\in\mathbb{R}_{\mathcal{F}}$ *satisfying (8) exists, we say that* $f$ *is generalized Hukuhara differentiable (gH-differentiable for short) at* $x_0$*.*

**Theorem 21** *The gH-differentiability concept and the weakly generalized (Hukuhara) differentiability given in Definition 19 coincide.*

**PROOF.** The proof is similar to the proof of a corresponding result in [39]. Indeed, let us suppose that $f$ is gH-differentiable (as in Definition 20). By Proposition 6, iii), for any sequence $h_n \searrow 0$, for $n$ sufficiently large, at least two of the Hukuhara differences $f(x_0+h_n) \ominus_H f(x_0)$, $f(x_0) \ominus_H f(x_0+h_n)$, $f(x_0) \ominus_H f(x_0-h_n)$, $f(x_0-h_n) \ominus_H f(x_0)$ exist. As a conclusion we have $A_{n_0}^{(1)} \cup A_{n_0}^{(2)} \cup A_{n_0}^{(3)} \cup A_{n_0}^{(4)} = \{n \in \mathbb{N}; n \geq n_0\}$ for any $n_0 \in \mathbb{N}$. The rest is obtained by observing that $\frac{E_n^{(j)}}{(-1)^{j+1}h_n} = \frac{f(x_0+h_n) \ominus_{gH} f(x_0)}{h_n}$, written with gH-difference this time. Reciprocally, if we assume $f$ to be weakly generalized (Hukuhara) differentiable then since at least two of the sets $A_{n_0}^{(1)}, A_{n_0}^{(2)}, A_{n_0}^{(3)}, A_{n_0}^{(4)}$ are infinite $\lim_{h \to 0} \frac{1}{h}[f(x_0+h) \ominus_{gH} f(x_0)] = \lim_{h_n \searrow 0} \frac{E_n^{(j)}}{(-1)^{j+1}h_n}$ for at least two indices from $j \in \{1,2,3,4\}$, so $f$ is gH-differentiable. As a conclusion weakly generalized (Hukuhara) differentiability is equivalent to gH-differentiability.□

**Example 22** *Let $f(x) = p(x)a$ where $p$ is a crisp differentiable function and $a \in \mathbb{R}_{\mathcal{F}}$, then it follows relatively easily that the gH-derivative exists and it is $f'_{gH}(x) = p'(x)a$.*

As we have seen in conditions (5) and in equation (6), both *gH*-difference and *g*-difference are based on the *gH*-difference for each $\alpha$-cut of the involved fuzzy numbers; this level characterization is obviously inherited by the *gH*-derivative, with respect to the level-wise *gH*-derivative.

**Definition 23** *Let $x_0 \in ]a,b[$ and $h$ be such that $x_0 + h \in ]a,b[$, then the level-wise gH-derivative (LgH-derivative for short) of a function $f : ]a,b[ \to \mathbb{R}_{\mathcal{F}}$ at $x_0$ is defined as the set of interval-valued gH-derivatives, if they exist,*

$$f'_{LgH}(x_0)_\alpha = \lim_{h \to 0} \frac{1}{h} \left( [f(x_0+h)]_\alpha \ominus_{gH} [f(x_0)]_\alpha \right). \tag{9}$$

*If $f'_{LgH}(x_0)_\alpha$ is a compact interval for all $\alpha \in [0,1]$, we say that $f$ is level-wise generalized Hukuhara differentiable (LgH-differentiable for short) at $x_0$ and the family of intervals $\{f'_{LgH}(x_0)_\alpha | \alpha \in [0,1]\}$ is the LgH-derivative of $f$ at $x_0$, denoted by $f'_{LgH}(x_0)$.*

Clearly, *LgH-differentiability*, and consequently level-wise continuity, *is a necessary condition for gH-differentiability*; but from (5), it is not sufficient.

The next result gives the analogous expression of the fuzzy gH-derivative in terms of the derivatives of the endpoints of the level sets. This result extends the result given in [6] (Theorem 5) and it is a characterization of the gH-differentiability for an important class of fuzzy functions.

**Theorem 24** *Let $f : ]a,b[ \to \mathbb{R}_{\mathcal{F}}$ be such that $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$. Suppose that the functions $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are real-valued functions, differentiable w.r.t. $x$, uniformly w.r.t. $\alpha \in [0,1]$. Then the function $f(x)$ is gH-*

*differentiable at a fixed* $x \in ]a,b[$ *if and only if one of the following two cases holds:*
*a)* $(f_\alpha^-)'(x)$ *is increasing,* $(f_\alpha^+)'(x)$ *is decreasing as functions of* $\alpha$, *and* $\left(f_1^-\right)'(x) \le \left(f_1^+\right)'(x)$, *or*
*b)* $(f_\alpha^-)'(x)$ *is decreasing,* $(f_\alpha^+)'(x)$ *is increasing as functions of* $\alpha$, *and* $\left(f_1^+\right)'(x) \le \left(f_1^-\right)'(x)$.
*Also,* $\forall \alpha \in [0,1]$ *we have*

$$\left[f'_{gH}(x)\right]_\alpha = [\min\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}, \max\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}] \tag{10}$$

**PROOF.** Let $f$ be gH-differentiable and assume that $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable. Clearly, gH-differentiability implies LgH differentiability; then we have

$$\left[f'_{gH}(x)\right]_\alpha = [\min\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}, \max\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}].$$

Now suppose that for fixed $x \in [a,b]$, the differences $(f_\alpha^+)'(x) - (f_\alpha^-)'(x)$ change sign at a fixed $\alpha_0 \in (0,1)$. Then $\left[f'_{gH}(x)\right]_{\alpha_0}$ is a singleton and, for all $\alpha$ such that $\alpha_0 \le \alpha \le 1$, also $\left[f'_{gH}(x)\right]_\alpha$ is a singleton because $\left[f'_{gH}(x)\right]_\alpha \subseteq \left[f'_{gH}(x)\right]_{\alpha_0}$; it follows that, for the same values of $\alpha$, $(f_\alpha^+)'(x) - (f_\alpha^-)'(x) = 0$, which is a contradiction with the fact that $(f_\alpha^+)'(x) - (f_\alpha^-)'(x)$ changes sign. We then conclude that $(f_\alpha^+)'(x) - (f_\alpha^-)'(x)$ cannot change sign with respect to $\alpha \in [0,1]$.
To prove our conclusion, we distinguish three cases according to the sign of $\left(f_1^+\right)'(x) - \left(f_1^-\right)'(x)$
- If $\left(f_1^-\right)'(x) < \left(f_1^+\right)'(x)$, then $(f_\alpha^+)'(x) - (f_\alpha^-)'(x) \ge 0$ for every $\alpha \in [0,1]$ and

$$\left[f'_{gH}(x)\right]_\alpha = [\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)];$$

since f is gH-differentiable, the intervals $[(f_\alpha^-)'(x), (f_\alpha^+)'(x)]$ should form a fuzzy number, i.e., for any $\alpha > \beta$, $[(f_\alpha^-)'(x), (f_\alpha^+)'(x)] \subseteq [\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)]$ which shows that $(f_\alpha^-)'(x)$ is increasing and $(f_\alpha^+)'(x)$ is decreasing as a function of $\alpha$.
- If $\left(f_1^-\right)'(x) > \left(f_1^+\right)'(x)$, then $(f_\alpha^+)'(x) - (f_\alpha^-)'(x) \le 0$ for every $\alpha \in [0,1]$ and, in this case,

$$\left[f'_{gH}(x)\right]_\alpha = [\left(f_\alpha^+\right)'(x), \left(f_\alpha^-\right)'(x)]$$

so that $[(f_\alpha^+)'(x), (f_\alpha^-)'(x)] \subseteq [\left(f_\beta^+\right)'(x), \left(f_\beta^-\right)'(x)]$, for any $\alpha > \beta$, which shows that $(f_\alpha^-)'(x)$ is decreasing and $(f_\alpha^+)'(x)$ is increasing as a function of $\alpha$.
- In the third case, we have $\left(f_1^-\right)'(x) = \left(f_1^+\right)'(x)$; if $(f_{gH})'(x) \in \mathbb{R}$ is a crisp number, the conclusion is obvious; if this is not the case, then we may have either $\left(f_0^-\right)'(x) < \left(f_0^+\right)'(x)$ or $\left(f_0^-\right)'(x) > \left(f_0^+\right)'(x)$ and, taking $\alpha_0 =$

$\inf\{\alpha|\,(f_\alpha^-)'(x) = (f_\alpha^+)'(x)\}$, we have correspondingly that $(f_\alpha^-)'(x) \le (f_\alpha^+)'(x)\}$ or $(f_\alpha^-)'(x) \ge (f_\alpha^+)'(x)\}$ for all $\alpha \in (0,1)$, because the differences cannot change sign w.r.t. $\alpha$. We conclude that $(f_\alpha^-)'(x)$ and $(f_\alpha^+)'(x)$ are monotonic w.r.t. $\alpha$.

