Morse Theory via the Flow Category

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Abstract

Classical Morse theory studies the topology of manifolds through Morse functions defined on the manifolds. A main result is that a Morse function generates a CW-complex homotopic to the underlying manifold. Cohen, Jones, and Segal [9] have shown in 1995 that it is also possible to store the Morse theoretic information in a flow category. The classifying space of the flow category is homotopic to the underlying manifold if the function is a Morse function, and even homeomorphic if the function is a Morse-Smale function. We show how to define the flow category for weak Morse functions, these being smooth functions with isolated critical points. The classifying space of the flow category of a weak Morse function is shown to be homotopic to the underlying manifold. It is possible to define a flow category for a class of dynamical systems which need not come from a gradient, but exhibit gradient-like behavior. The axioms of this class excludes recurrent behavior in the system. The classifying space is shown to be homotopic to the underlying space. After this we will consider decompositions of general dynamical systems. Morse decompositions of dynamical systems give rise to isolating block decompositions. The isolating block decompositions allow us to define the flow category for any (real-time) dynamical system. The flow category is dependent on the Morse decomposition, and the isolating block decomposition subordinate to it, but the classifying space of the this flow category is shown to be homotopic to the underlying space, independent of this data.

Contents

Chapter 1

Introduction

1.1 Classical Morse Theory

Consider the landscape of figure 1.1. This landscape is the graph of a height function $f : [0,1]^2 \to \mathbb{R}$. In figure 1.2 we plotted sublevel sets $M_c = f^{-1}((-\infty, c))$ for various non-critical values of c. One realizes that the topology of the sublevel sets does not change as long as c does not pass a critical value of f. When c does cross a critical value the topology changes. Morse theory is the study of this phenomenon.

The above observation can also be made on compact manifolds M. One considers a smooth function $f : M \to \mathbb{R}$ and one studies sublevel sets as before. In figure 1.3 we plotted the sublevel sets of the height function of the torus embedded in $\mathbb{R}^3$. The height function returns the third coordinate of an embedding of the torus in $\mathbb{R}^3$. This function has four critical points. We have shown the sublevel sets of some values between critical points. More generally, if a manifold has a non-trivial homotopy type, the sublevelset $M_\infty = M$ has a non-trivial homotopy type and therefore f *must* have critical points. The Morse inequalities [2] are a concise formulation of this, relating

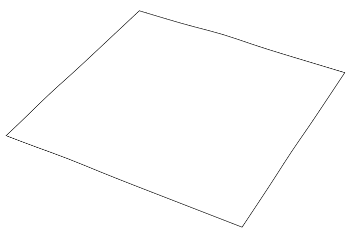

Figure 1.1: The landscape

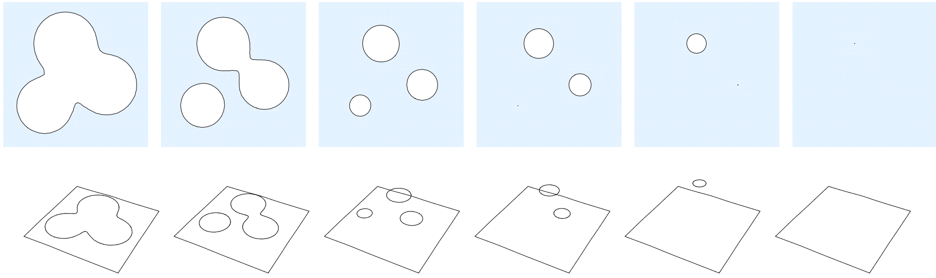

Figure 1.2: We plot the sublevel sets of the graph of the landscape. The topology of the sublevel sets changes when the sublevel set crosses a critical point of the landscape.

the minimum number of critical points of a function to the homology of the underlying manifold.

One can prove that the homotopy type of a sublevel set changes exactly by attaching an n-cell where n is given by the nature of the critical point, i.e. depending wether the critical point is a minimum, maximum, or a saddle point. One builds up a CW-complex in this manner, which captures the homotopy type of the manifold. For this to work the function f needs to satisfy certain properties, which are contained in the concept of a Morse function. Morse functions and their properties are studied in depth in [2].

Definition 1.1. Let M be a smooth manifold. A Morse function $f : M \to \mathbb{R}$ is a smooth function such that the critical points of f are non degenerate, i.e. at every point $x \in M$ where $d_x f = 0$ we have that $\det H_x f \neq 0$.

It is possible to show, see [2, Theorem 5.31], that the class of Morse functions lies open and dense in $C^r(M, \mathbb{R})$ for any $2 \leq r \leq \infty$. One can form an interesting twist

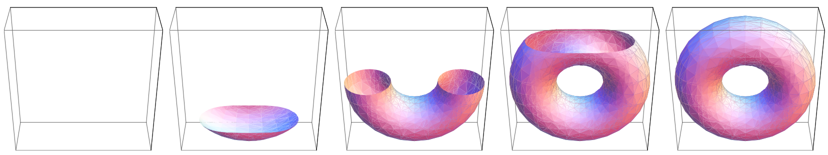

Figure 1.3: We plot the sublevel sets of the height function on the torus embedded in $\mathbb{R}^3$. The topology of the sublevel set changes when we pass a critical value of the height function. There are four critical points of the height function. If we are below the first critical point, the sublevel set is the empty set. If we pass the minimum of the height function, the sublevel set is homeomorphic to the disc. When we pass the first saddle point, we obtain a tube. After we pass the third critical value, the sublevel set equals the torus with a hole in it. When the maximum of the height function is reached, the whole torus is obtained.

on the idea that the critical points of a function are constrained by the homology of the underlying space. One can show that not only the number of critical points of a Morse function is constrained by the homology of the manifold; the function, if chosen generically, also gives means for computing the homology. One needs to define some extra technical apparatus, like boundary operators, but it is possible to do. The generic condition the function needs to satisfy is known as the Morse-Smale transversality condition.

Definition 1.2. Let M be a Riemannian manifold. A Morse function $f : M \to \mathbb{R}$ satisfies the *Morse-Smale transversality condition* if and only if the stable and unstable manifolds of f intersect transversely, i.e.

$$W^u(q) \pitchfork W^s(p) \tag{1.1}$$

for all critical points p, q. For all point $x \in W^u(q) \cup W^s(p)$, the tangent space of T_xM is spanned by the tangent spaces $T_xW^u(q)$ and $T_xW^s(p)$

$$T_xM = T_xW^u(q) \oplus T_xW^s(p). \tag{1.2}$$

A Morse function which satisfies the Morse-Smale transversality condition is called a *Morse-Smale* function.

1.2 A Categoric View of Morse Theory

Enter Cohen, Jones, and Segal. In an unpublished paper [9] they propose another way to store Morse theoretic information. The information of the gradient flow of the Morse function is not stored in a CW-complex, but in a topological category, which they call the flow category. The classifying space of the flow category, is homotopic to the underlying manifold if the function is a Morse function. They also prove the stronger result that the classifying space is homeomorphic to the underlying manifold if the function satisfies the Morse-Smale transversality condition.

Theorem 1.3 (Cohen, Jones, and Segal)**.** *Let $f : M \to \mathbb{R}$ be a Morse function defined on a closed Riemannian manifold M, and $\mathfrak{C}_f$ the flow category of this function. Then*

- *The classifying space of the flow category is homotopic to M:*

$$B\mathfrak{C}_f \simeq M. \tag{1.3}$$

- *If f is generic, i.e. it satisfies the Morse-Smale transversality condition, then the classifying space of the flow category is homeomorphic to M:*

$$B\mathfrak{C}_f \cong M. \tag{1.4}$$

Their paper did not push the weakest assumptions on f for which the first result holds. It is not necessary to start with a Morse function, we only need that the critical points are isolated. In chapter 3 we prove the homotopy part of their result for the weaker assumptions, also filling in some details omitted in the original paper.

The functions we study are weak Morse functions. These are smooth functions, which have isolated critical points, i.e. we can find neighborhoods around each critical point, such that in this neighborhood there is only one critical point.

We will not focus our attention on the second part of the theorem, that is equation (1.4). The result does not generalize to dynamical systems apparently.

1.3 Dynamical Systems

The ideas of Cohen et al. are applicable to a class of continuous dynamical systems on metric spaces which are sufficiently gradient-like. This is the class of strongly gradient-like systems, see definition 5.3. In chapter 5 we show that we can construct a flow category for this class of dynamical systems. For the flow category we show that we can prove the following theorem.

Theorem 1.4. *Let $\mathcal{D}$ be a strongly gradient-like dynamical system on a metric space S, and $\mathfrak{C}_{\mathcal{D}}$ its flow category. The classifying space of the flow category $\mathfrak{C}_{\mathcal{D}}$ is homotopic to the underlying metric space S*

$$B\mathfrak{C}_{\mathcal{D}} \simeq S. \tag{1.5}$$

This result directly generalizes theorem 3.1. The gradient flow of a weak Morse function determines a strongly gradient-like dynamical system. Another class of functions often studied for their nice properties is the class of Morse-Bott functions. We also show that the gradient flow of this class of functions gives rise to a strongly gradient-like dynamical system. The statement is much broader however, we do not even need a differentiable structure on the underlying manifold, a general compact metric space is enough.

Morse theory has also made an impact on general dynamical system theory. In general, the behavior of dynamical systems is extremely complex. A part of the behavior of the dynamical system can have properties of a gradient system. Recurrent behavior is not possible on this part of the dynamics. This can be modeled using Morse decompositions. Morse decompositions generalize the idea of critical points in a gradient system. In chapter 7 we attack general dynamical systems, using Morse decompositions. These Morse decompositions induce isolating block decompositions, which are closed neighborhoods of the Morse decompositions. These isolating blocks can be used to define a flow category for the dynamical system. This flow category depends on both the Morse decomposition, and the choice of the isolating block decomposition subordinate to it. However the homotopy type of the classifying space does not depend on either. We prove:

Theorem 1.5. *Let $\mathcal{D}$ be a dynamical system on a compact metric space S, M be a Morse decomposition of the dynamical system, and N be an isolating block decompo-*

sition subordinate to M*. The homotopy type of the classifying space of the flow category* $\mathfrak{C}_{\mathcal{D}}^{M,N}$ *is an invariant for dynamical systems, Morse decompositions and isolating block decompositions subordinate to this, and is of the same type as the underlying metric space* S*.*

$$B\mathfrak{C}_{\mathcal{D}}^{M,N} \simeq S. \tag{1.6}$$

1.4 Organization

The thesis is organized as follows. In chapter 2 we recall the notions of category and topological category. The classifying space of a topological category is defined. Some examples of classifying spaces are computed. We reformulate and prove the homotopy part of the theorem of Cohen et al. [9] in chapter 3 for a larger class of functions than they studied. We also correct some minor mistakes, and fill in some details omitted in the original paper. We continue in chapter 4 with some examples of weak Morse functions, their flow categories and the corresponding classifying spaces.

In chapter 5 we define a class of dynamical systems which allow for the construction of a flow category. We prove that the classifying space of the flow category has the homotopy type of the underlying metric space. Some examples of this theorem are studied in chapter 6.

In chapter 7 we attack general dynamical systems. We show that Morse decompositions give rise to isolating block decompositions, which allow for the definition of a flow category. The homotopy type of the flow category is an invariant for the underlying space, and equivalent to the homotopy type of the underlying space. Examples of this theorem are studied in chapter 8.

Chapter 2

Classifying Spaces

2.1 Introduction

Category theory is the abstract study of mathematical structures. A way to understand categories is to use the classifying space. The classifying space is a topological space which one associates to a a category. The space captures some information of the category. In this chapter we recall the notion of categories, topological categories, and we show how the classifying space is constructed. In the last section we discuss some examples.

2.2 Categories

Mathematics is the study of structure. Important are maps which preserve the structure under consideration. One can think for example of sets and functions, of vector spaces and linear maps, of topological spaces and continuous maps, or of groups and group homomorphisms. It is a deep insight, and has taken mathematicians a long time to realize, that the structure preserving maps illuminate the structure one studies more that the structures themselves do. What it does, it defines. Category theory is a profound way of expressing and generalizing this insight. In this section we recall the notion of a category and some basic properties of categories are studied. By no means this is exhaustive, or enough prerequisites to follow this thesis. We merely use it to refresh the mind and fix notation. A classic reference on this subject is Mac Lane [18], and one can consult the freely available primer of Hillman [14].

Definition 2.1. A *Category* $\mathfrak{C}$ consists of two collections

- the collection of *objects* $\mathrm{Ob}(\mathfrak{C})$ of $\mathfrak{C}$,
- the collection of *morphisms* $\mathrm{Hom}(\mathfrak{C})$ of $\mathfrak{C}$,

and four rules

1. a rule cod assigning to a morphism γ the *codomain* of γ, which is an object $\mathrm{cod}\,\gamma$ of $\mathfrak{C}$.

2. a rule dom assigning to a morphism γ the *domain* of γ, which is an object $\mathrm{dom}\,\gamma$ of $\mathfrak{C}$.

3. a rule id assigning to an object X the *identity*, which is a morphism id_X with the property $\mathrm{cod}\,\mathrm{id}_X = X$ and $\mathrm{dom}\,\mathrm{id}_X = X$.

4. a rule $\circ$, called *composition*, assigning to a *composable pair of morphisms* (α, β), that is a pair of morphisms with the property $\mathrm{dom}\,\beta = \mathrm{cod}\,\alpha$, a new morphism $\beta \circ \alpha$ with

$$\begin{aligned} \mathrm{dom}(\beta \circ \alpha) &= \mathrm{dom}\,\alpha \\ \mathrm{cod}(\beta \circ \alpha) &= \mathrm{cod}\,\beta. \end{aligned} \tag{2.1}$$

The collections and the rules form a category if the following two properties are satisfied

- composition is associative, i.e. $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$ for all composable morphisms,
- the identity can be removed or inserted in any string of compositions. The two properties

$$\begin{aligned} \gamma \circ \mathrm{id}_{\mathrm{dom}(\gamma)} &= \gamma, \\ \mathrm{id}_{\mathrm{cod}(\gamma)} \circ \gamma &= \gamma, \end{aligned} \tag{2.2}$$

  hold for all morphisms γ.

Remark 2.2. A morphism γ with $\mathrm{dom}\,\gamma = X$ and $\mathrm{cod}\,\gamma = Y$ will be written in the following equivalent forms $\gamma : X \to Y$, $X \xrightarrow{\gamma} Y$, and $Y \xleftarrow{\gamma} X$ interchangeably. These notations specify the same morphism. We sometimes omit the composition symbol $\circ$ when no ambiguity arises. Thus $\beta \circ \alpha$ will be written as $\beta\alpha$ or $\beta(\alpha)$ indiscriminately. Morphisms are also known as arrows.

A useful feature of category theory is the simplicity with which one can move between different layers of abstraction. One can study morphisms between categories. These morphisms are known as functors.

Definition 2.3. A *functor* $F : \mathfrak{C} \to \mathfrak{D}$ consists of two rules denoted by the same symbol

$$\begin{aligned} F &: \mathrm{Ob}(\mathfrak{C}) \to \mathrm{Ob}(\mathfrak{D}) \\ F &: \mathrm{Hom}(\mathfrak{C}) \to \mathrm{Hom}(\mathfrak{D}), \end{aligned} \tag{2.3}$$

which satisfy the intertwining properties

- for all morphisms γ, F intertwines the structural rules

$$\begin{aligned} \mathrm{dom}(F\gamma) &= F(\mathrm{dom}\,\gamma) \\ \mathrm{cod}(F\gamma) &= F(\mathrm{cod}\,\gamma) \end{aligned} \tag{2.4}$$

- for all composable pairs (α, β) of morphisms in $\mathfrak{C}$, F intertwines composition

$$F(\beta \circ \alpha) = (F\beta) \circ (F\alpha). \tag{2.5}$$

- F maps identities to identities

$$F\,\mathrm{id}_X = \mathrm{id}_{FX}\,. \tag{2.6}$$

  This holds for all objects X.

Remark 2.4. Functors take commutative diagrams to commutative diagrams. For example, let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor, and assume the diagram in $\mathfrak{C}$

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ & \searrow^{\nu\circ\mu} & \downarrow{\scriptstyle \nu} \\ & & Z \end{array} \tag{2.7}$$

commutes in $\mathfrak{C}$. The axioms of a functor makes that we can apply F to the whole diagram. The resulting diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\mu)} & F(Y) \\ & \searrow^{F(\nu\circ\mu)} & \downarrow{\scriptstyle F(\nu)} \\ & & F(Z) \end{array} \tag{2.8}$$

is a commutative diagram in $\mathfrak{D}$.

If we have two functors $F_i : \mathfrak{C} \to \mathfrak{D}$ we can also do the preceding in a commutative way. This is the idea of a natural transformation.

Definition 2.5. A *natural transformation* $\mathcal{N}$ between two functors $F : \mathfrak{C} \to \mathfrak{D}$, and $G : \mathfrak{C} \to \mathfrak{D}$, written $\mathcal{N} : F \dot{\to} G$, is a rule associating to each object $X \in \mathrm{Ob}(\mathfrak{C})$ a morphism $\mathcal{N}_X \in \mathrm{Hom}(\mathfrak{D})$, such that the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\mathcal{N}_X} & G(X) \\ {\scriptstyle F(\gamma)}\downarrow & & \downarrow{\scriptstyle G(\gamma)} \\ F(Y) & \xrightarrow[\mathcal{N}_Y]{} & G(Y), \end{array} \tag{2.9}$$

commutes for all morphisms $\gamma : X \to Y$ in $\mathfrak{C}$. The morphism $\mathcal{N}_X$ is called the *component* of $\mathcal{N}$ at X.

A natural transformation is a "functor between functors". This is made precise in the next proposition.

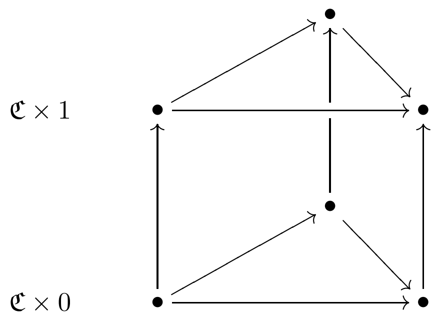

Figure 2.1: A sketch of the category $\mathfrak{C} \times \mathfrak{2}$. The category consists of two copies of $\mathfrak{C}$. We can move to the upper layer via the morphism $\uparrow$.

Proposition 2.6. *Let $F_0, F_1 : \mathfrak{C} \to \mathfrak{D}$ be two functors, and $\mathcal{N} : F_0 \dot{\to} F_1$ a natural transformation between them. The natural transformation $\mathcal{N}$ is isomorphic to a functor $N : \mathfrak{C} \times \mathfrak{2} \to \mathfrak{C}$. Here $\mathfrak{2}$ is the category with two objects, denoted $0, 1$, and one non-identity morphism $\uparrow : 0 \to 1$.*

Proof. The category $\mathfrak{C} \times \mathfrak{2}$ can best be understood using figure 2.1. Let $X \in \mathrm{Ob}(\mathfrak{C})$, $(\gamma : X \to Y) \in \mathrm{Hom}(\mathfrak{C})$, and $i \in \{0, 1\}$. Define the functor N via the equations

$$\begin{aligned} N(X, i) =& F_i(X) \\ N(\gamma, \mathrm{id}_i) =& F_i(\gamma) \\ N(\gamma, \uparrow) =& \mathcal{N}_Y \circ F_0(\gamma) = F_1(\gamma) \circ \mathcal{N}_X. \end{aligned} \tag{2.10}$$

The latter identity holds because $\mathcal{N}$ is a natural transformation. One easily verifies that N is a functor. These equations express that applying N to the commuting diagram

$$\begin{array}{ccc} (X,0) & \xrightarrow{(\mathrm{id}_X, \uparrow)} & (X,1) \\ {\scriptstyle (\gamma, \mathrm{id}_0)}\downarrow & & \downarrow{\scriptstyle (\gamma, \mathrm{id}_1)} \\ (Y,0) & \xrightarrow[(\mathrm{id}_Y, \uparrow)]{} & (Y,1) \end{array} \tag{2.11}$$

gives the commutative diagram

$$\begin{array}{ccc} F_0(X) & \xrightarrow{\mathcal{N}_X} & F_1(X) \\ {\scriptstyle F_0(\gamma)}\downarrow & & \downarrow{\scriptstyle F_1(\gamma)} \\ F_0(Y) & \xrightarrow[\mathcal{N}_Y]{} & F_1(Y) \end{array} \quad . \tag{2.12}$$

Any natural transformation $\mathcal{N}$ defines the functor N uniquely, and any functor $N : \mathfrak{C} \times \mathfrak{2} \to \mathfrak{C}$ defines a unique natural transformation $\mathcal{N}$. □

Two properties of categories are basic. We define them here.

Definition 2.7. A category $\mathfrak{C}$ is *small* if $\mathrm{Ob}(\mathfrak{C})$ and $\mathrm{Hom}(\mathfrak{C})$ are sets, and not proper classes [18]. A category is *concrete* if $X \in \mathrm{Ob}(\mathfrak{C})$ is a set with structure and $\gamma \in \mathrm{Hom}(\mathfrak{C})$ is a function preserving that structure for all objects and morphisms.

Example 2.8. The category of all sets $\mathfrak{S}et$, and the category of all groups $\mathfrak{G}rp$ are large and concrete. In the first case morphisms are functions between sets, and in the latter case morphisms are homomorphisms, i.e. functions preserving the group structure. The category $\mathfrak{2}$ is small and non-concrete.

Remark 2.9. The categories we will consider are small. Therefore the structural rules are actually maps. From here on we will speak of the structural maps, instead of structural rules.

2.3 Topological Categories

Categories can be enriched by letting the collections of objects and morphisms be objects in some other category, e.g. a concrete category is a category enriched by $\mathfrak{S}et$. We do not consider general enrichements of categories, only categories enriched by $\mathfrak{T}op$, the category of topological spaces and continuous functions. The axioms of a $\mathfrak{T}op$-enriched category boil down to the definition that follows.

Definition 2.10. A *Topological Category* is a small category $\mathfrak{C}$ satisfying the additional axioms

- $\mathrm{Ob}(\mathfrak{C})$ and $\mathrm{Hom}(\mathfrak{C})$ are topological spaces.
- The four structural maps are continuous. These are
  - The identity map $\mathrm{id} : \mathrm{Ob}(\mathfrak{C}) \to \mathrm{Hom}(\mathfrak{C})$
  $$X \mapsto \mathrm{id}_X\,. \tag{2.13}$$
  - The composition law $\circ : \mathrm{Hom}(\mathfrak{C}) \times \mathrm{Hom}(\mathfrak{C}) \to \mathrm{Hom}(\mathfrak{C})$
  $$(\gamma_1, \gamma_2) \mapsto \gamma_2 \circ \gamma_1. \tag{2.14}$$
  - The domain map $\mathrm{dom} : \mathrm{Hom}(\mathfrak{C}) \to \mathrm{Ob}(\mathfrak{C})$
  $$(\gamma : X \to Y) \mapsto X. \tag{2.15}$$
  - The codomain map $\mathrm{cod} : \mathrm{Hom}(\mathfrak{C}) \to \mathrm{Ob}(\mathfrak{C})$
  $$(\gamma : X \to Y) \mapsto Y. \tag{2.16}$$

Remark 2.11. Every small category can be topologically enriched to a topological category. We endow the sets $\mathrm{Ob}(\mathfrak{C})$ and $\mathrm{Hom}(\mathfrak{C})$ with the discrete topology, i.e. all subsets of these sets are open. All the structural maps are trivially continuous, thus the category is topological. Of course this is not the unique topology that turns the category into a topological category.

The natural morphisms between topological categories are continuous functors.

Definition 2.12. A *continuous functor* $F : \mathfrak{C} \to \mathfrak{D}$ is a functor between two topological categories satisfying the with the property that the maps $F : \mathrm{Ob}\,\mathfrak{C} \mapsto \mathrm{Ob}\,\mathfrak{D}$ and $F : \mathrm{Hom}\,\mathfrak{C} \mapsto \mathrm{Hom}\,\mathfrak{D}$ are continuous.

A perk of category theory is the ability to maneuver between different layers of abstraction. Category theory also allows us to study totalities of structures, e.g. the category of all sets and the category of all small categories. We are interested in the category of all topological categories.

Definition 2.13. The *category of topological categories* $\mathfrak{T}opCat$ is the large category whose objects are topological categories, and whose morphisms are continuous functors.

Remark 2.14. A topological category is a small category; Ob, and Hom are topological spaces, and therefore sets. In contrast $\mathfrak{T}opCat$ is not small, i.e. $\mathrm{Hom}(\mathfrak{T}opCat)$ is a proper class. For a discussion on small and large categories, consult Mac Lane [18].

