VU University, Amsterdam
Bachelorthesis

De Rham Cohomology of smooth manifolds

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1 Introduction

When I was told I had to write a thesis at the end of my bachelorphase I thought long and hard on the subject. In the short period I have spend studying mathematics I have always enjoyed both topology and analysis, as such my idea was to write my thesis on a subject in either field. Unfortunately choosing turned out to be a difficult task. Though, I remembered a course in my second year that seemed to mix both fields into one subject, namely the theory of smooth manifolds and differential geometry. Since it has always fascinated me I decided I would write about something related to this. I chose the subject of de Rham cohomology because it is very obvious that it relies heavily on both topology as well as analysis. One might even say it creates a natural bridge between the two. Of course I realized I wanted my thesis to be readable by others that may not have the prerequired knowledge of manifolds or differential forms, that is why the first part of this thesis (the first 5 chapters) is an introduction to smooth manifolds and differential forms. In the chapter after that I wrote a short introduction to algebraic topology, in here I define chains/cochains and basic exactness properties. Very important is the Zig-zag Lemma that will be used a lot in the later chapters. Then finally in chapter 7, de Rham cohomology groups are defined, as well as some basic properties proved. I dedicated chapter 8 to some examples of calculation of the de Rham groups. An important lemma is Poincaré's lemma that calculates the de Rham groups of contractible spaces. Chapter 9 starts with a short introduction to singular cohomology, and goes on with a proof of de Rham's theorem, which states that for smooth manifolds singular cohomology is identical to de Rham cohomology. A similar proof is used in chapter 10, where I proved Poincaré duality, which gives a relation between de Rham cohomology and de Rham cohomology with compact support. That is as far as this thesis will go, so I hope you'll find this an interesting read!

2 Smooth manifolds

To understand the ideas behind the de Rham cohomology it is first important to understand the types of spaces we will be working with. Initially the types of spaces we will be working with will be smooth manifolds but later on we will also consider submanifolds of any $\mathbb{R}^n$. A smooth manifold can best be described as a topological space that is locally very much like the Euclidian space of a certain dimension. The 'smooth' part of the name will relate to the differentiability of the maps that connect our space to the matching Euclidian space.

2.1 Formal definition of a smooth manifold

If we want to define what a smooth manifold is we first need to look at what a not-necessarily-smooth manifold is,

Definition 2.1. A topological space M is called a *manifold of dimension* n if:

· M is Hausdorff (points can be seperated by open sets).

· M is second countable (M has a countable topological base)

· For all $p \in M$ there is an open neighbourhood $U \subset M$ such that U is homeomorphic to an open subset V of $\mathbb{R}^n$.

These properties may look like they narrow down the amount of spaces we could work with, but in fact, in order to produce some of the theory this thesis will discuss we will need a stricter definition. In order to achieve this we will need the concept of an atlas. As we have seen in the definition of a manifold we need for each point in M a neighbourhood U that is homeomorphic to an open subset of $\mathbb{R}^n$. We can make this more rigorous by the following definition;

Definition 2.2. Let M be a manifold of dimension n. A pair (U, ψ), where $U \in M$ is open and $\psi : U \to V \subset \mathbb{R}^n$ a homeomorphism to some open V, is called a *chart*.

Remark 2.1. I will abuse notation in this thesis by saying that $p \in (U, \phi)$ if (U, ϕ) is a chart of M and $p \in U$.

We can thus rewrite our third condition from Definition 2.1 as;

· For all $p : p \in (U, \phi)$ for some chart.

The collection of charts such that each $p \in M$ is in a chart is called an *atlas*. It is important to realise that an atlas characterizes a manifold. Now that we have found a way to describe manifold with a collection of sets and maps we can add the additional requirement of 'smoothness'.

Definition 2.3. An atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ is called *smooth*, if for all ϕ_i, ϕ_j we have that $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism between $\phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$.

We already know that all the ϕ_α's are homeomorphisms, a smooth atlas only adds a certain degree of smooth transitioning to the equation. Now there is one more problem with a proper definition of a smooth manifold, and that is the fact that a manifold is not generated by a unique atlas; there could be multiple different atlasses that produce the same manifold. This is why we need to introduce the concept of a *maximal atlas*. Before we can formally introduce this we need the concept of *compatibility of atlasses*.

Definition 2.4. Atlasses $\mathcal{A}$ and $\mathcal{A}'$ are called compatible if $\mathcal{A} \cup \mathcal{A}'$ (contains any union of charts from $\mathcal{A}$ and $\mathcal{A}'$) is again a smooth atlas.

Remark 2.2. The relation '$\mathcal{A} \sim \mathcal{B} \Leftrightarrow \mathcal{A}$ and $\mathcal{B}$ compatible' forms an equivalence relation on smooth atlasses.

Definition 2.5. A *maximal atlas* for a manifold M is the union of all smooth atlasses in one equivalence class. In other words; $\mathcal{A}_{max} = \bigcup\{\mathcal{B} : \mathcal{B} \in [\mathcal{A}]\}$ for some $\mathcal{A}$ is a maximal atlas.

The maximal atlas of a manifold M is also called the differentiable structure of M. Now we can finally define what a smooth manifold is.

Definition 2.6. A *smooth manifold* is a pair $(M, \mathcal{A}_{max})$ where M is a manifold and $\mathcal{A}_{max}$ a differentiable structure of M.

While this is the formal definition, it usually suffices to find any smooth atlas for a manifold M to determine smoothness.

Remark 2.3. $\mathbb{R}^n$ is an n-dimensional smooth manifold with atlas $(\mathbb{R}^n, id)$.

We need one more definition, mostly because we also want a working definition for manifolds that have some sort of boundary, if we take for example the half-sphere $(p_1, p_2, p_3) \in \mathbb{R}^3 : p_1^2 + p_2^2 + p_3^2 = 1$ and $p_3 \geq 0$, we can see that our definition of a smooth manifold doesn't work since we cannot find an (open) chart around our points on the boundary of the half-sphere.

Definition 2.7. M is called a *manifold with boundary of dimension* n if:

- M is Hausdorff (points can be seperated by open sets).
- M is second countable (M has a countable topological base)
- For all $p \in M$ there is an open neighbourhood $U \subset M$ such that U is homeomorphic to an open subset V of $\mathbb{H}^n$.

The ressemblences to our previous definition of a manifold are clear, the difference is that open subsets of the upper-half plane $\mathbb{H}^n$ can contain something that could pass as a boundary. More general we might need corners, for instance think of a tetrahedron, which in itself is locally very much like a three dimensional space but the corners make it so that there are no diffeomorphisms to a Euclidian space. However if we replace $\mathbb{H}^n$ in definition 2.7 with $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_i \geq 0\}$ we get what is called a *manifold with corners*.

2.2 Smooth maps between manifolds

Now that we've defined what a smooth manifold is exactly, we want some sort of definition for a 'smooth' map between manifolds. Of course each manifold is a topological space, so maps have to be at least continuous. It would be a nice idea to have a sort of differentiability of a map between manifolds, however differentiability is merely a concept of functions to Euclidian spaces. That is why we define a smooth maps as follows;

Definition 2.8. Let M be an m-dimensional and N an n-dimensional smooth manifold. A continuous map $f : M \to N$ is called smooth if for all $p \in M$, $p \in (U, \phi)$ there is a chart (V, ψ) of N such that;

- $f(U) \subset V$
- $\hat{f} := \psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is infinitely differentiable.

Remark 2.4. The previously mentioned function $\hat{f}$ is called the *coordinate representation* of f

Remark 2.5. If $f : M \to N$ is a smooth homeomorphism and f^{-1} is also a smooth map we call f a diffeomorphism, and we say that M and N are diffeomorphic.

We can now also take a more categorical approach to smooth manifolds, which is illustrated by the following lemma.

Proposition 2.1. The collection $\mathcal{M} = \{M : M \text{ is a smooth manifold }\}$ together with $\mathcal{F} = \{f : M_1 \to M_2 | M_1, M_2 \in \mathcal{M} \text{ and f smooth }\}$, defines the category `Man` where $\mathrm{Ob}(\mathtt{Man}) = \mathcal{M}$ and $\mathrm{Mor}(\mathtt{Man}) = \mathcal{F}$.

Proof. All we need to show is that there is a smooth identity function, and that the composition of two smooth maps is smooth again.

- The identity function from any manifold to itself is smooth since $id_M(p) = p$, so we can take only one chart (U, ϕ) for both p and $id_M(p)$. Now it follows that $\widehat{id_M} = \phi \circ id_m \circ \phi^{-1} = id_{\mathbb{R}^m}$ where m is the dimension of M. And we know that the identity function on Euclidian spaces is infinitely differentiable.
- Let M, N and L be smooth manifolds of dimension m, n and l respectively. Consider two smooth maps $f : M \to N$, $g : N \to L$. We want to show that $g \circ f : M \to L$ is a smooth map. So take $p \in (U, \phi)$ and (V, ψ) such that $f(U) \subset V$ and $\hat{f}$ infinitely differentiable. The same we can do for $f(p) \in (V', \psi')$ and a chart (W, θ) of L. Now consider $V \cap V'$ with the coordinate ψ. We know ψ coincides with ψ' on $V \cap V'$. Now consider;

  $$\phi \circ f \circ g \circ \theta = \phi \circ f \circ \psi^{-1} \circ \psi \circ g \circ \theta.$$

  Now the latter is a composition of infinite differentiable functions, and as such so is the left side. This proves that $f \circ g$ is smooth.

$\square$

3 Tangent spaces

As we have seen in the previous sections smooth manifolds can be characterized by an atlas, however these atlasses can be very complex and hard to understand, so ideally we would like a simple characteristic of manifolds that we can easily work with. This section deals with one of these simpeler characteristics, namely the one of linear approximation. Specifically this can be viewed as a generalizations of the linear approach to the graph of a function (tangent line of the graph of a function can be viewed as a linear approximation). There are a few different definitions, we will use the one that uses paths in our space.

3.1 Paths and tangent spaces

Definition 3.1. A *path* α in a manifold M is an smooth function $\alpha : (-1, 1) \to M$.

Now to turn the previous definition into "vectors" that are tangent to our manifold we're going to need some sort of differentiation. Of course we cannot take the derrivative of our path, since its image is contained in M and not in a Euclidian space, hence we need some way to link the image of the path to some Euclidian space. Of course we have such a tool, namely the charts of our M. We might not be able to differentiate on M, but we can define for each point p and a chart (U, ϕ) around it the following vector in $\mathbb{R}^{\dim(M)}$;

$$\left(\frac{d}{dt}(\phi \circ \alpha)\right)(0),$$

if of course we take a path α with $\alpha(0) = p$. Now this isn't exactly ideal to work with, thus we will define an additional structure on it through means of identification of paths that look linearly equal at a given point $p \in M$.

Definition 3.2. For a $p \in M$ with $p \in (U, \phi)$, we define an equivalence relation $\sim$ on paths with $\alpha(0) = \beta(0) = p$ as follows;
$\alpha \sim \beta \Leftrightarrow (\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0)$

Proposition 3.1. Definition 3.2 does not depend on your choice of chart.

Proof. Let $p \in (U, \phi)$ and $p \in (V, \psi)$ for some manifold M, and let α be a path defined on M with $\alpha(0) = p$. Consider now $U \cap V$, which is an open set in M, we know by the fact that we are dealing with a manifold that on $U \cap V$, ϕ and ψ are the same. So if we restrict ourselves to this smaller open set which contains our point p we can differentiate and get the same resulting vector.

□

Now surprisingly we can recognize an additional structure on the *tangent space* $T_pM := \{\alpha : \alpha(0) = p \text{ and } \alpha \text{ is a path }\}/_\sim$.

Lemma 3.1. T_pM is a real vectorspace with;

- $c[\alpha] := [c\alpha], \forall c \in \mathbb{R}$
- $[\alpha] + [\beta] = [\gamma]$ for some γ with $(\phi \circ \alpha)'(0) + (\phi \circ \beta)'(0) = (\phi \circ \gamma)'(0)$.

I will not present the proof, as it is just an exercise in checking all the axioms of a vectorspace. Now as one would expect, the dimension of this vector space is finite, which of course means it is isomorphic to a Euclidian space.

Theorem 3.1. Let M be an m dimensional smooth manifold, and $p \in M$ then; $T_pM \cong \mathbb{R}^m$.

Proof. The following map is an isomorphism for any chart (U, ϕ), $\Phi : T_pM \to \mathbb{R}^m$, $[\alpha] \mapsto (\phi \circ \alpha)'(0)$.

- Injectivity follows from the definition of the class $[\alpha]$.
- Surjectivity follows from considering curves $\alpha_i := \phi^{-1}(x + te_i)$.
- The group actions follow from the definition of the vector operations.

$\square$

3.2 Working towards a categorical approach

We have seen before that the smooth manifolds with smooth maps form a category. Now categories come with functors, and the objective of this subsection is to make the first step towards the Tangentbundle-functor of manifolds. Functors work on spaces but also on the morphisms of a space to another. Therefore we need to construct a special linear map for each smooth function. We will do this as follows;

Definition 3.3. Let $f : M \to N$ where M and N are smooth manifolds. The *pushforward* of f, or $f_* : T_pM \to T_{f(p)}N$ is defined as $[\alpha] \mapsto [f \circ \alpha]$.

Remark 3.1. Using the following definition we can now define a proper (canonical) basis for T_pM namely by using a chart at p, (U, ϕ) and then defining $\frac{\partial}{\partial x_i}|_p := (\phi^{-1})_*(e_i)$. Note that this definition still heavily relies on the chosen chart, hence the (canonical) part.

Now in order for the pushforward to be a proper functorial result we need the following lemma.

Lemma 3.2. Let $f : M \to N$, $g : N \to P$ smooth functions and $p \in M$. Then,

(i) f_* is a linear map.

(ii) $(g \circ f)_* = g_* \circ f_*$.

(iii) $(id_M)_* = id_{T_pM}$.

(iv) If f is a diffeomorphism then f_* is an isomorphism (of vectorspaces).

3.3 Tangent bundles

Now we have defined at each point of a manifold what the tangent space is, we want some sort of way to have one structure for the entire manifold. This is why we define the vectorbundle.