Reciprocally, let us consider the Banach space $B = \bar{C}[0,1] \times \bar{C}[0,1]$, where $\bar{C}[0,1]$ is the space of left continuous functions on $(0,1]$, right continuous at 0, with the uniform norm. For any fixed $x \in ]a,b[$, the mapping $j_x : \mathbb{R}_{\mathcal{F}} \to B$, defined by

$$j_x(f) = (f^-(x), f^+(x)) = \{(f_\alpha^-(x), f_\alpha^+(x))|\alpha \in [0,1]\},$$

is an isometric embedding. Assuming that, for all $\alpha$, the two functions $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable with respect to $x$, the limits

$$\left(f_\alpha^-\right)'(x) = \lim_{h\to 0} \frac{f_\alpha^-(x+h) - f_\alpha^-(x)}{h}$$
$$\left(f_\alpha^+\right)'(x) = \lim_{h\to 0} \frac{f_\alpha^+(x+h) - f_\alpha^+(x)}{h}$$

exist uniformly for all $\alpha \in [0,1]$. Taking a sequence $h_n \to 0$, we will have

$$\left(f_\alpha^-\right)'(x) = \lim_{n\to \infty} \frac{f_\alpha^-(x+h_n) - f_\alpha^-(x)}{h_n}$$
$$\left(f_\alpha^+\right)'(x) = \lim_{n\to \infty} \frac{f_\alpha^+(x+h_n) - f_\alpha^+(x)}{h_n},$$

i.e., $(f_\alpha^-)'(x), (f_\alpha^+)'(x)$ are uniform limits of sequences of left continuous functions at $\alpha \in (0,1]$, so they are themselves left continuous for $\alpha \in (0,1]$. Similarly the right continuity at 0 can be deduced.

Assuming that, for a fixed $x \in [a,b]$, the function $(f_\alpha^-)'(x)$ is increasing and the function $(f_\alpha^+)'(x)$ is decreasing as functions of $\alpha$, and that $\left(f_1^-\right)'(x) \le \left(f_1^+\right)'(x)$, then also $(f_\alpha^-)'(x) \le (f_\alpha^+)'(x)$ $\forall \alpha \in [0,1]$ and it is easy to see that the pair of functions $(f_\alpha^-)'(x)$, $(f_\alpha^+)'(x)$ fulfill the conditions in proposition 2 and the intervals $[(f_\alpha^-)'(x), (f_\alpha^+)'(x)]$, $\alpha \in [0,1]$ determine a fuzzy number. Now we observe that the following limit uniformly exists

$$\left[\lim_{h\to 0} \frac{f(x+h) \ominus_{gH} f(x)}{h}\right]_\alpha = \left[\lim_{h\to 0} \frac{f_\alpha^-(x+h) - f_\alpha^-(x)}{h}, \lim_{h\to 0} \frac{f_\alpha^+(x+h) - f_\alpha^+(x)}{h}\right]$$
$$= [\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)], \forall \alpha \in [0,1],$$

and it is a fuzzy number, so by Lemma 4 we obtain that $f$ is gH-differentiable. If $(f_\alpha^-)'(x)$ is decreasing, $(f_\alpha^+)'(x)$ is increasing as functions of $\alpha$, and $\left(f_1^+\right)'(x) \le \left(f_1^-\right)'(x)$, then also $(f_\alpha^+)'(x) \le (f_\alpha^-)'(x)$ $\forall \alpha \in [0,1]$ and, by proposition 2, the intervals $[(f_\alpha^+)'(x), (f_\alpha^-)'(x)]$, $\alpha \in [0,1]$ determine a fuzzy number. Observing that the following limit exists uniformly

$$\left[\lim_{h\to 0}\frac{f(x+h)\ominus_{gH} f(x)}{h}\right]_{\alpha} = \left[\lim_{h\to 0}\frac{f_{\alpha}^{+}(x+h)-f_{\alpha}^{+}(x)}{h}, \lim_{h\to 0}\frac{f_{\alpha}^{-}(x+h)-f_{\alpha}^{-}(x)}{h}\right]$$
$$= [\left(f_{\alpha}^{+}\right)'(x), \left(f_{\alpha}^{-}\right)'(x)], \forall\alpha\in[0,1],$$

and it is a fuzzy number, using Lemma 4 again, we obtain that $f$ is gH-differentiable.□

**Remark 25** *It is interesting to observe that conditions a) and b) require the monotonicity of $(f_{\alpha}^{-})'(x)$ and $(f_{\alpha}^{+})'(x)$ with respect to $\alpha$ in $[0,1]$. On the other hand, the monotonicity seems not sufficient, as in fact it is also necessary that $(f_{\alpha}^{+})'(x)$ and $(f_{\alpha}^{-})'(x)$ be left-continuous for $\alpha\in]0,1]$ and right continuous at $\alpha=0$. It follows that the relationship between the (level-wise) LgH-differentiability and the (fuzzy) gH-differentiability is not obvious. On the other hand, we know that $f_{\alpha}^{-}(x)$ and $f_{\alpha}^{+}(x)$ satisfy (for all $x$) the left-continuity for $\alpha\in]0,1]$ and right-continuity at $\alpha=0$. We can formalize the problem in terms of iterated limits as follows. For simplicity, denote with $g_{\alpha}(x)$ one of the two functions $f_{\alpha}^{-}(x)$ or $f_{\alpha}^{+}(x)$ and let $g_{\alpha}'(x)$ be its derivative with respect to $x$. We know that each $f_{\alpha}^{-}(x)$ or $f_{\alpha}^{+}(x)$ is left-continuous for $\alpha\in]0,1]$ (the case of right continuity for $\alpha=0$ is analogous), so is $g_{\alpha}(x)$, i.e. $\lim_{h\uparrow 0} g_{\alpha+h}(x)=g_{\alpha}(x)$. On the other hand, differentiability of $g_{\alpha+h}(x)$ with respect to $x$ means*

$$\lim_{k\to 0}\frac{g_{\alpha+h}(x+k)-g_{\alpha+h}(x)}{k}=g_{\alpha+h}'(x).$$

*Now, it is true that*

$$\begin{aligned} g_{\alpha}'(x) &= \lim_{k\to 0}\frac{g_{\alpha}(x+k)-g_{\alpha}(x)}{k} \\ &= \lim_{k\to 0}\frac{1}{k}\left(\lim_{h\uparrow 0} g_{\alpha+h}(x+k)-\lim_{h\uparrow 0} g_{\alpha+h}(x)\right) \\ &= \lim_{k\to 0}\left(\lim_{h\uparrow 0}\frac{g_{\alpha+h}(x+k)-g_{\alpha+h}(x)}{k}\right) \end{aligned}$$

*and to have $g_{\alpha}'(x)$ left continuous at $\alpha$ we need*

$$\begin{aligned} g_{\alpha}'(x) &= \lim_{h\uparrow 0} g_{\alpha+h}'(x) \\ &= \lim_{h\uparrow 0}\left(\lim_{k\to 0}\frac{g_{\alpha+h}(x+k)-g_{\alpha+h}(x)}{k}\right). \end{aligned}$$

*It follows that left continuity of $g_{\alpha}'(x)$ requires that the following iterated limit equality holds:*

$$\lim_{k\to 0}\left(\lim_{h\uparrow 0}\frac{g_{\alpha+h}(x+k)-g_{\alpha+h}(x)}{k}\right)=\lim_{h\uparrow 0}\left(\lim_{k\to 0}\frac{g_{\alpha+h}(x+k)-g_{\alpha+h}(x)}{k}\right). \quad (11)$$

*From a well known theorem on double and iterated limits, the existence of the double limit* $\lim\limits_{k\to 0, h\uparrow 0} \frac{g_{\alpha+h}(x+k)-g_{\alpha+h}(x)}{k}$ *in the* $(\alpha, x)$ *plane is sufficient, in our case, for (11) to be valid. As we can see from the previous Theorem 24, the existence of derivatives, uniformly for all level sets, is a sufficient condition to solve the problem discussed in the remark.*

According to Theorem 24, for the definition of gH-differentiability when $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are both differentiable, we distinguish two cases, corresponding to (i) and (ii) of (4).