Remark 2.15. The term topological category has two different meanings in the mathematical literature. The first notion of topological category is the one we have described in this section, small categories where the sets of objects and morphisms come equipped with a topology such that the four structural maps are continuous. The second one, found for example in Brümmer [7], is a generalization of the category of topological spaces $\mathfrak{T}op$. These topological categories have morphisms which have properties similar to the the morphisms in $\mathfrak{T}op$ i.e. continuous functions. Some examples include the category of metric spaces, or the category of topological spaces $\mathfrak{T}op$ itself.

2.4 The Classifying Space of a Topological Category

To every topological category we can associate the classifying space of the topological category. This is a topological space. The classifying space gives geometric understanding of the category. This construction is functorial. A continuous functor induces a continuous map between the associated classifying spaces. In this section we describe this construction.

Definition 2.16. Let $\boldsymbol{X} = (X_0, X_1, ..., X_l)$ be a finite sequence of objects of $\mathfrak{C}$. Such a sequence is called *admissible* if for all $0 \leq i \leq l-1$, $\mathrm{Hom}(X_i, X_{i+1})$ is nonempty. Then $l(\boldsymbol{X}) = l$ is the *length* of the sequence. Furthermore we define the space $\mathrm{Hom}(\boldsymbol{X})$ of morphisms of the admissible sequence $\boldsymbol{X}$ with length $l(\boldsymbol{X}) \geq 1$ to be

$$\mathrm{Hom}(\boldsymbol{X}) := \mathrm{Hom}(X_0, X_1) \times \mathrm{Hom}(X_1, X_2) \times ... \times \mathrm{Hom}(X_{l-1}, X_l). \tag{2.17}$$

It is endowed with the product topology. If $l(\boldsymbol{X}) = 0$ then

$$\mathrm{Hom}(\boldsymbol{X}) := \mathrm{Hom}(X_0, X_0). \tag{2.18}$$

The classifying space will be built up from limbs.

Definition 2.17. Let Δ^n be the standard n-*simplex*

$$\Delta^n := \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum_{i=0}^{n} x_i = 1 \right\}, \tag{2.19}$$

and the spaces $N_n\mathfrak{C}$ of a topological category $\mathfrak{C}$ be defined by

$$N_n\mathfrak{C} := \coprod_{l(\boldsymbol{X})=n} \operatorname{Hom}(\boldsymbol{X}). \tag{2.20}$$

The disjoint union runs over all admissible sequences of length n. The *limbs* $K\mathfrak{C}$ are defined by

$$K\mathfrak{C} := \coprod_{n=0}^{\infty} \Delta^n \times N_n\mathfrak{C}. \tag{2.21}$$

$K\mathfrak{C}$ is a topological space.

The last ingredient necessary for the definition of the classifying space is an equivalence relation. This equivalence relation is generated by four maps, which we define below.

Definition 2.18. The face map $\delta_i : \Delta^n \to \Delta^{n+1}$ and the degeneracy map $\sigma_i : \Delta^n \to \Delta^{n-1}$ are defined by

$$\begin{aligned} \delta_i(x_0, \dots, x_n) &= (x_0, x_1, \dots, x_{i-1}, 0, x_i, \dots, x_n) && \text{for } 0 \leq i \leq n+1 \\ \sigma_i(x_0, \dots, x_n) &= (x_0, x_1 \dots, x_i + x_{i+1}, x_{i+2}, \dots, x_n) && \text{for } 0 \leq i \leq n-1. \end{aligned}$$

What remains are the more complicated maps $d_i : N_n\mathfrak{C} \to N_{n-1}\mathfrak{C}$ and $s_i : N_n\mathfrak{C} \to N_{n+1}\mathfrak{C}$. For $l(\boldsymbol{X}) \geq 2$ we have

$$d_i : \operatorname{Hom}(X_0, \dots, X_l) \to \operatorname{Hom}(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_l), \tag{2.22}$$

defined by

$$d_i(\gamma_0, \gamma_1, \dots, \gamma_{l-1}) = \begin{cases} (\gamma_1, \dots, \gamma_{l-1}) & i = 0 \\ (\gamma_0, \dots, \gamma_i \circ \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_{l-1}) & 1 \leq i \leq l-1 \\ (\gamma_0, \dots, \gamma_{l-2}) & i = l \end{cases}. \tag{2.23}$$

Notice that γ_i is a morphism with $\operatorname{dom}\gamma_i = X_i$ and $\operatorname{cod}\gamma_i = X_{i+1}$. If $l(\boldsymbol{X}) = 1$ we have the degenerate case $d_0 : \operatorname{Hom}(X_0, X_1) \to \operatorname{Hom}(X_1)$ which maps $\gamma_0 \mapsto \mathrm{id}_{X_1}$, and $d_1 : \operatorname{Hom}(X_0, X_1) \to \operatorname{Hom}(X_0)$ which maps $\gamma_0 \mapsto \mathrm{id}_{X_0}$. The last map we need is

$$s_i : \operatorname{Hom}(X_0, \dots, X_l) \to \operatorname{Hom}(X_0, \dots, X_i, X_i, \dots, X_l), \tag{2.24}$$

defined by the equation

$$s_i(\gamma_0, \gamma_1, \dots, \gamma_{l-1}) = (\gamma_0, \dots, \gamma_{i-1}, \mathrm{id}_{X_i}, \gamma_i, \dots, \gamma_{l-1}). \tag{2.25}$$

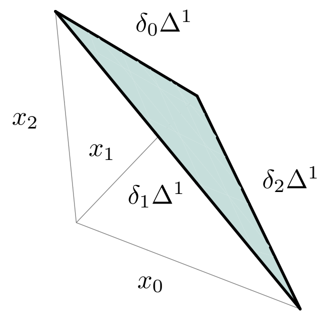

Figure 2.2: The maps $\delta_0, \delta_1, \delta_2$ are shown acting on Δ^1. They map the 1-simplex Δ^1 onto the boundary of the 2-simplex Δ^2 as indicated in the figure. Each map maps Δ^1 to a different edge of the 2-simplex Δ^2.

Remark 2.19. We have chosen to name all face and degeneracy maps by the same symbol. That is $\delta_i : \Delta^n \to \Delta^{n+1}$ are all denoted δ_i for all n. From context the dimensionality of the domains and images will be clear. One should realize that d_i goes in the opposite direction as δ_i; i.e. d_i maps a Hom space of an admissible sequence into a Hom space of a shorter admissible sequence. δ_i maps a low dimensional simplex into a higher dimensional simplex. The situation for s_i and σ_i is similar, but the directions are reversed. s_i maps an admissible sequence into a longer admissible sequence, while σ_i maps a simplex into a lower dimensional simplex.

In figure 2.3 we have drawn the zero, one, and two dimensional simplices. To better understand the maps δ and σ we have listed the action of $\delta : \Delta^1 \to \Delta^2$ in figure 2.2, and the action of $\sigma : \Delta^2 \to \Delta^1$ in figure 2.4. The face maps map lower dimensional simplices into the faces of a higher dimensional simplices. The degeneracy map project higher dimensional simplices onto lower dimensional simplices.

The maps d_i and s_i are best understood diagrammatically. Let $\boldsymbol{X}$, with $l(\boldsymbol{X}) \geq 2$ be an admissible sequence and $\boldsymbol{\gamma} \in \mathrm{Hom}(\boldsymbol{X})$. Then d_i acts on the string

$$\boldsymbol{\gamma} = \; a_0 \xrightarrow{\gamma_0} \cdots \xrightarrow{\gamma_{l-1}} a_l \tag{2.26}$$

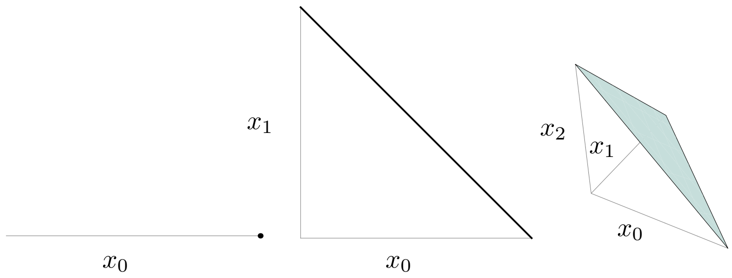

Figure 2.3: The blue dot, line, and triangle are the first three simplices. Later we will not consider the embedding of Δ^n into $\mathbb{R}^{n+1}$ but draw them as subspaces of $\mathbb{R}^n$. For computational purposes the embedding of Δ^n in $\mathbb{R}^{n+1}$ is convenient.

by

$$\begin{aligned} d_0(\boldsymbol{\gamma}) = &\ a_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{l-1}} a_l \\ d_i(\boldsymbol{\gamma}) = &\ a_0 \xrightarrow{\gamma_0} \cdots \xrightarrow{\gamma_{i-2}} a_{i-1} \xrightarrow{\gamma_i \circ \gamma_{i-1}} a_{i+1} \xrightarrow{\gamma_i} \cdots \xrightarrow{\gamma_{l-1}} a_l \\ d_l(\boldsymbol{\gamma}) = &\ a_0 \xrightarrow{\gamma_0} \cdots \xrightarrow{\gamma_{l-2}} a_{l-1}\ . \end{aligned} \tag{2.27}$$

The map s_i inserts an identity at the i-th spot

$$s_i(\boldsymbol{\gamma}) = \ a_0 \xrightarrow{\gamma_0} \cdots \xrightarrow{\gamma_{i-1}} a_i \xrightarrow{\mathrm{id}_{a_i}} a_i \xrightarrow{\gamma_i} \cdots \xrightarrow{\gamma_{l-1}} a_l\ . \tag{2.28}$$

Definition 2.20. The *Classifying Space* $B\mathfrak{C}$ associated to a topological category $\mathfrak{C}$ is the topological space

$$B\mathfrak{C} := K\mathfrak{C}/\sim, \tag{2.29}$$

where the equivalence relation $\sim$ in K is generated by stating

$$(x, d_i(\boldsymbol{\gamma})) \sim (\delta_i(x), \boldsymbol{\gamma}), \tag{2.30}$$

and

$$(x, s_i(\boldsymbol{\gamma})) \sim (\sigma_i(x), \boldsymbol{\gamma}). \tag{2.31}$$

Remark 2.21. The construction of the classifying space is also possible for ordinary, i.e. non-topological, categories. One builds up a simplicial set, called the nerve of the category, and then the classifying space is the geometric realization of this simplicial set. Equivalently one can turn the category into a topological category by imposing the discrete topology on $\mathrm{Hom}(\mathfrak{C})$ and $\mathrm{Ob}(\mathfrak{C})$, and one gets the classifying space outlined in this section. Both constructions are equivalent, and one obtains a CW-complex in this manner. For a proof of this and more information see Milnor [22].

A proper understanding of the classifying space construction involves some more terminology.

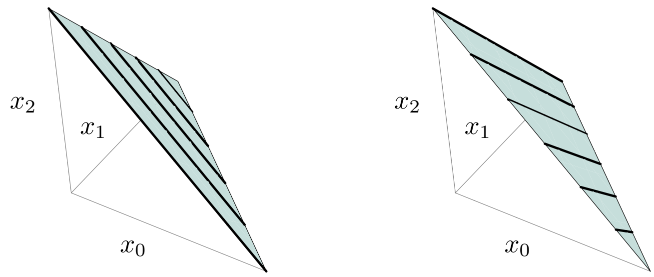

Figure 2.4: The left picture the identification σ_0 induces on Δ^2 is shown. A thick line is identified with a single point of Δ^1. On the right picture the identification of σ_1 is depicted in the same manner.

Definition 2.22. A point $x \in \Delta^n$ is *interior* if $x_i \neq 0$ for all $0 \leq i \leq n$. Let $\boldsymbol{X}$ be an admissible sequence. A chain $\boldsymbol{\gamma} \in \mathrm{Hom}(\boldsymbol{X})$ is said to be *degenerate* if $\gamma_i = \mathrm{id}_{X_i}$, otherwise it is *non-degenerate*. A point $(x, \boldsymbol{\gamma}) \in K\mathfrak{C}$ is *non-degenerate*, if $\boldsymbol{\gamma}$ is non-degenerate and x is interior.

Remark 2.23. Note that a chain cannot be degenerate if the admissible sequence does not stammer. In the categories we will consider, $\mathrm{Hom}(X, X)$ will always consist of a single morphism, the identity morphism. In this case the non-degeneracy of the chain is equivalent with the property that the admissible sequence does not stammer. Note that different non-degenerate elements of $K\mathfrak{C}$ are never equivalent.

The classifying space allows us to study a category geometrically. Objects are identified with points, morphisms with lines, triangular commuting diagrams with triangles etc. The construction of the classifying space identifies the edges of these objects with the elements from which they are constructed. We clarify this with an example. More examples are found at the end of this chapter in section 2.7.

Example 2.24. Let $\mathfrak{2}$ be the category with two elements, $0, 1$ and one non-identity morphism, the morphism $\uparrow\colon 0 \to 1$. The category is topologized by endowing $\mathrm{Ob}(\mathfrak{2})$ and $\mathrm{Hom}(\mathfrak{2})$ with the discrete topology. Most elements of the limbs

$$K\mathfrak{2} = \coprod_{i=0}^{\infty} \Delta^i \times \left(\coprod_{l(\boldsymbol{X})=i} \mathrm{Hom}(\boldsymbol{X}) \right) \tag{2.32}$$

are degenerate, c.f. definition 2.22. All elements with $i > 1$ are, because there are only two different objects. In the classifying space, all spaces with dimension $i > 1$ will be identified with spaces of dimension 0, or 1. For $i = 0$ we only have two points, these are the objects 0 and 1. For $i = 1$ we have a single admissible sequence which is not degenerate, this is the sequence

$$\boldsymbol{X} = \{0, 1\} \quad \text{and} \quad \boldsymbol{\gamma} \in \mathrm{Hom}(\boldsymbol{X}) \quad \text{is} \quad \boldsymbol{\gamma} = \{\uparrow\}. \tag{2.33}$$

One computes

$$d_0(\boldsymbol{\gamma}) = 0 \qquad \text{and} \qquad d_1(\boldsymbol{\gamma}) = 1. \tag{2.34}$$

The endpoints are identified with the domain and codomain. The resulting classifying space is a line. This is geometrically what we would expect the category to be. In the example section 2.7 we compute the classifying space of $\mathfrak{3}$, which is a triangle. This generalizes, for the category $\mathfrak{n}$ the classifying space can be computed to be homeomorphic to the $n-1$ simplex

$$B\mathfrak{n} \cong \Delta^{n-1}. \tag{2.35}$$

Remark 2.25. The classifying space is a way to study categories in a topological manner. We can lift topological properties to categories. For example we can define a category to be connected, if and only if the classifying space of the category is connected. We will not pursue this idea further.

Remark 2.26. We have chosen to give a rather explicit description of the classifying space. We formulate the process more abstractly. A category $\mathfrak{C}$, determines a nerve $N\mathfrak{C}$ which is a simplicial set, or a simplicial space if the category is topological. The classifying space is the geometric realization of the nerve. All maps in this construction are functors. The diagram

$$\mathfrak{C} \xrightarrow{\text{Nerve Functor}} N\mathfrak{C} \xrightarrow{\text{Geometric Realization Functor}} B\mathfrak{C} \tag{2.36}$$

illuminates this. The abstract functorial approach is studied in [28].

2.5 Some Propositions

We prove some basic propositions involving classifying spaces and topology.

Proposition 2.27. *A continuous functor $F : \mathfrak{C} \to \mathfrak{D}$ induces a continuous map of classifying spaces $BF : B\mathfrak{C} \to B\mathfrak{D}$. The operation*

$$B : \mathfrak{T}opCat \to \mathfrak{T}op, \tag{2.37}$$

is a functor.

Proof. We prove this result in two steps. We first show that a continuous map between $K\mathfrak{C}$ and $K\mathfrak{D}$ is induced. Secondly we show that this induced map respects the equivalence relation. It follows that the resulting map descends to the quotient.

Step 1. If the chain $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_l)$ has the property $\boldsymbol{\gamma} \in \mathrm{Hom}(\boldsymbol{X})$ for an admissible sequence $\boldsymbol{X} = (a_0, \ldots, a_l)$ in $\mathfrak{C}$, then $F(\boldsymbol{\gamma}) := (F\gamma_0, \ldots, F\gamma_l) \in \mathrm{Hom}(F(\boldsymbol{X}))$ for the admissible sequence $F(\boldsymbol{X}) := (F(a_0), \ldots, F(a_l))$ in $\mathfrak{D}$, by the functorial properties of F. The map $KF : K\mathfrak{C} \to K\mathfrak{D}$ defined by

$$KF(x, \boldsymbol{\gamma}) = (x, F(\boldsymbol{\gamma})), \tag{2.38}$$

is a continuous map by the continuity properties of F. If $G : \mathfrak{D} \to \mathfrak{E}$ is another continuous functor, then $K(G \circ F) = (KG) \circ (KF)$ as is directly verified.

Step 2. We show that KF respects the equivalence relation $\sim$. Let $l(\boldsymbol{X}) \geq 2$. Observe that

$$F(d_i(\boldsymbol{\gamma})) = \begin{cases} (F\gamma_1, \dots, F\gamma_{l-1}) & i = 0 \\ (F\gamma_0, \dots, (F\gamma_i) \circ (F\gamma_{i-1}), F\gamma_{i+1}, \dots, F\gamma_{l-1}) & 1 \leq i \leq l-1 \\ (F\gamma_0, \dots, F\gamma_{l-2}) & i = l \end{cases}$$
$$= d_i(F(\boldsymbol{\gamma})).$$

For $l = 1$ we also have that $F \circ d_i = d_i \circ F$, which is a triviality. The map s_i also satisfies the intertwining property $Fs_i = s_i F$. This makes that the map KF respects the equivalence relation. The factorization BF of KF to the quotient is continuous. □

Milnor [23] proves the following fact.

Proposition 2.28. *Let $(x, \boldsymbol{\mu}) \in K\mathfrak{C}$. There exists a unique non-degenerate $(y, \boldsymbol{\nu}) \in K\mathfrak{C}$ such that*

$$(x, \boldsymbol{\mu}) \sim (y, \boldsymbol{\nu}). \tag{2.39}$$

The following theorem is true, see Segal [28] for a proof.

Theorem 2.29. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be topological categories, where either $\mathfrak{C}$ or $\mathfrak{D}$ has a finite number of objects. There exists a homeomorphism*

$$B(\mathfrak{C} \times \mathfrak{D}) \cong (B\mathfrak{C}) \times (B\mathfrak{D}). \tag{2.40}$$

In Segal [28] the theorem is formulated in a more general setting. The above result is the one we need however. The following proposition is crucial in what is to come.

Theorem 2.30. *Let $F_0, F_1 : \mathfrak{C} \to \mathfrak{D}$ be continuous functors. Suppose that there exists a natural transformation $\mathcal{N} : F_0 \dot{\to} F_1$. Then the induced maps $BF_0 : B\mathfrak{C} \to B\mathfrak{D}$ and $BF_1 : B\mathfrak{C} \to B\mathfrak{D}$ are homotopic.*

Proof. We can view $\mathcal{N}$ as a functor $N : \mathfrak{C} \times \mathfrak{2} \to \mathfrak{D}$. The category $\mathfrak{2}$ is finite, hence there is a splitting $B(\mathfrak{C} \times \mathfrak{2}) \simeq B\mathfrak{C} \times B\mathfrak{2}$. We computed in example 2.24 that $B\mathfrak{2} \cong [0, 1]$. Furthermore, $BN(X, 0) = (BF_0(X), 0)$, and $BN(X, 1) = (BF_1(X), 0)$. BN is a homotopy, hence $\mathcal{N}$ induces the homotopy between the maps BF_0 and BF_1. □

2.6 The Subdivision of a Topological Category

Sometimes we would like to view objects and morphisms of a topological category on the same level. This is possible, because every object $X \in \mathrm{Ob}(\mathfrak{C})$ gives rise to a morphism id_X. The subdivision category is a nice way of viewing objects and morphisms on the same footing. Roughly, objects in the subdivision category are morphisms in the original category. Morphisms in the subdivision category are “inclusions of composition”. If $\mu = \beta \circ \nu \circ \alpha$ for morphisms μ, ν, β, α in $\mathfrak{C}$, then there is a morphism $\mu \to \nu$ in $\mathrm{sd}\,(\mathfrak{C})$.

Convention 2.31. In this section we explicitly state to which category a structural map belongs. For example we write $\mathrm{dom}^{\mathfrak{C}}$ for the domain map in $\mathfrak{C}$ and we write $\mathrm{dom}^{\mathrm{sd}(\mathfrak{C})}$ for the domain map in $\mathrm{sd}\,(\mathfrak{C})$. The proofs in this section move between all the different categories, so it is clearer if we specify all these labels. In later chapters we will not do this, we will either fully work in $\mathfrak{C}$ or in $\mathrm{sd}\,(\mathfrak{C})$ so the distinction is redundant and clumsy.

Definition 2.32. The *subdivision category* $\mathrm{sd}\,(\mathfrak{C})$ of a topological category $\mathfrak{C}$ is a topological category, where

- The space of objects $\mathrm{Ob}(\mathrm{sd}\,(\mathfrak{C}))$ of the subdivision category equals the space of morphisms of the original category $\mathrm{Hom}(\mathfrak{C})$,

$$\mathrm{Ob}(\mathrm{sd}\,(\mathfrak{C})) = \mathrm{Hom}(\mathfrak{C}). \tag{2.41}$$

- The space of morphisms is the subspace of $\mathrm{Hom}(\mathfrak{C}) \times \mathrm{Hom}(\mathfrak{C})$, where $(\alpha, \beta) : \mu \to \nu$ is a morphism if and only if the diagram

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & x' \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ y & \xleftarrow[\beta]{} & y' \end{array} \tag{2.42}$$

  commutes in $\mathfrak{C}$.

- Let $(\alpha, \beta) : \mu \to \nu$. The maps $\mathrm{dom}^{\mathrm{sd}(\mathfrak{C})}$ and $\mathrm{cod}^{\mathrm{sd}(\mathfrak{C})}$ are the obvious maps defined by

$$\begin{aligned} \mathrm{dom}^{\mathrm{sd}(\mathfrak{C})}(\alpha, \beta) &= \mu \\ \mathrm{cod}^{\mathrm{sd}(\mathfrak{C})}(\alpha, \beta) &= \nu. \end{aligned} \tag{2.43}$$

- The identity map $\mathrm{id}^{\mathrm{sd}(\mathfrak{C})}$ maps $\gamma \in \mathrm{Ob}(\mathrm{sd}\,(\mathfrak{C}))$ to $\left(\mathrm{id}^{\mathfrak{C}}_{\mathrm{dom}^{\mathfrak{C}}(\gamma)}, \mathrm{id}^{\mathfrak{C}}_{\mathrm{cod}^{\mathfrak{C}}(\gamma)}\right)$/

- Let (α, β) and (α', β') be composable morphisms. Then the composition of the morphisms is defined by

$$(\alpha, \beta) \circ^{\mathrm{sd}(\mathfrak{C})} (\alpha', \beta') = (\alpha' \circ^{\mathfrak{C}} \alpha, \beta \circ^{\mathfrak{C}} \beta'). \tag{2.44}$$

Remark 2.33. The formula for the identity map is derived by studying the diagram

$$\begin{array}{ccc} \mathrm{dom}(\gamma) & \xrightarrow{\mathrm{id}_{\mathrm{dom}(\gamma)}} & \mathrm{dom}(\gamma) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathrm{cod}(\gamma) & \xrightarrow{\mathrm{id}_{\mathrm{cod}(\gamma)}} & \mathrm{cod}(\gamma) \end{array}, \tag{2.45}$$

and the composition law is best understood while pondering the diagram

$$\begin{array}{ccccc} a & \xrightarrow{\alpha} & a' & \xrightarrow{\alpha'} & a'' \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma'} & & \downarrow{\scriptstyle\gamma''} \\ b & \xleftarrow[\beta]{} & b' & \xleftarrow[\beta']{} & b'' \end{array} \quad . \tag{2.46}$$

The composition law measures "inclusion of composition". If a morphism can be written as compositions of other morphisms $\gamma = \gamma_n \circ \ldots \circ \gamma_0$ in $\mathfrak{C}$ then there exists morphisms $\gamma \to \gamma_i$ in $\mathrm{sd}\,(\mathfrak{C})$.

Of course we need to prove that the structure we have defined is a topological category. We use the topological structure of the underlying category.

Proposition 2.34. *The subdivision category* $\mathrm{sd}\,(\mathfrak{C})$ *is a topological category, for any topological category* $\mathfrak{C}$.