Definition 3.4. (E, M, π) is called a vectorbundle of rank k if E, M are topological spaces, and $\pi : E \to M$ a continous surjection and,

- $\pi^{-1}(p)$ has a k-dimensional real vectorspace structure for all p.
- For all $p \in M$ there is an open neighbourhood U and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$. Such that on $\pi^{-1}(U)$ we have that $\pi_M \circ \Phi = \pi$, where π_M is the projection of $M \times \mathbb{R}^k$ to M. And also for each q in U we have that the restiction of Φ to $\pi^{-1}(q)$ gives an isomorphism from $\{q\} \times \mathbb{R}^k$ to $\mathbb{R}^k$.

Remark 3.2. We call Φ a *local trivialisation.*

Remark 3.3. We call a vectorbundle E smooth if both E and M are smooth manifolds and π is in fact a smooth map.

Basically we can see the vectorbundle as 'attaching' a vectorspace $\mathbb{R}^k$ to each of our points in the base M. Now naturally we didn't choose the name of our base without reason, in fact we will apply the idea of a vectorbundle to manifolds. Namely, we will attach to each point $p \in M$ the tangent space T_pM.

Definition 3.5. Let M be a smooth manifold then the tangent bundle TM is defined as,

$$TM := \bigsqcup_{p \in M} T_pM.$$

Now we want to add some additional structure to our tangent bundle, for one we want to see that it is in fact a smooth vectorbundle, but even more than that we would like to see that this makes it a

Theorem 3.2. Let M be an m-dimensional smooth manifold with smooth atlas $(U_\alpha, \phi_\alpha)_{\alpha \in I}$. Then TM is a smooth vector bundle of rank m over M with

$$\pi : TM \to M, \qquad T_pM \mapsto p.$$

Proof. Since we know that $\pi^{-1}(p) = T_pM$ we know by definition that it has an m-dimensional real vectorspace structure for all p, all that is left to prove is to find local trivialisations for all the charts, and of course to prove smoothness.

Let (U, ϕ) be a chart, define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ as

$$\Phi\left(\sum_i X^i \frac{\partial}{\partial x_i}|_p\right) := \left(p, X^1, \ldots, X^m\right).$$

Clearly this satisfies $\pi_M \circ \Phi = \pi$. Now to prove smoothness we need to find a smooth atlas for TM. Take a chart (U, ϕ), define the following function $\tilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2m}$ as;

$$\tilde{\phi}(\sum_i X^i \frac{\partial}{\partial x_i}|_p) = \left(\phi(p), X^1, \ldots, X^m\right).$$

Now the image of this map is $\phi(U) \times \mathbb{R}^m$ which is an open subset of $\mathbb{R}^{2m}$. These maps will be our charts. Also notice that;

$$\tilde{\phi}^{-1}(x^1, \ldots x^m, v_1, \ldots v_m) = \sum_i v_i \frac{\partial}{\partial x^i}|_{\phi^{-1}(x)}.$$

Now to check smoothness take two different charts of M, (U, ϕ) and (V, ψ) this corresponds to charts on TM, $(\pi^{-1}(U), \tilde{\phi})$ and $(\pi^{-1}(V), \tilde{\psi})$. Now we would like to see what these maps do on $\tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \psi(U \cap V) \times \mathbb{R}^m$ and $\tilde{\phi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \phi(U \cap V) \times \mathbb{R}^m$ so first consider

$$\begin{aligned}\tilde{\psi} \circ \tilde{\phi}^{-1}(x^1, \ldots, x^m, v^1, \ldots v^m) &= \tilde{\psi}\left(\sum_i v_i \frac{\partial}{\partial x^i}|_{\phi^{-1}(x)}\right) \\ &= \left(\psi \circ \phi^{-1}(x), v^1, \ldots v^m\right).\end{aligned}$$

This map is just $\psi \circ \phi^{-1} \times id_{\mathbb{R}^m}$ which is clearly a diffeomorphism by the fact that ψ and ϕ came from a smooth atlas.

Thus TM is a smooth manifold with atlas $\left(\pi^{-1}(U_\alpha), \tilde{\psi_\alpha}\right)_{\alpha \in I}$. (To prove that TM is a (not smooth) manifold is a matter of simple point-set topology).

□

Now that we know that TM has a known structure we can look a little closer at smooth vectorbundles, and see what maps between spaces are.

Definition 3.6. We call a pair $(f_\#, f)$ a *smooth bundle map* between (E, M, π) and (E', M', π') if,

- $f : M \to M'$ and $f_\# : E \to E'$ are smooth maps.
- $f_\#|_{E_p} : E_p \to E'_{f(p)}$ is a linear map for all p.
- $\pi' \circ f_\# = f \circ \pi$.

Remark 3.4. The class of all smooth vectorbundles with smooth bundle maps forms a category `Bund`.

Theorem 3.3. $T : \texttt{Man} \to \texttt{Bund}$ is a covariant functor. With $T(M) := TM$ and $T(f) := (f_{\#}, f)$ such that $f_{\#}|_{E_p} = f_* : T_pM \to T_{f(p)}M'$ is the pushforward of f.

Proof. Note that $E_p = T_pM$.

(i) First let us consider what T does on the identity map, We know by Lemma 3.2 that $\mathrm{id}_* = \mathrm{id}$, and by the definition of T we get that for all E_p $f_{\#}|_{T_pM} = \mathrm{id}$ thus $f_{\#} = \mathrm{id}$. And obviously $(\mathrm{id}, \mathrm{id})$ is the identity map on the smooth vectorbundles.

(ii) Next we will show that $T(g \circ f) = T(g) \circ T(f)$; proof by commuting diagram.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "T"'] & M' \arrow[r, "g"] \arrow[d, "T"'] & M'' \arrow[d, "T"'] \\
TM \arrow[r, "(f_{\#}, f)"] & TM' \arrow[r, "(g_{\#}, g)"] & TM''
\end{tikzcd}$$

$\square$

4 Cotangent bundle and differential forms

4.1 Cotangent spaces

In the previous section we have seen that tangent spaces are naturally endowed with a vectorspace structure. In linear algebra it is common to speak of the dual space of a vectorspace, namely all the linear functions from the vectorspace to a field. In this thesis we will take the field to be $\mathbb{R}$. We thus obtain the following definition,

Definition 4.1. The cotangent space of a manifold M at a point p is defined as, $T_p^*M := (T_pM)^*$, in other words it is the dual space of the vector space T_pM.

Remark 4.1. The vectors in the dual space are commonly called *covectors*.

Now a general result in finite dimensional linear algebra is that the dual space is again a vector space with the same dimension as the original space. Another important result is the following lemma,

Lemma 4.1. Let V be an n-dimensional vectorspace and $\{v_1, \dots v_n\}$ a basis for V, then the covectors $\{\theta_i\}$ such that $\theta_i(v_j) = \delta_{i,j}$, form a basis for the dual space V^* of V.

Remark 4.2. We can apply the above definition to our vector spaces T_pM and T_p^*M as such,

$\{\frac{\partial}{\partial x_i}|_p\}_{i=1\dots n}$ is a basis for T_pM and $\{dx^j|_p\}_{j=1\dots n}$ is the basis for T_p^*M such that, $dx^j|_p\left(\frac{\partial}{\partial x_j}|_p\right) = \delta_{i,j}$.

We call $dx^i|_p$ *differentials.*

Analogue to the case where we determined the maps that are induced by the tangent functor we need some sort of "pushforward" for the cotangent spaces. We can almost do this, however the direction is reversed and we call this a *pullback*.

Definition 4.2. Let M and N be smooth manifolds, and $f : M \to N$ a smooth map. Then the *pullback* of f denoted as f^* is a function from $T_{f(p)}^*N \to T_p^*M$ which does the following

$$f^*\theta\left(\sum_i X^i \frac{\partial}{\partial x_i}|_{f(p)}\right) = \theta\left(f_* \sum_i X^i \frac{\partial}{\partial x_i}|_p\right).$$

Which is well defined since θ lives in the cotangent bundle on N and $\theta \circ f_*$ in the cotangent bundle of M.

Example 4.1. Let $f : M \to \mathbb{R}$ a smooth map, then on $p \in M$ we have a function $f_* : T_pM \to \mathbb{R}$ which is clearly an element of T_p^*M thus we can write it as $\sum_{i=1}^{n} \lambda_i dx^i|_p$. Now we can see what f_* does on a vector $\frac{\partial}{\partial x_j}|_p$:

$$f_* \frac{\partial}{\partial x_j}|_p = \sum_{i=1}^{n} \lambda_i dx^i|_p \frac{\partial}{\partial x_j}|_p = \lambda_j.$$

However we also have that $f_* \frac{\partial}{\partial x_j}|_p = f_*[\phi^{-1}(x+te_i)] = [f \circ \phi^{-1}(x+te_i)] = \frac{\partial \hat{f}}{\partial x_i}$. So all together we get,

$$f_* = \sum_{i=1}^{n} \frac{\partial \hat{f}}{\partial x_i} dx^i|_p.$$

4.2 Cotangent bundle

In the last section we have tried to put, for an m-dimensional smooth manifold M, all tangent spaces into one bigger space of dimension $2m$. We can do exactly the same with the cotangent spaces.

Definition 4.3. The *Cotangent bundle* of a smooth manifold M is defined as,

$$T^*M := \bigsqcup_{p \in M} T_p^*M.$$

Proposition 4.1. *For M an m-dimensional smooth manifold we have that T^*M is a smooth vectorbundle of rank m.*

Proof. See the proof of Theorem 3.2

Proposition 4.2. *$T^* : \texttt{Man} \to \texttt{Bun}$ is a contravarient functor with, $T^*(M) = T^*M$ and $T(f) = (f_\#, f)$ where $f_\#|_{T_p^*M} = f^*$.*

Proof. The proof is identical to that of the theorem about the functoriality of the functor T.

4.3 Smooth vector fields and smooth sections

From calculus we know what a vectorfield is, namely a function that sents every point in some Euclidian space to a vector. It is possible to do exactly that with manifolds and tangent bundles.

Definition 4.4. A *smooth vectorfield* is a smooth function $X : M \to TM$ such that $\pi \circ X = id_M$.

Remark 4.3. It follows from the definition that X sents each point p in M to a tangentvector in the tangentspace T_pM. In this way it relates to what was previously mentioned about Euclidian vector fields.

Of course now that we know that the cotangent spaces also form a vector-bundle we can repeat this definition. However, it turns out we will use this definition far more often and thus it gets a different name.

Definition 4.5. A *differential 1-form* θ is a smooth vector field from M to T^*M.

Remark 4.4. The space of all differential 1-forms is usually denoted $\Gamma^1(M)$

The fact that there is a 1 in the denotion of all the differential 1-forms makes it seem as though there is something as a differential k-form. Indeed we will see in the next chapter that these exist and are of great importance to the de Rahm Cohomology.

It is clear that the differential 1-forms and the vector fields have a lot in common, after all they were defined in almost exactly the same way. However there is one big difference, namely that of the pullback-property of the differential 1-forms.

Definition 4.6. Let $f : M \to N$ a smooth map and $X(p)$ a vectorfield, then we define the *pullback* on differential 1-forms as such, for $\theta \in \Gamma^1(N)$:

$$f^*\theta(X(p)) = \theta(f_*(X(p))) \in \Gamma^1(M).$$

Now if we attempt to do the same with vector fields and pushforwards we have a problem, namely if we have a function $f : N \to M$ we get that a vector field on N (which sents each point of N to a vector) might not be injective, and thus we get on the pushed-forward vector field that one point in M admits to multiple vectors, which does not create a vectorfield. Also, if f does create a vector field on M, all points in M should be hit by f, after all a vectorfield is a function from ALL of M to TM. Thus we need a smooth bijection in order for a vector field to be pushed forward, this is of course only the case if f is a diffeomorphism. While on differential 1-forms no restrictions are necessary.

5 Tensor products and differential k-forms

As mentioned before, we will now generalize the notion of a 1-form. We can see the 1-forms as something one dimensional. It turns out differential forms are very natural to integrate over. However we can only integrate maximum forms, for instance if we want to integrate a form over the sphere, it turns out we will need a 2-form, 2 being the dimension of the sphere. However we will need a few more definition before we can see what a differential k-form is.

5.1 Tensors

Definition 5.1. A *covariant k-tensor on V T* is a (multi)linear function $T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$. Where V is a vectorspace.

Remark 5.1. k in the previous definition is called the *rank* of the tensor.

Remark 5.2. If we take V in the definition to be a dual vector space we call the tensor *contravariant* on V.

We denote the collection of all covariant r-tensors on V, $T^r(V)$ and the set of all contravariant r-tensors $T_r(V)$. These spaces are vectorspaces. Now we would like to find a way to multiply two tensors to make a new one. Naturally this is hard to do in vectorspaces in general. However in $\mathbb{R}$ multiplication is as easy as it gets. Thus the *tensorproduct* is defined as follows,

Definition 5.2. Let $T : V^n \to \mathbb{R}$ and $S : W^m \to \mathbb{R}$ be two contravariant n-respectively m-tensors. Then
$T \otimes S(v_1, \ldots v_n, w_1, \ldots w_m) := T(v_1, \ldots v_n)S(w_1, \ldots w_m)$.

Remark 5.3. It is easy to see that now the tensorproduct of two covariant tensors on V is again a covariant tensor on V. With the rank of the new tensor the sum of the ranks of the original tensors.

Remark 5.4. Do notice that taking the tensor product is not a commutative action.

The tensorspaces of any vector space V are, as noted before, vectorspaces. These vectorspaces are n^r dimensional, where r is the rank of the tensors in the space and n the dimension of V. But since this is a finite dimensional vectorspace we can find a basis for $T^r(V)$,

Theorem 5.1. Let V be a vectorspace with dual basis $\{\theta^1, \ldots \theta^n\}$ then,

$$\{\theta^{i_1} \otimes \cdots \otimes \theta^{i_r} : i_j \in \{1, \ldots n\},$$

is a basis for $T^r(V)$.

