A proof of the existence of good nested lattices

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1 Introduction

We show the existence of a sequence of nested lattices $(\Lambda_1^{(n)}, \Lambda^{(n)})$ with $\Lambda^{(n)} \subset \Lambda_1^{(n)}$ such that both lattices are Rogers-good and Poltyrev-good. The sequence is indexed by the lattice dimension n. The existence of a sequence of lattices $\Lambda^{(n)}$ which are good in both senses has been shown earlier [1]. Also, the existence of nested lattices where the coarse lattice is good in both senses and the fine lattice is Poltyrev-good has also been shown [2]. We show that the same construction as used in [2] results in a fine lattice that in addition to being Poltyrev-good is also Rogers-good. Our proof is essentially identical to the one given in [1].

In Section 2, we describe the construction of a random ensemble of nested lattices in which the coarse lattice Λ is fixed and the fine lattice Λ_1 is constructed in a randomized manner. This construction is the same as that described in [2]. In Section 3, we show the main result which is that with high probability, a nested lattice (Λ_1, Λ) in this ensemble is such that both Λ_1 and Λ are Rogers and Poltyrev-good. A straightforward corollary of this result is that the fine lattice Λ_1 is also good for MSE quantization and this is shown in Section 4. We conclude with some comments on further nesting of lattices in Section 5. We use the notation of [4] for lattice related quantities.

2 Construction of the Lattice Ensemble

We describe the construction of the nested lattice first. We start with a coarse lattice Λ (the superscript is dropped from here on) which is both Rogers and Poltyrev-good. Let $\mathcal{V}$ be the Voronoi region of Λ and $\sigma^2(\mathcal{V})$ be the second moment per dimension of Λ [4]. Let the generator matrix of Λ be G_Λ, i.e., $\Lambda = G_\Lambda \cdot \mathbb{Z}^n$. Formally, Λ satisfies

- (Rogers-good) Let R_u and R_l be the covering and effective radius of the lattice Λ. Λ (more precisely, a sequence of such lattices) is called Rogers-good if its covering efficiency $\rho_{\text{cov}}(\Lambda) \to 1$.

- (Poltyrev-good) For any $\sigma^2 < \sigma^2(\mathcal{V})$, let $\mathbf{N}$ be a Gaussian random vector whose components are i.i.d $\mathcal{N}(0, \sigma^2)$. Then, Λ (more precisely, a sequence of such lattices) is called Poltyrev-good if

$$Pr(\mathbf{N} \notin \mathcal{V}) < \exp\{-n[E_p(\mu) - o_n(1)]\} \tag{1}$$

  where $\mu = \sigma^2(\mathcal{V})/\sigma^2$ is the VNR (volume to noise ratio) of the lattice Λ relative to $\mathcal{N}(0, \sigma^2)$ and $E_p(\mu)$ is the Poltyrev exponent [1].

We now construct the fine lattice Λ_1 using Loeliger's type-A construction [3]. Let k, n, p be integers such that $k \leq n$ and p is prime. Their precise magnitudes are described later. Let G be a $k \times n$ generating matrix with its elements chosen uniformly from $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$. The construction of the fine lattice is now described by

- Define the discrete codebook $\mathcal{C} = \{x : x = y \cdot G \text{ for some } y \in \mathbb{Z}_p^k\}$

- Lift $\mathcal{C}$ to $\mathbb{R}^n$ to form

$$\Lambda_1^{'} = p^{-1}\mathcal{C} + \mathbb{Z}^n \tag{2}$$

- $\Lambda_1 \triangleq G_\Lambda \cdot \Lambda_1^{'}$ is the fine lattice

Note that, by construction, $\Lambda \subset \Lambda_1$. We now show that a randomly chosen member from this ensemble of nested lattices is such that Λ_1 is both Rogers and Poltyrev-good. The fact that such random selection results in a fine lattice which is with high probability Poltyrev-good has already been shown [2]. We now show that a similar selection results in Rogers-good fine lattices as well. By union bound then, we will have proved our claim.