Proof. We directly verify that the subdivision is a small category, for which $\mathrm{Ob}(\mathrm{sd}(\mathfrak{C}))$ and $\mathrm{Hom}(\mathrm{sd}(\mathfrak{C})$ are topological spaces; we only establish the continuity of the structural maps.

We start of with the continuity of the composition law. Let (α, β) and (α', β') be two composable morphisms. Their composition is defined by

$$(\alpha, \beta) \circ^{\mathrm{sd}(\mathfrak{C})} (\alpha', \beta') = (\alpha' \circ \alpha, \beta \circ \beta'). \tag{2.47}$$

This is the product map of the two continuous maps $(\alpha, \alpha') \mapsto \alpha' \circ \alpha$ and $(\beta, \beta') \mapsto \beta' \circ \beta$. These are both continuous, since $\mathfrak{C}$ is a topological category. The product map of two continuous functions is continuous.

The identity map $\mathrm{id}^{\mathrm{sd}(\mathfrak{C})}$ maps $\gamma \in \mathrm{Ob}(\mathrm{sd}\,(\mathfrak{C}))$ to the product

$$\mathrm{id}^{\mathrm{sd}(\mathfrak{C})}(\gamma) = (\mathrm{id}^{\mathfrak{C}}_{\mathrm{dom}^{\mathfrak{C}}(\gamma)}, \mathrm{id}^{\mathfrak{C}}_{\mathrm{cod}^{\mathfrak{C}}(\gamma)}). \tag{2.48}$$

This is the product map of two continuous maps, $\mathrm{id}^{\mathfrak{C}}_{\mathrm{dom}^{\mathfrak{C}}(\gamma)}$, and $\mathrm{id}^{\mathfrak{C}}_{\mathrm{cod}^{\mathfrak{C}}(\gamma)}$ hence continuous.

The codomain map $\mathrm{cod}^{\mathrm{sd}(\mathfrak{C})}$ maps a pair $(\alpha, \beta) : \mu \to \nu$ to ν. The preimage of ν under $\mathrm{cod}^{\mathrm{sd}(\mathfrak{C})}$ are all morphisms (α', β') which have $\mathrm{cod}^{\mathfrak{C}}(\alpha') = \mathrm{dom}^{\mathfrak{C}}(\nu)$ and $\mathrm{dom}^{\mathfrak{C}}(\beta') = \mathrm{cod}^{\mathfrak{C}}(\nu)$. This is illuminated by the diagram

$$\begin{array}{ccc} \mathrm{dom}^{\mathfrak{C}}(\mu) = \mathrm{dom}^{\mathfrak{C}}(\alpha') & \xrightarrow{\alpha'} & \mathrm{dom}^{\mathfrak{C}}(\nu) = \mathrm{cod}^{\mathfrak{C}}(\alpha') \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} \\ \mathrm{cod}^{\mathfrak{C}}(\mu) = \mathrm{cod}^{\mathfrak{C}}(\beta') & \xleftarrow[\beta']{} & \mathrm{cod}(\mathfrak{C}(\nu))\,\mathrm{dom}^{\mathfrak{C}}(\beta')/ \end{array} \tag{2.49}$$

Thus

$$\left(\mathrm{cod}^{\mathrm{sd}(\mathfrak{C})}\right)^{-1}(\nu) = \bigcup_{\rho,\sigma\in\mathrm{Hom}(\mathrm{sd}(\mathfrak{C}))} \left(\mathrm{Hom}^{\mathfrak{C}}\left(\rho, \mathrm{cod}^{\mathfrak{C}}(\nu)\right), \mathrm{Hom}^{\mathfrak{C}}\left(\mathrm{dom}^{\mathfrak{C}}(\nu), \sigma\right)\right)$$

Where we understand that we skip a term if one of the Hom spaces is empty. The preimage of ν is a union of open sets, therefore open, hence the preimage of an open set $U \subset \mathrm{Ob}(\mathrm{sd}\,(\mathfrak{C}))$ is open, because it is a union of open sets.

Finally the domain map $\mathrm{dom}^{\mathrm{sd}(\mathfrak{C})}$. One verifies

$$\mathrm{dom}^{\mathrm{sd}(\mathfrak{C})}(\alpha,\beta) = \beta \circ^{\mathfrak{C}} \mathrm{cod}^{\mathrm{sd}(\mathfrak{C})}(\alpha,\beta) \circ^{\mathfrak{C}} \alpha. \tag{2.50}$$

This is a composition of the continuous projections $\pi_1(\alpha,\beta) = \alpha$, $\pi_2(\alpha,\beta) = \beta$ and the continuous maps $\mathrm{cod}^{\mathrm{sd}(\mathfrak{C})}$ and $\circ^{\mathfrak{C}}$. We conclude that $\mathrm{dom}^{\mathrm{sd}(\mathfrak{C})}$ is therefore continuous. All four structural maps are continuous, hence $\mathrm{sd}\,(\mathfrak{C})$ is a topological category. ☐

The main interest in subdivisions is that the homotopy type of the classifying space does not change. The following is proved in [10, Theorem 32] or in [20].

Proposition 2.35. *The classifying space of a topological category and its subdivision have the same homotopy type*

$$B\,\mathrm{sd}\,\mathfrak{C} \simeq B\mathfrak{C}. \tag{2.51}$$

2.7 Examples

In this section we discuss several examples of classifying spaces of simple categories. In the discussion of the examples, the identification maps are explicitly described, to ease the understanding of these maps.

In the categories we will consider, N_n will consist of only degenerate elements if n is big enough, there are no infinite non-degenerate chains. We will only draw the N_n which are not completely degenerate. If we draw N_n we usually omit all degenerate elements as well.

Example 2.36. In this example we compute the classifying space of $\mathfrak{3}$. This is the category of three elements, denoted $0, 1$, and 2. There is a unique morphism between the object x and x' if and only if $x \leq x'$. We denote these by $\gamma_{01} : 0 \to 1, \gamma_{02} : 0 \to 2$, and $\gamma_{12} : 1 \to 2$. Of course we also have three identity morphisms, which we write $\mathrm{id}_0 : 0 \to 0$ etc. This category is topologized by endowing the spaces $\mathrm{Ob}(\mathfrak{C})$ and $\mathrm{Hom}(\mathfrak{C})$ with the discrete topology. Now N_n for $n > 2$ consist only of degenerate chains. We depicted $\Delta^0 \times N_0\mathfrak{3}$, $\Delta^0 \times N_1\mathfrak{3}$, and $\Delta^0 \times N_2\mathfrak{3}$ in figure 2.5. We have labeled the faces of the chains, what d_i would induce. We encourage the reader to reproduce the pictures; this will aid understanding of the theory. We now are able to read off that the classifying space of $\mathfrak{3}$ is the $3-$simplex. This example generalizes to all the categories $\mathfrak{n}$, the categories with n elements, with unique morphisms if $x \leq x'$.

The second example is an example of a useful category, that is the category associated to a specific poset.

Example 2.37. A partially ordered set (or poset) is a set with a binary operation $\leq$ which is transitive, reflexive, and antisymmetric [13, section 14]. There exists a notion of a classifying space of a poset. This is the geometric realization of the order complex associated to the poset. On the other hand, we can associate a topological category to a poset as follows. Let $\mathrm{Ob}\,\mathfrak{C}$ be the underlying set of the poset, and let there be a unique morphism if $a \leq b$. Endow both $\mathrm{Ob}\,\mathfrak{C}$ and $\mathrm{Hom}\,\mathfrak{C}$ with the discrete topology. The classifying space obtained in this manner is homeomorphic to the classifying space obtained from the geometric realization of the order complex. The previous example is actually a poset.

Example 2.38. We study another example which arises in poset theory. Let B_n be the set of subsets of $[n] := \{1, 2, \ldots, n\}$. This is a poset under inclusion, i.e. for $x, y \in B_n$ we have $x \leq y$ if and only if $x \subset y$. The category $\mathfrak{B}_n$ is defined to have as objects

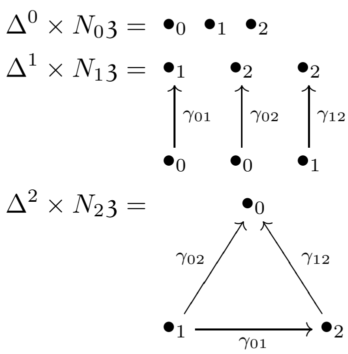

Figure 2.5: Some non-degenerate elements of the spaces $\Delta^n \times N_n 3$. The classifying space, the space we get after identifying the points with the same label, is homeomorphic to the $2-$simplex.

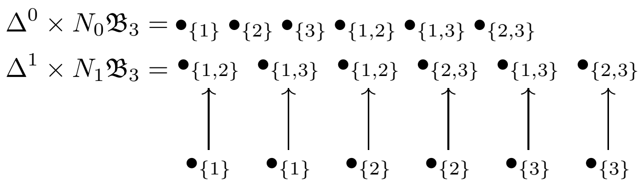

Figure 2.6: The limbs of $\mathfrak{B}_3$. Higher limbs are all degenerate.

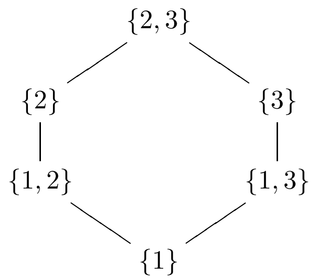

Figure 2.7: The classifying space $B\mathfrak{B}_3$ is homeomorphic to the circle S^1.

elements of B_n, excluding $\emptyset$ and $[n]$. There is a unique morphism $x \to y$ if $x \subset y$. Both $\mathrm{Ob}(\mathfrak{B}_n)$ and $\mathrm{Hom}(\mathfrak{B}_n)$ are endowed with the discrete topology.

Now we turn to the special case $n = 3$. There are six objects, and six morphisms. We depict the limbs in 2.6. We have have shown the resulting classifying space is a circle, as seen in figure 2.7. This is not a coincidence. The category we constructed is the barycentric subdivision of a $2-$simplex. We expect, if we remove the center point, to retrieve the boundary of the two simplex, which is homeomorphic to the circle. This works for all dimensions, i.e. $B\mathfrak{B}_n \cong S^{n-2}$ for $n > 1$.

In the third example we investigate the differences if the topologies of Ob and Hom vary.

Example 2.39. We define two categories, $\mathfrak{C}$ and $\mathfrak{D}$. Both categories have the same objects and morphisms

$$\mathrm{Ob}(\mathfrak{C}) = \mathrm{Ob}(\mathfrak{D}) = [0,1] \coprod [2,3] \quad \mathrm{Hom}(\mathfrak{C}) = \mathrm{Hom}(\mathfrak{D}) = [0,1], \tag{2.52}$$

viewed as sets. A morphism $x \in [0,1]$ maps x to $x + 2$. We define different topologies on the objects and morphisms of $\mathfrak{C}$ and $\mathfrak{D}$. For $\mathfrak{C}$ endow the collections of objects and morphisms with the discrete topology. Let $\mathrm{Ob}(\mathfrak{D})$ and $\mathrm{Hom}(\mathfrak{D})$ be endowed with the subspace topology of $\mathbb{R}$. The classifying space of $\mathfrak{C}$ consists of an infinite number of disjoint lines and the classifying space of $\mathfrak{D}$ is homeomorphic to the square $[0,1]^2 \subset \mathbb{R}^2$.

Our last example will actually be used in the proof of the homotopy theorems later in the thesis. We therefore formulate it as a proposition.

Proposition 2.40. *Let S be a topological space, and let $\mathfrak{S}$ be the topological category with*

$$\mathrm{Ob}(\mathfrak{S}) = S \quad \textit{and} \quad \mathrm{Hom}(\mathfrak{S}) = S \tag{2.53}$$

Where we interpret $x \in \mathrm{Hom}(\mathfrak{S})$ as id_x. Then the classifying space and the original space are homeomorphic,

$$B\mathfrak{S} \cong S. \tag{2.54}$$

Proof. All non-degenerate chains come from admissible sequences with length 0 thus

$$B\mathfrak{S} \cong \Delta^0 \times N_0\mathfrak{S}. \tag{2.55}$$

And we have

$$\Delta^0 \times N_0\mathfrak{S} \cong N_0\mathfrak{S} = S. \tag{2.56}$$

Thus the classifying space is homeomorphic to the space S. ☐

Chapter 3

The Flow Category of a Weak Morse Function

3.1 Introduction

Morse functions have nice properties. Around a critical point a Morse function is described by a quadratic polynomial. Sublevel sets $M_c = f^{-1}((-\infty, c])$ describe changes in homotopy of the underlying manifold. The gradient flow is heteroclinic, i.e. the alpha and omega limit sets of points are singletons. These properties all play a fundamental role in classical Morse theory. A wider class of functions, which contains the Morse functions, also have the latter property. This property is crucial for the proof of the homotopy part of the theorem. We call the class of functions with isolated critical points, which are allowed to be degenerate, weak Morse functions. We define the notion precisely in this chapter and we define the flow category a weak Morse function generates. The goal of this chapter is to prove the following theorem.

Theorem 3.1. *Let M be a Riemannian manifold, $f : M \to \mathbb{R}$ a weak Morse function. There exists a homotopy*

$$B\mathfrak{C}_f \simeq M, \tag{3.1}$$

where $\mathfrak{C}_f$ is the flow category of f.

3.2 Weak Morse Functions

Definition 3.2. Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is called *a weak Morse function* if it is smooth and all critical points of f are isolated. That is, for each critical point p, there exists a neighborhood $U_p \ni p$ such that p is the only critical point in this neighborhood.

A Morse function is always a weak Morse function. The notion of a weak Morse function is weaker than the notion of a Morse function. This sanctions the use of the name weak Morse.

Proposition 3.3. *A Morse function is a weak Morse function.*

Proof. The Morse lemma, c.f. [2, Lemma 3.11], shows that a critical point p with critical value c of a Morse function f, has a chart (U, φ) such that

$$f(\varphi^{-1}(x_1, \ldots, x_n)) = c - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2, \tag{3.2}$$

in this chart. It directly follows that the critical points of a Morse function are isolated, U isolates the critical point p from other critical points. □

The following property of weak Morse functions is critical, and its proof is standard.

Proposition 3.4. *If M is compact, a weak Morse function has a finite number of critical points. Conversely, a smooth function with a finite number of critical points is weak Morse.*

3.3 Heteroclinicity

Recall that if f is a function, then a *gradient flow line* or *orbit* through x is the maximal curve γ satisfying the differential equation

$$\dot{\gamma}(t) = -\nabla f(\gamma(t)) \qquad \gamma(s) = x \qquad \text{for some } s \in \mathbb{R}. \tag{3.3}$$

On compact manifolds gradient flows of Morse functions and weak Morse functions are heteroclinic, i.e. the limits $\lim_{\pm\infty} \gamma(t)$ exist. The following proposition is proved for Morse functions in Banyaga and Hurtubise [2]. The proof for weak Morse functions is completely analogous. We state the proof here for convenience.

Proposition 3.5. *Let $f : M \to \mathbb{R}$ be a weak Morse function on a closed Riemannian manifold. Then all gradient flow lines $\gamma : \mathbb{R} \to M$ are defined for all time, and begin and end at a critical points of f. That is*

$$\lim_{t\to\infty} \gamma(t) \qquad \textit{and} \qquad \lim_{t\to-\infty} \gamma(t) \tag{3.4}$$

exist, and both are critical points of f.

Proof. Let γ be a gradient flow line through $x \in M$. Since M is compact, the flow line γ is defined for all time. The compactness of M forces $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ to be bounded, because the image of a compact set is compact, and compact sets in $\mathbb{R}$ are bounded and closed. Thus for any subset $U \subset M$, the set $f(U)$ is bounded. We compute

$$\begin{aligned} \frac{df(\gamma(t))}{dt} &= df \frac{d\gamma(t)}{dt} \\ &= df(-\nabla f(\gamma(t))) \\ &= -\left|\left|\nabla f(\gamma(t))\right|\right|^2 \leq 0, \end{aligned} \tag{3.5}$$

hence f decreases along orbits. The boundedness of $f \circ \gamma$ and the preceding imply that

$$\lim_{t \to \pm\infty} \frac{df(\gamma(t))}{dt} = 0. \tag{3.6}$$

Now let $t_n^{\pm} \in \mathbb{R}$ be sequences with $\lim_{n\to\infty} t_n^{\pm} = \pm\infty$. Clearly $\gamma(t_n^{\pm}) \subset M$ are infinite sets of points on a compact manifold, hence they have accumulation points $q^{\pm}$. These accumulation points are critical points of f since $\lim_{n\to\infty} ||\nabla f(\gamma(t_n^{\pm}))|| = 0$. The critical points $q^{\pm}$ are isolated by assumption hence there exists neighborhoods $U^{\pm} \ni q^{\pm}$ where these are the only critical points in their neighborhoods. Now suppose that $\lim_{t\to\pm\infty} \gamma(t) \neq q^{\pm}$. Then there exists sequences $\tilde{t}_n^{\pm}$ with $\lim_{n\to\infty} \tilde{t}_n^{\pm} = \pm\infty$ and $\gamma(\tilde{t}_n^{\pm}) \in U^{\pm} \setminus V^{\pm}$ with V an open neighborhood of $q^{\pm}$. Therefore the sequence $\gamma(\tilde{t}_n^{\pm})$ must have accumulation points $\tilde{q}^{\pm} \in U^{\pm} \setminus V^{\pm}$. By the previous discussion these must be critical points. We assumed the critical points $q^{\pm}$ are the only critical points in $U^{\pm}$. Hence we conclude that $\lim_{t\to\pm\infty} \gamma(t) = q^{\pm}$. □

3.3.1 A Counterexample

One might conjecture that all gradient flows are heteroclinic. This is not true and we provide a counterexample in this section. We construct a C^2 function for which orbits do not have well defined limits as $t \to \infty$.

Example 3.6. We work in polar coordinates on $\mathbb{R}^2$. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(r,\theta) = \begin{cases} (r-1)^2(\sin(\frac{1}{r-1} + \theta) + 2) & \text{for } r > 1 \\ 0 & \text{for } r \leq 1. \end{cases} \tag{3.7}$$

Elementary analysis shows that f is C^2 everywhere, and C^∞ outside of a neighborhood of $r = 1$. The function is drawn in figure 3.1. The gradient flows spiral towards the an attractor, the circle $r = 1$. The flows themselves do not have well defined limits. One can choose sequences $t_n^x \in \mathbb{R}$ with $\lim_{n\to} t_n^x = \infty$, such that $\lim_{n\to\infty} \gamma_y(t_n) = x$ for any $x \in S^1$, and all y with $||y|| > 1$. All points on the circle are approached as $t \to \infty$.

3.4 The Flow

We will encode the information stored in the gradient flow of a function in a category. The objects of the category will be critical points, and the morphisms in the category will be unions of reparameterized orbits extending between equilibria. We also need to be able to compose orbits. We want to use concatenation as composition of orbits. The orbits are defined for all time, and this is an obstruction for a simple concatenation. Therefore we reparameterize the orbits, in such that it only takes a finite amount to trace the image of the orbit. We exploit the heteroclinicity we have proved before, and we will use some ODE theory. A slicker argument, one which generalizes to metric spaces, is used in chapter 5.

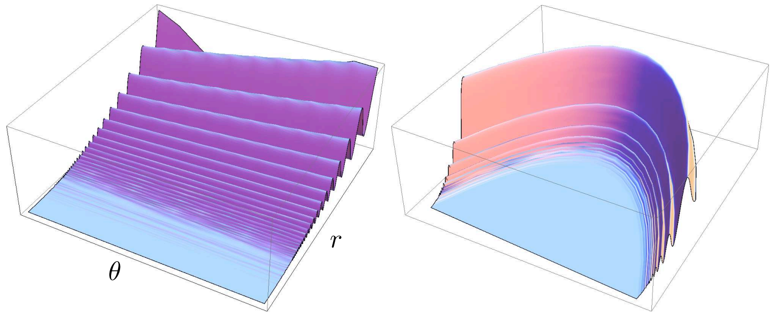

Figure 3.1: The function f sketched in polar coordinates on the left, on the right hand side we plotted a part of the function as observed in $\mathbb{R}^2$. Notice how the gradient flows spiral to the attracting circle $r = 1$. There is an infinite number of wobbles just outside of the attracting circle $r = 1$. Orbits of the gradient flow do not have well defined limits as $t \to \infty$. All points on the circle are approached by all orbits.

3.4.1 The Reparameterization of the Flow

In the following we understand M to be a closed, i.e. compact without boundary, Riemannian manifold and $f : M \to \mathbb{R}$ a weak Morse function. Let $\gamma_{ab} : \mathbb{R} \to M$ be a gradient flow line from the critical point a to the critical point b. The curve γ_{ab} satisfies the gradient flow equation

$$\frac{d\gamma_{ab}(t)}{dt} = -\nabla f(\gamma_{ab}(t)). \tag{3.8}$$

We want to find a reparameterization of the flow γ_{ab} which must be a path ω_{ab} : $[f(b), f(a)] \to M$ with the properties

- ω_{ab} is smooth,
- $\operatorname{im}\omega_{ab} = \overline{\operatorname{im}\gamma_{ab}}$, with the bar denoting topological closure,
- The value of f coincides with the parametrization along the orbit

$$f(\omega_{ab}(t)) = t. \tag{3.9}$$

In contrast to γ_{ab}, which starts at a and ends at b, the reparameterized flow line ω_{ab} starts at the critical point b and ends at the critical point a. Furthermore a differentiation of equation (3.9) shows

$$\begin{aligned} \frac{d}{dt}t = 1 = \frac{d}{dt}f(\omega_{ab}(t)) \\ = df(\omega_{ab}(t))\frac{d\omega_{ab}(t)}{dt} \\ = \langle \nabla f(\omega_{ab}(t)), \frac{d\omega_{ab}(t)}{dt}\rangle_{\omega_{ab}(t)}, \end{aligned} \tag{3.10}$$

that ω_{ab} ought to obey the differential equation

$$\frac{d\omega_{ab}(t)}{dt} = \frac{\nabla f(\omega_{ab}(t))}{||\nabla f(\omega_{ab}(t))||^2}. \tag{3.11}$$

We should note here that

$$\frac{d\omega_{ab}(t)}{dt} = \frac{\nabla f(\omega_{ab}(t))}{||\nabla f(\omega_{ab}(t))||^2} + V(\omega_{ab}(t)), \tag{3.12}$$

also solves (3.10) for any vector field V orthogonal to ∇f. However if we have a component which is orthogonal to f we will never satisfy the property $\operatorname{im}\omega_{ab} = \overline{\operatorname{im}\gamma_{ab}}$. Therefore we conclude that $V = 0$ for the paths that we are interested in. We want to show these paths exist, and that every non-critical point lies on a unique reparameterized path.

Proposition 3.7. *For each orbit* $\gamma_{ab} : \mathbb{R} \to M$ *satisfying*

$$\frac{d\gamma_{ab}(t)}{dt} = -\nabla f(\gamma_{ab}(t)), \tag{3.13}$$

and

$$\lim_{t\to-\infty} \gamma_{ab}(t) = a \qquad \lim_{t\to\infty} \gamma_{ab}(t) = b, \tag{3.14}$$

there exists a unique smooth reparameterized flow line $\omega_{ab}(t) : [f(b), f(a)] \to M$ *satisfying*

- $\operatorname{im}\omega_{ab} = \overline{\operatorname{im}\gamma_{ab}}$,
- *The value of f coincides with the parametrization, i.e.* $f(\omega_{ab}(t)) = t$.

Proof. The flow line satisfies the following property. If $\nabla f(\gamma_{ab}(t)) = 0$ for some t, then $\nabla f(\gamma_{ab}(t)) = 0$ for all $t \in \mathbb{R}$. In this case, we have trivially proven the theorem by the observation that a reparameterization is given by $\omega_{ab}(f(x)) := x$, the curve consisting of a single point. Therefore we assume without loss of generality that ∇f is nonzero on the image of γ_{ab}. Let

$$s := \frac{1}{2}(f(a) + f(b)). \tag{3.15}$$

The function f is strictly decreasing on $\operatorname{im}\gamma_{ab}$, with endpoint values of f being $f(a)$ and $f(b)$, therefore there exists a unique $x \in \operatorname{im}\gamma_{ab}$ with $f(x) = s$. Hence there is a unique $\tilde{s} \in \mathbb{R}$ with $\gamma_{ab}(\tilde{s}) = x$.