5.2 Symmetric and alternating tensors

The fact that tensors are multilinear functions does not mean we can interchange any arbitrary argument with another. However if the tensor can we call it *symmetrical*. If we can interchange any arbitrary argument with another and the result is -1 times the unchanged tensor, we call the tensor *alternating*.

Now clearly not every tensor is symmetrical or alternating, however we can always make a tensor symmetrical or alternating in the following way,

Definition 5.3. Sym $T = \frac{1}{r!}\sum_{\sigma\in S_r} {}^{\sigma}T$.

Now this definition needs some explaination, first of all what ${}^{\sigma}T$ is. Basically it is T where we interchange the arguments according to the cycle σ. We sum over all the different cycles and we then divide by the amount of cycles in S_r. In a way what we are doing is taking the average over all permutated tensors. Now it follows that any interchanging of arguments, does nothing on the tensor Sym T, because $(ij)S_r = S_r$.

In the same way we can define a function that makes a tensor alternating.

Definition 5.4. Alt $T = \frac{1}{r!}\sum_{\sigma\in S_r} \epsilon(\sigma)\, {}^{\sigma}T$.

Where ϵ is the signfunction for permutations. Now that we have the Alt function we can define one of the most important operations in differential geometry.

Definition 5.5. Let ψ and θ be tensors on V, then
$\psi \wedge \theta = \text{Alt } \psi \otimes \theta$.

This operation is called the *wedge product*. The set of all alternating r-tensors on V is usually denoted $\Lambda^r(V)$

5.3 Some algebra on $\Lambda^r(V)$

Since we will be working with the space of alternating tensors a lot we will use this section to do some computations that will make it easier to work with it.

Lemma 5.1. Let T, S, R be t-, s- respectively r-tensors then,

(i) $T \wedge (S \wedge R) = (T \wedge S) \wedge R$.

(ii) $(T + S) \wedge R = T \wedge R + S \wedge R$.

(iii) $T \wedge S = (-1)^{ts} S \wedge T$.

(iv) $T \wedge T = 0$.

Proof. We will proof only the most important part of the lemma, namely $T \wedge T = 0$; For one forms this is trivial since $\psi \wedge \psi = \psi \otimes \psi - \psi \otimes \psi = 0$. Now for higher forms the same principle applies, for every permutation we get in the sum there is the antipodal permutation which we can get to by switching around arguments (and since it's alternating tensors we just multiply the result by -1). This will end up analogously to the result for 1 forms with 0.

$\square$

Lemma 5.2. For V a vectorspace with covector basis $\{\theta^1, \dots, \theta^n\}$, $\Lambda^r(V)$ is a (sub)vectorspace (of $T^r(V)$) with basis

$$\{\theta^{i_1} \wedge \cdots \wedge \theta^{i_r} : 1 \leq i_1 < \cdots < i_r \leq n\}.$$

Remark 5.5. Notice that there is a strict inequality in the definition of the basis. This is, of course, because of property (iv) of Lemma 5.1.

Remark 5.6. With Lemma 5.1 and the basis defined as above we can see that for a vectorspace with dimension n we get that $\Lambda^{n+i}(V) = 0$ for all $i \in \mathbb{N}$.

5.4 Tensor bundles

Now naturally, our next step is to apply the tensor theory to vectorspaces we know and have worked with, namely tangent spaces of manifolds. We of course take a look at the covariant r-tensors of the tangentspace at a point $p \in M$; $T^r(T_pM)$. As we know these spaces are vectorspaces themselves and as such it is possible to define a vectorbundle;

Definition 5.6. $T^rM := \bigsqcup_{p \in M} T^r(T_pM)$

As with the tangent bundle we can also show that this space is in fact a smooth vector bundle of rank $m!$ over M. We call it the *covariant r-tensor bundle*. A useful identity is $T^1M = T^*M$. We can now, as we did before define *tensor field*, which are basically smooth sections in T^rM.

Definition 5.7. A smooth *tensor field* is a smooth function $\sigma : M \to T^rM$ such that $\pi \circ \sigma = id_M$.

If we restrict ourselves to all the alternating tensors, rather than all of them we end up with the tensorbundle Λ^rM which is a subvectorbundle of T^rM and also our main objective in this section.

Definition 5.8.

$$\Lambda^rM = \bigsqcup_{p \in M} \Lambda^r(T_pM)$$

Just like we can pullback smooth vector fields we can also pullback tensorfields.

Definition 5.9. Let $f : M \to N$ be smooth and $X_i \in T_pM$, we have $f^* : T^k(T_{f(p)}N) \to T^k(T_pM)$ defined as,

$$f^*\theta(X_1, \dots, X_k) = \theta(f_*X_1, \dots, f_*X_k).$$

Remark 5.7. This new notion of pullback is more general than the one from Definition 4.6 since $T^*(T_pM) = T^1(T_pM)$.

Lemma 5.3. Let $f : M \to N$, $g : N \to P$ be smooth functions. $p \in M$. $\theta \in T^k(T_{f(p)}N)$ and $\psi \in T^l(T_{f(p)}N)$. Then,

(i) f^* is a linear map.

(ii) $f^*(\theta \otimes \psi) = f^*\theta \otimes f^*\psi$.

(iii) $(g \circ f)^* = f^* \circ g^*$.

(iv) $id^* = id$.

(v) f^* induced a smooth bundle map in the obvious way.

Now finally we have the means to define *differential r-forms*,

Definition 5.10. A differential r-form on a manifold M is a smooth section on $\Lambda^r M$. And the space of all differential r-forms is called $\Gamma^r(M)$.

6 Differential forms

This section is devoted to differential forms and operations on them. As mentioned before differential forms are crucial in studying the de Rham Cohomology of a manifold. We have already seen that differential forms are smooth sections of covariant tensorbundles but they also have a very physical intepretation, namely they will turn out to be very 'natural' to integrate over a manifold.

6.1 Contractions and exterior derivatives

The first operation we will take a look at is contracting a differential form, this operations sends a differential k-form to a differential $k-1$ form, in a quite natural way.

Definition 6.1. The contraction (with X) of a differential k-form θ, is a linear function $i_X : \Gamma^k \to \Gamma^{k-1}$ where X is a smooth section $M \to TM$ defined by,

$$i_X\theta := \theta\left(X, \cdot, \cdots, \cdot\right)$$

This basically means we fix the first coordinate of our smooth section and thus leave $k-1$ 'free' arguments. The contraction admits to a few properties,

Remark 6.1. Let $\sigma \in \Gamma^r(M), \omega \in \Gamma^s(M)$ and $X : M \to TM$ a smooth section.

- Contracting is also lineair in it's smooth section argument.
- $i_X \circ i_X = 0$.
- $i_X\left(\sigma \wedge \omega\right) = \left(i_X\sigma\right) \wedge \omega + (-1)^r \sigma \wedge \left(i_X\omega\right)$

Remark 6.2. This last property is also called anti-derivation, for reasons that will soon be made clear.

The next operation on differential is probably the most important for the De Rham Cohomology. We will formulate the existence of these operators as a theorem. But first a bit of notation.

$$\omega_I dx^I = \omega_{i_1 \ldots \imath_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m}, I = (i_1 \ldots i_k).$$

Theorem 6.1. Let M be a smooth manifold. Then there are linear maps $d_k : \Gamma^k(M) \to \Gamma^{k+1}(M)$ for all $k \geq 0$ such that:

(i) If $f \in \Gamma^0(M)$ then $d_0 f = f_*$.

(ii) If $\omega \in \Gamma^k(M)$ and $\eta \in \Gamma^l(M)$ then

$$d_{k+l}(\omega \wedge \eta) = d_k\omega \wedge \eta + (-1)^k \omega \wedge d_l\eta.$$

(iii) $d_{i+1} \circ d_i = 0 \quad \forall i \geq 0$.

Proof. We will proof this theorem in two steps.

1 First assume M has an atlas with exactly one chart, and let $(x^1, \dots, x^m)$ be it's coordinates. We can now define d_k as such;

$$d_k(\sum_J \omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}) = \sum_J \sum_{i=1}^m \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

Or: $d(f dx^I) = df \wedge dx^I$. However this only works for increasing indices I since they form the basis of our $\Gamma^k(M)$. This map is clearly linear (by linearity of partial derivation), and it satisfies (i) by $f_* = \sum_{i=1}^m \frac{\partial f}{\partial_x} dx^i$.

Now $d(f dx^I) = df \wedge dx^I$ works for not only increasing indices but, in fact, for all indices. To see this consider the permutation σ that send an index J to an increasing index I. We thus get

$$d(f dx^J) = \epsilon(\sigma) d(f dx^I) = \epsilon(\sigma) df \wedge dx^I = df \wedge dx^J.$$

Now we can prove (ii), let $\omega = f dx^I, \eta = g dx^J$

$$\begin{aligned} d(\omega \wedge \eta) &= d(fg dx^I \wedge dx^J) \\ &= d(fg) \wedge dx^I \wedge dx^J \\ &= (g df + f dg) \wedge dx^I \wedge dx^J \\ &= (df \wedge dx^I) \wedge (g dx^J) + (-1)^k (f dx^I) \wedge (dg \wedge dx^J) \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \end{aligned}$$

By linearity we can extend this to any differential form.

For (iii) consider first, for a 0-form f:

$$\begin{aligned} d(df) &= \sum_{i=1}^k \sum_{j=1}^k \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j \\ &= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j = 0. \end{aligned}$$

Now by (ii) we get that

$$d(d\omega) = d(\sum_J d\omega_J \wedge dx^J) = \sum_J d(d\omega_J) \wedge dx^J = 0.$$

2 Now let M be any manifold. We have now shown that we can define a differential operator for all charts. Now we would like to see that they coincide on overlapping charts, so let U and V be two charts. My claim is that $d_U|_{U \cap V} = d_V|_{U \cap V}$, thus we can find a single d that satisfies the properties mentioned above for the whole M rather than just a single chart. Namely by

$$(d\omega)_p = (d_U \omega|_U)_p) \quad p \in U.$$

□

Lemma 6.1. The above mentioned differential operator is unique.

Proof. Let d and d' be two operators that satisfy the conditions of Theorem 6.1. Note that by (i) we get that on 0-forms $d = d'$, after all $df = f_* = d'f$. Now by induction assume that the property holds for differential $n-1$ forms. Look now at $d(fdx^{i_1}|_p \wedge \cdots \wedge dx^{i_n}|_p) = df \wedge dx^{i_1}|_p \wedge \cdots \wedge dx^{i_n}|_p = d'f \wedge dx^{i_1}|_p \wedge \cdots \wedge dx^{i_n}|_p = d'(fdx^{i_1}|_p \wedge \cdots \wedge dx^{i_n}|_p)$. Now by linearity this extends to arbitrary differential forms.

□

Proposition 6.1. The differential operator commutes with pullbacks of maps, let $f : M \to N$

$$f^* d\omega = df^*\omega.$$

for $\omega \in \Gamma^k(N)$

Proof. Let $\omega \in \Gamma^k(N)$, we need only check the above locally since d works locally. So take $p \in N$ Now by linearity of both f^* and d we need only check this property for $gdx^{i_1}|_p \wedge \cdots \wedge dx^{i_k}|_p$. Now,

$$\begin{aligned} f^*d\left(gdx^{i_1}|_p \wedge \cdots \wedge dx^{i_k}|_p\right) &= f^*\left(dg \wedge dx^{i_1}|_p \wedge \cdots \wedge dx^{i_k}|_p\right) \\ &= d(g \circ f) \wedge d(x^{i_1} \circ f)|_p \wedge \cdots \wedge d(x^{i_k} \circ f)|_p \\ &= d\left((g \circ f)d(x^{i_1} \circ f)|_p \wedge \cdots \wedge d(x^{i_k} \circ f)|_p\right) \\ &= df^*\left(gdx^{i_1}|_p \wedge \cdots \wedge dx^{i_k}|_p\right). \end{aligned}$$

□

6.2 Integrating over topforms

One of the (possibly) surprising facts about differential forms is, that there's a very natural way to integrate over them. In fact the construction of differential forms started all because of the need for a unique way to integrate over a manifold. Normally integrating with two different parametrizations could result in two different integrals, something you would like to avoid.

Definition 6.2. Let M be an n-dimensional smooth manifold, and let $\{E_1, \dots E_n\}$ and $\{E'_1, \dots E'_n\}$ be two bases for some $p \in M$. We say that two basis are *consistently oriented* if the transitionmatrix between them has positive determinant. This forms an equivalence relation, and the equivalence classes are called *orientations*. We call a manifold with a given basis for all p an *oriented manifold*. All the bases that are in the equivalence class of the given basis are called *positively oriented* and those that are not are called *negatively oriented*.

Remark 6.3. A manifold M is called *orientable* if there exists an orientation for it. There are manifolds that are non-orientable, such as the *Möbius Strip*.

An important property of orientable manifolds is that there always exists a non-vanishing topform that is positively oriented at each point. (The value of the form evaluated at a positively oriented basis is > 0). We call such a form an *orientation form.*

We need the orientation to determine the sign of the integral, after all we would like integration over the top half of any euclidian space of a positive function to be positive.

Definition 6.3. Let U be an open subset of some euclidian space and $f dx^1 \wedge \cdots \wedge dx^m$ a (compactly supported) topform on U. Then we say;

$$\int_U f dx^1 \wedge \cdots \wedge dx^m := \int_U f dx^1 \cdots dx^m$$

i.e. normal Lebesgue integration.

Definition 6.4. Let M be an n-dimensional oriented manifold with one coordinate chart (U, ϕ), and let $\omega \in \Gamma^n(M)$. Then we say

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega.$$

Where we use the previous definition to evaluate the integral over a differential form on a subset of a Euclidian space.

Remark 6.4. Note that the previous definition does not rely on the choice of coordinate (We will not proof this as I do not think it gives great insight).

We can now define the integral over any manifold of a differential form with compact support. Still we need the compact support as will become clear in the definition. Note that if $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M this is an open cover for every subset of M, as such also for the compact support of a differential form. But by compactness only a finite amount of charts is needed to cover it. As such consider $(U_i, \phi_i)_{i=1}^N$ to be this finite collection. And $\{\psi_i\}$ to be it's partition of unity.