3 Proof

To show Rogers-goodness, we show that a random fine lattice (with high probability) covers all the points inside the Voronoi region $\mathcal{V}$ of the coarse lattice with a covering efficiency that asymptotically reaches unity. We do this by first showing that almost every point in $\mathcal{V}$ is covered with high probability by a subset of the fine lattice points. We then show that increasing the number of points in the fine lattice decreases the number of uncovered points at a certain rate till no points remain uncovered. We then show that the covering efficiency of this construction asymptotically approaches unity.

Number the points of the fine lattice Λ_1 that lie inside $\mathcal{V}$. Let $\Lambda_1(i)$ be the ith such point for $i = 0, 1, \ldots, p^k - 1$. Since the whole space is tiled by regions congruent to $\mathcal{V}$, we restrict attention to only $\mathcal{V}$. Let A^* then denote A mod $\mathcal{V}$ for any set A.

Proposition 1. *The random ensemble described in Section 2 satisfies the following properties.*

- $\Lambda_1(0) = \mathbf{0}$ *deterministically.*

- $\Lambda_1(i)$ *is equally likely to be any of the points in $p^{-1}\Lambda \cap \mathcal{V}$.*

- *For any $i \neq j$, $(\Lambda_1(i) - \Lambda_1(j))^*$ is uniformly distributed over $p^{-1}\Lambda \cap \mathcal{V}$.*

Proof: A brief sketch of the proof of these facts is presented here. First, we note that since there is a one-to-one correspondence between the points of the lattices $\Lambda_1^{'}$ and Λ_1, it suffices to prove the assertions for the lattice $\Lambda_1^{'}$.

- Clearly, $0^n \in \mathcal{C}$ for any choice of the random matrix G in the construction described in Section 2. Hence the point 0^n belongs to the lattices $\Lambda_1^{'}$ and Λ_1 deterministically. Without loss of generality, we can name this point as $\Lambda_1(0)$ and hence the first property is proved.

- Let $\mathcal{C}(i)$ denote the codeword corresponding to the lattice point $\Lambda_1^{'}(i)$. It is easy to see that $\Lambda_1(i)$ is uniformly distributed over the p^n points in $p^{-1}\Lambda \cap G_\Lambda \cdot$CUBE where CUBE $= [0,1)^n$. Because every lattice point has a one-to-one correspondence with both $G_\Lambda \cdot$ CUBE and $\mathcal{V}$ and both these sets tile $\mathbb{R}^n$, it must be that they both have the same volume. This also implies that $|p^{-1}\Lambda \cap G_\Lambda \cdot \text{CUBE}| = |p^{-1}\Lambda \cap \mathcal{V}| = p^n$. Thus, the points in these two sets can be put in one-to-one correspondence with one another. By this correspondence and the fact that $p^{-1}\Lambda \cap G_\Lambda \cdot$ CUBE, it follows that $\Lambda_1(i)$ is uniformly distributed over $p^{-1}\Lambda \cap \mathcal{V}$.

- Once again, the same argument as above can be used. It is easy to show that $(\Lambda_1(i) - \Lambda_1(j))$ is uniformly distributed over the p^n points of $p^{-1}\Lambda \cap G_\Lambda \cdot$ CUBE for $i \neq j$. Using the one-to-one correspondence between this region and $p^{-1}\Lambda \cap \mathcal{V}$, it must be that $(\Lambda_1(i) - \Lambda_1(j))^*$ is uniformly distributed over the p^n points of $p^{-1}\Lambda \cap \mathcal{V}$.

This completes the proof of the properties of the lattice ensemble.

If we use the lattice points $\Lambda_1 \cap \mathcal{V}$ as codewords, then the effective rate of such a code would be $R = \frac{k}{n} \log p$. In what follows, we will be interested in keeping this code rate fixed as $n \to \infty$. Thus $p^k \to \infty$ as $n \to \infty$. We also remark that the following proof works for any $R > 0$.

