

ENUMERATION OF INVERSION SEQUENCES AVOIDING TRIPLES OF RELATIONS

WENQIN CAO, EMMA YU JIN, AND ZHICONG LIN

ABSTRACT. Inversion sequences are in natural bijection with permutations and have surprising connections with Lecture hall polytopes and partitions. Recently, Martinez and Savage carried out the systematic study of inversion sequences avoiding triples of relations. They established many connections with known integer sequences and highlighted several interesting conjectures, some of which have already been solved. In this paper, we address the remaining enumeration conjectures posed by them.

1. INTRODUCTION

An integer sequence (e_1, e_2, \dots, e_n) is an *inversion sequence of length n* if $0 \leq e_i < i$ for all $1 \leq i \leq n$. Inversion sequences and its generalization have remarkable connections with Lecture hall polytopes and partitions [15]. Let \mathbf{I}_n be the set of all inversion sequences of length n . There is a natural bijection Θ between \mathfrak{S}_n , the set of all *permutations* of $[n] := \{1, 2, \dots, n\}$, and \mathbf{I}_n defined for $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$ by

$$\Theta(\pi) = (e_1, e_2, \dots, e_n), \quad \text{where } e_i := |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

We view permutations and inversion sequences as words over \mathbb{N} . A word $W = w_1w_2 \dots w_n$ is said to avoid the word (or pattern) $P = p_1p_2 \dots p_k$ ($k \leq n$) if there exists $i_1 < i_2 < \dots < i_k$ such that the subword $w_{i_1}w_{i_2} \dots w_{i_k}$ of W is order isomorphism to P . For example, the word $W = 32421$ contains the pattern 231, because the subword $w_2w_3w_5 = 241$ of W has the same relative order as 231. However, W is 101-avoiding. For a set \mathcal{W} of words, the set of words in \mathcal{W} avoiding patterns P_1, \dots, P_r is denoted by $\mathcal{W}(P_1, \dots, P_r)$.

Pattern avoidance in permutations has already been extensively studied in the literature (see the book survey by Kitaev [7]), while the systematic study of patterns in inversion sequences was initiated only recently in [4] and [11], where inversion sequences avoiding patterns of length 3 are exploited. Martinez and Savage [12] further considered a generalization of pattern avoidance to a fixed triple of binary relations (ρ_1, ρ_2, ρ_3) . For each relation triple $(\rho_1, \rho_2, \rho_3) \in \{<, >, \leq, \geq, =, \neq, -\}^3$, they studied the set $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ consisting of those $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \rho_1 e_j$, $e_j \rho_2 e_k$ and $e_i \rho_3 e_k$. Here the relation " $-$ " on a set S is all of $S \times S$, i.e., $x - y$ for all $x, y \in S$. For example, $\mathbf{I}_n(<, >, =) = \mathbf{I}_n(010)$ and $\mathbf{I}_n(>, -, \leq) = \mathbf{I}_n(101, 102)$.

The study of inversion sequences avoiding relation triples turns out to be unexpected fruitful and interesting. The pioneering work of Martinez and Savage [12] classifies all the 343 possible relation triples into 98 equivalence classes of patterns, which were conjectured to have exactly 63 *Wilf-equivalence* classes. Among these 63 Wilf classes, 30 Wilf classes were suspected to have connections with integer sequences in the OEIS, the **O**n-**L**ine **E**ncyclopedia of **I**nteger **S**equences [13]

Date: April 5, 2018.

Key words and phrases. Inversion sequences, Pattern avoidance, Ordered trees, Lattice paths, Set partitions.

founded by N.J.A. Sloane in 1964. Many of these connections were either established in [12] or first conjectured in [12] and latter solved by others in [2, 3, 6, 8–10, 18]. To be more precise, Bindi, Guerrini and Rinaldi [2] constructed a bijection between set partitions of $[n]$ and $\mathbf{I}_n(=, \geq, -)$, which also restricts to a bijection between *non-crossing partitions* of $[n]$ and $\mathbf{I}_n(\geq, -, \geq)$. Bouvel, Guerrini, Rechnitzer and Rinaldi [3] proved that $(>, \geq, -)$ -avoiding inversion sequences are counted by *Semi-Baxter numbers*. Via the obstinate kernel method: Kim and Lin [8] showed that $\mathbf{I}_n(\geq, \geq, >)$ is counted by the n -th *Baxter number*; Lin [10] and independently Bindi–Guerrini–Rinaldi [2] proved that $\mathbf{I}_n(\geq, \geq, \geq)$ has the same cardinality as set partitions of $[n]$ avoiding *enhanced 3-crossings*. The latter result was also proved bijectively by Yan [18] via 01-fillings of triangular shape. Moreover, several classical statistics on pattern avoiding inversion sequences that are enumerated by *Large Schröder numbers* and *Euler numbers* were studied in [8, 9].

Nevertheless, those connections highlighted in [12, Table 2] with a “no” in column 3 are still unsolved. In this paper, we address the remaining enumeration conjectures posed in [12, Table 2], leaving only those related to the OEIS sequence [13, A098746] open. In particular, several connections between pattern avoiding inversion sequences and restricted ordered trees, underdiagonal lattice paths and restricted set partitions are established.

In the rest of this paper, we deal with the pattern $(\geq, -, \geq)$ in Section 2, the pattern $(-, -, \geq)$ in Section 3, the pattern $(>, -, \leq)$ in Section 4, the pattern $(>, \leq, -)$ in Section 5, the patterns $(=, \geq, -)$ and $(\geq, -, =)$ in Section 6, and the pattern $(-, -, =)$ in Section 7.

Statistics on inversion sequences. For an inversion sequence $e = (e_1, e_2, \dots, e_n) \in \mathbf{I}_n$, we define five classical statistics:

- $\text{asc}(e) = |\{i \in [n-1] : e_i < e_{i+1}\}|$, the number of *ascents* of e ;
- $\text{dist}(e) = |\{e_1, e_2, \dots, e_n\} \setminus \{0\}|$, the number of *distinct positive entries* of e ;
- $\text{rmin}(e) = |\{i \in [n] : e_i < e_j \text{ for all } j > i\}|$, the number of *right-to-left minima* of e ;
- $\text{zero}(e) = |\{i \in [n] : e_i = 0\}|$, the number of *zero entries* in e ;
- $\text{satu}(e) = |\{i \in [n] : e_i = i - 1\}|$, the number of *saturated entries* of e .

It is known (cf. [5]) that “asc” and “dist” are *Eulerian* statistics on \mathbf{I}_n , while “rmin”, “zero” and “satu” are *Stirling* statistics on \mathbf{I}_n . These five statistics will play important roles throughout this paper.

2. THE PATTERN $(\geq, -, \geq)$ AND ORDERED TREES

In the first version of their paper [12, Section 2.14], Martinez and Savage suspected that the number of $e \in \mathbf{I}_n(\geq, -, \geq)$ with $\text{asc}(e) = n - 1 - k$ is equal to the number of ordered trees with n edges and with k interior vertices (non-root, non-leaf) adjacent to a leaf. The following result confirms their conjecture when comparing with the formula for the latter object in OEIS [13, A108759].