**Definition 26** *Let* $f : [a,b] \longrightarrow \mathbb{R}_\mathcal{F}$ *and* $x_0 \in ]a,b[$ *with* $f_\alpha^-(x)$ *and* $f_\alpha^+(x)$ *both differentiable at* $x_0$*. We say that*
- $f$ *is (i)-gH-differentiable at* $x_0$ *if*

$$\text{(i.)} \qquad \left[f'_{gH}(x_0)\right]_\alpha = [\left(f_\alpha^-\right)'(x_0), \left(f_\alpha^+\right)'(x_0)], \forall\alpha \in [0,1] \tag{12}$$

- $f$ *is (ii)-gH-differentiable at* $x_0$ *if*

$$\text{(ii.)} \qquad \left[f'_{gH}(x_0)\right]_\alpha = [\left(f_\alpha^+\right)'(x_0), \left(f_\alpha^-\right)'(x_0)], \forall\alpha \in [0,1]. \tag{13}$$

It is possible that $f : [a,b] \longrightarrow \mathbb{R}_\mathcal{F}$ is gH-differentiable at $x_0$ and not (i)-gH-differentiable nor (ii)-gH-differentiable, as illustrated by the following example, taken from [34].

**Example 27** *Consider* $f\ :]-1,1[\longrightarrow \mathbb{R}_f$ *defined by the* $\alpha-$*cuts (it is 0-symmetric)*

$$[f(x)]_\alpha = \left[-\frac{1}{(1+|x|)(1+\alpha)}, \frac{1}{(1+|x|)(1+\alpha)}\right] \tag{14}$$

*i.e.* $f_\alpha^-(x) = -\frac{1}{(1+|x|)(1+\alpha)}$ *and* $f_\alpha^+(x) = \frac{1}{(1+|x|)(1+\alpha)}$*. The level sets are as in Fig. 2.*

*For all* $x \neq 0$ *and all* $\alpha \in [0,1]$*, both* $f_\alpha^-$ *and* $f_\alpha^+$ *are differentiable and satisfy conditions of Theorem 24; at the origin* $x = 0$ *the two functions* $f_\alpha^-$ *and* $f_\alpha^+$ *are not differentiable; they are, respectively, left and right differentiable but left derivative and right derivative are different, in fact*

$$(f_\alpha^-)'(x) = \begin{cases} -\frac{1}{(1-x)^2(1+\alpha)} & x < 0 \\ \nexists & x = 0 \\ \frac{1}{(1+x)^2(1+\alpha)} & x > 0 \end{cases} \quad \text{and } (f_\alpha^+)'(x) = \begin{cases} \frac{1}{(1-x)^2(1+\alpha)} & x < 0 \\ \nexists & x = 0 \\ -\frac{1}{(1+x)^2(1+\alpha)} & x > 0 \end{cases}.$$

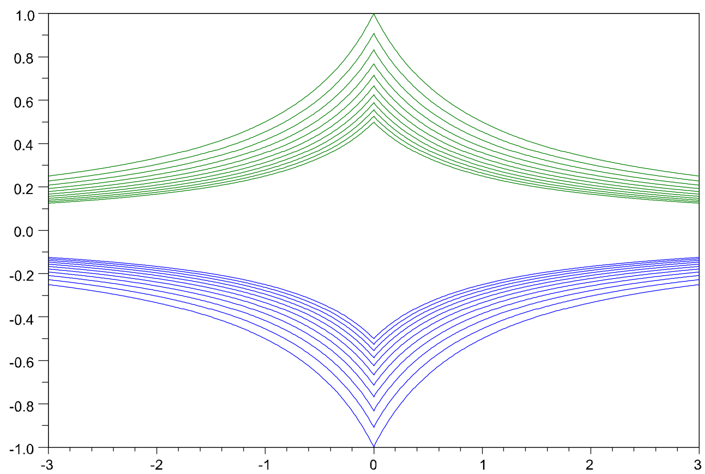


Figure 2. The level sets of the function in (14), $\alpha = 0, 0.1, ..., 1$

*Now, for the gH-difference and $h \neq 0$ we have*

$$\begin{aligned}
&\frac{[f(h) \ominus_{gH} f(0)]_\alpha}{h} = \\
&= \frac{1}{h}\left[-\frac{1}{(1+|h|)(1+\alpha)}, \frac{1}{(1+|h|)(1+\alpha)}\right] \ominus_{gH} \left[-\frac{1}{(1+\alpha)}, \frac{1}{(1+\alpha)}\right] \\
&= \frac{1}{h(1+\alpha)}\left[\min\left\{\frac{|h|}{(1+|h|)}, \frac{-|h|}{(1+|h|)}\right\}, \max\{idem\}\right] \\
&= \frac{1}{h(1+\alpha)}\left[\frac{-|h|}{(1+|h|)}, \frac{|h|}{(1+|h|)}\right] \\
&= \frac{1}{(1+\alpha)}\left[\frac{-1}{(1+|h|)}, \frac{1}{(1+|h|)}\right]
\end{aligned}$$

*It follows that the limit exists*

$$f'_{gH}(0) = \lim_{h \longrightarrow 0} \frac{[f(h) \ominus_{gH} f(0)]_\alpha}{h} = \left[\frac{-1}{(1+\alpha)}, \frac{1}{(1+\alpha)}\right]$$

*and $f$ is gH-differentiable at $x = 0$ but $f_\alpha^-$ and $f_\alpha^+$ are not differentiable at $x = 0$ for all $\alpha$; observe that $f$ is (i)-gH-differentiable if $x < 0$ and is (ii)-gH-differentiable if $x > 0$. (see Fig. 3).*

**Remark 28** *It is easy to see that the gH-differentiability concept introduced above is more general than the GH-differentiability in Definition 18. Indeed,*

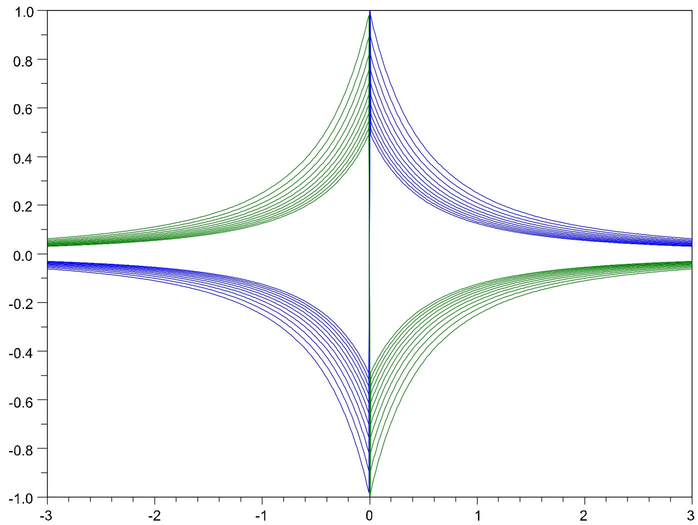


Figure 3. The level sets of the gH-derivative of the function in (14)

*consider the function* $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$,

$$f(x) = \begin{cases} (-1,0,1) \cdot (1 - x^2 \sin \frac{1}{x}) \text{ if } x \neq 0 \\ (-1,0,1), \text{ otherwise} \end{cases} .$$

*It is easy to check by Theorem 24 that $f$ is gH-differentiable at $x = 0$ and $f'_{gH}(0) = 0$. Also, we observe that $f$ is not GH-differentiable since there does not exist $\delta > 0$ such that $f(h) \ominus_H f(0)$ or $f(-h) \ominus_H f(0)$ exist for all $h \in (0, \delta)$.*

The following properties are obtained from Theorem 24.

**Proposition 29** *If $f : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ is gH-differentiable (or right or left gH-differentiable) at $x_0 \in [a, b]$ then it is level-wise continuous (or right or left level-wise continuous) at $x_0$.*

**PROOF.** If $f : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ is gH-differentiable at $x_0$ and $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$ let $[f'(x_0)]_\alpha = [g_\alpha^-(x_0), g_\alpha^+(x_0)]$ where

$$g_\alpha^-(x_0) = \lim_{h \to 0} \min \left\{ \frac{f_\alpha^-(x_0 + h) - f_\alpha^-(x_0)}{h}, \frac{f_\alpha^+(x_0 + h) - f_\alpha^+(x_0)}{h} \right\}$$
$$g_\alpha^+(x) = \lim_{h \to 0} \max \left\{ \frac{f_\alpha^-(x_0 + h) - f_\alpha^-(x_0)}{h}, \frac{f_\alpha^+(x_0 + h) - f_\alpha^+(x_0)}{h} \right\}.$$

Then for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all values of $h$ with