Part I: Almost complete covering

Fix an $r > 0$ to be chosen later. Fix an arbitrary $x \in \mathcal{V}$. Let $S_1(x)$ be the set of all points in $p^{-1}\Lambda \cap \mathcal{V}$ that are within a distance $(r - d)$ of x, i.e.,

$$S_1(x) = (p^{-1}\Lambda \cap (x + (r - d)\mathcal{B}))^* \tag{3}$$

Here, $\mathcal{B}$ denotes a ball of unit radius and d is the covering radius of Voronoi region of the lattice $p^{-1}\Lambda$. The probability that x is covered by the ith point of the fine lattice Λ_1 is given by

$$Pr(x \in (\Lambda_1(i) + (r - d)\mathcal{B})^*) = \frac{|S_1(x)|}{p^n} \tag{4}$$

We use the following lower bound on the cardinality of $S_1(x)$. Define $S_2(x)$ as

$$S_2(x) = \{y \in p^{-1}\Lambda : (y + p^{-1}\mathcal{V})^* \cap (x + (r - 2d)\mathcal{B})^* \neq \phi\} \tag{5}$$

Clearly $S_2(x) \subset S_1(x)$ and the cardinality of $S_2(x)$ can be bounded by $V_\mathcal{B}(r - 2d)/\text{Vol}(p^{-1}\mathcal{V})$. Thus a lower bound on the probability that x is $(r - d)$ covered by $\Lambda_1(i)$ is

$$Pr(x \in (\Lambda_1(i) + (r - d)\mathcal{B})^*) \geq \frac{V_\mathcal{B}(r - 2d)}{|\mathcal{V}|} \quad \text{for } i = 1, \ldots, p^k - 1 \tag{6}$$

Note that we exclude $i = 0$ from consideration since $\Lambda_1(0) = \mathbf{0}$ deterministically. Let η_i be the indicator random variable that indicates whether x is covered by $\Lambda_1(i)$ for $i = 1, \ldots, p^k - 1$. Let χ be the total number of points in $\Lambda_1 \cap \mathcal{V}$ that cover x. Then $\mathbb{E}(\chi)$ is given by

$$\mathbb{E}(\chi) = \sum_{i=1}^{p^k - 1} \mathbb{E}(\eta_i) \tag{7}$$

$$\geq (p^k - 1)\frac{V_\mathcal{B}(r - 2d)}{|\mathcal{V}|} \tag{8}$$

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We have $|\mathcal{V}|/|\mathcal{V}_1| = p^k$ and that $p^k \to \infty$ as $n \to \infty$. Thus, $\mathbb{E}(\chi)$ can be written as

$$\mathbb{E}(\chi) \geq c_n \frac{V_\mathcal{B}(r - 2d)}{|\mathcal{V}_1|} = c_n \left(\frac{r - 2d}{r_{\Lambda_1}} \right)^n \tag{9}$$

where r_{Λ_1} is the effective radius of the Voronoi region $\mathcal{V}_1$ of the fine lattice Λ_1 and $c_n = 1 - e^{-nR} \to 1$.

Using the pairwise independence of η_is (which follows from the 3rd property of the ensemble), we have

$$\mathrm{Var}(\chi) = \sum_{i=1}^{p^k - 1} \mathrm{Var}(\eta_i) \leq \sum_{i=1}^{p^k - 1} \mathbb{E}(\eta_i^2) = \sum_{i=1}^{p^k - 1} \mathbb{E}(\eta_i) = \mathbb{E}(\chi) \tag{10}$$

From Chebyshev's inequality, for any $\nu > 0$,

$$Pr\{|\chi - \mathbb{E}(\chi)| > 2^\nu \sqrt{\mathbb{E}(\chi)}\} \leq \frac{\mathrm{Var}(\chi)}{4^\nu \mathbb{E}(\chi)} \leq 4^{-\nu} \tag{11}$$

Let $\mu(\nu) \triangleq \mathbb{E}(\chi) - 2^\nu \sqrt{\mathbb{E}(\chi)}$. Then, we have $Pr(\chi < \mu(\nu)) \leq 4^{-\nu}$. If $\mu(\nu) > 1$, then $4^{-\nu}$ also bounds the probability that none of the points of $p^{-1}\Lambda$ cover x.