Theorem 2.1. *For $n \geq 1$ and $\lceil n/2 \rceil \leq k < n - 1$, we have*

$$(2.1) \quad |\{e \in \mathbf{I}_n(\geq, -, \geq) : \text{asc}(e) = k - 1\}| = \frac{1}{n+1} \binom{n+1}{2k-n} \binom{2n-k}{n-k}.$$

Proof. We will apply the following decomposition of $(\geq, -, \geq)$ -avoiding inversion sequences. For any $e \in \mathbf{I}_n(\geq, -, \geq)$, we distinguish two cases:

- If $e_n = n - 1$, then

$$\text{asc}(e) = \text{asc}(e_1, \dots, e_{n-1}) + \chi(n \neq 1).$$

Here $\chi(\mathbf{S})$ equals 1, if the statement \mathbf{S} is true; and 0, otherwise.

- If $k = \max\{i : e_i = i - 1\} < n$, then it is straightforward to show that e can be decomposed into two smaller inversion sequences: $f = (e_1, \dots, e_{k-1}, e_{k+1})$ in $\mathbf{I}_k(\geq, -, \geq)$ and $g = (e_{k+2} - k, e_{k+3} - k, \dots, e_n - k)$ in $\mathbf{I}_{n-1-k}(\geq, -, \geq)$ (possibly empty). This decomposition is reversible and satisfies the property:

$$\text{asc}(e) = \text{asc}(f) + \text{asc}(g) + 1 - \chi(k = n - 1) + \chi(e_{k-1} \geq e_{k+1} \text{ or } k = 1).$$

Let $\tilde{\mathbf{I}}_n(\geq, -, \geq) := \{e \in \mathbf{I}_n(\geq, -, \geq) : n = 1 \text{ or } e_{n-1} < e_n\}$ and let $C(x, t), A(x, t)$ be the generating functions of inversion sequences respectively from $\mathbf{I}_n(\geq, -, \geq)$ and $\tilde{\mathbf{I}}_n(\geq, -, \geq)$, counted by the length (variable t) and asc (variable x):

$$\begin{aligned} C(x, t) &:= \sum_{n \geq 1} t^n \sum_{e \in \mathbf{I}_n(\geq, -, \geq)} x^{\text{asc}(e)} = t + (1+x)t^2 + (4x+x^2)t^3 + \dots, \\ A(x, t) &:= \sum_{n \geq 1} t^n \sum_{e \in \tilde{\mathbf{I}}_n(\geq, -, \geq)} x^{\text{asc}(e)} = t + xt^2 + (2x+x^2)t^3 + \dots. \end{aligned}$$

Translating the above decomposition of $(\geq, -, \geq)$ -avoiding inversion sequences into generating functions (we omit the variables x, t) gives

$$\begin{cases} B := C - A \\ A = t + txC + txA^2 + tx^2AB = t + txC + txA(A + xB), \\ B = t(A + xB) + tx(A + xB)B = t(A + xB)(1 + xB). \end{cases}$$

We are going to solve this system of equations. Let $E = A + xB$, then

$$C = A + B = t + txC + tE(1 + x(A + B)) = t + txC + tE(1 + xC),$$

which is equivalent to

$$(2.2) \quad E = \frac{C - t - txC}{t(1 + xC)}.$$

On the other hand, we have $E = A + xB = t + txC + tE(x + xE)$. Substituting (2.2) into this equation yields the algebraic equation for C :

$$t(1 + xC)(tx^2C^2 + 2txC + t + 1) - C = 0$$

Setting $\tilde{C} = t(xC + 1)$, then the above equation is further simplified to

$$x\tilde{C}^3 + (tx - 1)\tilde{C} + t = 0,$$

which is equivalent to

$$t = \frac{\tilde{C}(1 - x\tilde{C}^2)}{1 + x\tilde{C}}.$$

By applying the *Lagrange inversion formula* (cf. [17, Theorem 5.4.2]), we get (2.1). □

3. THE PATTERN $(-, -, \geq)$ AND RESTRICTED ORDERED TREES

It was observed by Martinez and Savage [12, Section 2.10] that the sequences $e \in \mathbf{I}_n(-, -, \geq)$ are those for which $e_i > \max\{e_1, \dots, e_{i-2}\}$ for $i = 3, \dots, n$. For $e \in \mathbf{I}_n$, this is equivalent to the conditions

$$(3.1) \quad e_i > \max\{e_{i-2}, e_{i-3}\} \quad \text{for } 3 \leq i \leq n,$$

where we use the convention $e_0 = 0$. For example,

$$\mathbf{I}_3(-, -, \geq) = \{(0, 0, 1), (0, 0, 2), (0, 1, 1), (0, 1, 2)\}.$$

Using this characterization, they derived a 4-parameter recurrence that enabled them to conjecture that

$$\{|\mathbf{I}_n(-, -, \geq)|\}_{n \geq 1} = \{1, 2, 4, 10, 25, 68, 187, 534, 1544, \dots\}$$

is the integer sequence A049125 in OEIS [13]. As described by David Callan, this integer sequence enumerates ordered trees with n edges in which every interior vertices has at most one leaf child. This section is devoted to a proof of the above assertion algebraically.

For an inversion sequence $e \in \mathbf{I}_n$, an index i , $1 \leq i \leq n$, is said to be a *saturated* (resp. *double saturated*) position of e if $e_i = i - 1$ (resp. if $e_i = i - 1$ and $e_{i+1} = i$). Let NDS_n be the set of all inversion sequences $e \in \mathbf{I}_n(-, -, \geq)$ with **no double saturated** positions. Let NDS'_n the set of all inversion sequences $e \in NDS_n$ satisfying $e_n < n - 1$. Note that $NDS'_1 = \emptyset$.

Our decomposition of $(-, -, \geq)$ -avoiding inversion sequences is a bit intricate. For any sequence $e \in \mathbf{I}_n(-, -, \geq)$, we need to distinguish two cases:

- If $e \in NDS_n$.
- Otherwise $e \in \mathbf{I}_n(-, -, \geq) \setminus NDS_n$. Let k be the leftmost double saturated position of e . Then e can be decomposed into two sequences:

$$(e_1, \dots, e_{k-1}) \in NDS'_{k-1} \quad \text{and}$$

$$(e_{k+1} - k, e_{k+2} - k, \dots, e_n - k) \in \mathbf{I}_{n-k}(-, -, \geq).$$

This decomposition is reversible.

Let us introduce $\widetilde{\mathbf{I}}_n(-, -, \geq) := \{e \in \mathbf{I}_n(-, -, \geq) : n = 1 \text{ or } e_{n-1} < e_n\}$ and $\widetilde{NDS}_n := NDS_n \cap \widetilde{\mathbf{I}}_n(-, -, \geq)$. Define five generating functions

$$A(t) := 1 + \sum_{n \geq 1} |\mathbf{I}_n(-, -, \geq)| t^n = 1 + t + 2t^2 + 4t^3 + 10t^4 + \dots,$$

$$\widetilde{A}(t) := \sum_{n \geq 1} |\widetilde{\mathbf{I}}_n(-, -, \geq)| t^n = t + t^2 + 3t^3 + 6t^4 + \dots,$$

$$B(t) := \sum_{n \geq 1} |NDS_n| t^n = t + t^2 + 2t^3 + 5t^4 + \dots,$$

$$\widetilde{B}(t) := \sum_{n \geq 1} |\widetilde{NDS}_n| t^n = t + 2t^3 + 2t^4 + \dots,$$

$$C(t) := \sum_{n \geq 2} |NDS'_n| t^n = t^2 + t^3 + 4t^4 + \dots.$$

Since $|NDS_n| = |NDS'_n| + |NDS'_{n-1}|$, the above decomposition gives

$$(3.2) \quad \begin{cases} B(t) = t + (1+t)C(t), \\ A(t) = 1 + B(t) + t(1+C(t))(A(t)-1), \\ \tilde{A}(t) = \tilde{B}(t) + t(1+C(t))\tilde{A}(t). \end{cases}$$

Furthermore, a sequence $e = (0, 0, e_3, \dots, e_n) \in NDS_n$ ($n \geq 2$) can be decomposed as follows.