$|h| < \delta_\varepsilon$ we have (simultaneously)

$$g_\alpha^-(x_0) - \varepsilon < \min\left\{\frac{f_\alpha^-(x_0+h) - f_\alpha^-(x_0)}{h}, \frac{f_\alpha^+(x_0+h) - f_\alpha^+(x_0)}{h}\right\} \quad (15)$$
$$< g_\alpha^-(x_0) + \varepsilon$$

and

$$g_\alpha^+(x_0) - \varepsilon < \max\left\{\frac{f_\alpha^-(x_0+h) - f_\alpha^-(x_0)}{h}, \frac{f_\alpha^+(x_0+h) - f_\alpha^+(x_0)}{h}\right\} \quad (16)$$
$$< g_\alpha^+(x_0) + \varepsilon.$$

Suppose $f_\alpha^-(x)$or $f_\alpha^+(x)$ are not continuous w.r.t. $x$ for some $\alpha \in [0,1]$; then $\underset{h\to 0}{lim}(f_\alpha^-(x_0+h) - f_\alpha^-(x_0)) \neq 0$ or $\underset{h\to 0}{lim}(f_\alpha^+(x_0+h) - f_\alpha^+(x_0)) \neq 0$ and so one of the two functions $\frac{f_\alpha^-(x_0+h) - f_\alpha^-(x_0)}{h}$ or $\frac{f_\alpha^+(x_0+h) - f_\alpha^+(x_0)}{h}$ is unbounded for small $|h|$ and this contradicts inequalities 15 or 16; so $f_\alpha^-(x)$or $f_\alpha^+(x)$ are continuous for all $\alpha \in [0,1]$ and $f$ is level-wise continuous.□

**Proposition 30** *The (i)-gH-derivative and (ii)-gH-derivative are additive operators, i.e., if $f$ and $g$ are both (i)-gH-differentiable or both (ii)-gH-differentiable then*
*(i)* $(f+g)'_{(i)-gH} = f'_{(i)-gH} + g'_{(i)-gH}$,
*(ii)* $(f+g)'_{(ii)-gH} = f'_{(ii)-gH} + g'_{(ii)-gH}$.

**PROOF.** Consider (i) and suppose that $f$ and $g$ are both (i)-gH-differentiable; then, for every $\alpha \in [0,1]$ we have, $[f']_\alpha = [(f_\alpha^-)', (f_\alpha^+)']$ and $[g']_\alpha = [(g_\alpha^-)', (g_\alpha^+)']$ with $(f_\alpha^-)' \leq (f_\alpha^+)'$ and $(g_\alpha^-)' \leq (g_\alpha^+)'$; it follows that

$$\begin{aligned}[(f+g)']_\alpha &= [(f_\alpha^- + g_\alpha^-), (f_\alpha^+ + g_\alpha^+)]' \\ &= [(f_\alpha^- + g_\alpha^-)', (f_\alpha^+ + g_\alpha^+)'] \\ &= [(f_\alpha^-)' + (g_\alpha^-)', (f_\alpha^+)' + (g_\alpha^+)'] \\ &= [(f_\alpha^-)', (f_\alpha^+)'] + [(g_\alpha^-)', (g_\alpha^+)'] \\ &= [f']_\alpha + [g']_\alpha;\end{aligned}$$

the case of $f$ and $g$ both (ii)-gH-differentiable is similar.□

**Remark 31** *From Proposition 30, it follows that (i)-gH-derivative and (ii)-gH-derivative are semi-linear operators (i.e. additive and positive homogeneous). They are not linear in general since we have $(kf_{gH})'_{(i)-gH} = k(f_{gH})'_{(ii)-gH}$, if $k < 0$.*

## 4 Generalized fuzzy differentiability (g-differentiability)

Based on the g-difference introduced in Definition 7, we propose the following g-differentiability concept, that further extends the gH-differentiability.

**Definition 32** *Let $x_0 \in ]a,b[$ and $h$ be such that $x_0 + h \in ]a,b[$, then the g-derivative of a function $f : ]a,b[ \to \mathbb{R}_{\mathcal{F}}$ at $x_0$ is defined as*

$$f'_g(x_0) = \lim_{h \to 0} \frac{1}{h}[f(x_0+h) \ominus_g f(x_0)]. \tag{17}$$

*If $f'_g(x_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (17) exists, we say that $f$ is generalized differentiable (g-differentiable for short) at $x_0$.*

**Remark 33** *Let us observe that the g-derivative is the most general among the previous definitions. Indeed, $f(x_0+h) \ominus_g f(x_0) = f(x_0+h) \ominus_{gH} f(x_0)$ whenever the gH-difference on the right exists. An example of a function that is g-differentiable and not gH-differentiable will be given later in Example 39.*

In the following theorem we prove that the g-derivative is well defined for a large class of fuzzy valued functions. Also we prove a characterization and a practical formula for the g-derivative.

**Theorem 34** *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be such that $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$. If $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable real-valued functions with respect to $x$, uniformly for $\alpha \in [0,1]$, then $f(x)$ is g-differentiable and we have*

$$\left[f'_g(x)\right]_\alpha = \left[\inf_{\beta \ge \alpha} \min\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}, \sup_{\beta \ge \alpha} \max\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}\right]. \tag{18}$$

**PROOF.** By Proposition 8 we have

$$\begin{aligned} \frac{1}{h}[f(x+h) \ominus_g f(x)]_\alpha = \frac{1}{h}[&\inf_{\beta \ge \alpha} \min\{f(x+h)_\beta^- - f(x)_\beta^-, f(x+h)_\beta^+ - f(x)_\beta^+\}, \\ &\sup_{\beta \ge \alpha} \max\{f(x+h)_\beta^- - f(x)_\beta^-, f(x+h)_\beta^+ - f(x)_\beta^+\}]. \end{aligned}$$

Since $f_\alpha^-(x), f_\alpha^+(x)$ are differentiable we obtain

$$\begin{aligned} &\lim_{h \to 0} \frac{1}{h}[f(x+h) \ominus_g f(x)]_\alpha \\ &= \left[\inf_{\beta \ge \alpha} \min\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}, \sup_{\beta \ge \alpha} \max\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}\right] \end{aligned}$$

for any $\alpha \in [0,1]$. Also, let us observe that if $f_\alpha^-$, $f_\alpha^+$ are left continuous with respect to $\alpha \in (0,1]$ and right continuous at 0, considering a sequence $h_n \to 0$, the functions

$$\frac{f_\alpha^-(x+h_n) - f_\alpha^-(x)}{h_n}, \frac{f_\alpha^+(x+h_n) - f_\alpha^+(x)}{h_n}$$

are left continuous at $\alpha \in (0,1]$ and right continuous at 0. Also, the functions

$$\inf_{\beta \geq \alpha} \min \left\{ \frac{f_\beta^-(x+h_n) - f_\beta^-(x)}{h_n}, \frac{f_\beta^+(x+h_n) - f_\beta^+(x)}{h_n} \right\}$$

and

$$\sup_{\beta \geq \alpha} \max \left\{ \frac{f_\beta^-(x+h_n) - f_\beta^-(x)}{h_n}, \frac{f_\beta^+(x+h_n) - f_\beta^+(x)}{h_n} \right\}$$

fulfill the same properties. Then it follows that

$$\inf_{\beta \geq \alpha} \min\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}, \sup_{\beta \geq \alpha} \max\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}$$

are left continuous for $\alpha \in (0,1]$ and right continuous at 0. It is easy to check that $\inf_{\beta \geq \alpha} \min\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}$ is increasing w.r.t. $\alpha \in [0,1]$ and $\sup_{\beta \geq \alpha} \max\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}$ is decreasing w.r.t. $\alpha \in [0,1]$; by Proposition 2 they define a fuzzy number. As a conclusion, the level sets $\left[f_g'(x)\right]_\alpha$ define a fuzzy number, and so, by Lemma 4, the derivative $f_g'(x)$ exists in the sense of the g-derivative.□

The next result provides an expression for the g-derivative and its connection to the interval gH-derivative of the level sets. According to the result that the existence of the gH-differences for all level sets is sufficient to define the g-difference, the uniform LgH-differentiability is sufficient for the g-differentiability.