Definition 6.5. Let M, ω be as above. Then

$$\int_M \omega = \sum_{i=1}^N \int_M \psi_i \omega.$$

Now later on we will need integration in a more general sense, namely integrating forms over manifolds with boundary or even manifolds with corners. In these cases not a lot changes apart from the open sets (in Euclidian spaces) that could differ when dealing with boundaries or corners.

Now a final lemma which will be used later on;

Lemma 6.2. Let M be an orientable manifold, and ω_0 an orientation form.

$$\int_M \omega_0 > 0.$$

7 Cochains and cohomologies

This chapter is all about basic algebraic topology, since the de Rham Cohomology will turn out to be a *cohomology theory* it is important to see what that means. Therefore we will work towards this concept step by step.

7.1 Chains and cochains

In this subsection we will not be talking about manifolds at all. Mostly we will talk of modules (over a general ring R or $\mathbb{Z}$).

Definition 7.1. A sequence $C_\bullet = (C_n, \partial_n \mid n \in \mathbb{Z})$ of modules C_n and homomorphisms $\partial_n : C_n \to C_{n-1}$ is called a *chain complex*, if for all $n \in \mathbb{Z}$ we have that $\partial_{n-1} \circ \partial_n = 0$ holds.

Remark 7.1. The ∂_n functions are usually called the *boundary operators* or *differentials*.

Remark 7.2. A chain complex is usually visualised in a diagram as such,

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

An important observation is that since $\partial_{n-1} \circ \partial_n = 0$ we immediately get that $\operatorname{Im}(\partial_n) \subset \operatorname{Ker}(\partial_{n-1})$. Note that these are both submodules of C_n

Definition 7.2. The *n-cycles* of a chain complex $C_\bullet$ is;

$$Z_n(C_\bullet) = \{ \operatorname{Ker}(\partial_n)\}.$$

Definition 7.3. The *n-boundaries* of a chain complex $C_\bullet$ is;

$$B_n(C_\bullet) = \{ \operatorname{Im}(\partial_{n+1})\}.$$

Definition 7.4. The *n-th homology module* of a chain complex $C_\bullet$ is;

$$H_n(C_\bullet) = Z_n/B_n.$$

Definition 7.5. Let $C_\bullet = (C_n, c_n)$ and $D_\bullet = (D_n, d_n)$ be chain complexes. Then a chainmap $f_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $f_n : C_n \to D_n$ such that;

$$\begin{tikzcd} C_n \arrow[r, "c_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "d_n"] & D_{n-1} \end{tikzcd}$$

commutes for all n.

Now naturally it is possible that after taking the homology quotient nothing is left. This is a special kind of chaincomplex,

Definition 7.6. We call a chaincomplex(sequence) *exact* if $Z_n = B_n$ for all n

Remark 7.3. An exact sequence has only trivial homology modules.

Remark 7.4. The homology module is nothing more than a quotient, so a chainmap can be defined on homology level as $f_n([c]) = [f_n(c)]$.

Because we will naturally apply the above definition to topological spaces, specifically manifolds we would like homologies to be invariant under homotopy. The first step into this might seem a bit weird but it turns out this definition is what we're looking for.

7.2 Cochains

Sometimes working with decreasing indices doesn't quite cut it. Therefore we define a *cochain complex* to be a chaincomplex with increasing indices. Analoguous you can define a cochain to consist of the modules of R-linear maps from your modules to R, together with special (*boundary*) maps $d^n : C^n \to C^{n+1}$.

Remark 7.5. The *co* part of cochain can be related to the dual of a module in the same way a vector space relates to it's dual. This, together with a fitting boundary operator would also give us a cochain. For now however the definition as above is more useful and easier to work with.

Definition 7.7. A *chain homotopy* s from $f : C^\bullet \to D^\bullet$ to $g : C^\bullet \to D^\bullet$ is a sequence $s^n : C^n \to D^{n-1}$ such that, if $c^\bullet$ and $d^\bullet$ are the boundary maps for $C^\bullet$ and $D^\bullet$ respectively, then;

$$d^{n-1} \circ s^n + s^{n+1} \circ c^n = g^n - f^n.$$

Remark 7.6. $f^\bullet$ and $g^\bullet$ are called (chain)homotopic maps.

Theorem 7.1. If $f^\bullet$ and $g^\bullet$ are homotopic cochainmaps they are equal on the homology modules of the domaincomplex.

Proof. To see that this is satisfied consider $[\omega] \in H_n(C^\bullet)$, now

$$\begin{aligned} f^n[\omega] - g^n[\omega] &= (f^n - g^n)[\omega] \\ &= \left(d^{n-1} \circ s^n + s^{n+1} \circ c^n\right)[\omega] \\ &= [d^{n-1} \circ s^n \omega] + [s^{n+1} \circ c^n \omega] \\ &= 0 + [s^{n+1} \circ 0] = 0. \end{aligned}$$

The first term is zero since it is mapped into the image of d^{n-1} which on homology level is 0. The second part is 0 because ω is in the $\mathrm{Ker}(c^n)$ by definition. Thus we conclude that $f^\bullet$ and $g^\bullet$ are equal on homology level.

□

7.3 A few useful lemmas

Before we get into the De Rham Cohomology it may be useful to state a few lemmas regarding (co-)chains. These will occur often in proofs and it is therefore essential that they are mentioned.

First, a few observations regarding injectivity and surjectivity of maps in a chain. Consider the following exact sequence,

$$0 \xrightarrow{0} A \xrightarrow{a} B$$

from the fact that this sequence is exact we can deduct that a is an injective function, after all $\text{Ker}(a) = \text{Im}(0)$, and R-linear maps with trivial kernel are injective. In the same way you can prove that for

$$B \xrightarrow{b} C \xrightarrow{0} 0$$

it can be deducted in the same way as before that b has to be surjective if the sequence is exact.

Remark 7.7. It can also be proven that the reverse holds, if a injective or b surjective then the sequences are exact.

Remark 7.8. The 0-map will from hereon be omitted since it is the only possible map there could be in that spot.

Combining the above two sequences into one gives us a special kind of exact sequence.

Definition 7.8. A exact sequence of the form,

$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$$

is called a *short exact sequence*, often abbreviated by *ses*.

Remark 7.9. We can extend the concept of a short exact sequence of modules to a short exact sequence of (co)chain complexes, namely by saying

$$0 \longrightarrow A_\bullet \xrightarrow{a_\bullet} B_\bullet \xrightarrow{b_\bullet} C_\bullet \longrightarrow 0$$

is short exact if,

$$0 \longrightarrow A_n \xrightarrow{a_n} B_n \xrightarrow{b_n} C_n \longrightarrow 0$$

is short exact for all n.

Another important sequence is the following.We will omit the proof and merely refer to the previous results.

$$0 \longrightarrow D \xrightarrow{\cong} E \longrightarrow 0$$

Now we will discuss some important lemmas in algebraic topology which will be useful in the future.

Lemma 7.1. *(The Five Lemma)* Consider the following commutative diagram with exact rows,

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "a"] & B \arrow[r, "\beta"] \arrow[d, "b"] & C \arrow[r, "\gamma"] \arrow[d, "c"] & D \arrow[r, "\delta"] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\gamma'"] & D' \arrow[r, "\delta'"] & E' \end{tikzcd}$$

If a, b, d and e are isomorphisms then so is c.

Proof. First we will prove injectivity of c. Consider $x \in C$ such that $c(x) = 0$. Then $\gamma' \circ c(x) = d \circ \gamma(x) = 0$. This implies that $\gamma(x) \in \operatorname{Ker}(d)$, by injectivity of d we now get that $\gamma(x) = 0$ thus $x \in \operatorname{Ker}(\gamma) = \operatorname{Im}(\beta)$. Thus there is a y such that $x = \beta(y)$. Now $c \circ \beta(y) = \beta' \circ b(y) = 0$, thus $b(y) \in \operatorname{Ker}(\beta') = \operatorname{Im}(\alpha')$. Now we get that $b(y) = \alpha'(z')$, by surjectivity of a we obtain $b(y) = \alpha' \circ a(z)$ for a unique $z \in A$. By commutativity, $b(y) = b \circ \alpha(y)$. Thus finally $x = \beta \circ \alpha(z) = 0$ by exactness.

Now for surjectivity consider $x' \in C'$. Look at $\gamma'(x')$ by surjectivity of d we get that there is a $y \in D$ such that $d(y) = \gamma'(x')$ now by exactness $\delta' \circ d(y) = e \circ \delta(y) = 0$. By injectivity of e we thus get that $\delta(y) = 0$ and $y \in \operatorname{Ker}(\delta) = \operatorname{Im}(\gamma)$. Thus there exist an $x \in C$ such that $\gamma(x) = y$. Now by $\gamma' \circ c(x) = d \circ \gamma(x) = d(y)$. Now $c(x)$ and x' both map to $d(y)$ under γ', we can thus consider the difference, $\gamma'(c(x) - x') = 0$ to obtain $(c(x) - x') \in \operatorname{Ker}(\gamma') = \operatorname{Im}(\beta')$. Thus there is a $v' \in B'$ such that $\beta'(v') = c(x) - x'$, by surjectivity of b we get a $v \in B$ such that $b(v) = v'$ thus, $\beta' \circ b(v) = c \circ \beta(v) = c(x) - x' \Rightarrow c(x - \beta(v)) = x'$ and thus there is an element of C such that it maps to x' for all $x' \in C'$, this proves surjectivity.

□

Lemma 7.2. *(Splitting Lemma)*
Let $0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$ be a short exact sequence. Then equivalent are,

(i) The image of a is a direct summand of B.

(ii) There is a homomorphism $r : B \to A$ such that $r \circ a = id$.

(iii) There is a homomorphism $s : C \to B$ such that $b \circ s = id$.

We call such a sequence a *splitting sequence*.

Lemma 7.3. *(Zigzag Lemma)* Given a ses of cochains

$$0 \longrightarrow A^\bullet \xrightarrow{a^\bullet} B^\bullet \xrightarrow{b^\bullet} C^\bullet \longrightarrow 0$$

then for all n there exists a *connecting homomorphism* $\delta : H^n(C^\bullet) \to H^{n+1}(A^\bullet)$, such that the following sequence is exact,

$$\cdots \xrightarrow{\delta} H^n(A^\bullet) \xrightarrow{a^n} H^n(B^\bullet) \xrightarrow{b^n} H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(A^\bullet) \xrightarrow{a^{n+1}} \cdots$$

Proof. Consider the following commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A^n \arrow[r, "a^n"] \arrow[d, "d"] & B^n \arrow[r, "b^n"] \arrow[d, "d"] & C^n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r, "a^{n+1}"] \arrow[d, "d"] & B^{n+1} \arrow[r, "b^{n+1}"] \arrow[d, "d"] & C^{n+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{n+2} \arrow[r, "a^{n+2}"] & B^{n+2} \arrow[r, "b^{n+2}"] & C^{n+2} \arrow[r] & 0
\end{tikzcd}$$

We would like to make δ the map $\left(a^{n+1}\right)^{-1} \circ d \circ (b^n)^{-1}$, which works on the level of cohomologies due to the fact that $a^\bullet$ and $b^\bullet$ commute with d and thus send boundaries to boundaries and cycles to cycles (more on that later). Now the question is of course if this map as we defined above is well defined, and independant of choice.

So take an element $\gamma_n \in C^n$, because b^n is surjective there exists a β_n such that $b^n\beta_n = \gamma_n$. But since we are only interested in δ working on the cohomology level it suffices to consider γ_n such that $d\gamma_n = 0$. Hence by commutivity we get that, $db^n\beta_n = b^{n+1}d\beta_n = 0$. Thus $d\beta_n \in \operatorname{Ker}(b^{n+1}) = \operatorname{Im}(a^{n+1})$, thus there exists a (unique by injectivity) α_{n+1} such that $a^{n+1}\alpha_{n+1} = d\beta_n$. Now we get by commutivity of the diagram that $a^{n+2}d\alpha_{n+1} = da^{n+1}\alpha_{n+1} = dd\beta_n = 0$, thus $d\alpha_{n+1} \in \ker a^{n+2}$ but since a^{n+2} is injective it follows that $d\alpha_{n+1} = 0$ which of course means that the procedure we've followed ends up at a representative of a cohomology class.

However we are not completely done, we have to show that the output doesn't depend on the choice we made for β_n (upto an element of the form $d\alpha_n'$). Furthermore it is not yet clear that δ respects the homology structure. So let us start with choosing a different β_n'.

- Consider $\beta_n - \beta_n'$ since both map to the same point under b^n we get that $b^n(\beta_n - \beta_n') = 0$. By exactness there exists a α_n such that $a^n\alpha_n = \beta_n - \beta_n'$. Now by commutativity $d(\beta_n - \beta_n') = a^{n+1}d\alpha_n$. By previous results we know there exist x, x' such that $d\beta_n = a^{n+1}x$ and $d\beta_n' = a^{n+1}x'$. Now consider; $a^{n+1}(x - x' - d\alpha_n) = 0$, by injectivity we then get $x - x' = d\alpha_n$. Thus on homology level $x - x' = 0$, and thus maps to 0.
- Now take an element $\gamma_n = d\gamma_n \in C_n$. Now by surjectivity of b^{n-1} we obtain a β_{n-1} such that $b^{n-1}\beta_{n-1} = \gamma_{n-1}$. Now by commutativity of the

diagram we get that $d\beta_{n-1} = \beta_n$. Thus $d\beta_n = 0$. Now if we follow the defining process of δ we at some point obtain $a^{n+1}\alpha_{n+1} = d\beta_n = 0$, by injectivity this means that $\alpha_{n+1} = 0$. Thus we see that all elements in $[\gamma_n]$ (whose difference is a boundary) are mapped onto the same $[\alpha_{n+1}]$ which implies well definedness of δ.