- If there does not exist $k \geq 3$ such that $e_k = k - 1$, then $(e_3 - 1, e_4 - 1, \dots, e_n - 1) \in \mathbf{I}_{n-2}(-, -, \geq)$.
- Otherwise, let $k \geq 3$ be the leftmost saturated position of e . We need to consider three cases:
 - (i) If $k = n$, then $(e_3 - 1, e_4 - 1, \dots, e_{n-1} - 1) \in \mathbf{I}_{n-3}(-, -, \geq)$.
 - (ii) If $3 \leq k < n$ and $e_{k+1} < k - 1$, then e can be decomposed into two smaller sequences:

$$(e_3 - 1, e_4 - 1, \dots, e_{k-1} - 1, e_{k+1} - 1) \in \widetilde{\mathbf{I}}_{k-2}(-, -, \geq) \quad \text{and}$$

$$(0, 0, e_{k+2} - k + 1, e_{k+3} - k + 1, \dots, e_n - k + 1) \in NDS_{n-k+1}.$$

This decomposition is reversible.

- (iii) If $3 \leq k < n$ and $e_{k+1} = k - 1$, then e can be decomposed into two smaller sequences:

$$(e_3 - 1, e_4 - 1, \dots, e_{k-1} - 1) \in \mathbf{I}_{k-3}(-, -, \geq) \quad \text{and}$$

$$(0, 0, e_{k+2} - k + 1, e_{k+3} - k + 1, \dots, e_n - k + 1) \in NDS_{n-k+1}.$$

This decomposition is again reversible.

Translating the above decomposition into generating functions gives

$$(3.3) \quad \begin{cases} B(t) = t + (t^2 + t^3)A(t) + (t\tilde{A}(t) + t^2A(t))(B(t) - t) \\ \tilde{B}(t) = t + t^2\tilde{A}(t) + t^3A(t) + (t\tilde{A}(t) + t^2A(t))(\tilde{B}(t) - t). \end{cases}$$

Solving the system of equations (3.2) and (3.3) (cf. using Maple) leads to the following algebraic equation for $A(t)$.

Theorem 3.1. *The generating function $A(t)$ for $(-, -, \geq)$ -avoiding inversion sequences satisfies the algebraic equation:*

$$t^2 A^3(t) + (t^2 - t)A^2(t) + (2t - 1)A(t) + 1 = 0.$$

Equivalently, $tA(t)$ is the composition inverse of $t(1 + t - t^2)(1 + t)^{-2}$.

Comparing with the definition of the integer sequence [13, A049125] in OEIS we arrive at the following equinumerosity, as was conjectured by Martinez and Savage [12].

Corollary 3.2. *The number of $(-, -, \geq)$ -avoiding inversion sequences of length n equals that of ordered trees with n edges in which every interior vertices has at most one leaf child.*

It would be interesting to find a bijective proof of Corollary 3.2.

4. THE PATTERN $(>, -, \leq)$ AND $(4123, 4132, 4213)$ -AVOIDING PERMUTATIONS

The integer sequence A106228

$$\{1, 1, 2, 6, 21, 80, 322, 1347, 5798, 25512, \dots\}$$

in the OEIS [13] are defined as the coefficients of the algebraic generating function

$$(4.1) \quad A(t) = 1 + \frac{tA(t)}{1 - tA(t)^2}.$$

Recently, Albert, Homberger, Pantone, Shar and Vatter [1, Section 3.2] showed that this integer sequence enumerates permutations avoiding the patterns $(4123, 4132, 4213)$. Based on calculations, Martinez and Savage [12, Section 2.21] conjectured that inversion sequences avoiding the triple of relations $(>, -, \leq)$ are also counted by A106228. The following result confirms their conjecture affirmatively.

Theorem 4.1. *The cardinality of $\mathbf{I}_n(>, -, \leq)$ equals the number of permutations of length n avoiding $(4123, 4132, 4213)$.*

The rest of this section is devoted to an algebraic proof of Theorem 4.1. It would be interesting to see whether there is a bijective proof of this result with further refinement.

Inversion sequences avoiding $(>, -, \leq)$ are these that avoid the patterns 101 and 102, whose set is denoted by $\text{Inv}(101, 102)$. For an inversion sequence e , denote $|e|$ the length of e and $\max(e) = \max\{e_i : 1 \leq i \leq |e|\}$ the value of maximal entries of e . For any inversion sequence $e \in \text{Inv}(101, 102)$ with $\max(e) = k$, if $e_n = k$ is the rightmost maximal entry of e (equivalently, $e_n = k > e_{n+1}$) and $|e| = n + m$, then by definition, the subsequence $(e_i : 1 \leq i \leq n)$ is weakly increasing and $e_j \leq k - 1$ for all $n + 1 \leq j \leq n + m$ (if any). The case for $k = 0$ is trivial, so we consider the case $k \geq 1$. Let us introduce

$$\mathcal{A}_{n,k,m} = \{e \in \text{Inv}(101, 102) : |e| = n + m, e_n = \max(e) = k > e_{n+1}\}$$

and define the generating functions

$$\begin{aligned} \mathcal{A}(t; q, w) &= \sum_{n=k+1}^{\infty} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} |\mathcal{A}_{n,k,m}| (tw)^n t^m q^k, \\ \mathcal{M}(t; q, w) &= \mathcal{A}(t; q, w) - \mathcal{H}(t; q, w) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} |\mathcal{A}_{k+1,k,m}| (tw)^{k+1} t^m q^k. \end{aligned}$$

So the generating function $\mathcal{A}(t)$ of $\text{Inv}(101, 102)$, counted by the length (variable t), is given by $\mathcal{A}(t) = (1 - t)^{-1} + \mathcal{A}(t; 1, 1)$.

In order to derive a functional equation for $\mathcal{A}(t; q, w)$, we divide the set

$$\mathcal{T} = \bigcup_{n,k \neq 0, m} \mathcal{A}_{n,k,m}$$

into the following disjoint subsets:

$$\begin{aligned}\mathcal{T}_1 &= \bigcup_{n,k \neq 0,m} \{e \in \mathcal{A}_{n,k,m} : e_{n-1} = k\}, \\ \mathcal{T}_2 &= \bigcup_{n,k \neq 0,m} \{e \in \mathcal{A}_{n,k,m} : e_{n-1} < k, (k-1) \notin \{e_{n+i} : 1 \leq i \leq m\}\}, \\ \mathcal{T}_3 &= \bigcup_{n,k \neq 0,m} \{e \in \mathcal{A}_{n,k,m} : e_{n-1} < k, (k-1) \in \{e_{n+i} : 1 \leq i \leq m\}\}.\end{aligned}$$

Lemma 4.2. *The generating function $\mathcal{A}(t; q, w)$ satisfies*

$$(4.2) \quad \begin{aligned}\mathcal{A}(t; q, w) &= \frac{q(wt)^2}{(1-wt)(1-t)} + wt\mathcal{A}(t; q, w) + \frac{qtw^2}{1-w}\mathcal{H}(t; qw, 1) \\ &\quad + \frac{q(1-w-tw)}{1-w}\mathcal{H}(t; q, w).\end{aligned}$$

Proof. We count the generating function for each \mathcal{T}_i separately.