**Theorem 35** *Let $f : ]a,b[ \to \mathbb{R}_\mathcal{F}$ be uniformly LgH-differentiable at $x_0$. Then $f$ is g-differentiable at $x_0$ and, for any $\alpha \in [0,1]$,*

$$[f_g'(x_0)]_\alpha = cl \left( \bigcup_{\beta \geq \alpha} f_{LgH}'(x_0)_\beta \right).$$

**PROOF.** Let $x_0 \in ]a,b[$ and $h$ be such that $x_0 + h \in ]a,b[$, and denote, for $\alpha \in [0,1]$, the intervals

$$\begin{aligned}
\lambda_\alpha(h) &= \frac{1}{h}\left([f(x_0+h)]_\alpha \ominus_{gH} [f(x_0)]_\alpha\right), \\
l_\alpha &= \lim_{h\to 0} \lambda_\alpha(h) = f'_{LgH}(x_0)_\alpha, \\
\Lambda_\alpha(h) &= cl\left(\bigcup_{\beta\geq\alpha} \lambda_\beta(h)\right) = \frac{1}{h}\left([f(x_0+h)]_\alpha \ominus_g [f(x_0)]_\alpha\right), \\
L_\alpha &= cl\left(\bigcup_{\beta\geq\alpha} l_\beta\right).
\end{aligned}$$

Let $\Lambda(h)$ and $L$ be the fuzzy numbers having the intervals $\{\Lambda_\alpha(h)|\alpha \in [0,1]\}$ and $\{L_\alpha|\alpha \in [0,1]\}$ as level-cuts, respectively. The fuzzy numbers $\Lambda(h)$ and $L$ are well defined. Indeed, as it was shown in the previous Theorem 34 the level sets $\{\Lambda_\alpha(h)|\alpha \in [0,1]\}$ and $\{L_\alpha|\alpha \in [0,1]\}$ verify the conditions in Proposition 2. We will show that the following limit exists

$$\lim_{h\to 0} \Lambda(h) = L$$

and, consequently, that the g-derivative of $f$ at $x_0$ exists and equals $L$.
Denoting the intervals $\Lambda_\alpha(h) = [\Lambda^-_\alpha(h), \Lambda^+_\alpha(h)]$ and $L_\alpha = [L^-_\alpha, L^+_\alpha]$ we have

$$\begin{aligned}
\Lambda^-_\alpha(h) &= \inf_{\beta\geq\alpha} \lambda^-_\beta(h),\ \Lambda^+_\alpha(h) = \sup_{\beta\geq\alpha} \lambda^+_\beta(h) \\
L^-_\alpha &= \inf_{\beta\geq\alpha} l^-_\beta,\ L^+_\alpha = \sup_{\beta\geq\alpha} l^+_\beta
\end{aligned}$$

and, from the uniform LgH-differentiability of $f$, we have that for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\begin{aligned}
|h| < \delta_\varepsilon \quad &\Longrightarrow \quad l^-_\alpha - \frac{\varepsilon}{4} < \lambda^-_\alpha(h) < l^-_\alpha + \frac{\varepsilon}{4} \quad \textit{for all } \alpha \in [0,1] \\
|h| < \delta_\varepsilon \quad &\Longrightarrow \quad l^+_\alpha - \frac{\varepsilon}{4} < \lambda^+_\alpha(h) < l^+_\alpha + \frac{\varepsilon}{4} \quad \textit{for all } \alpha \in [0,1].
\end{aligned}$$

On the other hand, from the definition of inf and sup, we also have that, for arbitrary $\varepsilon > 0$ and for all $\alpha$ and all $h$, there exist $\beta_1 \geq \alpha$, $\beta_2 \geq \alpha$, $\beta_3 \geq \alpha$ and $\beta_4 \geq \alpha$, such that $\Lambda^-_\alpha(h) > \lambda^-_{\beta_1}(h) - \frac{\varepsilon}{4}$, $L^-_\alpha > l^-_{\beta_2} - \frac{\varepsilon}{4}$, $\Lambda^+_\alpha(h) < \lambda^+_{\beta_3}(h) + \frac{\varepsilon}{4}$, $L^+_\alpha < l^+_{\beta_4} + \frac{\varepsilon}{4}$.
It follows that, for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, if $|h| < \delta_\varepsilon$ and for

all $\alpha \in [0,1]$,

$$\begin{aligned}
\Lambda_\alpha^-(h) > \lambda_{\beta_1}^-(h) - \frac{\varepsilon}{4} > l_{\beta_1}^- - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} \geq L_\alpha^- - \frac{\varepsilon}{2}, \\
L_\alpha^- > l_{\beta_2}^- - \frac{\varepsilon}{4} > \lambda_{\beta_2}^-(h) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} \geq \Lambda_\alpha^-(h) - \frac{\varepsilon}{2}, \\
\Lambda_\alpha^+(h) < \lambda_{\beta_3}^+(h) + \frac{\varepsilon}{4} < l_{\beta_3}^+ + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq L_\alpha^+ + \frac{\varepsilon}{2}, \\
L_\alpha^+ < l_{\beta_4}^+ + \frac{\varepsilon}{4} < \lambda_{\beta_4}^+(h) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \Lambda_\alpha^+(h) + \frac{\varepsilon}{2}
\end{aligned}$$

and, consequently, for the same values of $h$,

$$\begin{aligned}
\|\Lambda(h) \ominus_g L\| &= \sup_{\alpha \in [0,1]} \|\Lambda_\alpha(h) \ominus_{gH} L_\alpha\|_* \\
&= \sup_{\alpha \in [0,1]} \max\{\left|\Lambda_\alpha^-(h) - L_\alpha^-\right|, \left|\Lambda_\alpha^+(h) - L_\alpha^+\right|\} \\
&\leq \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

It follows that $\lim_{h\to 0} \Lambda(h) = L$.□

The next Theorem shows a minimality property for the g-derivative.

**Theorem 36** *Let $f$ be uniformly LgH-differentiable. Then $f'_g(x)$, for a fixed $x$, is the smallest fuzzy number $w \in \mathbb{R}_\mathcal{F}$ (in the sense of fuzzy inclusion) such that $f'_{LgH}(x)_\alpha \subseteq [w]_\alpha$ for all $\alpha \in [0,1]$.*

**PROOF.** The result is similar to the minimality property for the g-difference. For the proof let us observe first that from Theorem 35, we have $([f(x)]_\alpha)'_{gH} \subseteq [f'_g(x)]_\alpha, \forall \alpha \in [0,1]$. Let us consider now $w \in R_\mathcal{F}$ such that $([f(x)]_\alpha)'_{gH} \subseteq [w]_\alpha$. Then for $\beta \geq \alpha$ we have

$$([f(x)]_\beta)'_{gH} \subseteq [w]_\beta \subseteq [w]_\alpha,$$

and we get

$$cl \bigcup_{\beta \geq \alpha} ([f(x)]_\beta)'_{gH} \subseteq [w]_\alpha,$$

i.e., $[f'_{LgH}(x)]_\alpha \subseteq [w]_\alpha$.□

From the example after Definition 26, the converse of Theorem 34 is not valid; in fact, we may have $f_\alpha^-(x)$, $f_\alpha^+(x)$ not necessarily differentiable in $x$ for all $\alpha$.

The most important cases of differentiability, from an application point of view, are those in (i.) and (ii.) in Definition 26, since these cases are easily characterized using real-valued functions and used in the study of fuzzy differential equations ([4]).

It is an interesting, non-trivial problem to see how the switch between the two cases (i.) and (ii.) in Definition 26 can occur. We will assume, for the rest of this section, that $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable w.r.t. $x$ for all $\alpha$.

**Definition 37** *We say that a point $x \in ]a,b[$ is an l-critical point of $f$ if it is a critical point for the length function $len([f(x)]_\alpha) = f_\alpha^+(x) - f_\alpha^-(x)$ for some $\alpha \in [0,1]$.*

If $f$ is gH-differentiable everywhere in its domain the switch at every level should happen at the same time, i.e, $\frac{d}{dx} len([f(x)]_\alpha) = (f_\alpha^+(x) - f_\alpha^-(x))' = 0$ at the same point $x$ for all $\alpha \in [0,1]$.

**Definition 38** *We say that a point $x_0 \in ]a,b[$ is a switching point for the gH-differentiability of $f$, if in any neighborhood $V$ of $x_0$ there exist points $x_1 < x_0 < x_2$ such that*
*type-I switch) at $x_1$ (12) holds while (13) does not hold and at $x_2$ (13) holds and (12) does not hold, or*
*type-II switch) at $x_1$ (13) holds while (12) does not hold and at $x_2$ (12) holds and (13) does not hold.*

Obviously, any switching point is also an *l-critical* point. Indeed, if $x_0$ is a switching point then $[(f_\alpha^-)'(x_0), (f_\alpha^+)'(x_0)] = [(f_\alpha^+)'(x_0), (f_\alpha^-)'(x_0)]$ and so $\left(f_0^+\right)'(x_0) = \left(f_0^-\right)'(x_0)$ and $len(f(x_0))' = 0$. Clearly, not all *l-critical* points are also switching points.

If we consider the g-derivative, the switching phenomenon is much more complex as it is shown in the following example.