We want to find a smooth diffeomorphism $\lambda : (f(b), f(a)) \to \mathbb{R}$ such that the curve $\omega_{ab} : (f(b), f(a)) \to M$ defined by

$$\omega_{ab}(t) := \gamma_{ab}\left(\lambda(t)\right), \tag{3.16}$$

can be extended (to the full domain $[f(b), f(a)]$), such that ω_{ab} satisfies the properties of the proposition. If we would be able to find such a diffeomorphism, it ought to satisfy the differential equation

$$\frac{d\lambda(t)}{dt} = \frac{-1}{||\nabla f(\gamma_{ab}(\lambda(t)))||^2}, \tag{3.17}$$

as the chain rule shows, after a differentiation of (3.16) to t

$$\begin{aligned}\frac{d\omega(t)}{dt} =& \frac{d}{dt}\gamma_{ab}(\lambda(t)) \\ =& \dot{\gamma}_{ab}(\lambda(t))\dot{\lambda}(t) \\ =& -\nabla f(\gamma_{ab}(\lambda(t)))\dot{\lambda}(t),\end{aligned} \tag{3.18}$$

if we want ω_{ab} to satisfy (3.11). The right hand side of (3.17) is smooth, hence by the theory of ordinary differential equations there is a unique maximal solution $\lambda : I \to \mathbb{R}$ to this equation, with initial condition $\lambda(s) = \tilde{s}$. Of course we have that $\nabla(f) \neq 0$ on $\operatorname{im}\gamma_{ab}$.

We first prove that the image of λ is in fact the whole of $\mathbb{R}$, then we will conclude that the maximal domain is $(f(b), f(a))$. From this it will follow that ω_{ab} defined above satisfies the properties that we want. By compactness of M, and the smoothness of f we have

$$||\nabla f||^2 < R \tag{3.19}$$

for some $R > 0$. This gives the following bound on $\frac{d\lambda}{dt}$

$$\frac{d\lambda(t)}{dt} < -\frac{1}{R}. \tag{3.20}$$

We have two cases. The domain of λ is bounded, or it is not. We study these cases separately. First, if the domain of λ is unbounded from above or below the image cannot be bounded from below or above respectively. This directly follows from equation (3.20).

Now we argue that even if the domain is bounded from above or below the image still cannot be bounded from below or above respectively. These results combined show that $\operatorname{im}\lambda = \mathbb{R}$. Suppose that the domain is bounded from above by q, and the image is bounded from below, and call the biggest lower bound r. Then we compute the limit

$$\lim_{t\to q}\lambda(t) = r. \tag{3.21}$$

Now we solve the differential equation (3.17) with boundary initial condition $\tilde{\lambda}(q) = r$. By the local existence and uniqueness theorem, this solution exists, and agrees with λ on $(q-\delta, q)$ for some $\delta > 0$. Hence we conclude that λ can be extended to $q + \delta$ for some $\delta > 0$. This extended λ has a lower lower bound, since $\frac{d\lambda}{dt}$ is strictly decreasing. This shows that if the domain is bounded from above, the image cannot be bounded from below. A similar reasoning shows that the same if the domain is bounded from below, than the image cannot be bounded from above.

This gives that the image of λ is the whole of $\mathbb{R}$. Now we show that ω_{ab} has the reparameterization property that we wanted.

$$\begin{aligned} f(\omega_{ab}(t)) - f(\omega_{ab}(s)) &= \int_s^t \frac{d}{d\tau} f(\omega_{ab}(\tau)) d\tau \\ &= \int_s^t d_{\omega_{ab}(\tau)} f \frac{d\omega_{ab}(\tau)}{d\tau} d\tau \\ &= \int_s^t \langle \nabla f(\omega_{ab}(\tau)), \frac{d\omega_{ab}(\tau)}{d\tau} \rangle d\tau. \end{aligned} \tag{3.22}$$

Using equations (3.17) and (3.18) we compute

$$\begin{aligned} f(\omega_{ab}(t)) - f(\omega_{ab}(s)) &= \int_s^t \langle \nabla f(\omega_{ab}(\tau)), \frac{d}{d\tau} \gamma_{ab}(\lambda(\tau)) \rangle d\tau \\ &= \int_s^t \langle \nabla f(\omega_{ab}(\tau)), \frac{\nabla f(\omega_{ab}(\tau))}{||\nabla f(\omega_{ab}(\tau))||^2} \rangle d\tau \\ &= \int_s^t d\tau = t - s. \end{aligned} \tag{3.23}$$

We have $\omega_{ab}(s) = \gamma_{ab}(\tilde{s}) = x$ and $f(x) = s$ by construction, so we can conclude that $f(\omega(t)) = t$ on the whole domain on which λ is defined. This directly shows that the domain of λ cannot be larger than $(f(b), f(a))$ since the differential equation (3.16) would be ill defined. If the domain of λ is smaller than $(f(b), f(a))$ then we can extend λ contradicting the assumption that the domain was maximal.

The only thing that remains is extending ω_{ab} to $f(b)$ and $f(a)$. This is done by imposing $\omega_{ab}(f(b)) = b$ and $\omega_{ba}(f(a)) = a$. The extended ω_{ab} is still a continuous function. □

We will consider spaces of reparameterized curves. A natural topology on function spaces is the compact open topology.

Definition 3.8. Let X and Y be topological spaces. The *compact-open topology* on the space of continuous functions from X to Y, $\mathcal{C}(X, Y)$, is the topology generated by the subbasis

$$D(C, U) = \{f \in \mathcal{C}(X, Y) \mid f(C) \subset U\}, \tag{3.24}$$

where $C \subset X$ is compact, and $U \subset Y$ is open.

More information on the compact-open topology can be found in Munkres [25]. We can now define the spaces of piecewise flow lines.

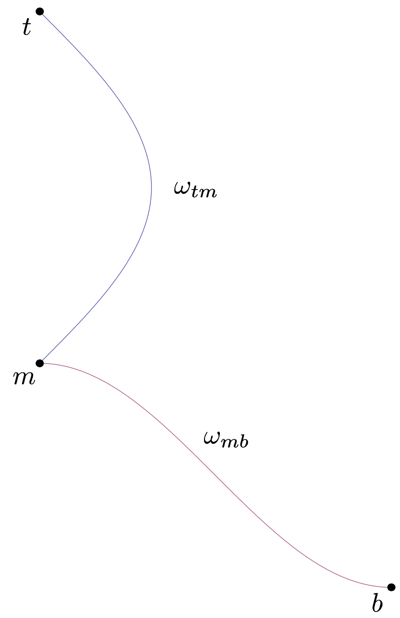

Figure 3.2: t, m, and b are three critical points of a weak Morse function, and γ_{tm}, and γ_{mb} are reparameterized orbits through them. The curves γ_{tm} and γ_{mb} are smooth, however their composition $\gamma_{mb} \circ \gamma_{tm}$ is merely continuous. For almost all flows this phenomenon occurs.

Remark 3.9. In the spaces we consider in this thesis, the compact topology is equivalent to the topology of uniform convergence. An aim is to generalize the theorems we prove in this thesis, to more general systems, where the topology of uniform convergence is ill-defined. Therefore we will work with the slightly more general compact-open topology.

Definition 3.10. Let a and b be different critical points of f. The *space of piecewise flow lines* $\mathrm{Hom}\,(a, b)$ is defined to be the space of all curves $\omega_{ab} : [f(b), f(a)] \to M$ satisfying (3.11) on all points of $t \in (f(b), f(a))$ where $\omega_{ab}(t)$ is not a critical point for f and having the limits $\omega_{ab}(f(b)) = b$, and $\omega_{ab}(f(a)) = a$. We topologize $\mathrm{Hom}\,(a, b)$ as a subspace of the space of continuous maps $\mathcal{C}([f(b), f(a)], M)$ endowed with the compact open topology. The space of piecewise flow lines $\mathrm{Hom}(a, a)$ has only element; the curve defined by $\omega_{aa}(f(a)) = a$.

These spaces are in a sense, which we make more explicit in the propositions 3.13 and 3.15, the closure of the space of all flow lines from a to b. Notice that the flows in $\mathrm{Hom}(a, b)$ need not be smooth, but are piecewise flow lines, hence they are piecewise smooth and continuous everywhere, see figure 3.2.

The next proposition is crucial, however we have not found a proof of this result.

Proposition 3.11. *Let $x \in M$. Then the assignment operator*

$$\text{asgn} : M \to \coprod_{a,b \in \text{Crit}(f)} \text{Hom}(a,b), \tag{3.25}$$

which assigns to x the reparameterized curve ω_{ab} through x is continuous.

For the compactness of $\text{Hom}(a,b)$ we make use of a version of Arzelà-Ascoli's theorem [25, Theorem 47.1].

Theorem 3.12 (Arzelà-Ascoli)**.** *Let X be a topological space, and (Y,d) a metric space. Let $\mathcal{C}(X,Y)$ be the space of continuous functions $X \to Y$ endowed with the topology of uniform convergence on compact sets. Let $\mathcal{F} \subset \mathcal{C}(X,Y)$. If $\mathcal{F}$ is equicontinuous, and the set*

$$\mathcal{F}_a := \{g(a) \mid g \in \mathcal{F}\} \tag{3.26}$$

has compact closure for all $a \in X$, then $\mathcal{F}$ has compact closure in $\mathcal{C}(X,Y)$.

We use this to prove that the Hom spaces are compact.

Proposition 3.13. $\text{Hom}\,(a,b)$ *is compact for all critical points a and b.*

Proof. M is a metric space, it is a Riemannian manifold and its topology corresponds to the topology generated by the metric

$$d(x,y) := \inf L(\gamma). \tag{3.27}$$

Where the infimum runs over smooth γ, which connect x and y, and $L(\gamma)$ is the length of the curve measured by the Riemannian metric cf. [17, Proposition 1.2]. It is a fact that the compact-open topology is equivalent to the topology of uniform convergence on compact sets [25, Theorem 46.8] if the space is a compact metric space. The set $\text{Hom}(a,b)$ is a subset of all continuous maps $\mathcal{C}([f(b),f(a)],M)$ in the compact open topology. We show that the space is equicontinuous. Pick $t \in [f(b),f(a)]$, and $\epsilon > 0$. Form the open tubular neighborhood T of $f^{-1}(t)$ by

$$T = \bigcup_{x \in f^{-1}(t)} B_\epsilon(x). \tag{3.28}$$

Here $B_\epsilon(x)$ is the ϵ ball around x in M. The continuity of f shows the existence of a $\delta > 0$ such that $f^{-1}(B_\delta(t)) \subset T$, where the δ ball is in $[f(b),f(a)]$. Now for any $\omega_{ab} \in \text{Hom}(a,b)$ we have

$$\omega_{ab}(B_\delta(t)) \subset f^{-1}(B_\delta(t)), \tag{3.29}$$

because ω_{ab} is a reparameterized orbit, cf. proposition 3.7. Combining this with the fact that the right hand side is a subset of T, we have proven equicontinuity. Because M is a compact space we have that the sets $\text{Hom}(a,b)_t = \{\omega_{ab}(t) \mid \omega_{ab} \in \text{Hom}(a,b)\}$ have compact closure in M, any closed subset of a compact Hausdorff space is compact. We conclude that $\text{Hom}(a,b)$ has compact closure in $\mathcal{C}([f(b),f(a)],M)$.[1] □

[1] We need to proof that $\text{Hom}(a,b)$ is closed, then we have proven the result.

The spaces $\mathrm{Hom}(a, b)$ with the compact-open topology are somewhat abstract. For examples it is convenient if we can get some geometric understanding of the spaces $\mathrm{Hom}(a, b)$. A space that we can sketch in examples is the following

Definition 3.14. Let $t \in (f(b), f(a))$. Let $M_t(a, b)$ be the space

$$M_t(a, b) := W^s(b) \cap W^u(a) \cap f^{-1}(t), \tag{3.30}$$

where $W^u(a)$ and $W^s(b)$ are the stable and unstable manifolds of f,

$$\begin{aligned} W^s(b) &:= \{x \in M \mid \lim_{t\to\infty} \varphi_t(x) = b\} \\ W^u(a) &:= \{x \in M \mid \lim_{t\to-\infty} \varphi_t(x) = a\}. \end{aligned} \tag{3.31}$$

It is easy to sketch these spaces. Just look at all the flow lines beginning at a and ending at b. Take the image of all these flow lines, and draw a line orthogonal to all the curves. This is a space which is homeomorphic to $M_t(a, b)$.

Proposition 3.15. *Let $t \in]f(b), f(a)[$. The assignment operator restricted to $M_t(a, b)$ is an embedding into* $\mathrm{Hom}\,(a, b)$.

Proof. We will show that the assignment operator asgn is an embedding, restricted to $M_t(a, b)$. The assignment operator is continuous, see proposition 3.11, and is clearly injective restricted to $M_t(a, b)$, each point has a unique reparameterized flow line through it. To show that this map is an embedding in the sense of topology we have to show that it is a homeomorphism onto its image. The inverse operation $\mathrm{ev}_t : \mathrm{Hom}(a, b)|_{\mathrm{asgn}(M_t(a,b))} \to M_t(a, b)$, which sends a curve γ to $\gamma(t)$, is restriction of the evaluation map $\mathrm{ev} : [f(b, f(a)] \times \mathrm{Hom}(a, b) \to M$. The compact-open topology forces the evaluation map to be continuous [25]. Therefore the restriction is continuous as well. Hence asgn is an embedding. □

Proposition 3.16. *There exists an associative composition law*

$$\circ : \mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c), \tag{3.32}$$

which is continuous.

Proof. The composition law is just the obvious concatenation of the curves. Given $\omega_{ab} \in \mathrm{Hom}(a, b)$ and $\omega_{bc} \in \mathrm{Hom}(b, c)$ we can form $\omega_{ac} \in \mathrm{Hom}(a, c)$ by

$$\omega_{ac}(t) := \begin{cases} \omega_{bc}(t) & \text{for } f(c) \le t \le f(b) \\ \omega_{ab}(t) & \text{for } f(b) < t \le f(a)\,. \end{cases} \tag{3.33}$$

Clearly this function satisfies the defining differential equation on all non-critical points, since both ω_{ab} and ω_{bc} do. The limits also work out fine, because $\omega_{ac}(f(a)) = \omega_{ab}(f(a)) = a$ and this holds analogously for the endpoint $f(c)$. By construction this composition is associative, which clearly follows.

We want to check that $\circ : \mathrm{Hom}(a,b) \times \mathrm{Hom}(b,c) \to \mathrm{Hom}(a,b)$ is continuous in the compact open topology. The compact-open topology on $\mathrm{Hom}(a,b)$ is the topology generated by the subbasis

$$D(C,U) = \{\omega_{ab} \in \mathrm{Hom}(a,b) \mid \omega_{ab}(C) \subset U\}, \tag{3.34}$$

where $C \subset [f(b), f(a)]$ is compact and $U \subset M$ is open. For the continuity of the composition map we need to check that the preimage of a subbasis element is open. Let $C \subset [f(c), f(a)]$ be compact, and U open in M. Clearly we can decompose $C = C_1 \cup C_2$, with $C_1 \subset [f(b), f(a)]$ and $C_2 \subset [f(c), f(b)]$. One readily verifies

$$\circ^{-1} D(C,U) = D(C_1,U) \times D(C_2,U), \tag{3.35}$$

with $D(C_1,U)$ a subbasis element of $\mathrm{Hom}(a,b)$ and $D(C_2,U)$ a subbasis element of $\mathrm{Hom}(b,c)$. Hence the preimage of any subbasis element under $\circ$ is open. Therefore $\circ$ is a continuous map. □

Convention 3.17. From now on we will not study the original orbits, the notation ω_{ab} will get clumsy. Therefore we will use the notation γ to denote a general reparameterized orbit.

3.5 The Flow Category

We are now in the position to define the flow category. The topology we endow on the set of objects is discrete, cf. 3.19. This is a peculiarity because we look at the gradient flow of a weak morse function. In chapter 5 we will construct a flow category where the topology on the objects of the category is non-discrete.

Definition 3.18. The *flow category*, associated to the weak Morse function f, is the topological category $\mathfrak{C}_f$, which is defined by the following rules.

- The space of objects $\mathrm{Ob}(\mathfrak{C}_f) = \mathrm{Crit}(f)$ is the space of critical points, i.e. $x \in \mathrm{Ob}(\mathfrak{C}_f)$ implies $d_x f = 0$. It is topologized as a subspace of M.
- The space of morphisms

$$\mathrm{Hom}(\mathfrak{C}_f) = \coprod_{x,y \in \mathrm{Ob}(\mathfrak{C}_f)} \mathrm{Hom}(x,y), \tag{3.36}$$

  is the union of all spaces of piecewise reparameterized orbits.
- dom sends a morphism $\gamma \in \mathrm{Hom}(x,y)$ to the domain x.
- cod sends a morphism $\gamma \in \mathrm{Hom}(x,y)$ to the codomain y.
- id sends an object x to the constant orbit $\mathrm{id}_x \in \mathrm{Hom}(x,x)$, which satisfies

$$\mathrm{id}_x(f(x)) = x. \tag{3.37}$$

- The composition of morphisms is the concatenation $\circ$ defined in definition 5.14.

The topology on the space of objects is discrete. The topology we have described above generalizes to the case where the flow we study is not the gradient flow of a weak Morse function.

Proposition 3.19. *The topology on* $\mathrm{Ob}(\mathfrak{C}_f)$ *is discrete.*

Proof. There are only a finite number of objects due to proposition 3.4. All these critical points are isolated. The isolating neighborhoods are open sets, restricted to $\mathrm{Ob}(\mathfrak{C}_f)$ are singletons. Thus the topology of $\mathrm{Ob}(\mathfrak{C}_f)$ is discrete. □

Theorem 3.20. *The category* $\mathfrak{C}_f$ *is a topological category.*

Proof. It is clear that $\mathfrak{C}_f$ is a category. We show that $\mathfrak{C}_f$ satisfies the axioms of a *topological* category. By definition $\mathrm{Ob}(\mathfrak{C})$ and $\mathrm{Hom}(\mathfrak{C})$ are topological spaces. We only need to show that the involved maps are continuous.

The identity map is continuous. The preimage of an open $U \subset \mathrm{Hom}(\mathfrak{C})$ under the identity map is some set $V \subset \mathrm{Ob}(\mathfrak{C})$. $\mathrm{Ob}(\mathfrak{C})$ has the discrete topology, hence V is open. A similar reasoning shows that the maps cod and dom are open. The composition map is continuous by proposition 3.16. All structural maps are continuous, therefore $\mathfrak{C}_f$ is a topological category. □

The spaces $\mathrm{Hom}(a, b)$ are compact. This ensures that the classifying space is compact.

Proposition 3.21. *The classifying space* $B\mathfrak{C}_f$ *is compact.*

Proof. Proposition 3.13 shows that all the spaces $\mathrm{Hom}(a, b)$ are compact. All simplices δ^n are bounded and closed subsets of $\mathbb{R}^{n+1}$ hence compact. There are only a finite number of non-degenerate spaces $\Delta^n \times N_n\mathfrak{C}_f$. Thus the space of all non-degenerate limbs is compact. These are sufficient for the formation of the classifying space. The formation of the quotient over the limbs does not change compactness. The topology on the quotient is the smallest topology such that the projection map is continuous. The image of a compact space under a continuous map is compact. □

Remark 3.22. This proposition is also a direct corollary to theorem 1.3 in the case the function f is Morse-Smale.

3.6 The Subdivision of the Flow Category

We need precise control over the reparameterized flow lines. The subdivision from section 2.6 is a valuable tool for this purpose.

Remark 3.23. It is good to view the composition in the subdivision category of the flow category as inclusion of reparameterized flow lines. There exists a morphism $\gamma_1 \mapsto \gamma_2$ if and only if $\mathrm{im}\,\gamma_2 \subset \mathrm{im}\,\gamma_1$. Thus this category measures if the piecewise flows are included inside each other. Figure 3.3 clarifies this.

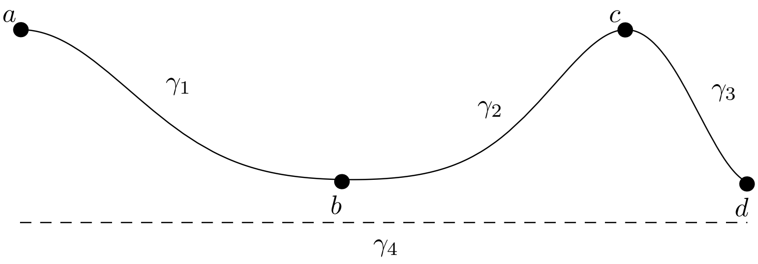

Figure 3.3: The subdivision category $\mathrm{sd}(\mathfrak{C}_f)$ measures inclusions of piecewise flow lines. In the figure we have three reparameterized flow lines γ_1, γ_2, γ_3, and their composition $\gamma_4 = \gamma_3 \circ \gamma_2 \circ \gamma_1$. There are morphisms $\gamma_4 \to \gamma_i$ for all $1 \leq i \leq 4$, because there is are inclusions of images $\mathrm{im}\,\gamma_i \subset \mathrm{im}\,\gamma_4$.

The subdivision is still a bit to crude. We want to keep track of the points of the curves, and relate them to the underlying manifold. To this end we define the tweaked subdivision, which is also a topological category

Definition 3.24. The flow category $\mathfrak{C}_f$ admits the *tweaked subdivision* $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$. This is the category where

- The space of objects, $\mathrm{Ob}(\overline{\mathrm{sd}}\,(\mathfrak{C}_f))$, consists of pairs (γ, x) with $\gamma \in \mathrm{Ob}(\mathrm{sd}(\mathfrak{C}_f))$ and $x \in \mathrm{im}\,\gamma$ a point on the curve. The space of objects is topologized as a subspace of $\mathrm{Ob}(\mathrm{sd}(\mathfrak{C}_f)) \times M$.
- The space of morphisms, $\mathrm{Hom}(\overline{\mathrm{sd}}\,(\mathfrak{C}_f))$ consists of triples $(\alpha, \beta)_x$, which are morphisms between (γ, x) and (γ', x') if and only if $x = x'$ and there exists a morphism $(\alpha, \beta) : \gamma \to \gamma'$ in $\mathrm{sd}\,(\mathfrak{C}_{\mathcal{D}})$. The space is topologized as a subspace of $\mathrm{Hom}(\mathrm{sd}\,(\mathfrak{C}_f)) \times M$.
- Let (γ, x) be an object, and $(\alpha, \beta)_x : (\gamma, x) \to (\gamma', x)$ be a morphism. The identity, domain, and codomain maps are the obvious maps defined by

$$\begin{aligned} \mathrm{id}^{\overline{\mathrm{sd}}(\mathfrak{C}_f)}(\gamma, x) &:= (\mathrm{dom}^{\mathfrak{C}_f}\gamma, \mathrm{cod}^{\mathfrak{C}_f}\gamma)_x \\ \mathrm{dom}^{\overline{\mathrm{sd}}(\mathfrak{C}_f)}(\alpha, \beta)_x &:= (\mathrm{dom}^{\mathfrak{C}_f}(\alpha, \beta), x) = (\gamma, x) \\ \mathrm{cod}^{\overline{\mathrm{sd}}(\mathfrak{C}_f)}(\alpha, \beta)_x &:= (\mathrm{cod}^{\mathfrak{C}_f}(\alpha, \beta), x) = (\gamma', x). \end{aligned} \tag{3.38}$$

- If $(\alpha, \beta)_x$ and $(\alpha', \beta')_{x'}$ are composable, then the composition is defined by

$$(\alpha, \beta)_x \circ^{\overline{\mathrm{sd}}(\mathfrak{C}_f)} (\alpha', \beta')_{x'} = (\alpha' \circ^{\mathfrak{C}_f} \alpha, \beta \circ^{\mathfrak{C}_f} \beta')_x. \tag{3.39}$$

  Note that the morphisms are composable only if $x = x'$.

Remark 3.25. The construction of the tweaked subdivision can only be performed for the flow category. It is not a categorical construction, we use the precise structure of the objects and the morphisms of the underlying flow category. This is in contrast to the construction of the subdivision in section 2.6, which is a categorical construction.