First, the generating function for \mathcal{T}_1 is $wt\mathcal{A}(t; q, w)$, since there is an easy one-to-one correspondence between \mathcal{T}_1 and \mathcal{T} , that is, for any $e \in \mathcal{T}_1 \cap \mathcal{A}_{n+1,k,m}$, removing the rightmost maximal $e_{n+1} = k$ gives a sequence in $\mathcal{A}_{n,k,m}$.

Second, the generating function for \mathcal{T}_2 is $q\mathcal{H}(t; q, w)$, since there is a bijection between $\mathcal{T}_2 \cap \mathcal{A}_{n,k,m}$ and $\mathcal{A}_{n,k-1,m}$, for any $n \geq k+1$. More precisely, for any $e \in \mathcal{T}_2 \cap \mathcal{A}_{n,k,m}$, since $e_{n-1} \leq k-1$ and all entries after e_n are not equal to $(k-1)$, according to the definition of $\text{Inv}(101, 102)$, after replacing k by $(k-1)$ on the n -th position of e , we obtain a sequence in $\mathcal{A}_{n,k-1,m}$ with $n \neq k$.

Third, the generating function for \mathcal{T}_3 is

$$\frac{q(wt)^2}{(1-wt)(1-t)} + \frac{qtw^2}{1-w}\mathcal{H}(t; qw, 1) - \frac{qtw}{1-w}\mathcal{H}(t; q, w).$$

For $k=1$, the set $\mathcal{T}_3 \cap \mathcal{A}_{n,1,m}$ contains all sequences $(0, \dots, 0, 1, 0, \dots, 0)$ and the corresponding generating function is $q(wt)^2(1-wt)^{-1}(1-t)^{-1}$. For any fixed $k \geq 2$, $n \geq k+2$ and $k+2 \leq i \leq n$, there is a bijection between

$$\{e \in \mathcal{T}_3 \cap \mathcal{A}_{i,k+1,n+m-i+1} : e_{n+1} = k > e_{n+2}\} \text{ and } \{(e, i) : e \in \mathcal{A}_{n,k,m}\}.$$

For any $e \in \mathcal{T}_3 \cap \mathcal{A}_{i,k+1,n+m-i+1}$, we claim that the subsequence $(e_{i-1}, e_{i+1}, \dots, e_{n+1})$ is weakly increasing. Note that $e_{i-1} \leq e_{i+1}$. Because if $e_{i-1} > e_{i+1}$, then all entries after e_{i+1} must be less than e_{i-1} in order to avoid patterns 101 and 102, which implies $k \leq e_{i-1} - 1$ and it contradicts the assumption that $e_{i-1} \leq k$. Similarly, we can show $e_j \leq e_{j+1}$ for all $i+1 \leq j \leq n$. Since $e_{n+1} = k > e_{n+2}$, removing the unique entry $(k+1)$ from e results in a new sequence in $\mathcal{A}_{n,k,m}$. Conversely, for any pair (e, i) such that $e \in \mathcal{A}_{n,k,m}$ and $k+2 \leq i \leq n$, we insert $(k+1)$ right after the $(i-1)$ -th entry of e , yielding a sequence from $\mathcal{T}_3 \cap \mathcal{A}_{i,k+1,n+m-i+1}$ such that the rightmost k is located on the $(n+1)$ -th position. In terms of the generating functions, we do the substitution

$$w^n \rightarrow w^{k+2} + \dots + w^n = \frac{w^{k+2} - w^{n+1}}{1-w}$$

in the generating function $qt\mathcal{H}(t; q, w)$, that is,

$$\frac{qtw^2}{1-w}\mathcal{H}(t; qw, 1) - \frac{qtw}{1-w}\mathcal{H}(t; q, w),$$

as desired.

Combining all the above three cases, we get (4.2), which completes the proof. \square

Next we turn to derive a functional equation for the generating function $\mathcal{M}(t; q, w)$.

Lemma 4.3. *The generating function $\mathcal{M}(t; q, w)$ satisfies*

$$(4.3) \quad \mathcal{M}(t; q, w) = \frac{qtw^2}{1-qw} \mathcal{A}(t; qw, 1) - \frac{qtw^2}{1-qw} \mathcal{A}(t; 1, qw) + \frac{q(tw)^2}{(1-t)(1-twq)}.$$

Proof. The generating function for all sequences $(0, \dots, 0, i, 0, \dots, 0)$ such that i is on the $(i+1)$ -th position is $q(tw)^2(1-t)^{-1}(1-twq)^{-1}$.

Let

$$\mathcal{A}'_{i+1, i, n+m-i} = \mathcal{A}_{i+1, i, n+m-i} \setminus \underbrace{\{(0, \dots, 0, i, 0, \dots, 0)\}}_{\text{length}=i+1}.$$

For $k \geq 1$ and $k+1 \leq i \leq n$, there is a bijection between

$$\{e \in \mathcal{A}'_{i+1, i, n+m-i} : \max\{e_j : j \neq i+1\} = k, e_{n+1} = k > e_{n+2}\} \text{ and } \{(e, i) : e \in \mathcal{A}_{n, k, m}\}.$$

For any $e \in \mathcal{A}'_{i+1, i, n+m-i}$ such that the second largest entry is k , we simply remove the unique i from e to get a new sequence in $\mathcal{A}_{n, k, m}$. Conversely, for any pair (e, i) such that $e \in \mathcal{A}_{n, k, m}$ and $n \geq i \geq k+1$, we insert i right after the i -th entry of e to get a new sequence in $\mathcal{A}'_{i+1, i, n+m-i}$ whose rightmost second largest entry k is located on the $(n+1)$ -th position. In terms of the generating functions, we do the substitution

$$w^n q^k \rightarrow tw^{k+2} q^{k+1} + tw^{k+3} q^{k+2} + \dots + tw^{n+1} q^n = tw \frac{(qw)^{k+1} - (qw)^{n+1}}{1-qw}$$

in the generating function $\mathcal{A}(t; q, w)$, that is,

$$\frac{qtw^2}{1-qw} \mathcal{A}(t; qw, 1) - \frac{qtw^2}{1-qw} \mathcal{A}(t; 1, qw).$$

Combining the above two cases, we conclude that (4.3) holds. \square

We are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We aim to show that the ordinary generating function $\mathcal{A}(t)$ of $\text{Inv}(101, 102)$ satisfies

$$(4.4) \quad \mathcal{A}(t) = 1 + t\mathcal{A}(t)(1 - t\mathcal{A}^2(t))^{-1},$$

which will finish the proof after comparing with (4.1).