Now last of all we need to prove exactness of the sequence in the lemma,

$$\cdots \xrightarrow{\delta} H^n(A^\bullet) \xrightarrow{a^n} H^n(B^\bullet) \xrightarrow{b^n} H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(A^\bullet) \xrightarrow{a^{n+1}} \cdots$$

As for exactness of the sequence we will suffice with proving exactness at $H^n(A)$, as the proof for $H^n(B)$ is trivial and the proof at $H^n(C)$ similar.
Take an element $\delta[c] \in H^n(A)$ and apply a^n to this element. By following the defining process we see that $a^n\delta[c] = [d\beta_{n-1}] = [0]$. Thus $\mathrm{Im}(\delta) \subset \mathrm{Ker}(a^n)$. The other way around let $[\alpha_n]$ be an element in $\mathrm{Ker}(a^n)$. Thus we have that $a^n\alpha_n = d\alpha_{n-1}$. We can (as a sketch) inverse the boundary operator to see;

$$((a^n)^{-1} \circ d \circ \left(b^{n-1}\right)^{-1})^{-1} = b^{n-1} \circ d^{-1} \circ a^n.$$

Now since we know that $a^n\alpha_n = d\alpha_{n-1}$ we can find that the image we seek is $b^{n-1}\alpha_n - 1$. And thus finally we get $\mathrm{Ker}(a^n) \subset \mathrm{Im}(\delta)$.

$\square$

8 The de Rham cohomology

We are now in a postition to define the de Rham Cohomology.

8.1 The definition

Now we want to apply the previous chapter of basic homologytheory to manifolds. For this we need, as seen before, R-modules. We will use for R-modules in this case real vectorspaces, namely the vector spaces or differential k-forms $\Gamma^k(M)$, for all $k = 1 \dots m$, if m is the dimension of the manifold.

Definition 8.1. A differential form θ is called *closed* if $d\theta = 0$. And a differential k-form ω is exact if there exist a differential $(k-1)$-form ψ such that $\omega = d\psi$.

Remark 8.1. Note that because $d \circ d = 0$ we have that all exact forms are also closed.

Lemma 8.1. Let M be a smooth manifold, the following diagram is a cochain;

$$\cdots \xrightarrow{d} \Gamma^{n-1}(M) \xrightarrow{d} \Gamma^n(M) \xrightarrow{d} \Gamma^{n+1}(M) \xrightarrow{d} \cdots$$

Remark 8.2. The n-cycles are exactly the closed n-forms on M. And the n-boundaries are the exact n-forms on M.

Remark 8.3. We also use the fact that $\Gamma^n = 0$ for $n > \dim(M)$ and $n < 0$.

Proof. Note that $\Gamma^n(M)$ is a real vector space, thus an $\mathbb{R}$-module. And d is a $\mathbb{R}$-linear map. Furthermore $d \circ d = 0$ which is the same as saying $\operatorname{Im}(d) \subset \operatorname{Ker}(d)$. Thus we are dealing with a cochain.

□

Definition 8.2. The p-th *de Rham cohomology group* is equal to the p-th cohomology groups of the cochain in Lemma 8.1. This is usually denoted $H^p_{\mathrm{dR}}(M)$.

Lemma 8.2. Let M and N be smooth manifolds and $f : M \to N$ a smooth map then the pullback sends closed forms on N to closed forms on M and exact forms on N to exact forms on M. Thus f has a pullback $f^* : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$.

Proof. Let θ be a closed form on N then $d\theta = 0$. Now by Proposition 6.1 we have

$$0 = f^* d\theta = df^*\theta$$

thus $f^*\theta$ is closed. Now let ω be an exact form, thus $\omega = d\psi$ then we get, again by Proposition 6.1,

$$f^*\omega = f^* d\psi = df^*\psi$$

and thus $f^*\omega$ is closed.

$\square$

Remark 8.4. The pullback as defined above is sometimes denoted $H^p_{\mathrm{dR}}(M)(f)$.

With the remark we can now prove the functoriality of the de Rham Cohomology functor;

Theorem 8.1. *$H^p_{\mathrm{dR}}(\cdot)$ is a contravariant functor from* `MAN` *to* $\mathbb{R}$-`MOD`.

Remark 8.5. Remember that $\mathbb{R}$-`MOD` is the category of all modules over the ring $\mathbb{R}$.

Proof. It will be sufficed to prove that $(F \circ G)^* = G^* \circ F^*$ and $id^* = id$. The first part follows almost immediatly from the commuting property of pullbacks with the differential operator. The second part will follow later on in the chapter of homotopy invariance.

8.2 Homotopy Invariance

The definition of the de Rham cohomology groups may be clear now but it is not clear yet how to compute them or even work with them. That is why we will use the next sections to make it easier to compute these groups. First of all we will prove homotopy invariance.

We have seen before, that chainmaps that are homotopic induce the same maps on homology level. So what we want to find for each homotopy

$$H : M \times \mathbb{I} \to N,$$

a (chain) homotopy operator $(h_n) : \Gamma^n(N) \to \Gamma^{n-1}(M)$. This will suffice because of Theorem 7.1. Often we shall omit the subscript as with the differential operator to make it a little less confusing.

Definition 8.3. Two smooth maps $f, g : M \to N$ are called *smoothly homotopic* if there excists a smooth map $H : M \times \mathbb{I} \to N$ such that $H(x, 0) := H_0(x) = f(x)$ and $H(x, 1) := H_1(x) = g(x)$.

Lemma 8.3. *Let M be a smooth manifold and $i_0 : M \hookrightarrow M \times \mathbb{I}$ the inclusion on the zero level and $i_1 : M \hookrightarrow M \times \mathbb{I}$ the inclusionmap on the 1 level. Then there exists a chainhomotopy h between i_0^* and i_1^*.*

Proof. We shall define the chainhomotopy as such, let $\omega \in \Gamma^k(M \times \mathbb{I})$, and let $X_1, \dots, X_{k-1} \in T_pM$;

$$(h\omega)_p = \int_0^1 \omega\left(\frac{\partial}{\partial t}, X_1, \dots, X_{k-1}\right) dt$$

Which basically boils down to integrating over a coefficient function. Now to check that $h(d\omega) + d(h\omega) = i_1^*\omega - i_0^*\omega$. Consider two cases, one in which there is a dt term in the differential form, and one where there isn't.

- Case 1; $\omega = f(x,t)dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$.

$$\begin{aligned} d(h\omega) &= d\left(\left(\int_0^1 f(x,t)dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}\right) \\ &= \sum_j \left(\frac{\partial}{\partial x^j}\int_0^1 f(x,t)dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \\ &= \left(\int_0^1 \frac{\partial f}{\partial x^j}(x,t)\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}. \end{aligned}$$

Now for the other term because $dt \wedge dt = 0$;

$$\begin{aligned} h(d\omega) &= h\left(\frac{\partial f}{\partial x^j} dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}\right) \\ &= \int_0^1 \frac{\partial f}{\partial x^j}(x,t) i_{\frac{\partial}{\partial t}}\left(dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}\right) dt \\ &= -\left(\int_0^1 \frac{\partial f}{\partial x^j}(x,t)dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \\ &= -d(h\omega) \end{aligned}$$

Now it follows that the sum of the terms above adds up to zero. But since $i_1^* dt = i_0^* dt = 0$ the homotopic equivalence relation holds.

- Case 2; $\omega = f(x,t)dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Now because $i_{\frac{\partial}{\partial t}}\omega = 0$ we get that $h(d\omega) = 0$, as for the other part,

$$\begin{aligned} h(d\omega) &= h\left(\frac{\partial f}{\partial t} dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \text{some terms without } dt\right) \\ &= \left(\int_0^1 \frac{\partial f}{\partial t}(x,t)dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= (f(x,1) - f(x,0))dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned}$$

Now since $f(x,s)dx^{i_1} \wedge \cdots \wedge dx^{i_k} = i_s^*\omega$ we get that;

$$(f(x,1) - f(x,0))dx^{i_1} \wedge \cdots \wedge dx^{i_k} = i_1^*\omega - i_0^*\omega.$$

Which proves the homotopic equivalence relation $h(d\omega) + d(h\omega) = i_1^*\omega - i_0^*\omega$.

$\square$

Theorem 8.2. Let M and N be smooth manifolds and $f, g : M \to N$ be smoothly homotopic maps. Then,

$$f^* = g^* : H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M).$$

Proof. First note that if $i_t : M \hookrightarrow M \times \mathbb{I}$ then $H \circ i_0 = f$ and $H \circ i_1 = g$, and let h be the homotopy operator of Lemma 8.3. Define

$$\bar{h} = h \circ H^* : \Gamma^n(N) \to \Gamma^{n-1}(M).$$

For any $\omega \in \Gamma^n(N)$ we have,

$$\begin{aligned}
\bar{h}(d\omega) + d(\bar{h}\omega) &= h(H^* d\omega) + d(hH^*\omega) = hd(H^*\omega) + dh(H^*\omega) \\
&= i_1^* H^*\omega - i_0^* H^*\omega \\
&= (H \circ i_1)^*\omega - (H \circ i_0)^*\omega = G^*\omega - F^*\omega.
\end{aligned}$$

Now by Lemma 7.1 we get the desired result.

□

Definition 8.4. Let $f : M \to N$ and $g : N \to M$ be smooth maps between smooth manifolds, then if $f \circ g$ is smoothly homotopic to the identitymap on N and $g \circ f$ is smoothly homotopic to the identity map on M, we say that M and N are (smoothly) homotopic equivalent, sometimes denoted as $M \simeq N$.

Theorem 8.3. Let M and N be manifolds such that $M \simeq N$. Then $H^n_{\mathrm{dR}}(M) \cong H^n_{\mathrm{dR}}(N)$ for all n.

Proof. $M \simeq N$ implies that there are $f : M \to N$ and $g : N \to M$ such that $f \circ g \simeq id$ and $g \circ f \simeq id$. But since we know that homotopic equivalent maps induce the same map on de Rham cohomology level, we get

$$f^* \circ g^* = (g \circ f)^* = id^* = id$$

And also

$$g^* \circ f^* = (f \circ g)^* = id^* = id$$

So clearly these maps are eachothers inverse which implies that f^* is an isomorphism.

□

8.3 The Mayer-Vietoris sequence

Another important tool we may use in finding cohomology groups of manifolds is the Mayer-Vietoris sequence. This tool looks a bit like the Van Kampen theorem in homotopy theory and is used in a similar way. For the prove of existence we will use the Zigzag Lemma (7.3)

Theorem 8.4. *(Mayer-Vietoris)* Let M be a smooth manifold, and U and V open subsets of M such that $U \cup V = M$, then for all n there is a connecting homomorphism $\delta : H^n_{\mathrm{dR}}(U \cap V) \to H^{n+1}_{\mathrm{dR}}(M)$ such that the following sequence is exact:

$$\cdots \xrightarrow{\delta} H^n_{\mathrm{dR}}(M) \overset{k^*\oplus l^*}{\to} H^n_{\mathrm{dR}}(U) \oplus H^n_{\mathrm{dR}}(V) \overset{i^*-j^*}{\to} H^n_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} H^{n+1}_{\mathrm{dR}}(M) \overset{k^*\oplus l^*}{\to} \cdots$$

where i, j, k and l are all inclusion maps as such,

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"] & U \arrow[d, "k"] \\ V \arrow[r, "l"] & M \end{tikzcd} \qquad \begin{tikzcd} \Gamma^n(M) \arrow[r, "k^*"] \arrow[d, "l^*"] & \Gamma^n(U) \arrow[d, "i^*"] \\ \Gamma^n(V) \arrow[r, "j^*"] & \Gamma^n(U \cap V) \end{tikzcd}$$

it may be useful to see that the pullback of inclusions is nothing but the restriction of a differentialform.

Now before we start the proof of this powerful tool we need an additional powerful tool. Namely the existence of a *partition of unity* for every open cover of a manifold.

Definition 8.5. Let $(U_\alpha)_{\alpha \in I}$ be an open cover of a smooth manifold M. A collection (smooth) functions $\{\psi_\alpha : M \to \mathbb{R}\}_{\alpha \in I}$ is called a (smooth) partition of unity if:

(i) $0 \leq \psi_\alpha \leq 1$ for all α.

(ii) The support of ψ_α is contained in U_α.

(iii) Each point $p \in M$ has a neighbourhood that intersects only a finite number of $\mathrm{supp}(\psi_\alpha)$.

(iv) $\sum_{\alpha \in I} \psi_\alpha(x) = 1$ for all $p \in M$.

The following theorem is most important, but the proof is rather technical so we will suffice with just mentioning the theorem.

Theorem 8.5. If M is a smooth manifold, any open cover induces a smooth partition of unity.

Now we can prove the Mayer-Vietoris theorem.

Proof. By the zigzag lemma it will suffice to prove that the following is short exact:

$$0 \to \Gamma^n(M) \overset{k^*\oplus l^*}{\to} \Gamma^n(U) \oplus \Gamma^n(V) \overset{i^*-j^*}{\to} \Gamma^n(U \cap V) \to 0$$

- First we will prove exactness at $\Gamma^n(M)$, which means we have to show that $k^* \oplus l^*$ is injective. So take σ such that $(k^* \oplus l^*)\,\sigma = (\sigma|_U, \sigma|_V) = (0, 0)$ but since $U \cup V = M$ this implies that $\sigma = 0$ and this proves injectivity.

- To prove exactness at $\Gamma^n(U) \oplus \Gamma^n(V)$ first consider

  $(i^* - j^*) \circ (k^* \oplus l^*)\, \omega = (i^* - j^*)\, (\omega|_U, \omega|_V) = \omega|_{U \cap V} - \omega|_{U \cap V} = 0.$

  Thus $\mathrm{Im}(k^* \oplus l^*) \subset \mathrm{Ker}(i^* - j^*)$.

  Now for the other side consider $(\omega, \omega') \in \mathrm{Ker}(i^* - j^*)$ then $i^*\omega = j^*\omega'$ and thus $\omega|_{U \cap V} = \omega'|_{U \cap V}$ this means there exists a form σ on M such that $\sigma|_U = \omega$ and $\sigma|_V = \omega'$ now clearly $(\omega, \omega') = (k^* \oplus l^*)\sigma$ and thus $\mathrm{Im}(k^* \oplus l^*) \supset \mathrm{Ker}(i^* - j^*)$. This proves exactness.