Proposition 3.26. *The tweaked subdivision* $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$ *is a topological category.*

Proof. All the structural maps in $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$ are the restrictions of the structural maps in $\mathrm{sd}\,(\mathfrak{C}_f)$. To spell this out

$$\begin{aligned} \mathrm{id}^{\overline{\mathrm{sd}}(\mathfrak{C}_f)} &:= \mathrm{id}^{\mathrm{sd}(\mathfrak{C}_f)} \times \mathrm{id}^M \Big|_{\mathrm{Ob}(\mathrm{sd}(\mathfrak{C}_f))} \\ \mathrm{dom}^{\overline{\mathrm{sd}}(\mathfrak{C}_f)} &:= \mathrm{dom}^{\mathrm{sd}(\mathfrak{C}_f)} \times \mathrm{id}^M \Big|_{\mathrm{Hom}(\mathrm{sd}(\mathfrak{C}_f))} \\ \mathrm{cod}^{\overline{\mathrm{sd}}(\mathfrak{C}_f)} &:= \mathrm{cod}^{\mathrm{sd}(\mathfrak{C}_f)} \times \mathrm{id}^M \Big|_{\mathrm{Hom}(\mathrm{sd}(\mathfrak{C}_f))} \end{aligned} \tag{3.40}$$

The composition is the restriction of the continuous map

$$(((\alpha,\beta),x),((\mu,\nu),y)) \mapsto ((\alpha,\beta)\circ^{\mathrm{sd}(\mathfrak{C}_f)}(\mu,\nu),x), \tag{3.41}$$

to the subspace of $\mathrm{Hom}(\overline{\mathrm{sd}}\,(\mathfrak{C}_f)) \times \mathrm{Hom}(\overline{\mathrm{sd}}\,(\mathfrak{C}_f))$ where $x = y$. All structural maps are continuous, hence $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$ is a topological category. □

The following theorem is proved in [9, Lemma 6.3]

Proposition 3.27. *The classifying spaces of* $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$ *and* $\mathfrak{C}_f$ *are homotopic,*

$$B\overline{\mathrm{sd}}\,(\mathfrak{C}_{\mathcal{D}}) \simeq B\mathfrak{C}_{\mathcal{D}}. \tag{3.42}$$

3.7 Proof of the Homotopy Theorem

We have developed all the tools needed to prove theorem 3.1. We outline the strategy of proof first. We define a topological category $\mathfrak{M}$ whose classifying space is obviously homeomorphic to M. We construct two continuous functors $\Theta : \overline{\mathrm{sd}}\,(\mathfrak{C}_f) \to \mathfrak{M}$ and $\Gamma : \mathfrak{M} \to \overline{\mathrm{sd}}\,(\mathfrak{C}_f)$. The composition $\Theta \circ \Gamma$ is the identity functor on $\mathfrak{M}$, while the composition $\Gamma \circ \Theta$ is naturally transformable to the identity functor on $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$. In view of proposition 2.30 we can conclude that this natural transformation induces a homotopy on the level of classifying spaces. We are now able to proof theorem 3.1.

Proof. We define the topological category $\mathfrak{M}$. The space of objects $\mathrm{Ob}(\mathfrak{M})$ equals M. The only morphisms are the identity morphisms, topologically $\mathrm{Hom}(\mathfrak{M}) = \mathrm{Ob}(\mathfrak{M})$. The classifying space $B\mathfrak{M}$ is homeomorphic to M, which is proven in proposition 2.40.

The continuous functor $\Theta : \overline{\mathrm{sd}}\,(\mathfrak{C}_f) \to \mathfrak{M}$ is defined by

$$\Theta(\gamma, x) = x, \tag{3.43}$$

acting on objects, on morphisms it is defined through

$$\Theta(\alpha,\beta)_x = \mathrm{id}_x\,. \tag{3.44}$$

It maps a pair of a reparameterized orbit γ and a point x on this reparameterized orbit to the point x. Any morphism is mapped to the identity morphism in $\mathfrak{M}$. This functor is continuous because of the following reasoning. The functor Θ acting on objects is the restriction of the projection $\mathrm{Ob}(\mathrm{sd}(\mathfrak{C}_f)) \times M$, which maps $(\gamma, x) \mapsto x$ to the subspace $\mathrm{Ob}(\overline{\mathrm{sd}}\,(\mathfrak{C}_\mathcal{D})) \subset \mathrm{Ob}(\mathrm{sd}(\mathfrak{C}_f)) \times M$. This projection is clearly continuous. The same holds for Θ acting on morphisms. Hence Θ is a continuous functor.

The continuous functor $\Gamma : \mathfrak{M} \to \overline{\mathrm{sd}}\,(\mathfrak{C}_f)$ maps a point x

$$\Gamma(x) = (\gamma_x, x), \tag{3.45}$$

to the reparameterized orbit through x, keeping track of the point x as well. Thus γ_x is the unique reparameterized orbit through x as described in proposition 5.10. Γ maps a morphism in $\mathfrak{M}$, i.e. id_x, to the identity morphism

$$\Gamma(\mathrm{id}_x^{\mathfrak{M}}) = \mathrm{id}_{(\gamma_x,x)}^{\overline{\mathrm{sd}}(\mathfrak{C}_f)} = (\mathrm{id}_{s(\gamma_x)}^{\mathrm{sd}(\mathfrak{C}_f)}, \mathrm{id}_{e(\gamma_x)}^{\mathrm{sd}(\mathfrak{C}_f)})_x. \tag{3.46}$$

This functor is continuous acting on objects, because it is the restriction of the assignment operator to $\mathrm{sd}\,(\mathfrak{C}_\mathcal{D})$. The functor acting on morphisms is also continuous, because it is the composition of continuous maps asgn and $\mathrm{id}^{\mathrm{sd}(\mathfrak{C}_f)}$.

The composition acts

$$\begin{aligned} \Theta \circ \Gamma(x) &= \Theta(\gamma_x, x) = x \\ \Theta \circ \Gamma(\mathrm{id}_x) &= \Theta(\mathrm{id}_{(\gamma_x,x)}) = \mathrm{id}_x\,. \end{aligned} \tag{3.47}$$

as the identity functor on $\mathfrak{M}$. The induced map $B\Theta \circ \Gamma = \mathrm{id}_{B\mathfrak{M}}$ is the identity map on $B\mathfrak{M}$.

Now we study the reverse composition. Recall that any object of $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$ has a unique decomposition

$$(\gamma, x) = (\beta \circ \gamma_x \circ \alpha, x). \tag{3.48}$$

Here γ_x is the reparameterized orbit through x as described in proposition 3.7. We now define a natural transformation

$$\mathcal{N} : \mathrm{id}_{\overline{\mathrm{sd}}(\mathfrak{C}_f)} \dot{\to} \Gamma \circ \Theta, \tag{3.49}$$

whose components map a piecewise reparameterized orbit γ to the decomposing morphisms (α, β). That is

$$\mathcal{N}(\gamma, x) = (\alpha, \beta)_x \qquad \text{where } (\alpha, \beta)_x : (\gamma, x) \mapsto (\gamma_x, x). \tag{3.50}$$

The transformation $\mathcal{N}$ is natural between the identity functor on $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$ and the functor $\Gamma \circ \Theta$. Let (γ, x) and (γ', x') be objects in $\overline{\mathrm{sd}}\,(\mathfrak{C}_f)$, with a morphism (δ, ϵ) between them. That is $x = x'$ and $\gamma = \epsilon \circ \gamma' \circ \delta$. The morphism γ' has a decomposition

$$\gamma' = \beta' \circ \gamma_x \circ \alpha'. \tag{3.51}$$

Because all decompositions involved are unique, we have $\beta = \epsilon \circ \beta'$ and $\alpha = \alpha' \circ \delta$. These observations can be written in a commutative diagram

$$\begin{array}{ccc} (\gamma, x) & \xrightarrow{(\alpha,\beta)} & (\gamma_x, x) \\ {\scriptstyle (\delta,\epsilon)_x}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{(\gamma_x,x)}} \\ (\gamma', x) & \xrightarrow[(\alpha',\beta')]{} & (\gamma_x, x) \end{array} \quad . \tag{3.52}$$

We verify the identities

$$\begin{aligned} \mathcal{N}(\gamma, x) &= (\alpha, \beta)_x \\ \mathcal{N}(\gamma', x) &= (\alpha', \beta')_x \\ \Gamma \circ \Theta(\gamma, x) &= (\gamma_x, x) \\ \Gamma \circ \Theta(\gamma', x) &= (\gamma_x, x) \\ \Gamma \circ \Theta(\delta, \epsilon)_x &= \mathrm{id}_{(\gamma_x, x)} \, . \end{aligned} \tag{3.53}$$

We conclude that diagram 3.52 is equivalent to

$$\begin{array}{ccc} \mathrm{id}(\gamma, x) & \xrightarrow{\mathcal{N}(\gamma,x)} & \Gamma \circ \Theta(\gamma, x) \\ \big\downarrow{\scriptstyle \mathrm{id}_{(\delta,\epsilon)_x}} & & \big\downarrow{\scriptstyle \Gamma\circ\Theta(\delta,\epsilon)_x} \\ \mathrm{id}(\gamma', x) & \xrightarrow[\mathcal{N}(\gamma',x)]{} & \Gamma \circ \Theta(\gamma', x) \end{array} \quad . \tag{3.54}$$

This diagram commutes for all morphisms $(\delta, \epsilon)_x$. We have proven that $\mathcal{N}$ is a natural transformation from the identity functor to the composition $\Gamma \circ \Theta$. In view of proposition 2.30 we conclude $B\overline{\mathrm{sd}}\,(\mathfrak{C}_f) \simeq B\,\mathrm{sd}(\mathfrak{C})$, and the theorem follows after we apply proposition 3.27. □

Chapter 4

Examples of the Theorem

4.1 Introduction

Mathematics is best understood through examples. We study some examples of the homotopy theorem 3.1. In some examples we will actually study Morse-Smale functions. It will be clear that we do not only get homotopies, but homeomorphisms in these cases. We will also see two phenomena which obstruct homeomorphisms in general. These are thickening, as seen in example 4.2 in the example of the deformed circle, and we also get sometimes redundant flaps. The latter happens in the example of the monkey saddle on the torus which can be found in example 4.4.

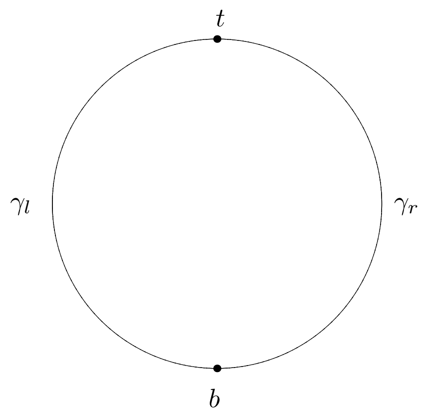

Figure 4.1: The height function on the circle has two critical points. At the top t, and at the bottom b.

4.2 Two Different Circles

The Gelfand principle asserts that it is best to introduce a new concept with the simplest non-trivial example. For Morse theory this is probably the circle, see figure 4.1, and the height function.

4.2.1 The Circle

Example 4.1. We view the circle S^1 as a submanifold of $\mathbb{R}^2$

$$S^1 := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} = 1\}. \tag{4.1}$$

The height function $f : S^1 \to \mathbb{R}$ returns the second variable.

$$f(x) = x_2. \tag{4.2}$$

This function clearly has two critical points, at the top $t = (0, 1)$ and the bottom $b = (0, -1)$. There are four reparameterized orbits; two are stationary, id_t and id_b which stay at the top and bottom of the circle. The other orbits can be computed to be

$$\begin{aligned} \omega_l(s) &= \Big(-\cos(-\frac{\pi}{2}t + \pi), \sin(-\frac{\pi}{2}t + \pi)\Big) \\ \omega_r(s) &= \Big(\cos(-\frac{\pi}{2}t + \pi), \sin(-\frac{\pi}{2}t + \pi)\Big). \end{aligned} \tag{4.3}$$

The spaces $\mathrm{Hom}(t, t)$ and $\mathrm{Hom}(b, b)$ consist of one point each, and the space $\mathrm{Hom}(t, b)$ consist of two disjoint points. One verifies that all N_n for $n > 1$ are degenerate. We list in figure 4.2 the non degenerate elements of the limbs.

The map d_i acts on $\mathrm{Hom}(t, b)$ as follows

$$\begin{aligned} d_0(\gamma_l) &= d_0(\gamma_r) = \mathrm{id}_b \\ d_1(\gamma_l) &= d_1(\gamma_l) = \mathrm{id}_t \end{aligned} \tag{4.5}$$

$$\begin{aligned} \Delta^0 \times N_0\mathfrak{C}_f &= \bullet_{\mathrm{id}_t} \ \bullet_{\mathrm{id}_b} \\ \Delta^1 \times N_1\mathfrak{C}_f &= \begin{array}{cc} \bullet_{\mathrm{id}_t} & \bullet_{\mathrm{id}_t} \\ \uparrow{\scriptstyle \gamma_l} & \uparrow{\scriptstyle \gamma_r} \\ \bullet_{\mathrm{id}_b} & \bullet_{\mathrm{id}_b} \end{array} \end{aligned} \tag{4.4}$$

Figure 4.2: The non degenerate limbs of the circle.

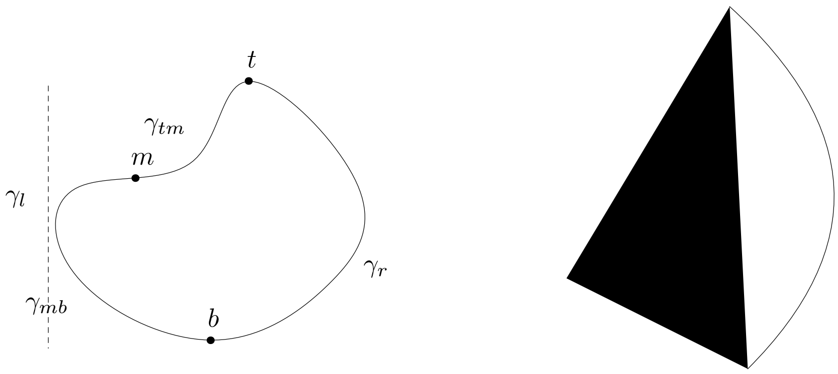

Figure 4.3: The deformed circle is shown on the left. The height function has three critical points, at the top t, at the middle m and at the bottom b. On the right we have drawn the classifying space. It consists of a $2-$simplex with an attached $1-$simplex. Note that the classifying space is *not* homeomorphic, merely homotopic to the deformed circle.

In this manner we arrived at the correct identifications. Hence we see that the end of the lines in $\Delta^1 \times \mathrm{Hom}(t, b)$ are identified with $\bullet_{\mathrm{id}_t}$ and $\bullet_{\mathrm{id}_b}$. The resulting space is seen to be homeomorphic, and in particular homotopic, to the circle. In this example we find a homeomorphism, because the function f is actually a Morse-Smale function.

4.2.2 The Deformed Circle

In the previous example we computed the classifying space and the flow category of a manifold and a Morse function on it. We now study an example where the function defining the flow category is strictly weak Morse.

Example 4.2. Consider the deformed circle depicted in 4.3, viewed as a submanifold of $\mathbb{R}^2$, and the height function f, which returns the second variable. The function has three critical points, t, m and b. The flow category consists of three objects, and there are *seven* morphisms. The spaces

$$\mathrm{Hom}(t,t), \quad \mathrm{Hom}(m,m), \quad \mathrm{Hom}(b,b), \quad \mathrm{Hom}(m,t), \quad \text{and}, \mathrm{Hom}(m,b) \tag{4.6}$$

all contain a unique morphism, which we denote by

$$\mathrm{id}_t, \quad \mathrm{id}_m, \quad \mathrm{id}_b, \quad \gamma_{tm}, \quad \text{and} \quad \gamma_{mb} \tag{4.7}$$

respectively. The space $\mathrm{Hom}(t, b)$ has two morphisms, which we denote by γ_l and γ_r. One morphism is the flow line on the right. And the other flow line is the piecewise flow line, which passes the critical point m. It is equivalent to the concatenation of the flow line in $\mathrm{Hom}(t, m)$ and the flow line in $\mathrm{Hom}(m, b)$. It is clear that the elements in $\Delta^n \times K_n$ are degenerate if $n > 2$. We have depicted non-degenerate limbs in 4.4. Once

$$
\begin{aligned}
\Delta^0 \times N_0\mathfrak{C}_f &= \bullet\mathrm{id}_t \ \bullet\mathrm{id}_m \ \bullet\mathrm{id}_b \\
\Delta^1 \times N_1\mathfrak{C}_f &= \begin{array}{cccc} \bullet\mathrm{id}_t & \bullet\mathrm{id}_t & \bullet\mathrm{id}_m & \bullet\mathrm{id}_t \\ \uparrow \gamma_l & \uparrow \gamma_r & \uparrow \gamma_{mb} & \uparrow \gamma_{mt} \\ \bullet\mathrm{id}_b & \bullet\mathrm{id}_b & \bullet\mathrm{id}_b & \bullet\mathrm{id}_m \end{array} \\
\Delta^2 \times N_2\mathfrak{C}_f &= \begin{array}{ccc} & \bullet\mathrm{id}_b & \\ \nearrow \gamma_l & & \nwarrow \gamma_{mt} \\ \bullet\mathrm{id}_m & \xrightarrow[\gamma_{bm}]{} & \bullet\mathrm{id}_t \end{array}
\end{aligned}
\tag{4.8}
$$

Figure 4.4: Some limbs of the height function on the deformed circle.

more we have drawn the identifications that need be made. We see that the classifying space of the flow category is a 2-simplex with an attached 1-simplex, cf. figure 4.3. The classifying space is not homeomorphic to the deformed circle, but simply homotopic. This phenomenon of thickening will occur in most situations where the function fails to be a Morse function.

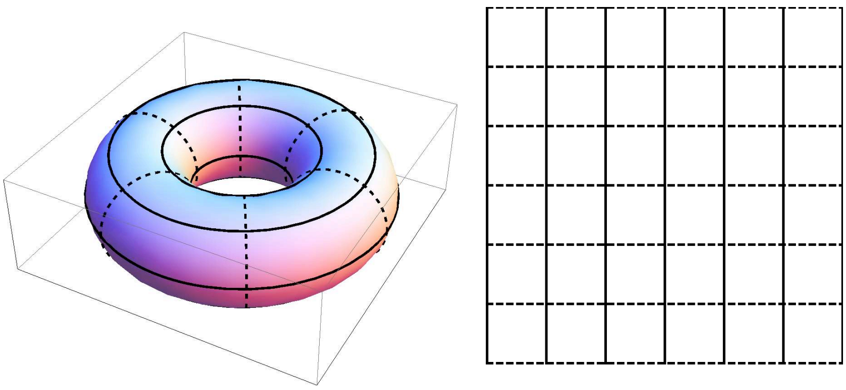

Figure 4.5: The torus, as a submanifold of $\mathbb{R}^3$, and the torus viewed as $\mathbb{R}^2/\mathbb{Z}^2$. Lines on the righthand side are identified with the circles on the left-hand side.

4.3 Three Functions on the Torus

It is also good to study some higher dimensional examples. Classically the torus has been *the* example for demonstrating Morse theory. We will study the flow categories and their classifying space for three different functions defined on the torus. These functions all have some different properties, illuminating the different aspects of the theory. The first function has the best properties, it is a Morse-Smale function. The classifying space is homeomorphic to the torus. The second example is a weak Morse function on the torus. We see that the classifying space is in a sense two dimensional, but has "extra flaps". The third example is the standard height function on the torus. The classifying space of the flow category is not homeomorphic to the torus. The classifying space has an higher dimension than the torus. The phenomenon of thickened spaces occurs.

Throughout this section we will view the torus, depicted in figure 1.3, as $\mathbb{R}^2/\mathbb{Z}^2$. A function on $\mathbb{R}^2$ which is periodic in both variables thus determines a function on the torus. The explicit map we use for the embedding of $\mathbb{R}^2/\mathbb{Z}^2$ in $\mathbb{R}^3$ is

$$\phi(\theta, \phi) = \{(2 + \cos(\theta))\cos(\phi), (2 + \cos(\theta))\sin(\phi), \sin(\theta)\}. \tag{4.9}$$

4.3.1 The Morse-Smale Function

Example 4.3. Consider the function $f : T \to \mathbb{R}$

$$f(x_1, x_2) = \sin(2\pi x_1) + \sin(2\pi x_2), \tag{4.10}$$

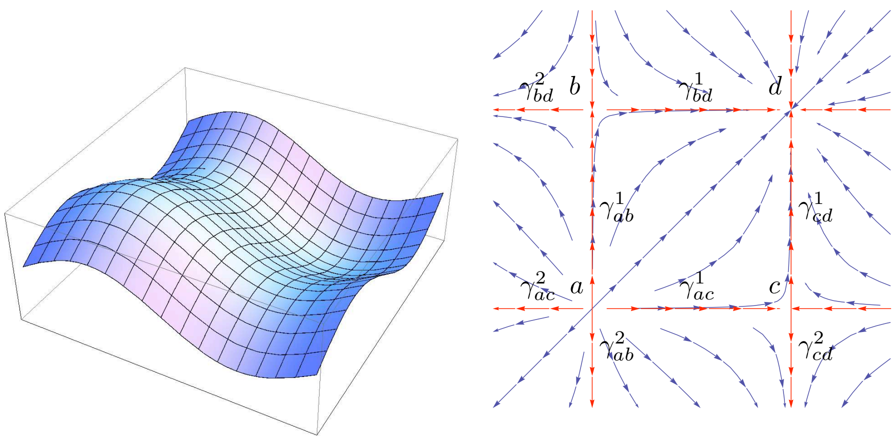

Figure 4.6: We have shown the Morse-Smale function f on the torus on the left hand side. On the right hand side we depicted the flow of gradient vector field.

which is a Morse-Smale function. The function is depicted in 4.6, and has four critical points, and an infinite number of morphisms. The objects of the flow category, i.e. the critical points, are labeled by

$$a = \left(\frac{1}{4}, \frac{1}{4}\right), \quad b = \left(\frac{1}{4}, \frac{3}{4}\right), \quad c = \left(\frac{3}{4}, \frac{1}{4}\right), \quad d = \left(\frac{3}{4}, \frac{3}{4}\right). \tag{4.11}$$

The spaces

$$\mathrm{Hom}(a, b), \quad \mathrm{Hom}(a, c), \quad \mathrm{Hom}(b, d), \quad \text{and} \quad \mathrm{Hom}(c, d), \tag{4.12}$$

all consist of two disjoint points, while the space $\mathrm{Hom}(a, d)$ contains an infinite number of morphisms. The topology defined on it allow us to view it as four disjoint lines, c.f. proposition 3.15. The edges of the lines are precisely the possible compositions in $\mathrm{Hom}(a, b, d)$ and $\mathrm{Hom}(a, c, d)$. All elements of $\mathrm{Hom}(\boldsymbol{X})$ with $l(\boldsymbol{X}) > 2$ are degenerate. These observations let us draw the limbs in figure 4.7. We reconstruct the classifying space in figure 4.8.

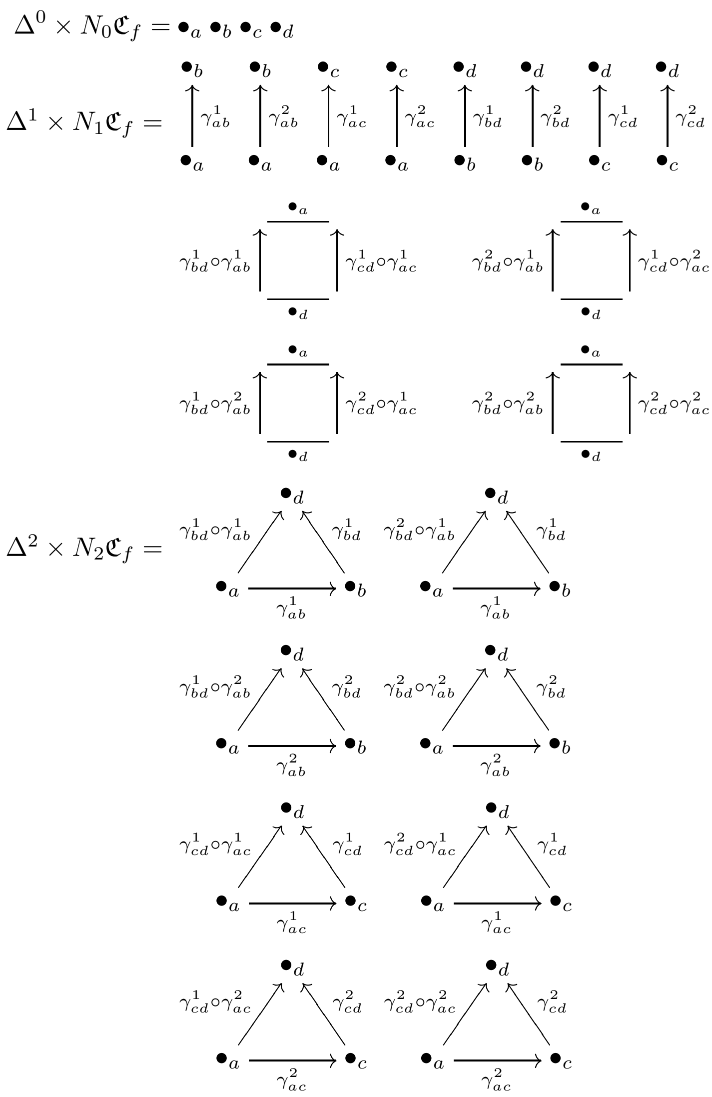

Figure 4.7: The limbs of the flow category of the Morse-Smale function defined on the torus. Four objects give four points, and an infinitude of morphisms give 8 disjoint lines, and four filled squares. There are eight non-degenerate commuting triangles which give rise to eight 2-simplices.