By definition $\mathcal{M}(t; q, w) = w\mathcal{M}(t; qw, 1)$. Combining (4.2) and (4.3), we obtain

$$(4.5) \quad \left(1 - wt - \frac{q(1-w-tw)}{1-w}\right) \mathcal{A}(t; q, w) = \frac{qt^2 w^2 (1-q)}{(1-t)(1-twq)(1-tw)} \\ + \frac{tqw^2(1-q)}{(1-qw)(1-w)} \mathcal{A}(t; qw, 1) \\ + \frac{tq^2 w^2}{1-qw} \mathcal{A}(t; 1, qw).$$

We apply the kernel method and set the coefficient of $\mathcal{A}(t; q, w)$ to be zero. Solving for q in terms of t and w gives $q = (1 - wt)(1 - w)(1 - w - wt)^{-1}$. Let δ be defined by

$$\delta = \delta(t, w) = qw = \frac{w(1 - wt)(1 - w)}{1 - w - wt}.$$

Substituting $q = \delta w^{-1}$ on the right hand side of (4.5) implies that

$$(4.6) \quad \mathcal{A}(t; \delta, 1) = -\frac{z(1 - \delta)(1 - w)}{(1 - t)(1 - t\delta)(1 - tw)} - \frac{\delta(1 - w)}{w - \delta} \mathcal{A}(t; 1, \delta).$$

We set $\delta(t, z) = w$ and solve $z = z(t, w)$, which satisfies

$$(4.7) \quad tz^3 - (t + 1)z^2 + (w + tw + 1)z - w = 0.$$

Replacing w by z and δ by w on both sides of (4.6) leads to

$$(4.8) \quad \mathcal{A}(t; w, 1) = -\frac{t(1 - w)(1 - z)}{(1 - t)(1 - tw)(1 - tz)} - \frac{w(1 - z)}{z - w} \mathcal{A}(t; 1, w).$$

Let z_1, z_2, z_3 be the three roots of (4.7), then substituting z_1, z_2 into (4.8) gives

$$\begin{aligned} \mathcal{A}(t; w, 1) &= -\frac{t(1 - w)(1 - z_1)}{(1 - z_1)(1 - tw)(1 - tz_1)} - \frac{w(1 - z_1)}{z_1 - w} \mathcal{A}(t; 1, w), \\ \mathcal{A}(t; w, 1) &= -\frac{t(1 - w)(1 - z_2)}{(1 - t)(1 - tw)(1 - tz_2)} - \frac{w(1 - z_2)}{z_2 - w} \mathcal{A}(t; 1, w). \end{aligned}$$

Solving this system of two equations yields

$$(4.9) \quad \mathcal{A}(t; 1, w) = -\frac{t}{w(1 - tw)} \frac{(z_1 - w)(z_2 - w)}{(1 - tz_1)(1 - tz_2)}.$$

Setting $w = 1$ in (4.9) gives

$$\mathcal{A}(t; 1, 1) = -\frac{t(z_1 - 1)(z_2 - 1)}{(1 - t)(1 - tz_1)(1 - tz_2)},$$

where z_1, z_2, z_3 satisfy $tz^3 - (t + 1)z^2 + (t + 2)z = 1$, that is, equation (4.7) when $w = 1$. We will next express $\mathcal{A}(t; 1, 1)$ in terms of z_3 and t . Since z_1, z_2, z_3 are the three roots of $tz^3 - (t + 1)z^2 + (t + 2)z = 1$, clearly $z_1 z_2 z_3 = t^{-1}$ and $z_1 + z_2 + z_3 = 1 + t^{-1}$, leading to

$$\mathcal{A}(t; 1, 1) = \frac{(tz_3^2 - z_3 + 1)}{(1 - t)(tz_3 - tz_3^2 - t)}.$$

It remains to check if $\mathcal{A}(t) = (1 - t)^{-1} + \mathcal{A}(t; 1, 1)$ satisfies the functional equation (4.4).

In view of $tz_3^3 - (t + 1)z_3^2 + (t + 2)z_3 = 1$, we find that $tz_3(z_3^2 - z_3 + 1) = (1 - z_3)^2$, by which we can further simplify the formula of $\mathcal{A}(t; 1, 1)$ and consequently,

$$\mathcal{A}(t) = \frac{1}{1 - t} + \mathcal{A}(t; 1, 1) = \frac{1}{1 - t} + \frac{(tz_3 - 1)}{(1 - z_3)(1 - t)} = \frac{z_3}{z_3 - 1}.$$

It is easily seen that $\mathcal{A}(t) = z_3(z_3 - 1)^{-1}$ satisfies (4.4), which completes the proof. \square

5. THE PATTERN $(>, \leq, -)$ AND UNDERDIAGONAL LATTICE PATHS

As observed by Martinez and Savage [12, Section 2.19], an inversion sequence e belongs to $\mathbf{I}_n(>, \leq, -)$ if and only if there exists some t , $1 < t \leq n$, such that

$$(5.1) \quad e_1 \leq \cdots \leq e_t > e_{t+1} > \cdots > e_n.$$

They further conjectured that $(>, \leq, -)$ -avoiding inversion sequences are counted by [13, A071356] and the polynomial $U_n(z) := \sum_{e \in \mathbf{I}_n(>, \leq, -)} z^{\text{dist}(e)}$ is palindromic and unimodal. In this section, we will prove their conjecture and show that $U_n(z)$ is γ -positive, which is stronger than palindromic and unimodal.

Let $h(z) = h_r z^r + h_{r+1} z^{r+1} + \cdots + h_s z^s$ be a real polynomial with $h_r \neq 0$ and $h_s \neq 0$. It is called *palindromic* (or symmetric) of *darga* n if $n = r + s$ and $h_{r+i} = h_{s-i}$ for all i . For example, the darga of $1 + z$ and z are 1 and 2, respectively. Any palindromic polynomial $h(z)$ of darga n can be written uniquely as

$$h(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k z^k (1+z)^{n-2k},$$

where $\gamma_k \in \mathbb{R}$. If $\gamma_k \geq 0$ then we say that it is γ -positive of darga n . It is clear that γ -positivity implies palindromicity and unimodality, because each term $z^k (1+z)^{n-2k}$ in the expansion is palindromic with (the same) center $\lfloor \frac{n}{2} \rfloor$. One typical example of γ -positive polynomial is the *Eulerian polynomial* $\sum_{e \in \mathbf{I}_n} z^{\text{dist}(e)}$. For many other γ -positive polynomials arising in enumerative and geometric combinatorics, we refer the reader to the excellent book exposition by Petersen [14].

Introduce the generating function

$$\mathcal{D}(t, z) := \sum_{n \geq 1} U_n(z) t^n.$$

The following result confirms the above conjectures of Martinez and Savage.

Theorem 5.1. *The generating function $\mathcal{D}(t, z)$ satisfies the algebraic equation*

$$(5.2) \quad \mathcal{D} = 2tz\mathcal{D}^2 + (tz + t)\mathcal{D} + t.$$

Consequently, the polynomial $U_n(z)$ is γ -positive, and therefore palindromic and unimodal.

It is an immediate consequences of Eq. (5.2) that $(>, \leq, -)$ -avoiding inversion sequences are counted by A071356 in the OEIS [13], which Emeric Deutsch notes also counts the number of *underdiagonal lattice paths* from $(0, 0)$ to the line $x = n$ using only steps $R = (1, 0)$, $V = (0, 1)$ and $D = (1, 2)$. The rest of this section is devoted to a proof of Theorem 5.1.

Recall that a *Dyck path* of length n is a lattice path in \mathbb{N}^2 from $(0, 0)$ to (n, n) using the *east step* $(1, 0)$ and the *north step* $(0, 1)$, which does not pass above the line $y = x$. The *height of an east step* in a Dyck path equals the number of north steps before this east step. For our purpose, we will represent a Dyck path as $d_1 d_2 \cdots d_n$, where d_i is the height of its i -th east step. See Fig. 1 for the drawing of Dyck path 001113667. Denote by \mathcal{D}_n the set of all Dyck paths of length n . For any Dyck path $D = d_1 d_2 \cdots d_n \in \mathcal{D}_n$, we are concerned with the following two statistics:

- $\text{last}(D) = d_n$, the *height of the last east step* of D ;
- $\text{turn}(D)$, the number of *turns* of D , where a turn is an east step that is followed immediately by a north step.