- Next is exactness at $\Gamma^n(U \cap V)$ which translates to nothing but proving that $i^* - j^*$ is surjective. So let $\nu \in \Gamma^n(U \cap V)$. Since $\{U, V\}$ is an open cover of M there exists a smooth partition of unity $\{\phi, \psi\}$. Now define $\eta \in \Gamma^n(U)$ as

  $$\eta = \begin{cases} \psi\nu & \text{on } U \cap V; \\ 0 & \text{on } U - \text{supp } \psi. \end{cases}$$

  $$\eta' = \begin{cases} -\phi\nu & \text{on } U \cap V; \\ 0 & \text{on } U - \text{supp } \phi. \end{cases}$$

  Then we obtain $(i^* - j^*)(\eta, \eta') = i^*\eta - j^*\eta' = \psi\nu - (-\phi\nu) = \nu$.

$\square$

9 Some computations of de Rham cohomology

In this section we will calculate the de Rham cohomology groups of a few well known manifolds. And some simple corollaries that follow from these facts.

A point as a topological space is usually denoted by *. If we want to determine $H^n_{\mathrm{dR}}(*)$ the first step is finding the spaces $\Gamma^n(*)$. We know that since * is 0-dimensional that $\Gamma^n(*) = 0$ for all $n > 0$. As for $\Gamma^0(*)$ this consists of all functions $* \to \mathbb{R}$ and as such $\Gamma^0(*) \cong \mathbb{R}$. Now we can determine all the de Rham groups;

Proposition 9.1.

$$H^n_{\mathrm{dR}}(*) = \begin{cases} \mathbb{R} & n = 0; \\ 0 & n > 0. \end{cases}$$

Proof. Consider the chaincomplex of $\Gamma^n(*)$'s.

$$0 \longrightarrow \mathbb{R} \longrightarrow 0 \longrightarrow \cdots$$

The homology groups now become $\frac{\mathrm{Ker}(\mathbb{R}\to 0)}{\mathrm{Im}(0\to\mathbb{R})} = \mathbb{R}$ for $n = 0$ and 0 for all the other n.

□

Lemma 9.1. *(Poincaré)* Let M be a contractible manifold.

$$H^n_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R} & n = 0, \\ 0 & n > 0. \end{cases}$$

Proof. If M is a contractible manifold we have that $M \simeq \{*\}$ now by Theorem 8.3 we obtain $H^n_{\mathrm{dR}}(M) \cong H^n_{\mathrm{dR}}(*)$.

□

Now the next thing to find is the de Rham groups of all the spheres. In order to obtain this we will need a lemma which is in a way overkill for what we will use it for. Nonetheless it is an important result that needs mentioning.

From topology we know what a disjoint union of sets is. The disjoint union of manifolds M_1 and M_2 is commonly denoted $M_1 \coprod M_2$, and is called the *coproduct of* M_1 *and* M_2, important to realize is that this coproduct is again a manifold. For we can take the union of any smooth atlas of M_1 with a smooth atlas of M_2, this new atlas is again smooth since there is no overlap of elements due to $M_1 \cap M_2 = \emptyset$. Now this relates to de Rham groups in the following way;

Proposition 9.2. De Rham Cohomology is additive, which means for $\{M_i\}_{i\in I}$ smooth manifolds that;

$$H^n_{\mathrm{dR}}\left(\coprod_{i\in I} M_i\right) \cong \bigoplus_{i\in I} H^n_{\mathrm{dR}}(M_i).$$

Proof. Let ι_i be the inclusion maps of $M_i \hookrightarrow M$ then the isomorphism is given by

$$\omega \mapsto (\iota_1^*\omega, \iota_2^*\omega, \dots) = (\omega|_{M_1}, \omega|_{M_2}, \dots).$$

Injectivity and surjectivity follow almost instantly.

Corollary 9.1. The de Rham groups of a space M that is smoothly homotopic to two points are;

$$H^n_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & n = 0, \\ 0 & n > 0. \end{cases}$$

Proposition 9.3. Let M be a connected smooth manifold, then;

$$H^0_{\mathrm{dR}}(M) = \mathbb{R}.$$

Proof. $H^0_{\mathrm{dR}}(M) = \left(Z^0(\Gamma^\bullet(M))\right) / \left(B^0(\Gamma^\bullet(M))\right)$. Now $\Gamma^{-1}(M) = 0$, so $B^0(\Gamma^\bullet(M)) = 0$. Thus,

$$H^0_{\mathrm{dR}}(M) = Z^0.$$

So all we need to look at is functions $f : M \to \mathbb{R}$ such that $df = 0$. Now since M is connected it follows that df needs to be a constant function, thus $Z^0 \cong \mathbb{R}$.

□

Now for a last lemma;

Lemma 9.2. Let

$$0 \longrightarrow A^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{m-1}} A^m \longrightarrow 0$$

be an exact sequence of finite vectorspaces. Then

$$\sum_{i=0}^{m} (-1)^i \dim(A^i) = 0.$$

Remark 9.1. This is a special case of the Euler characteristic of exact sequences.

Proof. Since we are working with finite vectorspaces we know that for all i,

$$\dim\ \mathrm{Im}(d_i) + \dim\ \mathrm{Ker}(d_i) = \dim A^i.$$

Furthermore we know that, since this is an exact sequence

$$\dim\ \mathrm{Im}(d_i) = \dim\ \mathrm{Ker}(d_{i+1}).$$

Now telescoping gives us the desired result.

□

Now we have enough tools to compute the de Rham groups of a circle.

Proposition 9.4.

$$H^n_{\mathrm{dR}}(\mathbb{S}^1) = \begin{cases} \mathbb{R} & n = 0, 1; \\ 0 & n > 1. \end{cases}$$

Proof. Denote N as the northpole of $\mathbb{S}^1$ and S as the southpole of $\mathbb{S}^1$. Let $U = \mathbb{S}^1 \; N$ and $V = \mathbb{S}^1 \; S$. We can now apply Mayer-Vietoris to obtain;

$$\begin{tikzcd}
0 \arrow[r] & H^0_{\mathrm{dR}}(\mathbb{S}^1) \arrow[r] & H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \arrow[r] & H^0_{\mathrm{dR}}(U \cap V) \arrow[d, "i^* - j^*"'] \\
0 & H^1_{\mathrm{dR}}(U \cap V) \arrow[l] & H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V) \arrow[l] & H^1_{\mathrm{dR}}(\mathbb{S}^1) \arrow[l]
\end{tikzcd}$$

As an exact sequence. Since $\mathbb{S}^1$ is connected, and U and V are contractible, and $U \cap V$ is homotopic equivalent to two points we can fill out the diagram as such.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "f"] & \mathbb{R} \oplus \mathbb{R} \arrow[d, "\delta"] \\
0 & 0 \arrow[l] & 0 \arrow[l] & H^1_{\mathrm{dR}}(\mathbb{S}^1) \arrow[l]
\end{tikzcd}$$

Now by Lemma 9.2 we can compute the dimension of $H^1_{\mathrm{dR}}(\mathbb{S}^1)$, namely we get;

$$-1 + 2 - 2 + \quad ? \quad - 0 = 0$$

Thus we obtain $\dim H^1_{\mathrm{dR}}(\mathbb{S}^1) = 1$ and as such $H^1_{\mathrm{dR}}(\mathbb{S}^1) \cong \mathbb{R}$.

$\square$

Now that we know what the de Rham groups of a circle are we can inductively compute the groups of <u>all</u> spheres.

Theorem 9.1. For $n > 0$;

$$H^q_{\mathrm{dR}}(\mathbb{S}^n) = \begin{cases} \mathbb{R} & q = 0, n; \\ 0 & \text{other}. \end{cases}$$

Proof. The main idea of the proof is that we assume the theorem holds for $n - 1$ and show that it then also holds for n. By Proposition 9.4 we already know this holds for $n = 1$.

So assume the theorem holds for $m = n - 1$ then consider N to be the northpole of $\mathbb{S}^n$ and S to be the southpole. Now define $U = \mathbb{S}^n \backslash S$ and $V =$

$\mathbb{S}^n \backslash N$. Since these two open sets cover $\mathbb{S}^n$ we can apply Mayer-Vietoris. Once more U and V are contractible spaces. However this time $U \cap V$ is not merely two points, it is in fact homotopic equivalent to $\mathbb{S}^{n-1}$. Now by the induction assumption we know what the de Rham groups of this sphere is. Additionally we know that $\mathbb{S}^n$ is connected thus has $H^0_{\text{dR}}(\mathbb{S}^n) \cong \mathbb{R}$.

Now at the parts of the Mayer-Vietoris sequence with $0 < q < n-1$ we get that it has the following shape;

$$0 \longrightarrow H^q_{\text{dR}}(\mathbb{S}^n) \longrightarrow 0 \longrightarrow 0.$$

Thus we get that for these values $H^q_{\text{dR}}(\mathbb{S}^n) \cong 0$. So all we need to look at is the following part of the Mayer-Vietoris sequence;

$$\begin{tikzcd}
0 \arrow[r] & H^{n-1}_{\text{dR}}(\mathbb{S}^n) \arrow[r] & H^{n-1}_{\text{dR}}(U) \oplus H^{n-1}_{\text{dR}}(V) \arrow[r] & H^{n-1}_{\text{dR}}(U \cap V) \arrow[d] \\
0 & H^n_{\text{dR}}(U \cap V) \arrow[l] & H^n_{\text{dR}}(U) \oplus H^n_{\text{dR}}(V) \arrow[l] & H^n_{\text{dR}}(\mathbb{S}^n) \arrow[l]
\end{tikzcd}$$

which filled in becomes;

$$\begin{tikzcd}
0 \arrow[r] & H^{n-1}_{\text{dR}}(\mathbb{S}^n) \arrow[r] & 0 \arrow[r] & \mathbb{R} \arrow[d] \\
0 & 0 \arrow[l] & 0 \arrow[l] & H^n_{\text{dR}}(\mathbb{S}^n) \arrow[l]
\end{tikzcd}$$

□

This gives us the desired result $H^n_{\text{dR}}(\mathbb{S}^n) \cong \mathbb{R}$ and $H^{n-1}_{\text{dR}}(\mathbb{S}^n) \cong 0$. Now the last space for which we will compute the De Rham groups will give us a peculiar conclusion.

Proposition 9.5.

$$H^q_{\text{dR}}(\mathbb{R}^n \backslash \{0\}) = \begin{cases} \mathbb{R} & q = 0, n-1; \\ 0 & \textit{other.} \end{cases}$$

Proof. We have that $\mathbb{R}^n \backslash \{0\} \simeq \mathbb{S}^{n-1}$, we can for instance use the the map,

$$(x,t) \mapsto \left((1-t)\frac{1}{\|x\|} + t \right) x.$$

So now when we use Theorem 8.3 and Theorem 9.1 the desired result follows.

□

Remark 9.2. Instead of 0, we could remove any point and this result would still hold.

Now what we can conclude by this result is that by *removing* a point from a space we have created differential forms that are exact but not closed. So in a way this is addition by substraction.

We can also compute the top cohomology group of an orientable manifold. This is in fact where the De Rham cohomology works a lot better than singular cohomology (which will be introduced later).

Proposition 9.6. Let M be a smooth, connected, orientable and compact n-manifold. Then

$$I : H^n_{\mathrm{dR}}(M) \to \mathbb{R},$$

the integration map is an isomorphism.

Proof. First of all we need to show that this map is well defined. In other words we want for a closed differential form ω that $\int_M \omega + d\eta = \int_M \omega$ for all (dimensionmatching) differential forms η. But by linearity of the integral and Stokes theorem $\int_M d\eta = 0$. As such the identity holds for all (closed) differential forms ω.

Next there is the issue of surjectivity, but since we know that when M is orientable there exists an orientation form ω_0 with the property that $\int_M \omega_0 = b > 0$ (see Lemma 6.2). We can use linearity of integration $I(a\omega_0) = aI(\omega_0) = ab$, and since b is non-zero we can make any real number in such a way.

For injectivity we use same the proof as in Corollary 11.1.

□

Remark 9.3. It turns out that if M is a smooth, connected, compact and NON-orientable manifold the top cohomology group will be 0. Thus we have found a way to check if a smooth, connected and compact manifold is orientable or not, by using de Rham groups.

10 The de Rham Theorem

One of the most common homology theories in algebraic topology is the Singular Homology, this homology is often the first step into the world of algebraic topology and hence the first thing learned in most basic courses of algebraic topology. The central question in this section will be *How does singular homology relate to de Rham cohomology?*, we will answer this question in steps. First a brief recap of what singular homology (and cohomology) is, then additional preperation working towards the main theorem of this section the de Rham Theorem, which will give a direct isomorphism between Singular Cohomology groups and the de Rham groups.

10.1 Singular Homology

Singular homology is an homology theory based on so called *singular simplices.*

Definition 10.1. Let $\mathbb{R}^\infty$ have the basis $e_0, e_1, \ldots$. The *standard p-simplex* is

$$\Delta_p = \left\{ \sum_{i=0}^{p} \lambda_i e_i : \sum \lambda_i = 1, 0 \leq \lambda_i \leq 1 \right\}$$

Remark 10.1. The λ_i's are called the barycentric coordinates.

Definition 10.2. For given $v_0, \ldots, v_n \in \mathbb{R}^q$, $[v_0, \ldots, v_n]$ is the map $\Delta_n \to \mathbb{R}^q$ which works as such; $\sum \lambda_i e_i \mapsto \sum \lambda_i v_i$. This is called an *affine singular n-simplex.*

The next definition gives a notation for 'mapping a lower standard simplex into one of higher dimension'.

Definition 10.3. The ith facemap F_i^p is the map $[e_0, \ldots, \hat{e}_i, \ldots, e_p] : \Delta_{p-1} \to \Delta_p$ where the hat on a vector means you 'leave it out'.

Definition 10.4. Let X be a topological space, a continous map $\sigma_p : \Delta_p \to X$ is called a *singular p-simplex.* The free abelian group over all p-simplices is denoted $S_p(X)$ and called the singular p-chain group.

Remark 10.2. So a p-chain of X is a formal sum of p-simplices.