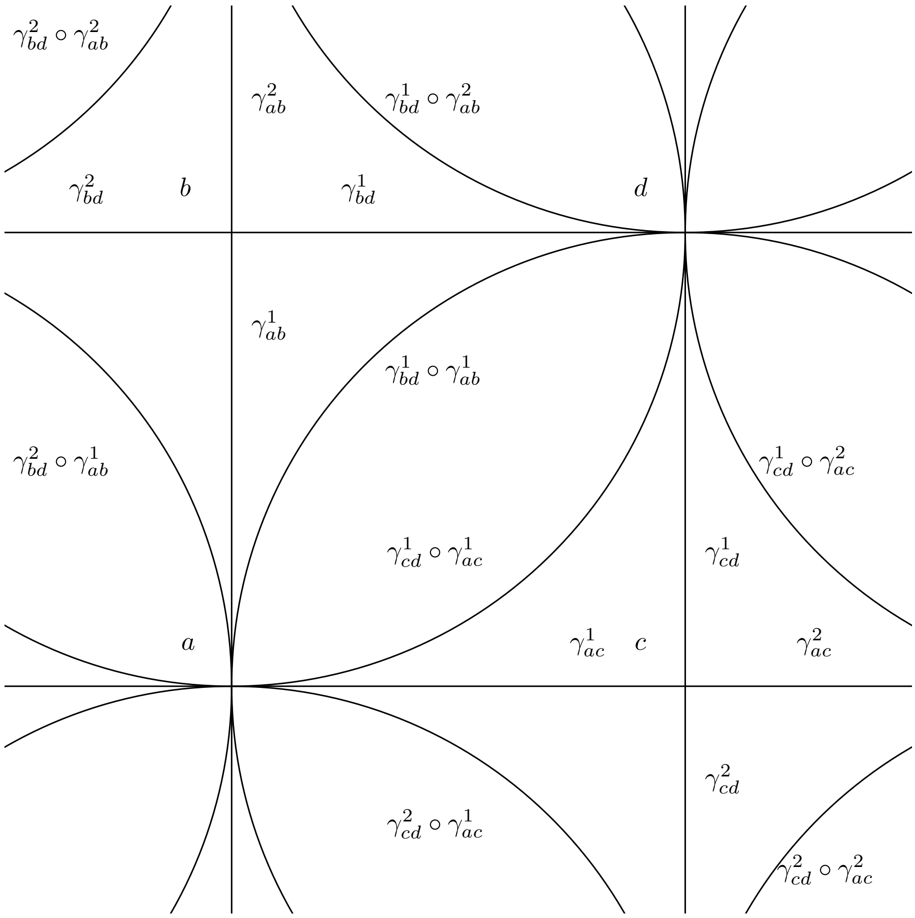

Figure 4.8: The classifying space of the flow category of the Morse-Smale function on the torus. We have drawn the classifying space embedded in $\mathbb{R}^2/\mathbb{Z}^2$. The spaces between the lines is filled. The classifying space is homeomorphic to the torus.

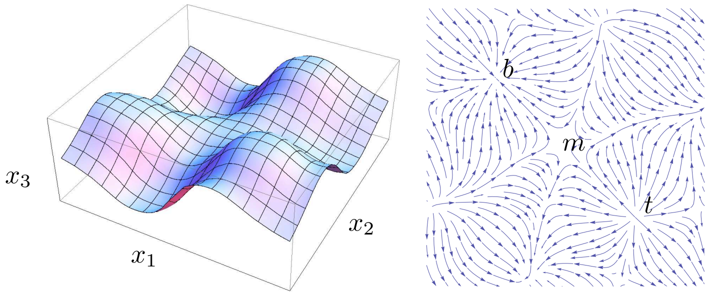

Figure 4.9: The monkey saddle on $\mathbb{R}^2/\mathbb{Z}^2$ is drawn on the left hand side. On the right hand side we have depicted the gradient flow.

4.3.2 The Monkey Saddle

Example 4.4. Consider the function drawn in figure 4.9. This is a function, which has a maximum at $t = (\frac{1}{4}, \frac{3}{4})$ and a minimum at $b = (\frac{3}{4}, \frac{1}{4})$. The middle point $m = (\frac{1}{2}, \frac{1}{2})$ is a monkey saddle, i.e. there exists a chart (U, φ) such that

$$f \circ \varphi^{-1}(x, y) = x^3 - 3xy^2, \tag{4.13}$$

in this chart. Clearly this is not a Morse function, since the critical point at m is degenerate. It is a weak Morse function however, the critical points are isolated. The spaces

$$\mathrm{Hom}(t, m), \qquad \text{and} \qquad \mathrm{Hom}(m, b) \tag{4.14}$$

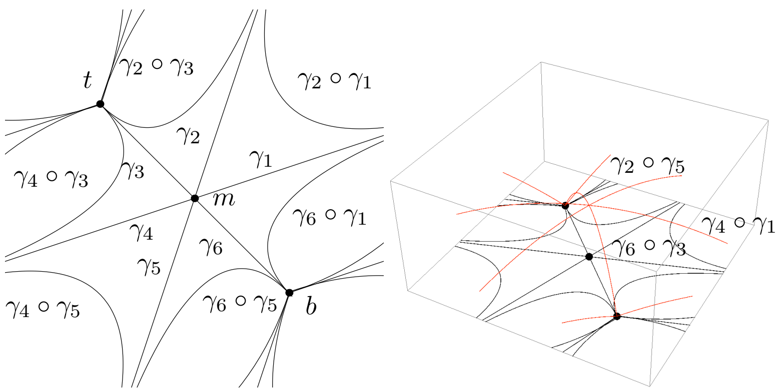

Figure 4.10: The classifying space of the monkey saddle cannot be nicely embedded into $\mathbb{R}^3$ or $\mathbb{R}^2/\mathbb{Z}^2$. On the left hand side we have shown a projection onto $\mathbb{R}^2/\mathbb{Z}^2$. On the right hand side we have a projection of the classifying space embedded in $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}$. This is not a nice embedding. In the original space the three red lines do not cross.

consist of three points each. We label these by γ_1,γ_3,γ_5, and γ_2,γ_4,γ_6, respectively. The space $\mathrm{Hom}(t, b)$ consists of three disjoint lines, cf. 3.13, and three disjoint points. The edges of the lines correspond to the compositions, which bound the lines. The three disjoint points are unnatural. They correspond to the compositions which cross the critical point m. These are

$$\gamma_2 \circ \gamma_5 \quad \gamma_4 \circ \gamma_1 \quad \gamma_6 \circ \gamma_3. \tag{4.15}$$

The disjoint points are an artifact which arise because the function is not Morse-Smale. It will obstruct the construction of a homeomorphism of the classifying space of the flow category to the torus. The limbs are computed in figure 4.11.

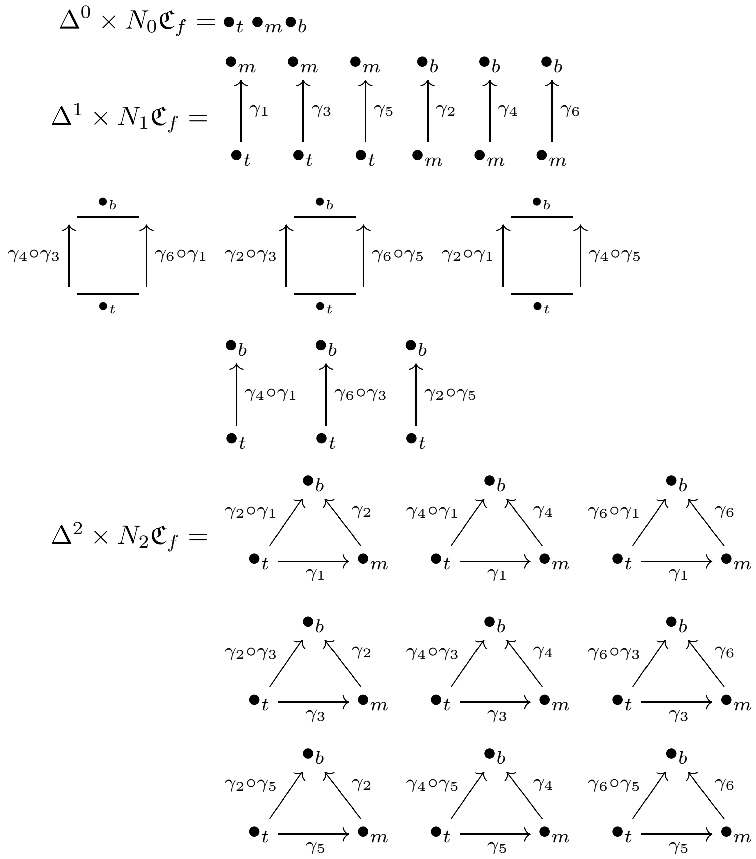

Figure 4.11: The non-degenerate limbs of the monkey saddle on the torus.

4.3.3 The Morse Function on the Torus

It is interesting that the standard example of Morse theory is the most difficult example we will discuss in this chapter. It is the height function on the torus. The tricky thing is that it isn't a Morse-Smale function. The height function returns the third coordinate of the torus embedded in $\mathbb{R}^3$, where the torus is standing up. The flow of the Morse function is depicted in figure 4.12. The function has 4 critical points. One maximum a, one minimum d, and two saddle points b, c. The $\mathrm{Hom}(a, b)$,$\mathrm{Hom}(b, c)$, and $\mathrm{Hom}(c, d)$ consist of two disjoint points. The space $\mathrm{Hom}(a, d)$ has two disjoint lines, and 4 disjoint points. These observations allow the computation of the non-degenerate limbs, in figure 4.13.

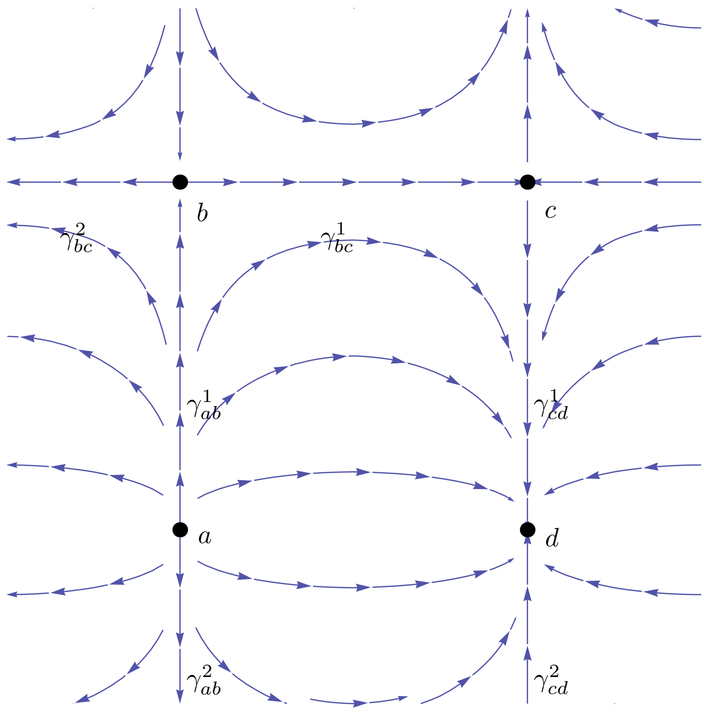

Figure 4.12: The flow of the height function on the torus. There are four critical points, and an infinite number of morphisms. The non-identity morphisms that are elements of Hom sets with a finite number of elements are named.

$$\Delta^0 \times N_0\mathfrak{C}_f = \bullet_a \ \bullet_b \ \bullet_c \ \bullet_d$$

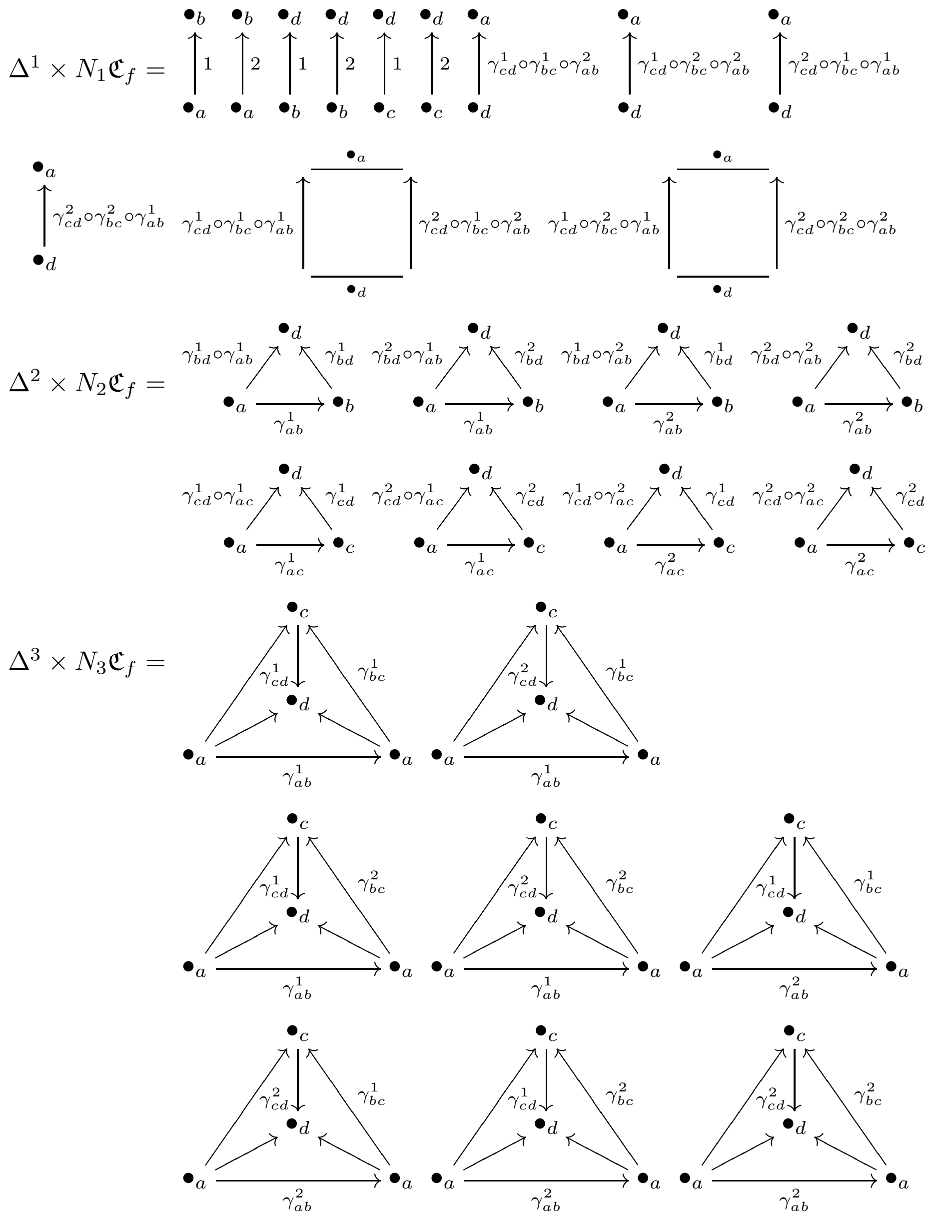

Figure 4.13: The limbs of the height function on the torus. We have chains of length 3, we have depicted the 3-simplices by triangles with a barycentric subdivision. These are projections of the 3-simplex to the plane. Some of the labels of the compositions miss in $\Delta^3 \circ N_3\mathfrak{C}_f$. All the simplices commute, therefore it is easy to complete the labeling of the simplices. If we would choose to label all the compositions the figures would become too cluttered.

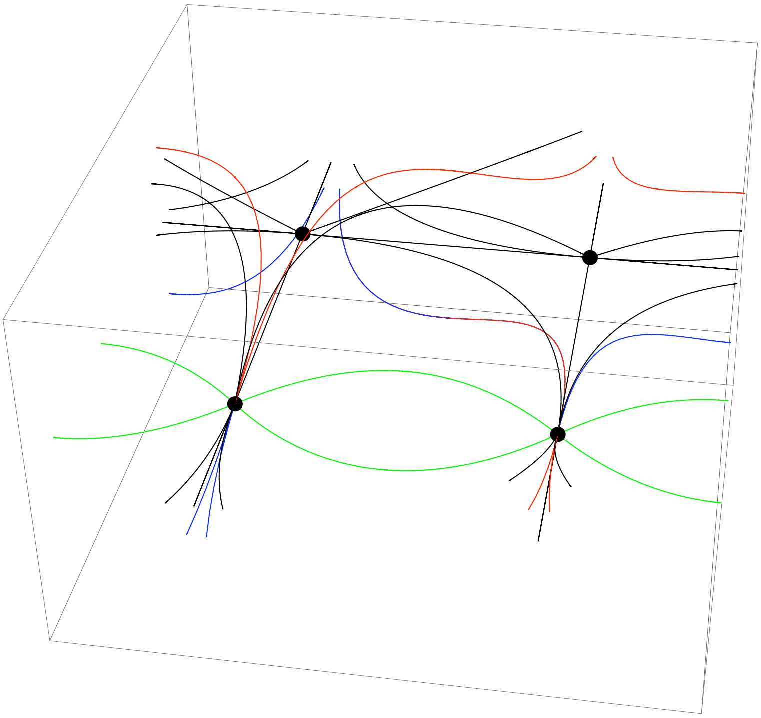

Figure 4.14: The classifying space of the height function on the torus. We have chosen to embed the classifying space in $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}$. The picture is hard to understand. The colored lines correspond to the threefold compositions. The green lines are the morphisms $\gamma^1_{cd} \circ \gamma^1_{bc} \circ \gamma^1_{ab}$, $\gamma^2_{cd} \circ \gamma^1_{bc} \circ \gamma^2_{ab}$, $\gamma^1_{cd} \circ \gamma^2_{bc} \circ \gamma^1_{ab}$, and $\gamma^2_{cd} \circ \gamma^2_{bc} \circ \gamma^2_{ab}$. The red and blue morphisms correspond to the other possible combinations of 3 morphisms. Note that we have to fill the space between these morphisms, and the morphisms whose compositions yield these. This is impossible to draw however. We end up with a 3-dimensional space that is homotopic to the torus, i.e. the plane in $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}$. Clearly it is not homeomorphic, the classifying space is three dimensional, while the torus is two dimensional.

Chapter 5

Dynamical Systems

We have spent quite some time studying gradient like systems. The main objective of this chapter is to generalize the preceding ideas to a large class of dynamical systems, for which the proofs of chapter 3 carry over verbatim. In this chapter we will show the proofs of statements which do not carry completely analogously.

We recall the notion of a dynamical system.

Definition 5.1. Let (S, d) be metric space. A *dynamical system* on S is a continuous map $\varphi : \mathbb{R} \times S \to S$ satisfying two properties

- $\varphi(0, x) = x$
- $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$,

for all $x \in S$ and all $t, s \in \mathbb{R}$. The map φ is called the flow. The flow is a continuous group action of the reals on S. The *orbit* or *trajectory* through x is the curve $\gamma_x : \mathbb{R} \to S$ defined by

$$\gamma_x(t) := \varphi(t, x). \tag{5.1}$$

The *set of equilibria* E is

$$E := \{x \in S \mid \varphi(t, x) = x \text{ for all } t \in \mathbb{R}\}. \tag{5.2}$$

E is topologized as a subspace of S. We will usually denote the time parameter with a subscript

$$\varphi_t(x) := \varphi(t, x). \tag{5.3}$$

Remark 5.2. We state in the definition *the* orbit through x. There is a free $\mathbb{R}$ action defined on the orbits. The images of $\gamma_t(x)$ and $\gamma_s(\gamma_t(x))$ are the same. There is a unique orbit with $\gamma_x(0) = x$. In this sense the orbits are unique.

The behavior of dynamical systems can be extremely complex. Recurrent behavior is something we cannot hope to capture in the flow category. Therefore we study a subclass of all dynamical systems where this behavior cannot occur. We also exclude almost recurrent behavior; all orbits converge to points in positive and negative time, cf. example 3.6.

Definition 5.3. A dynamical system is said to be *strongly gradient-like*, if the dynamical system satisfies the axioms

1. The space of equilibria E is non-empty and has only a finite number of connected components.

2. The flow is gradient-like, i.e. there exists a continuous *Lyapunov function* $f : S \to \mathbb{R}$, which is constant on connected components of E and strictly decreasing on all non-equilibrium trajectories. That is

$$f(\gamma_x(t)) \leq f(\gamma_x(t+s)), \tag{5.4}$$

for all $x \in S$, all $t \in \mathbb{R}$ and all positive times $s \in \mathbb{R}_{>0}$. Equality only holds if and only if $x \in E$.

3. The orbits γ_x are heteroclinic, i.e. both limits

$$\lim_{t\to\pm\infty} \gamma_x(t), \tag{5.5}$$

exist and are elements of E for all $x \in S$. The limits are denoted by

$$s(\gamma_x) := \lim_{t\to-\infty} \gamma_x(t) \qquad e(\gamma_x) := \lim_{t\to\infty} \gamma(t). \tag{5.6}$$

Some familiar mathematical constructions determine strongly gradient-like dynamical systems. A generalization of a Morse function is a Morse-Bott function.

Definition 5.4. Let M be a smooth Riemannian manifold. A smooth function $f : M \to \mathbb{R}$ is a *Morse-Bott* function if the set

$$\mathrm{Crit}(f) = \{x \in M \mid df(x) = 0\}, \tag{5.7}$$

of critical values consist of a disjoint union of connected submanifolds, and for each connected submanifold $C \subset \mathrm{Crit}(f)$, the Hessian, restricted tot the normal bundle of C is non-degenerate.

Corollary 5.5. *A Morse function f is in particular a Morse-Bott function.*

Proof. $\mathrm{Crit}(f)$ consist of disjoint critical points. These are in particular submanifolds. The Hessian at the critical points is non-degenerate by the Morse lemma. Hence f is a Morse-Bott function. □

Information on Morse-Bott functions can be found in for example [2]. The axioms of a strongly gradient-like dynamical system where modeled after the flow of a Morse-Bott function. In particular we have the following.

Proposition 5.6. *Let $f : M \to \mathbb{R}$ be a Morse-Bott function. The flow of the gradient vector field*

$$V := -\nabla f \tag{5.8}$$

is a strongly gradient-like dynamical system, where the Lyapunov function is the function f.

Convention 5.7. For the remainder of this chapter we denote by $\mathcal{D} = (S, d, \varphi, f)$ a strongly gradient-like dynamical system.

The objective of this chapter is the proof of the following theorem.

Theorem 5.8. *Let $\mathcal{D}$ be a strongly gradient-like dynamical system. The classifying space of the flow category $\mathfrak{C}_{\mathcal{D}}$ is homotopic to the underlying metric space S.*

$$B\mathfrak{C}_{\mathcal{D}} \simeq S. \tag{5.9}$$

The theorem states that a flow category of a strongly gradient-like dynamical system is constrained by the underlying topological space. One can view this as an invariant for the space. Of course this invariant is not different from the homotopy type of the space itself, but it might be easier to compute, since one has the freedom of choosing the dynamical system defined on it.

5.1 The Flow

The flow category associated to a strongly gradient-like dynamical system is inspired by the flow category of a weak Morse function. The objects in the category are equilibrium points, and the morphisms are reparameterized orbits of the flow. The space of objects is topologized in the subspace topology of S, and the space of morphisms is topologized as a subspace of the compact open topology. Composition in the flow category is concatenation of these reparameterized orbits. The reparameterization part of this chapter is done differently than in chapter 3, and the definition of the flow category contains some subtleties. After constructed the category all proofs carry over from chapter 3.

5.1.1 The Reparameterization of the Flow

Even though we do not assume that S is compact, the image of the Lyapunov function is compact. Strongly gradient-like dynamical systems are "closed and bounded in the direction of the flow".