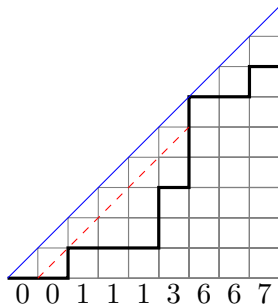


FIGURE 1. A Dyck path with its classical decomposition.

For example, if $D = 001113667$, then $\text{last}(D) = 7$ and $\text{turn}(D) = 5$. Define the generating function of Dyck paths with respect to these two statistics by:

$$D(t; x, u) := \sum_{n \geq 1} t^n \sum_{D \in \mathcal{D}_n} x^{\text{last}(D)} u^{\text{turn}(D)-1}.$$

We have the following functional equation for $D(t; x, u)$.

Lemma 5.2. *The generating function $D(t; x, u)$ satisfies*

$$(5.3) \quad D(t; x, u) = t + txuD(t; x, u) + txuD(tx; 1, u)D(t; x, u) + tD(t; x, u).$$

Proof. We will apply the usual decomposition of Dyck paths (see Fig. 1). Each Dyck path $D = d_1d_2 \dots d_n \in \mathcal{D}_n$ with $k = \min\{i \geq 2 : d_{i+1} = i \text{ or } i = n\}$ can be decomposed uniquely into a pair (D_1, D_2) of Dyck paths, where $D_1 = d_2d_3 \dots d_k \in \mathcal{D}_{k-1}$ (possibly empty) and $D_2 = (d_{k+1} - k)(d_{k+2} - k) \dots (d_n - k) \in \mathcal{D}_{n-k}$ (possibly empty). This decomposition is reversible and satisfies the following properties:

$$\text{last}(D) = \chi(D_2 \neq \emptyset) \cdot (k + \text{last}(D_2)) + \chi(D_2 = \emptyset) \cdot \text{last}(D_1)$$

and

$$\text{turn}(D) = \text{turn}(D_1) + \text{turn}(D_2) + \chi(D_1 = \emptyset).$$

In terms of generating function, this decomposition gives (5.3), as desired. \square

The key point to prove Theorem 5.1 lies in the following simple observation.

Lemma 5.3. *We have the following relation between $\mathcal{D}(t, z)$ and $D(t; x, u)$:*

$$(5.4) \quad \mathcal{D}(t, z) = D\left(t; 1 + tz, \frac{z + zt}{1 + tz}\right).$$

Proof. Recall from (5.1) that any $(>, \leq, -)$ -avoiding inversion sequence has the form

$$e_1 \leq \dots \leq e_t > e_{t+1} > \dots > e_n$$

for some $1 < t \leq n$. The first t entries can be considered as a Dyck path and the remaining $(n - t)$ entries are chosen from the set $\{0, 1, \dots, e_t - 1\}$, where a chosen entry contributes a new distinct entry if and only if it is chosen from $\{0, 1, \dots, e_t - 1\} \setminus \{e_1, e_2, \dots, e_{t-1}\}$. In terms of generating functions, this gives (5.4). \square

Proof of Theorem 5.1. Setting $x = 1$ in (5.3) and solving the resulting functional equation for $D(t; 1, u)$ gives

$$D(t; 1, u) = \frac{1 - t - tu - \sqrt{t^2u^2 - 2t^2u - 2tu + t^2 - 2t + 1}}{2tu}.$$

Plugging into (5.3) and solving the functional equation for $D(t; x, u)$ results in

$$D(t; x, u) = \frac{2t}{1 - tux + tx - 2t + \sqrt{t^2u^2x^2 - 2t^2ux^2 + t^2x^2 - 2tux - 2tx + 1}}$$

In view of relation (5.4), we have

$$\mathcal{D}(t, z) = \frac{2t}{1 - tz - t + \sqrt{tz(tz - 6t - 2) + (t - 1)^2}},$$

which is equivalent to (5.2).

It remains to show the γ -positivity of $U_n(z)$. By (5.2), we have the recursion for $U_n(z)$:

$$U_{n+1}(z) = (z + 1)U_n(z) + 2z \sum_{i=1}^{n-1} U_i(z)U_{n-i}(z) \quad \text{for } n \geq 1.$$

The γ -positivity of $U_n(z)$ then follows from this recursion and the facts that

- the product of a γ -positive polynomial of darga n with a γ -positive polynomial of darga m is γ -positive of darga $m + n$;
- the sum of two γ -positive polynomials of darga n is γ -positive of darga n .

This completes the proof of the theorem. \square

6. THE PATTERNS $(=, \geq, -)$, $(\geq, -, =)$ AND SET PARTITIONS

In [12, Section 2.17], Martinez and Savage made the following two conjectures:

- the number of $e \in \mathbf{I}_n(=, \geq, -)$ with k repeats, i.e., $n - 1 - \text{dist}(e) = k$, is given by A124323, the number of set partitions of $[n]$ with k blocks of size large than 1;
- there exists a bijection between $\mathbf{I}_n(=, \geq, -)$ and $\mathbf{I}_n(\geq, -, =)$, which preserves the quadruple $(\text{satu}, \text{max}, \text{zero}, \text{dist})$.

Bindi, Guerrini and Rinaldi [2] constructed recursively a bijection between set partition of $[n]$ and $\mathbf{I}_n(=, \geq, -)$, which proves the cardinality of $\mathbf{I}_n(=, \geq, -)$ is the n -th *Bell number*. In this section, we show that their bijection actually answers the first conjecture above and we will also construct the desired bijection for the second conjecture.

Let Π_n be the set of all set partitions of $\{0, 1, \dots, n - 1\}$. For a partition $\pi \in \Pi_n$ and an integer $0 \leq k \leq n - 1$, we say that k is *adjacent to itself* in π if k is the minimal element in its block; otherwise, k is *adjacent to j* in π , where

$$j = \max\{i : i < k \text{ and } i \text{ lies in the same block as } k\}.$$

Let $\text{adj}(k)$ denote the integer that k is adjacent to. The bijection $\Psi : \Pi_n \rightarrow \mathbf{I}_n(=, \geq, -)$ due to Bindi et al. [2] can be constructed recursively as

- $\Psi(\{0\}) = (0)$ and for $n \geq 2$ and $\pi \in \Pi_n$,
- if $k = \text{adj}(n - 1)$, then $\Psi(\pi) = (e_1, e_2, \dots, e_k, k, e_{k+1}, e_{k+2}, \dots, e_{n-1})$, where $\hat{\pi}$ is the partition π restricts to $\{0, 1, \dots, n - 2\}$ and $(e_1, e_2, \dots, e_{n-1}) = \Psi(\hat{\pi})$.

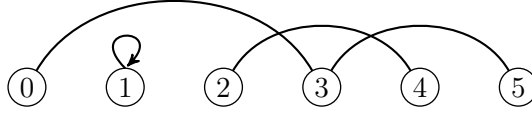


FIGURE 2. The arc diagram of $\{\{0, 3, 5\}, \{1\}, \{2, 4\}\}$.