**Example 39** *Let us consider the function $f(x)$ given level-wise for $x \in [0,1]$ as*

$$f_\alpha^-(x) = xe^{-x} + \alpha^2\left(e^{-x^2} + x - xe^{-x}\right)$$
$$f_\alpha^+(x) = e^{-x^2} + x + (1-\alpha^2)\left(e^{x} - x + e^{-x^2}\right)$$

*and pictured in Figure 4.*

*It is easy to see that it is g-differentiable but it is not gH-differentiable. The derivatives of $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are in Figure* **??** *and we see that it is (ii)-gH-differentiable on the sub-interval $[0, x_1]$ where $x_1 \approx 0.61$, is (i)-gH-differentiable on $(x_2, 1]$ where $x_2 \approx 0.71$ and is g-differentiable on the sub interval $[x_1, x_2]$. The g-derivative is represented in Figure 5.*

*We can see that the transition between (ii)-gH and (i)-gH differentiability is not simply at a single point. Instead we have a region where the transition happens.*

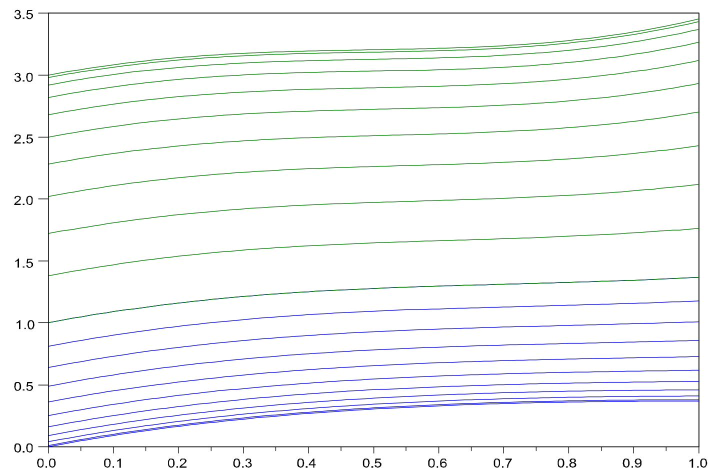


Figure 4. Level sets of the function defined in Example 39.

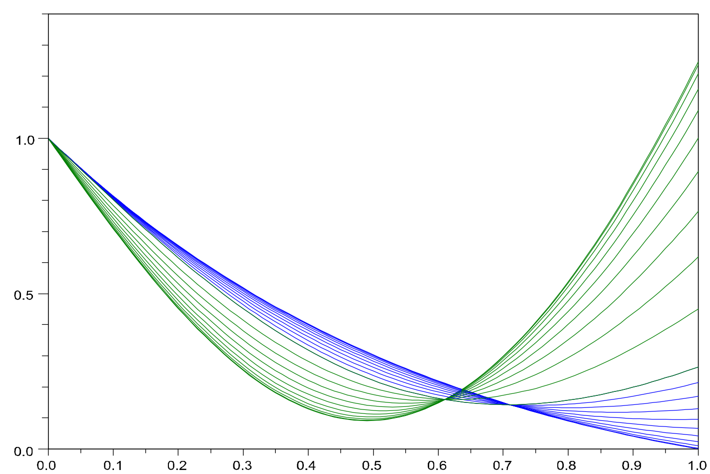


Figure 5. Example of a function that is g-differentiable but not gH-differentiable

**Definition 40** *We say that an interval* $S = [x_1, x_2] \subseteq ]a, b[$, *where* $f$ *is g-differentiable, is a transitional region for the differentiability of* $f$, *if in any neighborhood* $(x_1 - \delta, x_2 + \delta) \supset S$, $\delta > 0$, *there exist points* $x_1 - \delta < \xi_1 < x_1$ *and* $x_2 < \xi_2 < x_2 + \delta$ *such that*
*type-I region) at* $\xi_1$ *(12) holds while (13) does not hold and at* $\xi_2$ *(13) holds and (12) does not hold, or*
*type-II region) at* $\xi_1$ *(13) holds while (12) does not hold and at* $\xi_2$ *(12) holds and (13) does not hold.*

Similar to [39] we have a strong connection between the concepts of GH-differentiability, gH-differentiability and g-differentiability. The new concept of g-differentiability is more general than the other two concepts, but in practical investigations we may use gH- or GH- differentiabilities depending on the given application.

**Theorem 41** *Let $f:]a,b[\to \mathbb{R}_{\mathcal{F}}$ be a function $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$. The following statements are equivalent:*
*(1) $f$ is GH-differentiable,*
*(2) $f$ is gH-differentiable and the set of switching points is finite,*
*(3) $f$ is g-differentiable and the transitional regions are singletons and there are finitely many of them.*

**PROOF.** The proof of the equivalence between (1) and (2) is similar to the proof of Theorem 28 in [39]. It is easy to see that (2) implies (3). To prove that (3) implies (1) we can observe that except for the transitional regions, the cases (i.) and (ii.) in Definition 26 are satisfied. The set of transitional regions coincides with the set of switching points and these are now singletons. Since there are finitely many such switch-points we obtain that the function is GH-differentiable and the proof is complete.□

We end this section by considering the gH-derivative in terms of the CPS (crisp+profile+symmetric) decomposition of fuzzy numbers, introduced in [41] and [38]. Given a fuzzy-valued function $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ with level-cuts $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$, the CPS representation decomposes $f(x)$ in terms of the following three additive components

$$f(x) = \widehat{f}(x) + \widetilde{f}(x) + \overline{f}(x)$$

where $\widehat{f}(x) = [\widehat{f}^-(x), \widehat{f}^+(x)]$ is a (crisp) interval-valued function, $\widetilde{f}(x) = \{\widetilde{f}_\alpha(x)|\alpha\in[0,1]\}$ is a family of real valued (profile) functions $\widetilde{f}_\alpha : [a,b]\to\mathbb{R}$ and $\overline{f}(x)$ is a fuzzy valued function $\overline{f}:[a,b]\to\mathbb{R}_{\mathcal{F}}$ of 0-symmetric type $[\overline{f}(x)]_\alpha = [-\overline{f}_\alpha(x), \overline{f}_\alpha(x)]$; the three components are defined as follows

$$\begin{aligned}
\widehat{f}(x) &= [f(x)]_1, \text{ i.e., } \widehat{f}^-(x) = f_1^-(x) \text{ and } \widehat{f}^+(x) = f_1^+(x)\\
\widetilde{f}_\alpha(x) &= \frac{f_\alpha^-(x)+f_\alpha^+(x)}{2} - \frac{f_1^-(x)+f_1^+(x)}{2} \text{ for all } \alpha\in[0,1]\\
\overline{f}_\alpha(x) &= \frac{f_\alpha^+(x)-f_1^+(x)}{2} - \frac{f_\alpha^-(x)-f_1^-(x)}{2} \geq 0 \text{ for all } \alpha\in[0,1]
\end{aligned}$$

and are such that

$$[f(x)]_\alpha = [\widehat{f}^-(x), \widehat{f}^+(x)] + \widetilde{f}_\alpha(x) + [-\overline{f}_\alpha(x), \overline{f}_\alpha(x)], \text{ i.e.,} \tag{19}$$
$$f_\alpha^-(x) = \widehat{f}^-(x) + \widetilde{f}_\alpha(x) - \overline{f}_\alpha(x)$$
$$f_\alpha^+(x) = \widehat{f}^+(x) + \widetilde{f}_\alpha(x) + \overline{f}_\alpha(x).$$

Equations (19) define the CPS decomposition of $f(x) \in \mathbb{R}_\mathcal{F}$; for its properties we refer to [41] and [38]. Consider that $f(x)$ is a symmetric fuzzy number if and only if $\widetilde{f}_\alpha(x) = 0$ for all $\alpha \in [0,1]$ (the profile is identically zero). We call $\widehat{f}(x) + \overline{f}(x)$ the symmetric part of $f(x)$.

Assume that the lower and the upper functions $f_\alpha^-$ and $f_\alpha^+$ are differentiable w.r.t. $x$ for all $\alpha$; then also $\widehat{f}^-$ and $\widehat{f}^+$ are differentiable w.r.t. $x$; $\widetilde{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable w.r.t. $x$ for all $\alpha$. Obviously

$$(f_\alpha^-)'(x) = (\widehat{f}^-)'(x) + (\widetilde{f}_\alpha)'(x) - (\overline{f}_\alpha)'(x)$$
$$(f_\alpha^+)'(x) = (\widehat{f}^+)'(x) + (\widetilde{f}_\alpha)'(x) + (\overline{f}_\alpha)'(x)$$

and the level cuts of the gH-derivative of $f$ are given by

$$[f'_{gH}]_\alpha = (\widetilde{f}_\alpha)' + [\min\{(\widehat{f}^-)' - (\overline{f}_\alpha)', (\widehat{f}^+)' + (\overline{f}_\alpha)'\}, \max\{idem\}]. \tag{20}$$

From (20) we deduce some interesting facts.