Proposition 5.9. *The set $f(S)$ is compact. The smallest interval containing $f(S)$ is defined to be J.*

Proof. The image of an orbit along with its limit points, is a bounded and closed interval, i.e. compact. We directly verify

$$f(S) = \bigcup_{x \in S} f(\overline{\operatorname{im} \gamma_x}). \tag{5.10}$$

We see that $f(\overline{\operatorname{im} \gamma_x}) = [f(e(\gamma_x)), f(s(\gamma_x))]$. The number of connected components of E is finite, there are only a finite number of different closed and bounded intervals that can be formed. Hence the union in equation (5.10) is actually finite. A finite union of compact sets is compact. Hence $f(S)$ is compact. $\square$

We can reparameterize the orbits using the Lyapunov function. We do this because the reparameterized orbits are unique, in the sense that all points of S lie on a unique curve, in contrast with remark 5.2. This is not the case for the original orbits, there is a free $\mathbb{R}$ action defined on them. We are also able to concatenate the reparameterized curves, which we need for the construction of the flow category. One should compare the next proposition with proposition 3.7.

Proposition 5.10. *Let γ_x be the orbit through $x \in S \setminus E$. Define the interval $I :=]f(e(\gamma_x)), f(s(\gamma_x))[$. There exists a unique continuous curve $\gamma : \overline{I} \to S$ with the properties*

- $\operatorname{im}\gamma = \overline{\operatorname{im}\gamma_x}$
- $f(\gamma(t)) = t$.

If $x \in E$ then the function $\gamma : [f(x), f(x)] \to S$ defined by

$$\gamma(f(x)) = x \tag{5.11}$$

is a reparameterization of γ_x.

Proof. This proof makes use of Brouwer's theorem on invariance of domain [6]. This theorem states that a continuous injective map $g : U \to V$ is an open map if $U, V \subset \mathbb{R}^n$ are open. In particular it is a homeomorphism if it is surjective as well.

Define the map $\lambda : \mathbb{R} \to I$ by the equation

$$\lambda(t) := f(\gamma_x(t)) \tag{5.12}$$

This map is continuous, being a composition of two continuous functions. It is injective by the gradient-like property of f, $\lambda(t) > \lambda(s)$ if $t < s$. Because γ_x has well defined limits, for any $s \in I$ there exists a $y \in \operatorname{im}\gamma_x$ with $f(y) = s$. λ is surjective. By Brouwer's theorem on invariance of domain the map λ is a homeomorphism. The inverse of λ is well defined and continuous. Define the curve $\gamma : I \to S$ by

$$\gamma(t) = \gamma_x(\lambda^{-1}(t)). \tag{5.13}$$

The curve naturally extends continuously to the closure of I by imposing $\gamma(f(s(\gamma_x))) = s(\gamma_x)$ and $\gamma(f(e(\gamma_x))) = e(\gamma_x))$, using the fact that the orbits in a strongly gradient-like dynamical system are heteroclinic orbits. The curve satisfies the properties in the proposition. We reason that the curve is the only curve satisfying these properties. f is strictly decreasing along orbits, hence there is a unique $y \in \operatorname{im}\gamma_x$ with $f(y) = t$ for fixed t. Hence we can define the curve via these equations, and the previous reasoning shows that the curve is continuous. ☐

Remark 5.11. If $x \in E$ then the reparameterization γ of γ_x will also be written as $\gamma =: \mathrm{id}_x$. These curves will form the identity morphisms in the flow category we are about to define.

Definition 5.12. If γ is a reparameterization of the orbit γ_x then we define the *start* and *end* by

$$s(\gamma) := s(\gamma_x) \qquad \text{and} \qquad e(\gamma) := e(\gamma_x). \tag{5.14}$$

Remark 5.13. Note that this nomenclature is somewhat misleading. The direction of the reparameterized orbit is in the opposite direction of the flow. This is because of historical reasons. The Lyapunov function is a generalization of minus the indexing function for a gradient system. One studies the dynamical system

$$\dot{\gamma}(t) = -\nabla(f)(\gamma(t)). \tag{5.15}$$

This equation makes that f decreases along orbits. It is natural to reparameterize the orbits using the values of the indexing function. The minus sign in the above equation makes that the reparameterized orbits travel in the opposite direction of the original curves.

We have shown that it is possible to reparameterize the orbits. This reparameterization allows us to concatenate orbits.

Definition 5.14. Let (α, β) be a pair of two reparameterized orbits as defined in proposition 5.10. The pair is *composable* if $e(\alpha) = s(\beta)$. Then the *composition* or *concatenation* of α and β is

$$\beta \circ \alpha(t) = \begin{cases} \beta(t) & \text{for } t \in [f(e(\beta), f(s(\beta))] \\ \alpha(t) & \text{for } t \in [f(e(\alpha), f(s(\alpha))] \end{cases} \tag{5.16}$$

Definition 5.15. Let $x, y \in E$ be equilibrium points. The space of piecewise orbits $\mathrm{Hom}(x, y)$ consists of all continuous curves

$$\gamma : [f(y), f(x)] \to S, \tag{5.17}$$

which are piecewise reparameterizations of orbits of the dynamical system. That is, we can write $\gamma = \gamma_n \circ \gamma_{n-1} \circ \cdots \circ \gamma_1$ for reparameterized curves γ_i. The space $\mathrm{Hom}(x, y)$ is topologized by viewing it as a subspace of all continuous maps $\mathcal{C}([f(y), f(x)], S)$ in the compact-open topology. An element of $\mathrm{Hom}(x, x)$ is said to be an identity. Suppose γ is not an identity. The decomposition $\gamma = \gamma_n \circ \gamma_{n-1} \circ \cdots \circ \gamma_1$ is *minimal* if all γ_i are not identities, if this is not the case, the decomposition is degenerate. A non-identity piecewise reparameterization γ is *prime*, if the minimal decomposition of γ consist of one reparameterized orbit. A non-identity reparameterized orbit can be uniquely decomposed in prime reparameterized orbits.

Proposition 5.16. *The composition* $\circ : \mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$ *defined in definition 5.14 is a continuous map.*

Proof. The compact-open topology on $\mathrm{Hom}(x, y)$ is the topology generated by the subbasis

$$D(C, U) = \{\gamma \in \mathrm{Hom}(x, y) \mid \gamma(C) \subset U\}, \tag{5.18}$$

where $C \subset [f(y), f(x)]$ is compact and $U \subset S$ is open. For the continuity of the composition map we need to check that the preimage of a subbasis element is open.

Let $C \subset [f(z), f(x)]$ be compact, and U open in S. Clearly we can decompose $C = C_1 \cup C_2$, with $C_1 \subset [f(y), f(x)]$ and $C_2 \subset [f(z), f(y)]$. We verify

$$\circ^{-1} D(C, U) = D(C_1, U) \times D(C_2, U), \tag{5.19}$$

where $D(C_1, U)$ a subbasis element of $\mathrm{Hom}(x, y)$ and $D(C_2, U)$ a subbasis element of $\mathrm{Hom}(y, z)$. This is a subbasis element of the topology on $\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z)$ thus open. Hence the preimage of any subbasis element under $\circ$ is open. The map $\circ$ is a continuous. □

5.1.2 Properties of the Hom sets

The Hom sets of a weak Morse function have some nice properties. In chapter 3 we have proven continuity of the assignment operator in proposition 3.11, we have shown the compactness of $\mathrm{Hom}(a, b)$ in proposition 3.13, and we have shown in proposition 3.13 that intersection of the stable and unstable sets embed nicely into $\mathrm{Hom}(a, b)$. These results hold generally for strongly gradient-like dynamical systems. The proofs are analogous to the proofs in chapter 3. We therefore omit the proofs.

Proposition 5.17. *The assignment operator* $\mathrm{asgn} : S \to \bigcup_{x,y \in E} \mathrm{Hom}(x, y)$ *which maps a point x to the reparameterized orbit γ through x is continuous.*

Proposition 5.18. *If S is compact, then* $\mathrm{Hom}(a, b)$ *is compact for all* $a, b \in E$.

For the last proposition we need some definitions.

Definition 5.19. Let $a \in E$. The *stable set* $W^s(a)$ and *unstable set* $W^u(a)$ are defined by

$$\begin{aligned} W^s(a) =& \{x \in S \mid \lim_{t \to \infty} \varphi_t(x) = a\} \\ W^u(a) =& \{x \in S \mid \lim_{t \to -\infty} \varphi_t(x) = a\}. \end{aligned} \tag{5.20}$$

Let $a, b \in E$ and $t \in (f(b), f(a))$ The space $M_t(a, b) \subset S$ is defined by

$$M_t(a, b) = W^u(a) \cup W^s(b) \cup f^{-1}(t). \tag{5.21}$$

This allows us to formulate and prove

Proposition 5.20. *There exists an embedding* $\iota : M_t(a, b) \to \mathrm{Hom}(a, b)$.

5.2 The Flow Category

The definition of the flow category is inspired by the definition of the flow category of the gradient flow of a weak Morse function.

Definition 5.21. The flow category of $\mathcal{D}$ is the topological category $\mathfrak{C}_{\mathcal{D}}$ where

- The space of objects $\mathrm{Ob}(\mathfrak{C}_{\mathcal{D}}) = E$ is the space of equilibrium points, i.e. $x \in E$ implies $x \in \mathrm{Ob}(\mathfrak{C}_{\mathcal{D}})$

- The space of morphisms

$$\mathrm{Hom}(\mathfrak{C}_{\mathcal{D}}) = \coprod_{x,y \in \mathrm{Ob}(\mathfrak{C}_{\mathcal{D}})} \mathrm{Hom}(x,y), \tag{5.22}$$

  is the union of all spaces of piecewise reparameterized orbits.

- dom sends a morphism $\gamma \in \mathrm{Hom}(x,y)$ to its domain x.
- cod sends a morphism $\gamma \in \mathrm{Hom}(x,y)$ to its codomain y.
- id sends an object x to the constant orbit $\mathrm{id}_x \in \mathrm{Hom}(x,x)$, the identity at x, where $\mathrm{id}_x(f(x)) = x$.
- The composition of morphisms is the concatenation $\circ$ defined in definition 5.15.

Theorem 5.22. *The flow category $\mathfrak{C}_{\mathcal{D}}$ is a topological category.*

Proof. It is clear that the above defined structure is a category. We do have to prove that this category is a topological category. For this we have to show that the structural maps are continuous.

We first prove that the identity map $\mathrm{id} : x \mapsto \mathrm{id}_x$ is continuous. Let $D(C,U)$ be a subbasis element of the topology on $\mathrm{Hom}(\mathfrak{C}_{\mathcal{D}})$, thus C is compact in J, the image of the Lyapunov function, and U is open in S. The preimage of this basis element under the identity map is

$$\mathrm{id}^{-1}(D(C,U)) = \mathrm{Ob}(\mathfrak{C}_{\mathcal{D}}) \cap f^{-1}(C) \cap U. \tag{5.23}$$

f is constant on connected components of $\mathrm{Ob}(\mathfrak{C}_{\mathcal{D}}) = E$, hence the preimage of C under f is a union of these connected components, and is therefore open in E. The set $\mathrm{id}^{-1}(D(C,U))$ is an intersection of three open sets hence open.

Now we prove that the map $\mathrm{dom} : (\gamma : x \to y) \mapsto x$ is continuous. Let $U \subset \mathrm{Ob}(\mathfrak{C}_{\mathcal{D}})$ be open. We check that

$$\mathrm{dom}^{-1}(U) = \bigcup_{\substack{y \in \mathrm{Ob}(\mathfrak{C}_{\mathcal{D}}),\\ x \in U}} \mathrm{Hom}(x,y). \tag{5.24}$$

Each $\mathrm{Hom}(x,y)$ is open, thus the union is open as well. We conclude that the map dom is continuous. Clearly this proof can be done analogously for cod.

We proved in proposition 5.16 that the concatenation is a continuous map. $\mathfrak{C}_{\mathcal{D}}$ is a topological category. ☐

Completely analogous to proposition 3.21 we can prove that the classifying space of the flow category is a compact space.

Proposition 5.23. *The classifying space $B\mathfrak{C}_{\mathcal{D}}$ is compact.*

5.3 The Subdivision of the Flow Category

The subdivision category for the flow category can be best viewed via inclusion, cf. 3.3. There is a morphism $\mu \mapsto \nu$ if and only if $\operatorname{im} \nu \subset \operatorname{im} \mu$. The morphism is unique. This is a property for the flow category. In general one can define a topological category for which this property does not hold. We prove the uniqueness of the morphisms in $\operatorname{sd}(\mathfrak{C}_{\mathcal{D}})$ in the next proposition.

Proposition 5.24. *The subdivision* $\operatorname{sd}(\mathfrak{C}_{\mathcal{D}})$ *is a poset.*

Proof. The topological category $\operatorname{sd}(\mathfrak{C}_{\mathcal{D}})$ has much more structure than a poset. A poset in this context means that we can show that for every pair of objects $\mu, \nu \in \operatorname{Ob}(\operatorname{sd}(\mathfrak{C}_{\mathcal{D}}))$ there is at most one morphism $\mu \to \nu$, and that the existence of arrows $\mu \to \nu$ and $\nu \to \mu$ implies the equality $\mu = \nu$.

Let (α, β) be a morphism between μ and ν, then

$$\mu = \beta \circ \nu \circ \alpha. \tag{5.25}$$

This decomposition is unique. $\operatorname{im} \mu \cap E$ are finitely many points x_i. Take $y_i \in]f(x_{i+1}), f(x_i)[$. Then there is a unique orbit through $f^{-1}(y_i) \cap \operatorname{im} \mu$. This can be reparameterized uniquely by 5.10 to curves μ_i. Then $\mu = \mu_n \circ \mu_{n-1} \circ \ldots \circ \mu_0$. This decomposition is unique in the sense that we cannot make a finer decomposition without using identity orbits, i.e. an orbit whose image lies in E. We also have $\nu = \nu_k \circ \ldots \circ \nu_0$, and the same reasoning shows a decomposition $\nu_i = \mu_{m+i}$ for some fixed m. Then $\beta = \mu_n \circ \ldots \circ \mu_{m+k}$ and $\alpha = \mu_{i-1} \circ \ldots \circ \mu_0$. If $k = n$, β is the identity $\beta = \operatorname{id}_{\operatorname{cod}(\nu)}$, and if $m = 0$, α is the identity $\alpha = \operatorname{id}_{\operatorname{dom}(\nu)}$. From this we see that the decomposition of equation (5.25) is unique. This reasoning also shows that there cannot be both $\mu \to \nu$ and $\nu \to \mu$ unless $\mu = \nu$, in this case $m = 0$ and $k = n$, and the morphism $\mu \to \nu$ is the identity. The subdivision is a poset. $\square$

The subdivision is a construction which is defined for any (topological) category. In the proof of the homotopy theorem we will a modified version of this, the tweaked subdivision. This tweaked subdivision is a category which is only defined for the flow category, we use the explicit structure of the morphisms and objects of the flow category.

Definition 5.25. The flow category $\mathfrak{C}_{\mathcal{D}}$ admits the *tweaked subdivision* $\overline{\operatorname{sd}}(\mathfrak{C}_{\mathcal{D}})$. This is a topological category,

- As objects, pairs (γ, x) with $\gamma \in \operatorname{Ob}(\operatorname{sd}(\mathfrak{C}_{\mathcal{D}}))$ and $x \in \operatorname{im} \gamma$ a point on the curve. The space of objects is topologized as a subspace of $\operatorname{Ob}(\operatorname{sd}(\mathfrak{C}_{\mathcal{D}})) \times M$.

- There is a morphism $(\alpha, \beta)_x$ between (γ, x) and (γ', x') if and only if $x = x'$ and there exists a morphism $(\alpha, \beta) : \gamma \to \gamma'$ in $\operatorname{sd}(\mathfrak{C}_{\mathcal{D}})$. The space is topologized as a subspace of $\operatorname{Hom}(\operatorname{sd}(\mathfrak{C}_{\mathcal{D}})) \times M$.

- Let (γ, x) be an object, and $(\alpha, \beta)_x : (\gamma, x) \to (\gamma', x)$ be a morphism in $\overline{\mathrm{sd}}\,(\mathfrak{C}_\mathcal{D})$. The domain, codomain, identity map and composition are the obvious maps defined by

$$\begin{aligned} \mathrm{id}^{\overline{\mathrm{sd}}(\mathfrak{C}_\mathcal{D})}(\gamma, x) &= (\mathrm{dom}^{\mathfrak{C}_\mathcal{D}}\,\gamma, \mathrm{cod}^{\mathfrak{C}_\mathcal{D}}\,\gamma)_x \\ \mathrm{dom}^{\overline{\mathrm{sd}}(\mathfrak{C}_\mathcal{D})}(\alpha, \beta)_x &= (\gamma, x) \\ \mathrm{cod}^{\overline{\mathrm{sd}}(\mathfrak{C}_\mathcal{D})}(\alpha, \beta)_x &= (\gamma', x). \end{aligned} \tag{5.26}$$

- If $(\alpha, \beta)_x$ and $(\alpha', \beta')_{x'}$ are composable, then the composition is defined by

$$(\alpha, \beta)_x \circ^{\overline{\mathrm{sd}}(\mathfrak{C}_\mathcal{D})} (\alpha', \beta')_{x'} = (\alpha' \circ^{\mathfrak{C}_\mathcal{D}} \alpha, \beta \circ^{\mathfrak{C}_\mathcal{D}} \beta')_x. \tag{5.27}$$

  Note that the morphisms are composable only if $x = x'$.

Remark 5.26. Note that the construction of the tweaked subdivision can only be performed for the flow category. It is not a categorical construction, we use the precise structure of the morphisms. This is in contrast to the construction of the subdivision in section 2.6, which is a categorical construction.

The following proofs are completely analogous to the proofs of propositions 3.26 and 3.27.

Proposition 5.27. *The tweaked subdivision* $\overline{\mathrm{sd}}\,(\mathfrak{C}_\mathcal{D})$ *is a topological category.*

5.4 Proof of the Homotopy theorem

The proof of the homotopy theorem is completely analogous to the proof in chapter 3. We have proven the continuity of the maps involved. The construction of the natural transformation is the same. Therefore we have proven 5.8.

Chapter 6

Examples of Dynamical Systems

6.1 Introduction

We show some examples of strongly gradient-like dynamical systems. We show how to compute the classifying space, and see that the spaces are homotopic to the underlying metric space.

6.2 Annulus

Example 6.1. Consider the annulus $A \subset \mathbb{R}^2$, defined by

$$A := \{x \in \mathbb{R}^2 \mid 1 \leq ||x|| \leq 2\}. \tag{6.1}$$

The annulus is best described in polar coordinates. On this annulus we study the flow

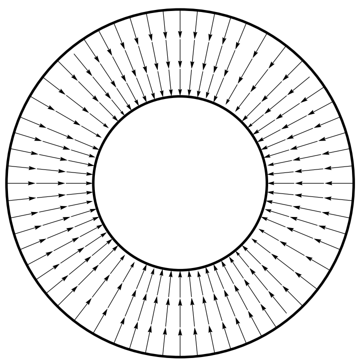

Figure 6.1: The flow φ on the annulus. All points are attracted to the circle $r = 1$ in forward time, and are attracted to $r = 2$ in backwards time.

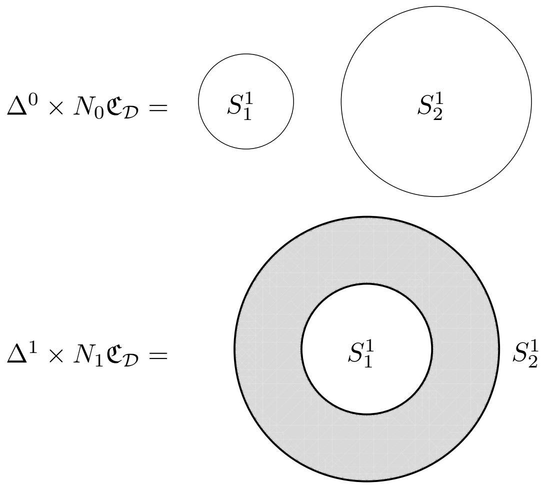

Figure 6.2: The non-degenerate limbs of the flow on the annulus

$\varphi : \mathbb{R} \times A \to A$ defined by the equation

$$\varphi_t(r,\theta) = \left(\frac{2(r-1) + (2-r)e^t}{r - 1 + (2-r)e^t}, \theta \right). \tag{6.2}$$

This flow is the flow corresponding to the differential equation

$$\begin{aligned} \dot{r} &= (r-1)(r-2) \\ \dot{\theta} &= 0, \end{aligned} \tag{6.3}$$

and is depicted in figure 6.1. It is clear that this cannot be described using the methods of chapter 3. The dynamical system $\mathcal{D}$ defined by the flow is strongly gradient-like. The set of equilibria consists of two connecting components. The attracting circle $S_1^1 = \{(r\cos(\theta, r\sin(\theta)) \in A \mid r = 1\}$, and the repelling circle $S_2^1 = \{(r\cos(\theta), r\sin(\theta)) \in A \mid r = 2\}$. A Lyapunov function $f : A \to \mathbb{R}$ is given by

$$f(r,\theta) = r. \tag{6.4}$$

One directly verifies that f satisfies the properties in definition 5.3. Furthermore, all orbits are heteroclinic.

$$\lim_{t\to\infty} \varphi_t(r,\theta) = (1,\theta) \quad \text{and} \quad \lim_{t\to-\infty} \varphi_t(r,\theta) = (2,\theta). \tag{6.5}$$

The dynamical system is strongly gradient-like. We show the limbs of the flow category $\mathfrak{C}_{\mathcal{D}}$ in figure 6.2. After we apply the equivalence relation we see a homeomorphism $B\mathfrak{C}_{\mathcal{D}} \cong A$. The theorem only ensures a homotopy, but apparently we get a homeomorphism.

6.3 Graphs

A directed graph can be seen as a dynamical system. On each vertex we define a flow which flows in the direction of the vertex. If the graph is acyclic, the dynamical system we have defined is strongly gradient-like. The classifying space has the same homotopy type as the graph embedded in Euclidean space.

Example 6.2. Consider the acyclic graph consisting of 5 vertices, and 5 edges, embedded in $\mathbb{R}^2$ as shown in figure 6.3. We add to this graph a dynamical system which respects the arrows on the edges. The coordinates of a vertex p are denoted by p_1 and p_2. Let (x, y) be a point on an edge connecting p to q. Then we have the differential equation

$$\begin{aligned} \dot{x} &= \operatorname{sgn}(p_1 - q_1)(x - p_1)(x - q_1) \\ \dot{y} &= \operatorname{sgn}(p_2 - q_2)(y - p_2)(y - q_2). \end{aligned} \tag{6.6}$$

The flow of this system of differential equations respects the direction of the arrow. An orbit starts at p and ends at q and completely lies in the vertex. For this system we can find a Lyapunov function. The height function, which returns the y coordinate is a Lyapunov function. The dynamical system with this Lyapunov function is a strongly gradient-like dynamical system. We have drawn the limbs in figure 6.4 and the classifying space in 6.5. The classifying space and the graph are homotopic to the circle. Therefore they are homotopic to each other.

This example probably extends to all directed acyclic graphs. We make the claim that we can embed any directed acyclic graph in $\mathbb{R}^3$, where the third coordinate respects the flow of the graph. It is easy to see that the differential equation (6.6) extends to such a dynamical system. Also note that the classifying space we construct here is different from the classifying space of the graph seen as a poset, see example 2.37.

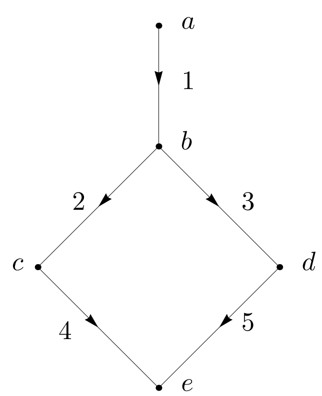

Figure 6.3: An acyclic directed graph embedded in $\mathbb{R}^2$. The function that returns the second coordinate on the edges and vertices is a Lyapunov function for this dynamical system.

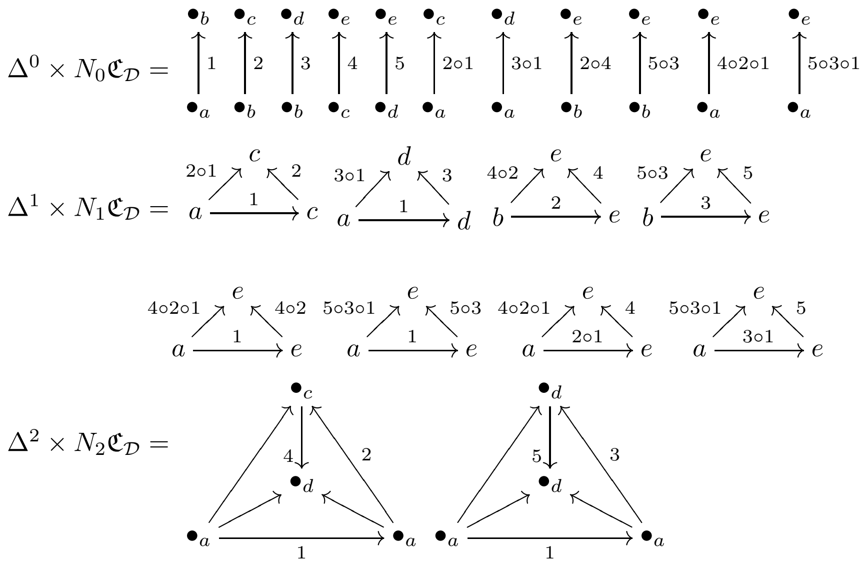

Figure 6.4: The limbs of the dynamical system on the graph. We have chosen not to label all the edges of the 3−simplex. The simplex completely commutes, so one can easily fill in the remaining labels of the edges.