Example 6.1. For partition $\pi = \{\{0, 3, 5\}, \{1\}, \{2, 4\}\} \in \Pi_6$ with its *arc diagram* depicted in Fig. 2, we can construct $\Psi(\pi)$ in the following steps:

$$(0) \rightarrow (0, \mathbf{1}) \rightarrow (0, 1, \mathbf{2}) \rightarrow (\mathbf{0}, 0, 1, 2) \rightarrow (0, 0, \mathbf{2}, 1, 2) \rightarrow (0, 0, 2, \mathbf{3}, 1, 2) = \Psi(\pi).$$

Proposition 6.2. Denote by $\text{bk}(\pi)$ and $\text{bk}_{\geq 2}(\pi)$ the number of blocks and the number of blocks of size larger than 1 of a partition π , respectively. Then, the bijection Ψ transforms the pair $(\text{bk}, \text{bk}_{\geq 2})$ on Π_n to the pair $(\text{rmin}, n - 1 - \text{dist})$ on $\mathbf{I}_n(=, \geq, -)$.

Proof. For any $\pi \in \Pi_n$, by the construction of Ψ , an integer is minimal in a block of size larger than 1 of π if and only if this integer appears two times in entries of $\Psi(\pi)$. Thus, $\text{bk}_{\geq 2}(\pi) = n - 1 - \text{dist}(\Psi(\pi))$. To show that $\text{bk}(\pi) = \text{rmin}(\Psi(\pi))$, we proceed by induction on n .

Suppose that $\hat{\pi} \in \Pi_{n-1}$ is the partition π restricts to $\{0, 1, \dots, n-2\}$ and $\text{bk}(\hat{\pi}) = \text{rmin}(\Psi(\hat{\pi}))$. If $n-1$ is adjacent to k in π , then k becomes a saturated value of $\Psi(\pi)$, which becomes a new right-to-left minimal letter of $\Psi(\pi)$ if and only if $k = n-1$, that is $\{n-1\}$ is a block of π . Therefore, $\text{bk}(\pi) = \text{rmin}(\Psi(\pi))$ holds and the proof is complete. \square

Theorem 6.3. There exists a bijection $\mathcal{R} : \mathbf{I}_n(=, \geq, -) \rightarrow \mathbf{I}_n(\geq, -, =)$, which preserves the quadruple $(\text{satu}, \text{max}, \text{zero}, \text{dist})$.

Proof. Notice that $\mathbf{I}_n(=, \geq, -) = \mathbf{I}_n(000, \mathbf{110})$, while $\mathbf{I}_n(\geq, -, =) = \mathbf{I}_n(000, \mathbf{101})$. The idea of the construction of \mathcal{R} is to replace iteratively occurrences of pattern 101 in an inversion sequence in $\mathbf{I}_n(000, \mathbf{110}) \setminus \mathbf{I}_n(000, \mathbf{101})$ with those of patterns 110.

Our \mathcal{R} when restricted to $\mathbf{I}_n(000, \mathbf{110}) \cap \mathbf{I}_n(000, \mathbf{101})$ is simply identity. So we only need to define the mapping \mathcal{R} from $\mathbf{I}_n(000, \mathbf{110}) \setminus \mathbf{I}_n(000, \mathbf{101})$ to $\mathbf{I}_n(000, \mathbf{101}) \setminus \mathbf{I}_n(000, \mathbf{110})$. Let $e \in \mathbf{I}_n(000, \mathbf{110}) \setminus \mathbf{I}_n(000, \mathbf{101})$. Clearly, e must contain the pattern 101 but avoid patterns 000 and 110. We apply the following *switch operation* to e :

- find the largest letter k such that k plays the role 1 in a pattern 101 of e ;
- switch the rightmost entry k with the rightmost entry smaller than k and lies between these two entries k .

We continue to apply the switch operation until there is no pattern 101 any more. Let $\mathcal{R}(e)$ be the resulting inversion sequence. For example, if $e = (0, 0, 1, 3, 2, 1, 4, 5, 7, 3, 4, 5) \in \mathbf{I}_{12}(000, \mathbf{110}) \setminus \mathbf{I}_{12}(000, \mathbf{101})$, then

$$\begin{aligned} (0, 0, 1, 3, 2, 1, 4, \mathbf{5}, 7, 3, \mathbf{4}, \mathbf{5}) &\rightarrow (0, 0, 1, 3, 2, 1, 4, \mathbf{5}, 7, \mathbf{3}, \mathbf{5}, 4) \rightarrow (0, 0, 1, 3, 2, 1, \mathbf{4}, \mathbf{5}, 7, 5, \mathbf{3}, \mathbf{4}) \\ &\rightarrow (0, 0, 1, \mathbf{3}, \mathbf{2}, \mathbf{1}, 4, 5, 7, 5, 4, \mathbf{3}) \rightarrow (0, 0, 1, \mathbf{3}, \mathbf{2}, \mathbf{3}, 4, 5, 7, 5, 4, 1) \rightarrow (0, 0, 1, 3, 3, 2, 4, 5, 7, 5, 4, 1) \end{aligned}$$

and we get $\mathcal{R}(e) = (0, 0, 1, 3, 3, 2, 4, 5, 7, 5, 4, 1) \in \mathbf{I}_{12}(000, \mathbf{101}) \setminus \mathbf{I}_{12}(000, \mathbf{110})$. To retrieve e , we just need to apply series of *inverse switch operation* with $\mathcal{R}(e) = \hat{e}$ as input:

- find the smallest letter k such that k plays the role 1 in a pattern 110 of \hat{e} ;

- switch the rightmost entry k with the leftmost entry that is smaller than k and lies after the rightmost entry k .

In each step of switch, the switch operation and inverse switch operation are inverse of each other. Therefore, \mathcal{R} is a bijection between $\mathbf{I}_n(000, 110)$ and $\mathbf{I}_n(000, 101)$. It is routine to check that \mathcal{R} preserves the quadruple $(\text{satu}, \text{max}, \text{zero}, \text{dist})$, which ends the proof. \square

7. THE PATTERN $(-, -, =)$ AND RESTRICTED SET PARTITIONS

In [12, Section 2.13], Martinez and Savage showed that

$$(7.1) \quad s(n, k) = (n - k)s(n - 1, k - 1) + (n - k - 1)s(n - 2, k - 1),$$

where $s(n, k) := |\{e \in \mathbf{I}_n(-, -, =) : \text{dist}(e) = k\}|$. They suspected that $(-, -, =)$ -avoiding inversion sequences are counted by the integer sequence [13, A229046] in OEIS. According to Alois P. Heinz, the sequence A229046 also enumerates set partitions such that the absolute difference between minimal elements of consecutive blocks (blocks are arranged so that the minimal elements are in increasing order) is always greater than 1. Let us denote such restricted set partitions of $[n]$ by RP_n . For instance, we have

$$\text{RP}_4 = \{1234, 12|34, 123|4, 124|3\}.$$

The following equidistribution is a generalization of the Martinez–Savage suspicion.

Theorem 7.1. *Let $\text{bk}(\pi)$ be the number of blocks of a set partition π . Then,*

$$(7.2) \quad |\{e \in \mathbf{I}_n(-, -, =) : \text{dist}(e) = n - k\}| = |\{\pi \in \text{RP}_{n+1} : \text{bk}(\pi) = k\}|.$$

Proof. Denote $\text{RP}_{n,k} = \{\pi \in \text{RP}_{n+1} : \text{bk}(\pi) = k\}$ and let $p(n, k)$ be its cardinality. We aim to show that

$$(7.3) \quad p(n, k) = kp(n - 1, k) + (k - 1)p(n - 2, k - 1),$$

which, after comparing with recursion (7.1), establishes the equidistribution (7.2).