Now as before, we can turn the graded groups Δ_p into a chaincomplex by defining a differential $\partial_p : \Delta_p(X) \to \Delta_{p-1}(X)$.

Definition 10.5. Let $\partial_p : \Delta_p(X) \to \Delta_{p-1}(X)$ be the homomorphism that works on the basis elements of $\Delta_p(X)$ as such, let σ be a p-simplex. And $\sigma^{(i)} := \sigma \circ F_i^p$.

$$\partial_p \sigma := \sum_{i=0}^{p} (-1)^i \sigma^{(i)}.$$

Lemma 10.1. $(\Delta_p, \partial_p)_{p \geq 0}$ is a chaincomplex.

Remark 10.3. The homology groups of this chaincomplex are called the *singular homology groups.*

Do note the distinct difference between the de Rham cohomology and singular homology that the de Rham complex is a complex over $\mathbb{R}$, since we're talking about real vectorspaces. On the other side singular homology are free abelian groups, so modules over $\mathbb{Z}$. So in order to properly link these two we need to make the singular chains work over the field $\mathbb{R}$ (among other things).

10.2 Singular cohomology

Two very common functors in algebraic topology are the $- \otimes A$ functor and the $\hom(-, A)$ functors. These functors, when applied to the singular chain groups, for A for instance an abelian group, or R-module, sometimes make the homology groups easier to compute. We will focus on the $\hom(-, A)$ functor now to define the singular cohomology.

Lemma 10.2. $\hom(-, A)$ is a functor from $\mathtt{Ab}$ to $\mathtt{Ab}$.

Now we want to turn the chaincomplex of singular chains into a cochain, but in order to do this we need a differential map.

Definition 10.6. $d_p : \hom(S_p(X), A) \to \hom(S_{p+1}(X), A)$ which works as such;

$$d_p f = f \circ \partial_{p+1}.$$

Lemma 10.3. $(\hom(S_p(X), A), d_p)_{p \geq 0}$ is a cochain complex.

Remark 10.4. The (co)homology groups of this cochain complex are called the *singular cohomology groups.*

10.3 Smooth simplices

So far we have considered singular simplices as continuous maps from the standard simplices into our topological space. However since we are only considering smooth manifolds in this thesis we will need to add differentiability requirements for the maps.

Definition 10.7. $C_p(X)$ is the *set* of all p-simplices.

Definition 10.8. $C_p^\infty(X)$ is the set of all *smooth* p-simplices.

Now repeating the process of the first subsection of this chapter we can define in the natural way, the *smooth singular homology groups* $H_p^\infty(M)$. Now it would be preferable if these two homologies coincide for any smooth manifold. Thankfully we can make such an identification. This identification uses a main theorem in differential topology by Whitney.

Theorem 10.1. *(Whitney Approximation Theorem)* Let M and N be smooth manifolds and $F : M \to N$ be a continous map. Then F is homotopic to a smooth map $\tilde{F} : M \to N$.

10.4 De Rham homomorphism

De Rham theorem states that de Rham cohomology groups are isomorphic to singular cohomology with coefficients in $\mathbb{R}$. We will prove this theorem in two steps. First we will define the *de Rham homomorphism* which maps the de Rham groups to the singular cohomology groups. Then we will show that this is a isomorphism if the space is a smooth manifold. (Note that singular homology can be computed for all topological spaces, while de Rham cohomology is only defined for manifolds)

Definition 10.9. Let σ be a (smooth) singular p-simplex and ω a closed p-form, then we define;

$$\int_\sigma \omega := \int_{\Delta_p} \sigma^* \omega.$$

The latter being the integral over a submanifold with corners of a differential-form on $\mathbb{R}^p$.

Remark 10.5. This integral is called *the intergral of ω over σ.*

Remark 10.6. We can also extend Definition 10.9 to any (smooth) p-chain by

$$\int_{\sum_{i=1}^k c_i \sigma_i} \omega := \sum_{i=1}^k c_i \int_{\sigma_i} \omega.$$

Now as with the regular integration over manifolds we want an analogous result of Stokes' Theorem for chains.

Theorem 10.2. *(Stokes' theorem)* Let c be a smooth q-chain in M a smooth manifold. And let ω be a smooth differential $(q-1)$-form on M. Then;

$$\int_{\partial c} \omega = \int_c d\omega.$$

Remark 10.7. Note that this is basically proves commutativity of differential operators.

Proof. We will prove it only for simplices, as it just extends by linearity to chains.

Let σ be a p-simplex. Then with Definition 10.9 and commutativity of pullbacks and differential operators(by Proposition 6.1) we get;

$$\int_\sigma d\omega = \int_{\Delta_q} \sigma^* d\omega = \int_{\Delta_q} d\sigma^* \omega = \int_{\partial \Delta_q} \sigma^* \omega.$$

Now since we know that $\partial \Delta_q = \sum_{i=0}^q (-1)^i \Delta_q \circ F_{i,q}$ we can deduce,

$$\int_{\partial \Delta_q} \sigma^* \omega = \int_{\sum_{i=0}^q (-1)^i \Delta_q \circ F_{i,q}} \sigma^* \omega.$$

Which by the remark of Definition 10.9 and parametrization is equal to;

$$\sum_{i=0}^{q}(-1)^i\int_{\Delta_q\circ F_{i,q}}\sigma^*\omega=\sum_{i=0}^{q}(-1)^i\int_{\Delta_{q-1}}F_{i,q}^*\sigma^*\omega=\sum_{i=0}^{q}(-1)^i\int_{\Delta_{q-1}}(\sigma\circ F_{i,q})^*\omega,$$

now once again using the definition we get

$$=\sum_{i=0}^{q}(-1)^i\int_{\sigma\circ F_{i,q}}\omega=\int_{\partial\sigma}\omega.$$

Definition 10.10. Let $[\omega]\in H^p_{\mathrm{dR}}(M)$ and $\tilde{c}\in[c]\in H^\infty_p(M)$. The *de Rham homomorphism* $\mathcal{I}:H^p_{\mathrm{dR}}(M)\to H^p(M;\mathbb{R})$ is the following map;

$$\mathcal{I}[\omega][c]=\int_{\tilde{c}}\omega.$$

Before we check that this is in fact an isomorphism we should first see if it's even well-defined.

Lemma 10.4. $\mathcal{I}$ is well-defined in $[\omega]$ and $[c]$.

Proof. Let $\omega=\nu+d\mu$. Then;

$$\mathcal{I}[\omega][c]=\int_{\tilde{c}}\omega=\int_{\tilde{c}}\nu+d\mu=\int_{\tilde{c}}\nu+\int_{\tilde{c}}d\mu=\int_{\tilde{c}}\nu=\mathcal{I}[\nu][c],$$

the integral over $d\mu$ is 0 because $\tilde{c}$ is in the singular p-cycles of M, thus $\partial\tilde{c}=0$, it now follows that the integral is 0 with Stokes' theorem.

Next let $c'=\tilde{c}+\partial d$. Then again by linearity over the chains of the integral and applying Stokes' theorem analogously we get again that $\mathcal{I}[\omega][c]=\mathcal{I}[\omega][c']$.

□

In the coming section we will also need naturality of the de Rham homomorphism. Now in algebraic topology a natural transformation is a 'function between functors', it involves a commuting diagram which will show up in the lemma.

Lemma 10.5. *(Naturality of the de Rham homomorphism)* Let $F:M\to N$ be a smooth function. The following diagram commutes;

$$\begin{tikzcd} H^p_{\mathrm{dR}}(N) \arrow[r, "F^*"] \arrow[d, "\mathcal{I}"'] & H^p_{\mathrm{dR}}(M) \arrow[d, "\mathcal{I}"'] \\ H^p(N;\mathbb{R}) \arrow[r, "F^*"] & H^p(M;\mathbb{R}) \end{tikzcd}$$

Proof. The lemma comes down to proving;

$$\mathcal{I}(F^*[\omega])[\sigma] = \mathcal{I}[\omega](F_*[\sigma]).$$

Now just writing out the definitions;

$$\mathcal{I}(F^*[\omega])[\sigma] = \int_{\Delta_p} \sigma^* F^* \omega = \int_{\Delta_p} (F \circ \sigma)^* \omega = \mathcal{I}[\omega][F \circ \sigma] = \mathcal{I}[\omega] F^*[\sigma].$$

□

10.5 de Rham theorem

Before we put down the de Rham theorem first a few useful definitions and lemmas.

Definition 10.11. A smooth manifold M is called a *de Rham manifold* if the de Rham homomorphism is an isomorphism on it.

Lemma 10.6. Every open convex subset U of $\mathbb{R}^n$ is de Rham.

Proof. By the Poincaré lemma for de Rham cohomology and singular cohomology we get that $H^q_{\mathrm{dR}}(U) = H^q(U;\mathbb{R}) = 0$ for $n > 0$ and $H^0_{\mathrm{dR}}(U) = H^0(M;\mathbb{R}) = \mathbb{R}$. So we need only show that $\mathcal{I} : H^0_{\mathrm{dR}}(U) \to H^0(U;\mathbb{R})$ is an isomorphism. Take $\Delta_0 = \{0\}$ (really any point would do)

But since we know that $H^0_{\mathrm{dR}}(U)$ consists of all constant functions $f : M \to \mathbb{R}$ we can calculate

$$\mathcal{I}[f][\sigma] = \int_{\Delta_0} \sigma^* f = (f \circ \sigma)(0) = f.$$

thus since $f \in \mathbb{R}$, U is de Rham.

□

Lemma 10.7. Let $\{U_i\}_{i\in I}$ be a collection of open, disjoint, de Rham subsets of M then $\coprod_{i\in I} U_i$ is de Rham.

Proof. We know that $H^p_{\mathrm{dR}}(\coprod_{i\in I} U_i) \cong \prod_{i\in I} H^p_{\mathrm{dR}}(U_i)$, the same can be said for $H^p(\coprod_{i\in I} U_i;\mathbb{R}) \cong \prod_{i\in I} H^p(U_i;\mathbb{R})$. Now by Lemma 10.5 the following diagram commutes;

$$\begin{tikzcd} H^p_{\mathrm{dR}}(\coprod_{i\in I} U_i) \arrow[r, "\cong"] \arrow[d, "\mathcal{I}"'] & \prod_{i\in I} H^p_{\mathrm{dR}}(U_i) \arrow[d, "\cong"'] \\ H^p(\coprod_{i\in I} U_i;\mathbb{R}) \arrow[r, "\cong"] & \prod_{i\in I} H^p(U_i;\mathbb{R}) \end{tikzcd}$$

$\square$

Now one final lemma before we can finally prove the de Rham theorem.

Lemma 10.8. Let U and V be open subsets of a smooth manifold M, then if U, V, and $U \cap V$ are de Rham. $U \cup V$ is also de Rham.

Proof. Consider the following Mayer-Vietoris sequences for de Rham cohomology and singular cohomology respectively;

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^p_{\mathrm{dR}}(U \cup V) \arrow[r, "\mathcal{I}"] \arrow[d] & H^p(U \cup V; \mathbb{R}) \arrow[d] \\
H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) \arrow[r, "\cong"] \arrow[d] & H^p(U; \mathbb{R}) \oplus H^p(V; \mathbb{R}) \arrow[d] \\
H^p_{\mathrm{dR}}(U \cap V) \arrow[r, "\cong"] \arrow[d] & H^p(U \cap V; \mathbb{R}) \arrow[d] \\
H^{p+1}_{\mathrm{dR}}(U \cup V) \arrow[r, "\mathcal{I}"] \arrow[d] & H^{p+1}(U \cup V; \mathbb{R}) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Now since we have a 1-2-1-2 pattern of isomorphisms we can apply the Five Lemma to see, that $H^p_{\mathrm{dR}}(U \cup V) \cong H^p(U \cup V; \mathbb{R})$. Thus showing that $U \cup V$ is de Rham.

$\square$

Now all that is left to prove is the following lemma from Bredon's *Topology and Geometry*;

Lemma 10.9. Let M be a smooth n-manifold. Suppose that $P(U)$ is a statement about open subsets of M, satisfying the following three properties:

(1) $P(U)$ is true for U diffeomorphic to a convex open subset of $\mathbb{R}^n$;

(2) $P(U), P(V), P(U \cap V) \Rightarrow P(U \cup V)$; and

(3) $\{U_\alpha\}$ disjoint, and $P(U_\alpha)$, all $\alpha \Rightarrow P(\bigcup U_\alpha)$.

Then $P(M)$ is true.

Proof. Since we know M has a countable basis, elements of which are are diffeomorphic to some open subset of $\mathbb{R}^n$. All we really need to show is that $P(U)$ for any open subset of $\mathbb{R}^n$. But since U is open in a Euclidian space it can be written as a countable union of open sets each of which are diffeomorphic to open balls ($\mathbb{R}^n$ is second countable with open balls). Now for all open balls W we have $P(W)$ since these are convex, furthermore the intersection of two open balls is convex once more. Thus by (2) the union of all these open balls is convex, so $P(U)$, which implies $P(M)$.

□

Theorem 10.3. *(de Rham)* $\mathcal{I} : H^p_{\mathrm{dR}}(M) \to H^p(M; \mathbb{R})$ is an isomorphism.

Proof. We need to show that all smooth manifolds are de Rham. But by Lemma 10.6, 10.8 and 10.7 we get that 'being de Rham' satisfies the conditions of Lemma 10.9. And thus we can conclude that every smooth manifold is de Rham.

□

11 Compactly supported cohomology and Poincaré duality

With the de Rham cohomology we have found a diffeomorphism invariant of smooth manifolds. However the de Rham cohomology does not give any kind of differentiation between contractible manifolds. Since by the Poincaré Lemma their (de Rham) cohomology groups are always identical. For instance it does not differentiate between $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$. This is why we will introduce a slightly altered version of the de Rham cohomology. Furthermore we will write about a famous relation between normal and compactly supported de Rham cohomology, namely the Poincaré duality.

11.1 Compactly supported de Rham cohomology

Definition 11.1. *A function $f : M \to \mathbb{R}$ is said to have compact support if* $\text{supp}(f) := \overline{\{p \in M : f(p) \neq 0\}}$ *is compact.*

Definition 11.2. $\Gamma_c^n(M) \subset \Gamma^n(M)$ is the set of all n-forms that have compact support(or with compactly supported coefficient functions).