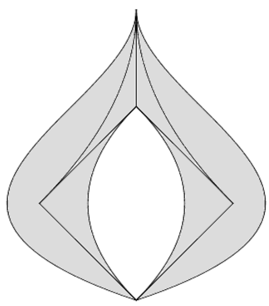

Figure 6.5: The classifying space of the graph in the plane. Note that we cannot embed this into the plane itself, it is a three dimensional space. The outermost lines lie in the dimension perpendicular to the plane. The classifying space is homotopic to the circle, as is the graph itself.

Chapter 7

Isolating Block Decompositions

We now turn our attention towards more general dynamical systems. In contrast to chapter 5 we allow the dynamical system to be exhibit non-gradient-like behavior. We only "forget" the non-gradient-like behavior, and focus on the gradient-like behavior of the system. This is done through Morse decompositions. Morse sets, i.e. elements of the decompositions, play the role of critical points in gradient-like systems. In contrast to the rest of the thesis we merely suggest an approach, proofs are omitted. The goal of this chapter is to attach to a dynamical system with a Morse decomposition a flow category and show that is reasonable to conjecture:

Conjecture 7.1. *Let (S, d) be a compact metric space, and $\mathcal{D}$ a dynamical system defined on it. Let M be a Morse decomposition of this dynamical system. Then there exists an isolating block decomposition N subordinate to M. An isolating block decomposition generates a flow category $\mathfrak{C}_{\mathcal{D}}^{M,N}$. The homotopy type of the classifying spaces of the flow categories are invariants of the space S and*

$$B\mathfrak{C}_{\mathcal{D}}^{M,N} \simeq S. \tag{7.1}$$

The facts on Morse decompositions, Lyapunov functions, and isolating neighborhood decompositions we state in this chapter can be found in an article of Robbin and Salamon [26] and the book on Conley theory of Kalies, Mischaikow, and Vandervorst [15] that will appear.

7.1 Attractors and Repellers

In this chapter we assume that S is a compact metric space, and that φ is the flow of a dynamical system $\mathcal{D}$. Not all orbits of a general dynamical system are heteroclinic. In order to attack the limiting behavior of orbits we use the concept of alpha and omega limit sets.

Definition 7.2. The *alpha* limits set of a point $x \in S$ is the set of all $y \in S$ such that there exists a sequence $t_n \in \mathbb{R}$ tending to minus infinity, i.e. $\lim_{n\to\infty} = -\infty$ such that

$$\lim_{n\to\infty} \varphi_{t_n}(x) = y. \tag{7.2}$$

Analogously, the *omega* limit set of a point $x \in S$ is the set of all $y \in S$ such that there exists a sequence $t_n \in \mathbb{R}$ tending to plus infinity, i.e. $\lim_{n\to\infty} = \infty$ such that

$$\lim_{n\to\infty} \varphi_{t_n}(x) = y. \tag{7.3}$$

Remark 7.3. An orbit is heteroclinic if the alpha and omega limit sets are singletons. In this case the orbit has proper limits as $t \to \pm\infty$, i.e. both limits

$$\lim_{t\to\pm} \varphi_t(x) \tag{7.4}$$

exists and are elements of S.

Fundamental in the study of dynamical systems are attractors and repellers. Recall these notions.

Definition 7.4. A compact set $N \subset S$ is a *trapping region* if it is forward invariant, i.e. $\varphi_t(N) \subset N$ for all positive times $t > 0$, and there exists a $T > 0$ such that $\varphi_T(N) \subset \text{int}(N)$. A *repelling region* is backward invariant, and there exists a $T < 0$ such that $\varphi_T(N) \subset \text{int}(N)$. A set $A \subset S$ is an *attractor* if there exists a trapping region $N \subset S$ with the property that the largest invariant subset in N is A. Similarly, a set $R \subset S$ is a *repeller* if there exists a repelling region $N \subset X$ such that R is the largest invariant subset in N.

The compactness of S enforces that an attractor comes equipped with a dual repeller. On the same footing, a repeller has a dual attractor.

Proposition 7.5. *Let $A \subset S$ be an attractor. The set*

$$A^* = \{x \in S \mid \omega(x) \cap A = \emptyset\}, \tag{7.5}$$

is a repeller, which we will call the dual repeller. *Similarly a repeller $R \subset S$ determines an attractor*

$$R^* = \{x \in S \mid \alpha(x) \cap R = \emptyset\}, \tag{7.6}$$

which we will call the dual attractor.

All orbits outside of the union of an attractor and its dual repeller 'start' at the repeller, and 'end' at the attractor. This behavior is modeled after gradient-like systems. The role of the gradient function is taken by a Lyapunov function. This function is like a height function for an attractor repeller pair. Lyapunov functions always exist.

Theorem 7.6. *Let (A, A^*) be an attractor repeller pair. Then there exists function $f : S \to [0, 1]$ with the properties:*

- *f is continuous.*
- *f is constant on the attractor and its dual repeller,*

$$f(A) = 0 \qquad \textit{and} \qquad f(A^*) = 1. \tag{7.7}$$

- *f strictly decreases along orbits*

$$f(\varphi_t(x)) \leq f(x), \tag{7.8}$$

for all $t > 0$, and all x. Equality holds if and only if $x \in A \cup A^*$. Such a function is called a Lyapunov function.

Remark 7.7. The construction of the Lyapunov function makes excessive use of the metric, and the fact that S is compact. While we have proved 5.8 for a class of dynamical systems, where we do not assume that S is a compact metric space, we do not expect to be able to formulate a flow category for general dynamical systems if S is non-compact. Most systems of interest do take place on metric spaces, and compact ones to boot. The assumptions are not limiting.

Attractors and repellers have the structure of a distributive lattice.

Definition 7.8. A set L equipped with two binary operations $\wedge, \vee : L \times L \to L$, called the *wedge* and *vee* respectively, is a *lattice* if the operations satisfy for all $a, b, c \in L$ the axioms:

- The operations are idempotent, i.e.

$$a \wedge a = a \vee a = a. \tag{7.9}$$

- The operations are commutative, i.e.

$$a \wedge b = b \wedge a \qquad a \vee b = b \vee a. \tag{7.10}$$

- The operations are associative, i.e.

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \qquad a \vee (b \vee c) = (a \vee b) \vee c. \tag{7.11}$$

- The operations are absorbent, i.e.

$$a \wedge (a \vee b) = a \vee (a \wedge b) = a. \tag{7.12}$$

The lattice is *distributive* if the lattice satisfies the additional axiom:

- The operations are distributive, i.e.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{7.13}$$

The lattice is *bounded* if the lattice satisfies the additional axiom:

- There exist *neutral* elements $0, 1 \in L$ such that

$$a \wedge 0 = 0, \quad a \vee 0 = a, \qquad \text{and } a \wedge 1 = a, \quad a \vee 1 = 1. \tag{7.14}$$

Theorem 7.9. *Attractors and repellers form a bounded distributive lattice with respect to set union and intersection. Lyapunov functions form a bounded distributive lattice with respect to pointwise minimum and maximum.*

7.2 Isolating Neighborhoods

The Lyapunov functions allow for the construction of isolating neighborhoods. Orbits outside of the attractors and repellers always pass the isolating neighborhoods at unique points. There exists Lyapunov functions which are constant on the isolating neighborhoods.

Theorem 7.10. *Let A, A^* be an attractor repeller pair. Let $0 < \epsilon < \frac{1}{2}$, and define the sets*

$$N_A^\epsilon = \{x \in S \mid f(x) \leq \epsilon\} \qquad N_{A^*}^\epsilon = \{x \in S \mid f(x) \geq 1 - \epsilon\} \tag{7.15}$$

These sets are compact and

$$\varphi_t(N_A^\epsilon) \subset \operatorname{int} N_A^\epsilon \qquad \varphi_{-t}(N_{A^*}^\epsilon) \subset \operatorname{int} N_{A^*}^\epsilon, \qquad \text{for all } t > 0, \tag{7.16}$$

and are isolating neighborhoods *of A and A^* respectively. There exists a modified Lyapunov function $g : S \to [0, 1]$ which has the property*

- *g is continuous.*
- *g is constant on the isolating neighborhoods.*

$$g(N_A^\epsilon) = 0 \qquad \text{and} \qquad g(N_{A^*}^\epsilon) = 1. \tag{7.17}$$

- *g strictly decreases along orbits*

$$g(\varphi_t(x)) \leq f(x) \tag{7.18}$$

for all $t > 0$, and all x. Equality holds if and only if $x \in N_A^\epsilon \cup N_{A^*}^\epsilon$.

Theorem 7.11. *The isolating neighborhoods form a bounded distributive lattice with respect to set union and intersection. Modified Lyapunov functions form a bounded distributive lattice with respect to pointwise minimum and maximum operations.*

The isolating neighborhoods are well behaved with respect to intersections with orbits.

Proposition 7.12. *Let $x \in S \setminus (N_A^\epsilon \cup N_{A^*}^\epsilon)$. There exists unique smallest time $\infty > \tau_+ > 0$ and biggest $-\infty < \tau_- < 0$ such that*

$$\phi_{\tau_+}(x) \in \partial N_A^\epsilon \qquad \phi_{\tau_-}(x) \in \partial N_{A^*}^\epsilon \tag{7.19}$$

that is, the orbits enter N_A^ϵ and leave $N_{A^*}^\epsilon$ at unique points.

7.3 Morse Decompositions

A tool for the study of the gradient like part of a dynamical system are Morse decompositions.

Definition 7.13. Let P be a finite poset. A finite collection $M = \{M(p) \subset S \mid p \in P\}$ of compact, non-empty, and pairwise disjoint invariant subsets of S, labeled by the poset P is a *Morse decomposition* if, for all $x \in S \setminus \left(\bigcup_{p\in P} M(p)\right)$, there exists $p, q \in P$, with $q < p$ such that

$$\alpha(x) \subset M(p) \qquad \text{and} \qquad \omega(x) \subset M(q). \tag{7.20}$$

Each $M(p)$ is called a *Morse Set*.

Remark 7.14. The Morse sets of a Morse decomposition are generalizations of the critical points of a gradient system. The nomenclature decomposition is somewhat strange. It is not really a decomposition, since a lot of points of S are not contained in the Morse decomposition. Robbin and Salamon [26] therefore speak of attractor networks. The name Morse decomposition has stuck however, and we conform to the literature.

Theorem 7.15. *A Morse decomposition is equivalent to a lattice of attractors, the lattice of Lyapunov functions, the lattice of isolating neighborhoods and the lattice of modified Lyapunov functions.*

Proposition 7.16. *Let $N = \{N(p) \mid p \in P\}$ be an isolating neighborhood decomposition subordinate to the Morse decomposition (M, P). Then the sum of the modified Lyapunov functions for all the isolating neighborhoods is a global Lyapunov function $f : S \to \mathbb{R}$ for the isolating neighborhood decomposition i.e.*

- *f is continuous*
- *f is strictly decreasing on orbits outside of the isolating neighborhood decomposition, i.e. it respects the order P.*
- *f is constant on the isolating neighborhoods.*

We will use the global Lyapunov function in the definition of the flow category in the next section. The isolating blocks are sketched in figure 7.1.

7.4 The Flow category

The objective of this section is to define a flow category. Objects in the category will be points of isolating neighborhoods. Morphisms are piecewise reparameterized shortened orbits, beginning and ending at the boundaries of isolating neighborhoods, and reparameterized using the Lyapunov function.

Proposition 7.17. *Each point $x \in S$ lies on a unique reparameterized shortened orbit γ_x. If $x \in E$ than the orbit is the trivial orbit which stays at x*

Definition 7.18. The flow category $\mathfrak{C}_{\mathcal{D}}^{M,N}$ is the topological category where

- The objects are points in N, topologized as a subspace of S

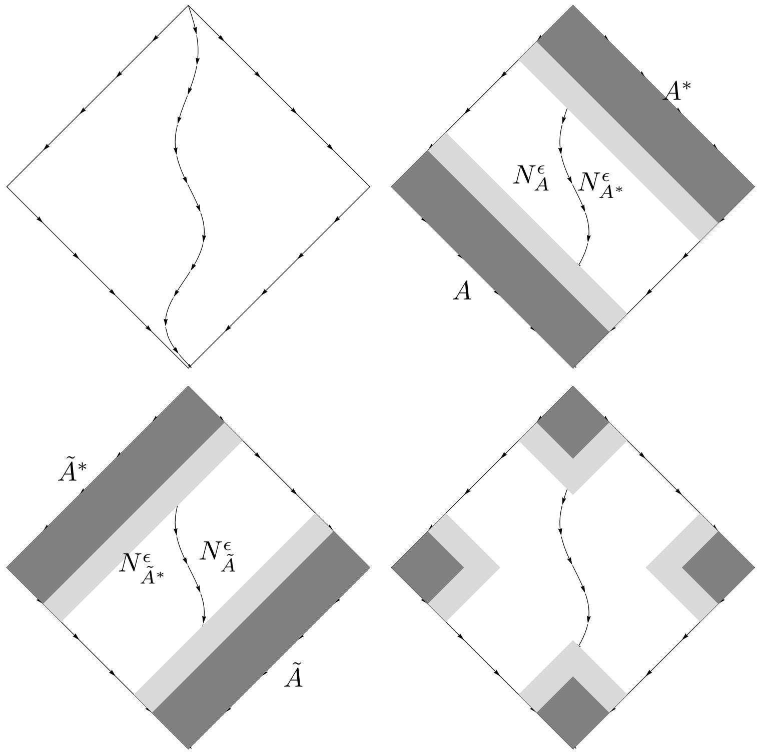

Figure 7.1: In the upper left corner we have drawn an example of a flow. We can identify two attractors, A and $\tilde{A}$ and their dual repellers A^* and $\tilde{A}^*$. We have drawn the isolating neighborhoods in light gray. The intersections of the attractors form a Morse decomposition. The intersections of the isolating neighborhoods form an isolating block decomposition. We have drawn these in the lower right corner.

- The morphisms are piecewise reparameterized shortened orbits, topologized as a subspace of the space of all continuous maps of the intervals to S, with the compact-open topology.
- Composition is concatenation of orbits.
- The maps dom, cod and id are the obvious maps.

We suspect the following.

Conjecture 7.19. *$\mathfrak{C}_{\mathcal{D}}^{M,N}$ is a topological category.*

7.5 The Homotopy Theorem

We expect that we can follow the steps outlined in chapters 3 and 5 to prove the homotopy theorem for the flow category we have just defined. That is, the following conjecture holds.

Conjecture 7.20. *Let (S, d) be a compact metric space, and $\mathcal{D}$ a dynamical system defined on it. Let M be a Morse decomposition of this dynamical system. Then there exists isolating block decompositions N subordinate to M. These isolating block decompositions generate a flow category $\mathfrak{C}_{\mathcal{D}}^{M,N}$. The homotopy types of the classifying spaces of these flow categories are invariants for the space S and*

$$B\mathfrak{C}_{\mathcal{D}}^{M,N} \simeq S. \tag{7.21}$$

Chapter 8

Examples of Isolating Block Decompositions

8.1 Flow on the Annulus

Example 8.1. This example is a variation of example 6.1. We consider the annulus $A \subset \mathbb{R}^2$ once more, and on this annulus we have the system of differential equations

$$\begin{aligned} \dot{r} &= (r-1)(r-2) \\ \dot{\theta} &= 1, \end{aligned} \tag{8.1}$$

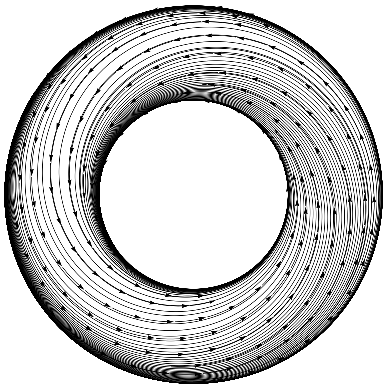

Figure 8.1: The annulus with the flow generated by the differential equations (8.1). The orbits are repelled by the outer circle and attracted to the inner circle.

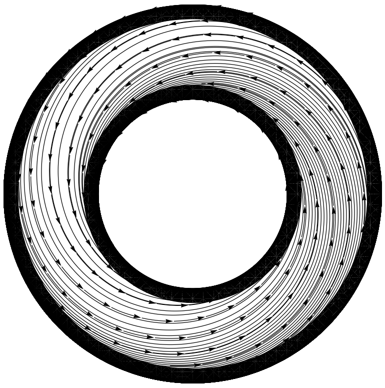

Figure 8.2: The isolating neighborhood decomposition with respect to the Morse decomposition contains two smaller annuli, which contain the morse sets, which are the inner and outer circle.

The system is equivalent to the system in equation (6.3) except that the θ coordinate is not constant, but linear. The flow of this system is given by

$$\varphi_t(r,\theta) = \left(\frac{2(r-1) + (2-r)e^t}{r-1+(2-r)e^t}, t+\theta \right), \tag{8.2}$$

and we have depicted it in figure 8.1. The system is not strongly gradient-like. The orbits do not have well defined limits as $t \to \pm\infty$. They keep approaching the attracting and repelling circles in forward and backward time. The boundary of the annulus are the attracting and repelling circles. These form a Morse decomposition of this dynamical system. The isolating neighborhood decomposition "thickens" the Morse decomposition. We have shown this isolating neighborhood decomposition in figure 8.2. The flow category with respect to this isolating neighborhood is well behaved. We can compute the limbs in figure 8.3. The resulting classifying space is homeomorphic to the original annulus. This is not something we would expect, we would expect only a homotopy. This raises the question if we can formulate a condition similar to the Morse-Smale transversality condition for general dynamical systems. Currently we do not have such a criterion.

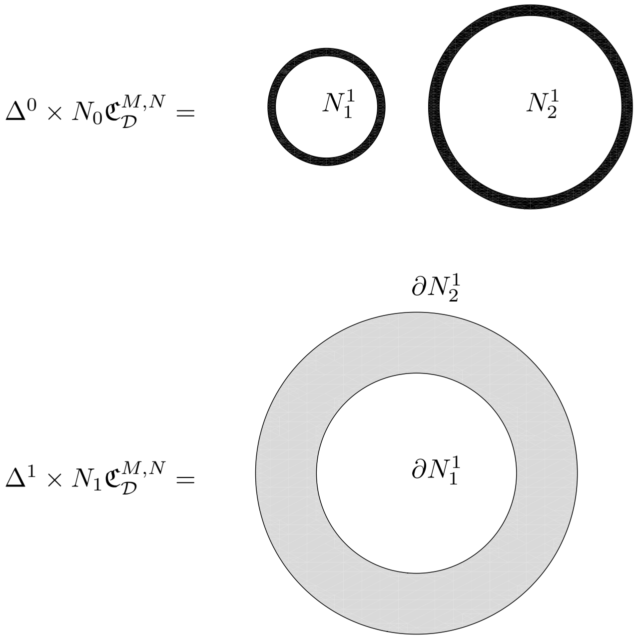

Figure 8.3: The limbs of the dynamical system on the annulus. This is actually similar to the limbs of the non-rotating dynamical system of the annulus, cf. 6.2. The difference is that the 0 limbs are thickened, because these are neighborhoods of the Morse decomposition, and the bounding circles of $\Delta^1 \times N_1\mathfrak{C}_{\mathcal{D}}^{M,N}$ are identified with the outer and inner circles of N_1^1 and N_2^1 respectively.

Chapter 9

Conclusion

In this thesis we have established some results regarding the topology of dynamical systems. We have shown, in various cases, that the homotopy type of a space is reflected in a dynamical system defined on the space. We have shown in different levels of generality, cf. theorems 3.1 and 5.8, that the homotopy type of the classifying space of the flow category is that of the underlying space. This is not the end of the story. We expect that much more information is encoded in the flow categories. We conclude the thesis with some conjectures and some ideas which require further study.

9.1 Conjectures

Conjecture 7.1 is the main conjecture that needs a proof. We do not seriously question the truth of this statement however. This is not the case of some of the conjectures we list below.

We have formulated the flow category for a general dynamical system using isolating block decompositions. We suggest that the choice of the blocks subordinate to the same Morse decomposition does not influence the classifying space of the flow category.

Conjecture 9.1. *The classifying space of the flow category $\mathfrak{C}_{\mathcal{D}}^{M,N}$ does not depend on the construction of the isolating block decomposition. There exists homeomorphisms between them*

$$\mathfrak{C}_{\mathcal{D}}^{M,N} \cong \mathfrak{C}_{\mathcal{D}}^{M,N'}. \tag{9.1}$$

Here N' is a different isolating block decomposition subordinate to M. If this is the case, than we can attach a classifying space to the Morse decomposition itself.

Remark 9.2. Note that we do not claim that we can construct a flow category of a Morse decomposition. We are able to attach a topological space to a Morse decomposition, and this topological space fulfills the role of the classifying space of the (non-existing) flow category.

Morse decompositions are finite, i.e. the poset indexing a Morse decomposition is assumed to be finite. There exists a notion of "infinite" Morse decompositions, chain recurrent sets. We suggest that we can attach a flow category to these chain recurrent sets.

Conjecture 9.3. *We can construct a flow category of a chain recurrent set, and the homotopy type of the classifying space is equivalent to the homotopy type of the underlying metric space.*

The Morse inequalities are a deep result in topology. They show that the topology of the manifold influences functions defined on the manifold. Can we extract similar information out of a flow category of a general dynamical system?

Conjecture 9.4. *Morse decompositions are ordered by a poset. We can look at all flows and equilibrium points below a certain $p \in P$, i.e. there is no non-trivial orbit ending at $M(p)$. These defines a subcategories $\mathfrak{C}^p$ of the flow category. We have injections $\mathfrak{C}^p \to \mathfrak{C}^q$ if $p < q$. Also on the level of classifying spaces the injection $B\mathfrak{C}^p \to B\mathfrak{C}^q$ holds. We can formulate Morse-like inequalities by studying the spaces*

$$B\mathfrak{C}^p / B\mathfrak{C}^q. \tag{9.2}$$

9.2 Ideas

We consider some ideas which we currently cannot formulate in a precise conjecture in this section.

Cohen, Jones and Segal [9] have shown that the Morse-Smale transversality condition is a condition that ensures that the classifying space of the flow category is homeomorphic, and not merely homotopic, to the underlying manifold if we look at a flow category of a gradient vector field. This condition does not naturally generalize to dynamical systems on metric spaces, where we have proved some results on flow categories. It would be good if we could find conditions that ensures that the homotopy is actually a homeomorphism in this case.

We have not studied dynamical systems that are not invertible. A wide class of natural dynamical systems, for example the flow of the heat equation, do not posses invertibility. These systems do obey uniqueness in forward time. This suggests that the information of these flows can be captured in operads. An operad is a generalization of a category, where the morphisms are allowed to have multiple inputs. The role of the nerve, which is a simplicial set (whose geometric realization is the classifying space) is filled by dendroidal sets. There exists a notion of geometric realization of a dendroidal set. It might be possible to capture the information of a non-invertible flow in these constructions, such that we can formulate homotopy results as with invertible dynamical systems.

The assumptions of a strongly gradient-like dynamical system enforces a system to be finite dimensional in the direction of the flow. Can we broaden the definitions to allow certain infinite dimensional systems?

How critical is the assumption of compactness? Can we formulate the theorems dropping this condition?

9.3 Acknowledgements

Writing this thesis has been wonderful. I was fortunate to have not only one, but two marvelous supervisors. Having on average more than one meeting each week was very inspiring. I'd like to thank Rob Vandervorst for the wonderful technical discussions and the ever growing list of ideas and I'd like to thank Heinz Hanßmann, for support and the discussions on how to do research in general, and a meticulous proofreading of my thesis.

Outside of my supervisors other people have helped me with technicalities in my thesis. Dropping by at room 603 each week stirring up some discussion was a lot of fun. Especially Jaap Eldering helped me to understand ODE's on manifolds. Furthermore Jan Jitse Venselaar and Sander Kupers deserve credit for feedback on various parts of my thesis.

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