To show (7.3), we introduce

$$a(n, k) = |\{\pi \in \text{RP}_{n+1,k} : \{n + 1\} \text{ is a block of } \pi\}|$$

and $b(n, k) = p(n, k) - a(n, k)$. Since $b(n, k)$ is the number of partitions in $\text{RP}_{n+1,k}$ such that $n + 1$ is not the only element of its block, we have

$$(7.4) \quad b(n, k) = kp(n - 1, k).$$

On the other hand, for each $\pi \in \text{RP}_{n+1,k}$ such that $\{n + 1\}$ is a block of π , we simply remove the block $\{n + 1\}$ to get a partition $\hat{\pi} \in \text{RP}_{n,k-1}$ such that $\{n\}$ is not a block of $\hat{\pi}$ (otherwise, the absolute difference between the minimal elements of the blocks $\{n\}$ and $\{n + 1\}$ of π is 1, a contradiction). It is clear that the mapping $\pi \mapsto \hat{\pi}$ is a one-to-one correspondence between

$$\{\pi \in \text{RP}_{n+1,k} : \{n + 1\} \text{ is a block of } \pi\} \text{ and } \{\pi \in \text{RP}_{n,k-1} : \{n\} \text{ is not a block of } \pi\}.$$

Therefore,

$$a(n, k) = b(n - 1, k - 1) = (k - 1)p(n - 2, k - 1),$$

where the second equality follows from (7.4). Combining this with (7.4) we get (7.3), which completes the proof. \square

The refinement $p(n, k)$ is now appears as [13, A298668] in OEIS. Recall that the *Stirling number of the second kind* $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ counts the set partitions of $[n]$ with k blocks. We have the following relation between $p(n, k)$ and $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$.

Proposition 7.2. *For any $1 \leq k \leq n$, we have*

$$p(n, k) = (k-1)! \left\{ \begin{smallmatrix} n-k+1 \\ k \end{smallmatrix} \right\}.$$

Proof. It is well known (cf. [16, Page 33]) that the Stirling numbers satisfy the recursion

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}.$$

It then follows that

$$\begin{aligned} S(n, k) &:= (k-1)! \left\{ \begin{smallmatrix} n-k+1 \\ k \end{smallmatrix} \right\} = k(k-1)! \left\{ \begin{smallmatrix} n-k \\ k \end{smallmatrix} \right\} + (k-1)! \left\{ \begin{smallmatrix} n-k \\ k-1 \end{smallmatrix} \right\} \\ &= kS(n-1, k) + (k-1)S(n-2, k-1). \end{aligned}$$

Comparing with (7.3), we see $S(n, k)$ and $p(n, k)$ share the same recurrence relation (and also initial value), thus $S(n, k) = p(n, k)$. This completes the proof. \square

ACKNOWLEDGEMENT

This work was supported by the National Science Foundation of China grant 11501244, the German Research Foundation DFG, JI 207/1-1 and the Austrian Research Fund FWF: START grant Y463, SFB project F50-02/03/10 and FWF-MOST (Austria-Taiwan) project P2309-N35. We appreciate the lecture rooms in the Univeristy of Vienna (Uni Wien) and the Vienna University of Technology (TU Wien) where our discussions were pursued.

REFERENCES

- [1] M.H. Albert, C. Homberger, J. Pantone, N. Shar and V. Vatter, Generating permutations with restricted containers, *J. Combin. Theory Ser. A*, **157** (2018), 205–232. [6](#)
- [2] A. Bindi, V. Guerrini and S. Rinaldi, Inversion Sequences and Generating trees, [talk](#) at Permutation Patterns 2017. [2](#), [12](#)
- [3] M. Bouvel, V. Guerrini, A. Rechnitzer, and S. Rinaldi, Semi-Baxter and strong-Baxter: two relatives of the Baxter sequence, [arXiv:1702.04529](#). [2](#)
- [4] S. Corteel, M. Martinez, C.D. Savage and M. Weselcouch, Patterns in Inversion Sequences I, *Discrete Math. Theor. Comput. Sci.*, **18** (2016), #2. [1](#)
- [5] S. Fu, E.Y. Jin, Z. Lin, S.H.F. Yan and R.D.P. Zhou, A new decomposition of ascent sequences and Eulerian–Stirling statistics, preprint (2018). [2](#)
- [6] V. Guerrini, On enumeration sequences generalising Catalan and Baxter numbers, Doctoral Dissertation, Università di Siena, 2017. [2](#)
- [7] S. Kitaev, *Patterns in permutations and words*, Springer Science & Business Media, 2011. [1](#)
- [8] D. Kim and Z. Lin, Refined restricted inversion sequences (extended abstract at FPSAC2017), *Sém. Lothar. Combin.*, **78B** (2017), Art. 52, 12 pp. [2](#)
- [9] Z. Lin and D. Kim, A sextuple equidistribution arising in Pattern Avoidance, *J. Combin. Theory Ser. A*, **155** (2018), 267–286. [2](#)
- [10] Z. Lin, Restricted inversion sequences and enhanced 3-noncrossing partitions, *European J. Combin.*, **70** (2018), 202–211. [2](#)
- [11] T. Mansour and M. Shattuck, Pattern avoidance in inversion sequences, *Pure Math. Appl. (P.U.M.A.)*, **25** (2015), 157–176. [1](#)

- [12] M. Martinez and C.D. Savage, Patterns in Inversion Sequences II: Inversion Sequences Avoiding Triples of Relations, *J. Integer Seq.*, **21** (2018), Article 18.2.2 ([arXiv:1609.08106v1](https://arxiv.org/abs/1609.08106v1)). [1](#), [2](#), [4](#), [5](#), [6](#), [10](#), [12](#), [14](#)
- [13] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, <http://oeis.org>, 2011. [1](#), [2](#), [4](#), [5](#), [6](#), [10](#), [14](#), [15](#)
- [14] T.K. Petersen, *Eulerian numbers*. With a foreword by Richard Stanley. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser/Springer, New York, 2015. [10](#)
- [15] C.D. Savage, The Mathematics of lecture hall partitions, *J. Combin. Theory Ser. A*, **144** (2016), 443–475. [1](#)
- [16] R. Stanley, *Enumerative combinatorics Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997. [15](#)
- [17] R. Stanley, *Enumerative combinatorics Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. [3](#)
- [18] S.H.F. Yan, Bijections for inversion sequences, ascent sequences and 3-nonnesting set partitions, *App. Math. Comput.*, **325** (2018), 24–30. [2](#)

(Wenqin Cao) SCHOOL OF SCIENCE, JIMEI UNIVERSITY, XIAMEN 361021, P.R. CHINA
E-mail address: wenqincao@jmu.edu.cn

(Emma Yu Jin) INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TU WIEN, WIEDNER HAUPTSTR.
8–10, 1040 WIEN, AUSTRIA
E-mail address: yu.jin@tuwien.ac.at

(Zhicong Lin) SCHOOL OF SCIENCE, JIMEI UNIVERSITY, XIAMEN 361021, P.R. CHINA
E-mail address: zhicong.lin@univie.ac.at