First, if $f$ is (i)-gH-differentiable or (ii)-gH-differentiable, the form of differentiability is decided by the derivative of the symmetric part $\widehat{f}(x) + \overline{f}(x)$.

Second, the two cases of Theorem 24 can be rewritten in terms of the components. In fact, $(f_\alpha^-)'(x)$ is increasing (or decreasing, respectively) w.r.t. $\alpha$ if and only if $\alpha < \beta$ implies $(\widetilde{f}_\alpha)'(x) - (\overline{f}_\alpha)'(x) \leq (\widetilde{f}_\beta)'(x) - (\overline{f}_\beta)'(x)$ (or $(\widetilde{f}_\alpha)'(x) - (\overline{f}_\alpha)'(x) \geq (\widetilde{f}_\beta)'(x) - (\overline{f}_\beta)'(x)$, respectively); $(f_\alpha^+)'(x)$ is increasing (or decreasing, respectively) w.r.t. $\alpha$ if and only if $\alpha < \beta$ implies $(\widetilde{f}_\alpha)'(x) + (\overline{f}_\alpha)'(x) \leq (\widetilde{f}_\beta)'(x) + (\overline{f}_\beta)'(x)$ (or $(\widetilde{f}_\alpha)'(x) + (\overline{f}_\alpha)'(x) \geq (\widetilde{f}_\beta)'(x) + (\overline{f}_\beta)'(x)$, respectively); then, the two cases in Theorem 24 become:

a) $\widehat{f}$ is (i)-gH-differentiable and, for $\alpha < \beta$, we have $\left|(\widetilde{f}_\alpha)'(x) - (\widetilde{f}_\beta)'(x)\right| \leq (\overline{f}_\alpha)'(x) - (\overline{f}_\beta)'(x)$; a necessary condition is that $(\overline{f}_\alpha)'$ is decreasing w.r.t. $\alpha$;

b) $\widehat{f}$ is (ii)-gH-differentiable and, for $\alpha < \beta$, we have $\left|(\widetilde{f}_\alpha)'(x) - (\widetilde{f}_\beta)'(x)\right| \leq (\overline{f}_\beta)'(x) - (\overline{f}_\alpha)'(x)$; a necessary condition is that $(\overline{f}_\alpha)'$ is increasing w.r.t. $\alpha$.

A third interesting situation is when functions $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable w.r.t. $x$ and w.r.t. $\alpha$, and the mixed second order derivatives $\frac{\partial^2 f_\alpha^-(x)}{\partial x \partial \alpha}$,

$\frac{\partial^2 f_\alpha^+(x)}{\partial x \partial \alpha}$ exist. It follows that the monotonicity conditions, according to Theorem 24, are

$$\frac{\partial^2 f_\alpha^-(x)}{\partial x \partial \alpha} \geq 0 \text{ and } \frac{\partial^2 f_\alpha^+(x)}{\partial x \partial \alpha} \leq 0$$

OR

$$\frac{\partial^2 f_\alpha^-(x)}{\partial x \partial \alpha} \leq 0 \text{ and } \frac{\partial^2 f_\alpha^+(x)}{\partial x \partial \alpha} \geq 0.$$

In terms of the CPS decomposition, we obtain (consider that $\widehat{f}^-(x)$ and $\widehat{f}^+(x)$ do not depend on $\alpha$)

$$\frac{\partial^2 \widetilde{f}_\alpha(x)}{\partial x \partial \alpha} - \frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha} \geq 0 \text{ and } \frac{\partial^2 \widetilde{f}_\alpha(x)}{\partial x \partial \alpha} + \frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha} \leq 0, \forall \alpha$$

OR

$$\frac{\partial^2 \widetilde{f}_\alpha(x)}{\partial x \partial \alpha} - \frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha} \leq 0 \text{ and } \frac{\partial^2 \widetilde{f}_\alpha(x)}{\partial x \partial \alpha} + \frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha} \geq 0, \forall \alpha$$

i.e.,

$$\frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha} \leq 0 \text{ and } \left| \frac{\partial^2 \widetilde{f}_\alpha(x)}{\partial x \partial \alpha} \right| \leq -\frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha}$$

OR

$$\frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha} \geq 0 \text{ and } \left| \frac{\partial^2 \widetilde{f}_\alpha(x)}{\partial x \partial \alpha} \right| \leq \frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha}.$$

**Remark 42** *For a symmetric fuzzy function, the monotonicity conditions for gH-differentiability are simplified; in this case, $\widetilde{f}_\alpha(x) = 0$ for all $\alpha$ and the monotonicity of $(\overline{f}_\alpha)'(x)$ w.r.t. $\alpha$ is sufficient: if*
*a) $\widehat{f}$ is (i)-gH-differentiable and $(\overline{f}_\alpha)'(x)$ is decreasing w.r.t. $\alpha$ (eventually $\frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha} \leq 0 \; \forall \alpha$);*
*or*
*b) $\widehat{f}$ is (ii)-gH-differentiable and $(\overline{f}_\alpha)'(x)$ is increasing w.r.t. $\alpha$ (eventually $\frac{\partial^2 \overline{f}_\alpha(x)}{\partial x \partial \alpha} \geq 0 \; \forall \alpha$);*
*then $f$ is gH-differentiable.*
*In particular, if $f(x)$ is a symmetric fuzzy number with $f_1^-(x) = f_1^+(x)$, then $f$ is gH-differentiable if and only if $(\overline{f}_\alpha)'(x)$ is monotonic w.r.t. $\alpha$.*

## 5 gH-derivative and the integral

In this section we examine the relations between gH-differentiability and the integral of fuzzy valued functions. A strongly measurable and integrably bounded fuzzy-valued function is called integrable [12]. The fuzzy Aumann integral of $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ is defined level-wise by

$$\left[\int_a^b f(x)\,dx\right]_\alpha=\int_a^b [f(x)]_\alpha\,dx,\ \ \alpha\in[0,1].$$

**Theorem 43** *Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ be continuous with $[f(x)]_\alpha=[f_\alpha^-(x),f_\alpha^+(x)]$. Then*

*(i) the function $F(x)=\int\limits_a^x f(t)dt$ is gH-differentiable and $F'_{gH}(x)=f(x)$,*

*(ii) the function $G(x)=\int\limits_x^b f(t)dt$ is gH-differentiable and $G'_{gH}(x)=-f(x)$.*

**PROOF.** We have $[F(x)]_\alpha=\left[\int\limits_a^x f(t)dt\right]_\alpha=[F_\alpha^-(x),F_\alpha^+(x)]$ and $[G(x)]_\alpha=\left[\int\limits_x^b f(t)dt\right]_\alpha=[G_\alpha^-(x),G_\alpha^+(x)]$. Then

$$\begin{array}{l}(F_\alpha^-)'(x_0)=\min\{f_\alpha^-(x_0),f_\alpha^+(x_0)\}=f_\alpha^-(x_0)\\(F_\alpha^+)'(x_0)=\max\{f_\alpha^-(x_0),f_\alpha^+(x_0)\}=f_\alpha^+(x_0)\end{array}$$

and

$$\begin{array}{l}(G^-)'_\alpha(x_0)=\min\{-f_\alpha^-(x_0),-f_\alpha^+(x_0)\}=-f_\alpha^+(x_0)\\(G^+)'_\alpha(x_0)=\max\{-f_\alpha^-(x_0),-f_\alpha^+(x_0)\}=-f_\alpha^-(x_0).\Box\end{array}$$

**Proposition 44** *If $f$ is GH-differentiable with no switching point in the interval $[a,b]$ then we have*

$$\int_a^b f'_{gH}(x)dx=f(b)\ominus_{gH}f(a).$$

**PROOF.** If there is no switching point in the interval $[a,b]$ then $f$ is (i) or (ii) differentiable as in Definition 26. Let us suppose for example that $f$ is (ii)-gH-differentiable (the proof for the (i)-gH-differentiability case being similar). We have

$$\begin{aligned}\left[\int_a^b f'_{gH}(x)dx\right]_\alpha&=\int_a^b[\left(f_\alpha^+\right)'(x),\left(f_\alpha^-\right)'(x)]dx\\&=\left[f_\alpha^+(b)-f_\alpha^+(a),f_\alpha^-(b)-f_\alpha^-(a)\right]\\&=f(b)\ominus_{gH}f(a).\Box\end{aligned}$$