Call $x \in \mathcal{V}$ remote from a set A if none of the points in A are within distance $(r - d)$ from x. Then, $\chi(x) < 1$ is the same as saying x is remote from Λ_1. Let Q be the set of points $x \in \mathcal{V}$ that are remote from Λ_1 and let $q \triangleq |Q|/|\mathcal{V}|$. Then,

$$|Q| = \int_\mathcal{V} \mathbf{1}_{(\chi(x) < 1)} dx \tag{12}$$

$$\leq \int_\mathcal{V} \mathbf{1}_{(\chi(x) < \mu(\nu))} dx \tag{13}$$

if $\mu(\nu) \geq 1$. Using the previously obtained bound, we then have $\mathbb{E}(q) \leq 4^{-\nu}$. From Markov's inequality, it then follows that

$$Pr(q > 2^\nu \mathbb{E}(q)) < 2^{-\nu} \tag{14}$$

and thus

$$Pr(q > 2^{-\nu}) < 2^{-\nu} \tag{15}$$

If we let $\nu \to \infty$ while still keeping $\mu(\nu) \geq 1$, we can let this probability decay to 0. This can be achieved by letting $\nu = o(\log n)$ and $\mathbb{E}(\chi) > n^\lambda$ for some $\lambda > 0$. But, we have $\mathbb{E}(\chi) \geq (p^k - 1)V_\mathcal{B}(r - 2d)/|\mathcal{V}|$. Thus, it is enough to choose r such that

$$\log\left(\frac{r - 2d}{r_{\Lambda_1}} \right) \geq \frac{\lambda}{n} \log n \tag{16}$$

For ease of reference, the choices of growth rates for the different variables are listed below and in the rest of the paper as well.

Choices I:

- ν goes to ∞ as $o(\log n)$.

- r is chosen such that $\mathbb{E}(\chi) > n^\lambda$ for some $\lambda > 0$.

With such a choice of parameters, for most lattices in the ensemble, almost all points of the region $\mathcal{V}$ are $(r-d)$ covered by points of the randomly chosen lattice Λ_1 with high probability. Note that, it suffices to choose $k=1$ even to reach this conclusion (in which case, p needs to grow exponentially). In what follows, we will restrict attention to covering only the points of the grid $p^{-1}\Lambda \cap \mathcal{V}$. We note that the bound obtained in equation (15) holds when q is interpreted as the fraction of uncovered points in $p^{-1}\Lambda \cap \mathcal{V}$ as well.

Part II: Complete covering

We now extend the analysis to provide complete covering of $\mathcal{V}$. The main idea is as follows. Any point $x \in \mathcal{V}$ is within a distance d from a point in $p^{-1}\Lambda \cap \mathcal{V}$. This simply follows from the definition of d as the covering radius of $p^{-1}\Lambda$. Thus, an $(r-d)$ covering of the points of $p^{-1}\Lambda$ will automatically result in an r covering of $\mathcal{V}$. Thus, we restrict our attention to the lattice $p^{-1}\Lambda \cap \mathcal{V}$ and attempt to cover only these lattice points in what follows. Correspondingly, we define $Q(A)$ to be the set of all lattice points $p^{-1}\Lambda \cap \mathcal{V}$ that are remote from the set A. Also, let x_i, $i = 0, \ldots, p^n - 1$ denote the ith point of the constellation $p^{-1}\Lambda \cap \mathcal{V}$.

Let $\Lambda_1[k_1]$ be the fine lattice obtained using the Loeliger construction while using only the first k_1 rows of the random matrix G. We saw in the previous section that any such k_1 would suffice to get an almost complete covering of $\mathcal{V}$. We will now demonstrate that the fraction of uncovered points squares when we go from $\Lambda_1[k_1]$ to $\Lambda_1[k_1 + 1]$ and thus when sufficient number of rows are added, the fraction of uncovered points becomes less than p^{-n} with high probability. Since there are only p^n points in $p^{-1}\Lambda$, this means that every point is covered. For this argument to work, we will need certain bounds on the magnitudes of the quantities involved which we shall provide later.