Now as was the case with the normal $\Gamma^\bullet(M)$, $\Gamma_c^\bullet(M)$ also forms a chaincomplex with the differential operator. We denote it's n-th cohomology group with $H_c^n(M)$. This group is called the *n-th compactly supported de Rham group.*
But why is this definition so different from the normal de Rham cohomology? After all, all compactly supported closed differential forms are also closed in the set of differential forms. The main difference is in the fact that when a differential form is exact, its "anti-derivative" need have compact support. And this is where the difference is nested.

Also note that on compact smooth manifolds all differential forms have compact support, therefore the definitions coincide as $\Gamma_c^q(M) = \Gamma^q(M)$.

Now you might wonder if compactly supported de Rham cohomology is still homotopy invariant. This next lemma will disprove this notion.

Lemma 11.1. $H_c^1(\mathbb{R}) \cong \mathbb{R}$.

Proof. Let $I : H_c^1(\mathbb{R}) \to \mathbb{R}$ be the map $\omega \mapsto \int_{\mathbb{R}} \omega$. This map is well defined because $\int_{-\infty}^{\infty} \frac{df}{dx} dx = \int_{-R}^{R} \frac{df}{dx} dx = f(R) - f(-R)$, and since f is compactly supported it is 0 outside of some compact set of $\mathbb{R}$ (compact sets are bounded in Hausdorff spaces) thus $f(R) = f(-R) = 0$ for R big enough. Next, surjectivity boils down to finding a compactly supported smooth function that integrates to 1. Finally all that is left is proving injectivity. What we need to prove is that if $\int_{\mathbb{R}} \omega = 0$ then $\omega = d\eta$. So consider $\omega = f dx$ the following function;

$$F(x) = \int_{-\infty}^{x} f(t) dt.$$

Now clearly $dF = fdx = \omega$, though we still need to show that $F(x)$ has compact support. Clearly if we choose R large enough we get $F(R) = \int_{-\infty}^{R} f(t)dt = \int_{-\infty}^{\infty} f(t)dt = 0$. But also if we choose $r < 0$ large enough we get $F(r) = \int_{-\infty}^{r} f(t)dt = \int_{-\infty}^{r} 0 = 0$. And thus we get that F has compact support and we have proven that I is an isomorphism.

□

Remark 11.1. Compactly supported de Rham cohomology is not homotopy invariant.

Proof. We know that $\mathbb{R} \simeq \{*\}$ but since the point is compact it's compactly supported de Rham groups are the same as it's regular de Rham groups. These differ from the compactly supported groups of $\mathbb{R}$ and as such compactly supported de Rham is not homotopy invariant.

□

Remember that there are no differential 2-forms on a 1-dimensional smooth manifold. As such the compactly supported de Rham cohomology groups of degree higher than the dimension of the manifold are 0. Next will calculate all topcohomology groups of Euclidian spaces.

Proposition 11.1. $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$ ($n \geq 0$, since $\mathbb{R}^0$ is compact.)

Proof. We will use the same basic proof as the previous lemma, only now we integrate the topform over $\mathbb{R}^n$ we will no longer prove surjectivity and well definedness and focus merely on injectivity. This boils down to saying 'if ω is a compactly supported differential n-form where $\int_{\mathbb{R}^n} \omega = 0$, then $\omega = d\eta$, with η compactly supported. Now we know that this is true for $n = 1$, now with induction assume it holds for $n = m - 1$. And consider a compactly supported m-form ω on $\mathbb{R}^m$ that integrates to 0. Since ω is compactly supported and $\mathbb{R}^n$ is a T_4-space we have open balls around the origin B and B' such that $\text{supp}(\omega) \subset B \subset \bar{B} \subset B'$. Furthermore, by the Poincaré lemma for regular de Rham cohomology there is a η_0 such that $d\eta_0 = \omega$. Now consider;

$$0 = \int_{\mathbb{R}^m} \omega = \int_{\bar{B}'} \omega = \int_{\bar{B}'} d\eta_0 = \int_{\partial B'} \eta_0.$$

Now because $\mathbb{R}^m \backslash \bar{B} \simeq \mathbb{S}^{m-1}$ and η_0 is an $(m-1)$-form that integrates to 0, we know that η_0 is exact on $\mathbb{R}^m \backslash \bar{B}$ by the inductionhypothesis. And as such we can find a γ such that $\eta_0 = d\gamma$. Where γ is an $(m-2)$-form on $\mathbb{R}^m \backslash \bar{B}$. If we now take ψ to be a function that is 1 on $\mathbb{R}^n \backslash B'$ and compactly supported in $\mathbb{R}^m \backslash \bar{B}$. We get that $\eta = \eta_0 - d(\psi\gamma)$ is a smooth funtion on $\mathbb{R}^m$. and furthermore satisfies $d\eta = d\eta_0 = \omega$. Since $d(\psi\gamma) = d\gamma = \eta_0$ on $\mathbb{R}^m \backslash B'$ we have that it is compactly supported.

□

Now that we know the top compactly supported de Rham groups we would like to know the other ones. As noted before the groups with index higher than the dimension of the manifold are 0. We shall see that all groups with index smaller than the dimension of the manifold will also be 0.

Proposition 11.2. $H^p_c(\mathbb{R}^n) = 0$ for all $0 \leq p < n$.

Proof. First consider $p = 0$. As we saw before the only differential 0-forms f in normal de Rham cohomology that give $df = 0$ are the constant functions, however only the function $f \equiv 0$ has compact support for $n > 0$. Thus $H^0_c(\mathbb{R}^n) = 0$. The basic idea is to make an isomorphism from $H^p_c(\mathbb{R}^n) \to H^0_c(\mathbb{R}^{n-p})$ by integrating out p coordinates. I will not go into details but this is the main idea. Note that you can also use this construction to proof the last proposition.

11.2 Mayer-Vietoris sequence for compactly supported de Rham cohomology

In section 7.3 we deduced the Mayer-Vietoris sequence for de Rham cohomology, this proved to be a valuable tool in computing the de Rham groups, so naturally we want an analogue for the compactly supported case. However the compact support part has a strange counterintuitive result, namely the existence of the special inclusion map for $U \hookrightarrow M$, $i_\# : \Gamma^n_c(U) \to \Gamma^n_c(M)$. That sends a compactly supported differential form on U to the same differential that is just 0 outside of U. (This can be done in a smooth way because of the compactly support of the form and since U is open (and M is Hausdorff)).

Lemma 11.2. The map $i_\#$ commutes with the differential.

Proof. We know that the differential sends $\omega \in \Gamma^n_c(U)$ to $d\omega \in \Gamma^{n+1}_c(U)$. So what $(i_\# \circ d)\omega$ would be $d\omega$ extended to 0 outside of U. Now if we first apply $i_\#$ we get;

$$i_\#\omega = \begin{cases} 0 & \text{on } M\backslash U; \\ \omega & \text{on } U. \end{cases}$$

Thus;

$$di_\#\omega = \begin{cases} d0 = 0 & \text{on } M\backslash U; \\ d\omega & \text{on } U. \end{cases}$$

□

Remark 11.2. $i_\#$ induced a map $i_* : H^p_c(U) \to H^p_c(M)$.

Theorem 11.1. *(Mayer-Vietoris for compactly support de Rham)* Let $U, V \subset M$ open, such that $U \cup V = M$. Then there exists a map δ_* such that the following sequence is exact.

$$\cdots \xrightarrow{\delta_*} H^p_c(U\cap V) \xrightarrow{i_*\oplus(-j_*)} H^p_c(U) \oplus H^p_c(V) \xrightarrow{k_*+l_*}$$

$$H^p_c(M) \xrightarrow{\delta_*} H^{p+1}_c(U\cap V) \xrightarrow{i_*\oplus(-j_*)} \cdots$$

where i, j, k and l are inclusion maps.

Since the proof is so much like the original Mayer-Vietoris proof, and it's mostly diagram chasing we will omit the proof.

Remark 11.3. There is also an MVS for the dual space of the compactly supported de Rham groups $H_c^p(M)^*$;

$$\cdots \xleftarrow{(\delta_*)^*} H_c^p(U \cap V) \xleftarrow{(i_*)^* - (j_*)^*} H_c^p(U) \oplus H_c^p(V) \xleftarrow{(k_*)^* \oplus (l_*)^*}$$

$$H_c^p(M) \xleftarrow{(\delta_*)^*} H_c^{p+1}(U \cap V) \xleftarrow{(i_*)^* - (j_*)^*} \cdots$$

11.3 Poincaré duality

In this section we will proof an important relation between normal and compactly supported de Rham groups. The proof will be similar to the proof of the de Rham theorem.

Definition 11.3. Consider the following map $\mathcal{PD} : \Gamma^p(M) \to \Gamma_c^{n-p}(M)^*$;

$$\mathcal{PD}(\omega)(\eta) = \int_M \omega \wedge \eta.$$

This is a linear map because the wedge product is bilinear, also this map commutes with the differentials d and d' (dual differential) because;

$$\mathcal{PD}(d\omega)(\eta) = \int_M d\omega \wedge \eta = \int_M d(\omega \wedge \eta) - \int_M \omega \wedge (-1)^p d\eta =$$
$$\mathcal{PD}(\omega)(-1^p d\eta) = d'\mathcal{PD}(\omega)(\eta),$$

since $d(\omega \wedge \eta)$ is a $(n+1)$ form on an n-manifold. This results into that $\mathcal{PD}$ induced the eponymous map on homology level. Now a few lemmas we will need to prove Poincaré duality. And a new definition.

Definition 11.4. A smooth (oriented) manifold is called *Poincaré* if $\mathcal{PD}$ is an isomorphism.

Lemma 11.3. Every open ball U of $\mathbb{R}^n$ is Poincaré.

Proof. We know that since U is contractible we get that $H^0_{\mathrm{dR}}(U) = \mathbb{R}$ and the other de Rham groups are 0. We also know that U is diffeomorphic to $\mathbb{R}^n$ thus $H_c^n(U) = \mathbb{R}$, and the other compactly supported de Rham groups are 0. Thus we need only check that $\mathcal{PD} : H^0_{\mathrm{dR}}(U) \to (H_c^n)^*$ is an isomorphism. Surjectivity follows easily since $\mathcal{PD}$ is nothing but multiplication by a constant function. And because U is oriented we have a non-vanishing topform. Basically the same argument as with the de Rham theorem. For injectivity consider f such that $\int_M f\eta = 0$ for all η, again by the existence of the non-vanishing form it is required that $f = 0$.

$\square$

Lemma 11.4. Let U, V be open sets such that U, V and $U \cap V$ are Poincaré. Then $U \cup V$ is Poincaré.

Proof. Consider the following diagram with MVS' as rows, and apply the Five Lemma:

$$\begin{tikzcd}
0 \arrow[r, "\cong"] \arrow[d] & 0 \arrow[d] \\
H^0_{\mathrm{dR}}(U \cup V) \arrow[r, "\mathcal{PD}"] \arrow[d] & H^0_c(U \cup V)^* \arrow[d] \\
H^0_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) \arrow[r, "\cong"] \arrow[d] & H^0_c(U)^* \oplus H^0_p(V)^* \arrow[d] \\
H^0_{\mathrm{dR}}(U \cap V) \arrow[r, "\cong"] \arrow[d] & H^0_c(U \cap V)^* \arrow[d] \\
0 & 0
\end{tikzcd}$$

$\square$

Remark 11.4. Note that you will need to have that $\mathcal{PD}$ commutes with the connecting homomorphism.

Lemma 11.5. Let $F : M \to N$, then the following diagram commutes.

$$\begin{tikzcd}
H^p_{\mathrm{dR}}(N) \arrow[r, "F^*"] \arrow[d, "\mathcal{PD}"'] & H^p_{\mathrm{dR}}(N) \arrow[d, "\mathcal{PD}"'] \\
H^{n-p}_c(N)^* \arrow[r, "(F_*)^*"] & H^{n-p}_c(M)^*
\end{tikzcd}$$

The proof is analogous to that of Lemma 10.5. In the same way we can prove the following lemma.

Lemma 11.6. Let $\{U_\alpha\}$ be a collection of open disjoint Poincaré sets then $\coprod_\alpha U_\alpha$ is Poincaré.

Now we can use Lemma 10.9, to once again show that;

Theorem 11.2. Let M be a smooth, orientable n-manifold. Then;

$$H^p_{\mathrm{dR}}(M) \cong H^{n-p}_c(M)^*.$$

Corollary 11.1. Let M be an orientable smooth compact 'n-manifold. Then;

$$\dim H^p_{\mathrm{dR}}(M) = \dim H^{n-p}_{\mathrm{dR}}(M).$$

Proof. Since M is compact we have that compactly supported de Rham cohomology is the same as normal de Rham cohomology. Furthermore since all de Rham cohomology groups of a compact manifold are finite dimensional we get that the dimension of a dual space of a finite dimensional vectorspace is equal to the original dimension.

$\square$

12 Conclusion

When I started writing this thesis I made a goal for myself that I wanted to understand, and be able to explain what exactly de Rham cohomology is and what it's importance is to the world of mathematics. I can now say without doubt that I have completed the first part, I understand the ideas behind the de Rham groups and in what way they work. As for the importance of the theory I will go back to my introduction where I stated that de Rham groups form a perfect example of the interaction between analysis and topology. For instance, if you know all about the differential forms of a manifold you can say something non-trivial about it's shape (is it diffeomorphic to a sphere, etc). Analogously, if you know about the shape of a manifold you can often conclude something relevant with respect to the functions on this manifold. I can certainly say that I enjoyed working on this thesis and that I truly learned a lot.

13 Bibliography

- *Introduction to Smooth Manifolds*, John M. Lee
- *From Calculus to Cohomology*, Ib H. Madsen
- *Differential Forms in Algebraic Topology*, Raoul Bott & Loring W. Tu
- *Topology and Geometry*, Glen E. Bredon
- *Introduction to differentiable manifolds*(lecture notes), Rob van der Vorst