**Theorem 45** *Let us suppose that function $f$ is gH-differentiable with $n$ switching points at $c_i$, $i = 1, 2, ..., n$, $a = c_0 < c_1 < c_2 < ... < c_n < c_{n+1} = b$ and exactly at these points. Then we have*

$$f(b) \ominus_{gH} f(a) = \sum_{i=1}^{n} \left[ \int_{c_{i-1}}^{c_i} f'_{gH}(x)dx \ominus_{gH} (-1) \int_{c_i}^{c_{i+1}} f'_{gH}(x)dx \right]. \qquad (21)$$

*Also,*

$$\int_a^b f'_{gH}(x)dx = \sum_{i=1}^{n+1} \left( f(c_i) \ominus_{gH} f(c_{i-1}) \right). \qquad (22)$$

*(summation denotes standard fuzzy addition in this statement).*

**PROOF.** The proof is similar to [39], [40].□

**Remark 46** *It is interesting to observe that, if the values $f(c_i)$ at all the $n$ switching points $c_i$, $i = 1, 2, ..., n$ are crisp (singleton), then we have $\int_a^b f'_{gH}(x)dx = f(b) - f(a)$ (the standard fuzzy difference); indeed, if $u \in \mathbb{R}_{\mathcal{F}}$ and $v \in \mathbb{R}$ we have $u \ominus_{gH} v = u - v$ and $v \ominus_{gH} u = v - u$. It follows that $\sum_{i=1}^{n+1} (f(c_i) \ominus_{gH} f(c_{i-1})) = \sum_{i=1}^{n+1} f(c_i) \ominus_{gH} f(c_{i-1}) = (f(b) - f(c_n)) + (f(c_n) - f(c_{n-1})) + ... + (f(c_2) - f(c_1)) + (f(c_1) - f(a)) = f(b) - f(a)$ (for the crisp terms we have $-f(c_i) + f(c_i) = 0$, $i = 1, 2, ..., n$).*

## 6 Conclusions and further work

We have investigated different new differentiability concepts for fuzzy number valued functions. The g-differentiability introduced here is a very general derivative concept, being also practically applicable. The next step in the research direction proposed here is to investigate fuzzy differential equations with g-differentiability and applications.

## References


[1] R.P. Agarwal, D. O'Regan, Existence for set differential equations via multivalued operator equations. Differential equations and applications. Vol. 5, 1–5, Nova Sci. Publ., New York, 2007.

[2] B. Bede, S.G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems, 151(2005), 581-599.

[3] B. Bede, Note on “Numerical solutions of fuzzy differential equations by predictor–corrector method”, Information Sciences, 178 (2008) 1917-1922.

[4] B. Bede, S.G. Gal, Solutions of fuzzy differential equations based on generalized differentiability, Communications in Mathematical Analysis, 9(2010), 22–41.

[5] A. M. Bica, Error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations, Information Sciences, 178, 5, 2008, 1279-1292.

[6] Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations Chaos, Solitons & Fractals, 38(2008), 112-119.

[7] Y. Chalco-Cano and H. Roman-Flores, Comparation between some approaches to solve fuzzy differential equations, Fuzzy Sets and Systems, doi:10.1016/j.fss.2008.10.002, 2008.

[8] Y. Chalco-Cano, H. Roman-Flores, M.D. Jimenez-Gamero, Generalized derivative and $\pi$-derivative for set-valued functions, Information Sciences 181 (2011) 2177–2188.

[9] T. Gnana Bhaskar, V. Lakshmikantham, Set differential equations and Flow invariance, J. Appl. Anal. 82 (2) (2003) 357–368.

[10] P. Diamond, Time-dependent differential inclusions, cocycle attractors and fuzzy differential equations, IEEE Trans. Fuzzy System, 7 (1999) 734-740.

[11] P. Diamond, Stability and periodicity in fuzzy differential equations, IEEE Trans. Fuzzy Systems, 8 (2000) 583-590.

[12] P. Diamond, P. Kloeden, Metric Spaces of Fuzzy Sets: Theory and Application, World Scientific, Singapore, 1994.

[13] P. Diamond, Brief note on the variation of constants formula for fuzzy differential equations, Fuzzy Sets and Systems 129 (2002) 65-71.

[14] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems 18 (1986) 31-42.

[15] E. Huellermeier, An approach to modelling and simulation of uncertain dynamical systems, Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems, 5 (1997), 117-137.

[16] M. Hukuhara, Integration des applications mesurables dont la valeur est un compact convex, Funkcial. Ekvac., 10 (1967) 205–229.

[17] L.J. Jowers, J.J. Buckley and K.D. Reilly, Simulating continuous fuzzy systems, Information Sciences 177 (2007) 436-448.

[18] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24(1987), 301-317.

[19] O. Kaleva, A note on fuzzy differential equations, Nonlinear Analysis 64 (2006), 895-900.

[20] V. Lakshmikantham, J.J. Nieto, Differential equations in metric spaces: An introduction and an application to fuzzy differential equations, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 10 (2003), 991-1000.

[21] V. Lakshmikantham, R. Mohapatra, Theory of Fuzzy Differential Equations and Inclusions, Taylor &Francis, London, 2003.

[22] S. Markov, A non-standard subtraction of intervals. Serdica 3 (1977) 359-370.

[23] S. Markov, Extended interval arithmetic. Compt. Rend. Acad. Bulg. Sci. 30, 9 (1977) 1239-1242.

[24] S. Markov, Existence and uniqueness of solutions of the interval differential equation $X' = f(t, X)$. Compt. Rend. Acad. Bulg. Sci. 31, 12 (1978) 1519-1522.

[25] S. Markov, Calculus for interval functions of a real variable. Computing 22 (1979) 325-337.

[26] S. Markov, On the Algebra of Intervals and Convex Bodies. Journal of Universal Computer Science 4, 1 (1998) 34-47.

[27] M.T. Mizukoshi, L.C. Barros, Y. Chalco-Cano, H. Roman-Flores, R.C. Bassanezi, Fuzzy differential equations and the extension principle, Information Sciences 177 (2007), 3627-3635.

[28] C.V. Negoita, D.A. Ralescu, Application of Fuzzy Sets to Systems Analysis, J. Wiley, 1975.

[29] J.J. Nieto, R. Rodriguez-Lopez, Euler polygonal method for metric dynamical systems, Information Sciences, 177 (2007), 4256-4270.

[30] J.J. Nieto, R. Rodriguez-Lopez, Hybrid metric dynamical systems with impulses, Nonlinear Analysis 64 (2006), 368-380.

[31] N.V. Plotnikova, Systems of linear differential equations with $\pi$-derivative and linear differential inclusions, Sbornik: Mathematics 196, 1677-1691, 2005.

[32] M. Puri, D. Ralescu, Differentials of fuzzy functions, J. Math. Anal. Appl., 91(1983), 552-558.

[33] R. Rodriguez-Lopez, Comparison results for fuzzy differential equations, Information Sciences, 178(2008), 1756-1779.

[34] L.J. Rodriguez-Munis, M. Lopez-Diaz, M.A. Gil, D.A. Ralescu, The s-differentiability of a fuzzy-valued mapping, Information Sciences 151 (2003) 283-299.

[35] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, 24(1987), 319-330.

[36] L. Stefanini, On the generalized LU-fuzzy derivative and fuzzy differential equations, Proceedings of the 2007 IEEE International Conference on Fuzzy Systems, London, July 2007, pp. 710-715.

[37] L. Stefanini, A generalization of Hukuhara difference, in D. Dubois , M.A. Lubiano, H. Prade, M.A. Gil, P. Grzegorzewski, O. Hryniewicz (Eds), *Soft Methods for Handling Variability and Imprecision*, Series on Advances in Soft Computing (vol 48), Springer, 2008.

[38] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets and Systems, 161 (2010) 1564–1584, doi:10.1016/j.fss.2009.06.009.

[39] L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Analysis 71 (2009) 1311-1328, doi:10.1016/j.na.2008.12.005.

[40] L. Stefanini, B. Bede, Some notes on generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Working Paper, University of Urbino, 2012. Available online at the *RePEc* repository, http://ideas.repec.org/f/pst233.html.

[41] L. Stefanini, M.L. Guerra, On fuzzy arithmetic operations: some properties and distributive approximations, *International Journal of Applied Mathematics*, 19 (2006) 171-199. An extended version is available at the *RePEc* repository, http://ideas.repec.org/f/pst233.html.

[42] L. Stefanini, L. Sorini, M.L. Guerra, Fuzzy numbers and Fuzzy Arithmetic, Chapter 12 in W. Pedrycz, A. Skowron, V. Kreinovich (eds), Handbook of Granular Computing, John Wiley & Sons, Ltd, 2008.

[43] C. Wu, S. Song, E.S. Lee, Approximate solutions, existence and uniqueness of the Cauchy problem of fuzzy differential equations, J. Math. Anal. Appl., 202(1996), 629-644.