Fix k_1 which grows faster than $(\log n)^2$. The necessity of this growth rate is explained later. Let x_j be the jth lattice point. Again, we exclude $j = 0$ from consideration. Let Q_i be the set of lattice points that remain uncovered by the lattice $\Lambda_1[k_1 + i]$, $i = 0, 1, \ldots, k_2 = k - k_1$. Correspondingly, define $q_i = |Q_i|/p^n$. Consider the set $S = (\Lambda_1[k_1] \cup (\Lambda_1[k_1] + p^{-1}\mathbf{g}_{k_1+1}))^*$ where $\mathbf{g}_i$ is the ith row of the random matrix G. Note that

$$\Lambda_1[k_1 + 1] = \bigcup_{m=0}^{p-1} (\Lambda_1[k_1] + m \cdot p^{-1}\mathbf{g}_{k_1+1})^* \tag{17}$$

and thus $S \subset \Lambda_1[k_1 + 1]$. This implies that $Q(\Lambda_1[k_1 + 1]) \subset Q(S)$ and $q_1 \leq |Q(S)|/p^n$. Since $\Lambda_1[k_1] + p^{-1}\mathbf{g}_{k_1+1}$ is an independent shift of $\Lambda_1[k_1]$, the probability that x_j is remote from $\Lambda_1[k_1] + p^{-1}\mathbf{g}_{k_1+1}$ is the same as the probability that x_j is remote from $\Lambda_1[k_1]$. Also note that, given a $\Lambda_1[k_1]$, q_0 is a deterministic function of $\Lambda_1[k_1]$. Therefore,

$$Pr(x_j \in Q(S) \mid \Lambda_1[k_1]) = Pr\left(x_j \in Q(S) \mid x_j \in Q(\Lambda_1[k_1]), \Lambda_1[k_1]\right) \cdot Pr\left(x_j \in Q(\Lambda_1[k_1]) \mid \Lambda_1[k_1]\right)$$
$$+ Pr(x_j \in Q(S) \mid x_j \notin Q(\Lambda_1[k_1]), \Lambda_1[k_1]) \cdot Pr(x_j \notin Q(\Lambda_1[k_1]) \mid \Lambda_1[k_1]) \tag{18}$$

Since $\Lambda_1[k_1] \subset S$, we have that $Pr(x_j \in Q(S) \mid x_j \notin Q(\Lambda_1[k_1]), \Lambda_1[k_1]) = 0$ and the second term in the expression above vanishes. Given $\Lambda_1[k_1]$, the event $\{x_j \in Q(\Lambda_1[k_1])\}$ is deterministic. Also, we have

$$Pr(x_j \in Q(S) \mid x_j \in Q(\Lambda_1[k_1]), \Lambda_1[k_1]) = Pr(x_j \in Q(\Lambda_1[k_1] + p^{-1}\mathbf{g}_{k_1+1}) \mid \Lambda_1[k_1]) \tag{19}$$
$$= Pr(x_j \in Q(\Lambda_1[k_1] + p^{-1}\mathbf{g}_{k_1+1})) \tag{20}$$
$$= q_0 \tag{21}$$

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where equation (21) follows from the independent nature of the shift. Thus, equation (18) can be written as

$$Pr(x_j \in Q(S) \mid \Lambda_1[k_1]) = q_0 \cdot Pr(x_j \in Q(\Lambda_1[k_1]) \mid \Lambda_1[k_1]) \tag{22}$$

$$= q_0 \cdot \mathbf{1}(x_j \in Q(\Lambda_1[k_1]) \mid \Lambda_1[k_1]) \tag{23}$$

Let $\eta_j^{'}$ be the indicator random variable denoting whether the jth grid point x_j belongs to $Q(S)$ conditioned on $\Lambda_1[k_1]$ for $j = 1, \ldots, p^n - 1$. Then, the cardinality of $Q(S)$ conditioned on $\Lambda_1[k_1]$ is given by $\sum_j \eta_j^{'}$. Therefore,

$$\mathbb{E}\left(\frac{|Q(S)|}{p^n} \,\middle|\, \Lambda_1[k_1]\right) = \frac{1}{p^n} \sum_{j=1}^{p^n-1} \mathbb{E}(\eta_j^{'} \mid \Lambda_1[k_1]) \tag{24}$$

$$= \frac{q_0}{p^n} \sum_{j=1}^{p^n-1} \mathbf{1}(x_j \in Q(\Lambda_1[k_1]) \mid \Lambda_1[k_1]) \tag{25}$$

$$= q_0^2 \tag{26}$$

where the last equality follows from the definition of q_0. A point worth noting is that in equation (26), q_0 is a function of the fine lattice $\Lambda_1[k_1]$. Using the relation $\mathbb{E}[X \mid \mathbb{E}(X \mid Y)] = \mathbb{E}(X \mid Y)$, we get the following.

$$\mathbb{E}\left(\frac{|Q(S)|}{p^n} \,\middle|\, q_0\right) = q_0^2 \tag{27}$$

This in turn implies that $\mathbb{E}(q_1 \mid q_0) \leq q_0^2$. Appealing to Markov inequality gives us (for any $\gamma > 0$)

$$Pr(q_1 > 2^\gamma \mathbb{E}(q_1 \mid q_0) \mid q_0) \leq 2^{-\gamma} \tag{28}$$

Combining this with the bound on $\mathbb{E}(q_1 \mid q_0)$, we get

$$Pr(q_1 \leq 2^{\gamma-2\nu} \mid q_0 \leq 2^{-\nu}) \geq 1 - 2^{-\gamma} \tag{29}$$

We use the previously derived bound for the probability of the event $\{q_0 \leq 2^{-\mu}\}$ from equation (15). Note that, even though this bound was derived for the case when q_0 refers to the fraction of uncovered points in $\mathcal{V}$, a similar argument works when q_0 is redefined to be the fraction of uncovered points in $p^{-1}\Lambda \cap \mathcal{V}$. By Bayes' rule, we finally arrive at

$$Pr(q_1 \leq 2^{\gamma-2\nu}) \geq (1 - 2^{-\gamma})(1 - 2^{-\nu}) \tag{30}$$

Iterating this procedure k_2 times gives us

$$Pr(q_{k_2} \leq 2^{2^{k_2}(\gamma-\nu)-\gamma}) \geq (1 - 2^{-\nu})(1 - 2^{-\gamma})^{k_2} \tag{31}$$

We now choose k_2 such that $2^{2^{k_2}(\gamma-\nu)-\gamma} < p^{-n}$. For this, it suffices to take $\gamma = \nu - 1$ and $2^{k_2} > n \log p - \nu + 1$. Using the earlier result that $\nu = o(\log n)$, it implies that we need to choose $k_2 \geq \lceil \log n + \log \log p \rceil$. This rate of growth for k_2 (and thus k) implies through $p^k = 2^{nR}$ that $\log p$ grows slower than $n/\log n$ and thus k_2 grows at least as fast as $\log n$. To ensure that $k_2 < k$, we need k to be growing faster than k_2, say as fast as $(\log n)^2$. We can then choose k_1 of the order $(\log n)^2$ to begin with. To ensure that the probability in equation (31) goes to 1 for this choice of k_2, we need to further restrict the rate of growth of ν while still ensuring that $\nu \to \infty$.

This can be accomplished by choosing $\nu = 2\log(\log n + \log\log p)$. With these choices, we have the probability of complete covering of $\mathcal{V}$ going to 1. From standard random coding arguments, it then follows that there exists a deterministic nested lattice (Λ, Λ_1) such that the lattice points Λ_1 r-cover $\mathbb{R}^n$ for the following choices of the parameters.

Choices II:

- $k = k_1 + k_2$ grows as fast as $(\log n)^2$.

- k_2 grows at least as fast as $\lceil \log n + \log\log p \rceil$.

- $\nu = 2\log(\log n + \log\log p)$ and $\gamma = \nu - 1$

The covering efficiency of the fine lattice can now be calculated.

$$p^{k_1+k_2} = \frac{n^\lambda |\mathcal{V}|}{V_\mathcal{B}(r - 2d)} p^{k_2} \tag{32}$$

However, we also have that $p^k = |\mathcal{V}|/V_\mathcal{B}(r_{\Lambda_1})$. Combining all this, we have

$$\frac{r}{r_{\Lambda_1}} = \sqrt[n]{\frac{V_\mathcal{B}(r)}{V_\mathcal{B}(r - 2d)} n^\lambda p^{k_2}} \tag{33}$$

$$\leq \left(\frac{r}{r - 2d}\right) \cdot n^{\frac{\lambda}{n}} \cdot 2^{(\log p \log n + \log p \log\log p + \log p)/n} \tag{34}$$

As $n \to \infty$, the right hand side should go to 1. It is easy to verify that the last 2 terms do indeed tend to 1. To show that the first term goes to 1, we need to show that $d \to 0$ as $n \to \infty$ for our choice of parameters. Since Λ is Rogers-good (which implies $p^{-1}\Lambda$ is Rogers-good as well), it has a covering efficiency asymptotically approaching 1. Thus the covering radius d of $p^{-1}\Lambda$ approaches $p^{-1}r_\Lambda$ as the lattice dimension $n \to \infty$. From the nesting ratio, we get

$$\frac{|\mathcal{V}|}{|\mathcal{V}_1|} = \left(\frac{r_\Lambda}{r_{\Lambda_1}}\right)^n = p^k = 2^{nR} \tag{35}$$

and hence d approaches $p^{-1}2^R r_{\Lambda_1}$. We know that (since k grows as $\log n + \log\log p$ and $p^k = 2^{nR}$) p grows as $o(n/\log n)$ and thus to ensure $d \to 0$, we need r_{Λ_1} to go to ∞ slower than p. Once could even take r_{Λ_1} to be constant in the above proof.

Choices III:

- r_{Λ_1} is chosen to be any positive constant. This in turn will imply that r_Λ is also constant.

Thus, we have shown that Λ_1 is an efficient covering lattice.

To summarize, we first showed that a small subset of the fine lattice is enough to cover most points of the Voronoi region of the coarse lattice with high probability. We then showed that by augmenting the fine lattice with more points, more fraction of the points were covered till all points were covered with high probability. We then showed that the chosen covering radius r is asymptotically the same as the effective radius of the fine lattice. Thus, the fine lattice is Rogers-good.

4   Goodness for MSE Quantization

It was shown in [5] that a lattice that is good for covering is necessarily good for quantization. This can be inferred from the following relation. For any lattice Λ

$$G(\Lambda) \leq G_n^* \cdot \frac{n+2}{n} \cdot (\rho_{\text{cov}}(\Lambda))^2 \tag{36}$$

where $G(\Lambda)$ is the normalized second moment of the lattice Λ and G_n^* is the normalized second moment of the n-dimensional sphere.

Since, we have shown that $\rho_{\text{cov}}(\Lambda_1) \to 1$ as $n \to \infty$ with high probability, it also follows that the fine lattice is good for MSE quantization with high probability.

5   Conclusion

We showed the existence of nested lattices (Λ_1, Λ), $\Lambda \subset \Lambda_1$, such that both lattices both Rogers and Poltyrev-good. By iterating this construction process, we can show the existence of good nested lattices with any finite level of nesting. More precisely, for any finite $m > 0$, one can show the existence of a nested lattice $(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$, $\Lambda_m \subset \cdots \subset \Lambda_1$ such that all the lattices $\Lambda_i, i = 1, \ldots, m$ are both Rogers-good and Poltyrev-good. Further, such nested lattices exist for any choice of the nesting ratios. By virtue of being Rogers-good, such lattices are also good for MSE quantization.

References

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