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Forcing of periodic points for orientation reversing twist maps via braids

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FORCING OF PERIODIC POINTS FOR ORIENTATION REVERSING TWIST MAPS VIA BRAIDS

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ABSTRACT. This paper contains some results about forcing of periodic solutions in two dimensional twist maps. We show that if such a map contains a four periodic point of special type then it is forced to be chaotic. The main idea behind the proof is to connect the evolution of a twist system with s special flow on braid diagrams and then with symbolic dynamics.

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0. INTRODUCTION

In this paper we study twist maps. The main reason for the recent interest in those maps is that they arise in studies of complicated dynamical systems. We can often discover their interesting properties by passing to twist maps. This is usually achieved by considering Poincare sections[1] (if it happens that the first return map has a twist property) or by decomposing an iterated map into twist components. In this sense many physical problems (recently there have also been many application of the dynamical systems in the other fields of sciences like biology or economy) can be investigated with the help of twist maps. We are especially interested in orientation reversing twist maps since intuition suggest that they are much more complicated than orientation preserving ones.

Our main aim is to prove that systems generated by iterating such maps can be very complicated. We will need just one assumption on our map – existence of a period four orbit of a special type. It is astonishing that this implies that such a system is then chaotic (we will call a system chaotic if it has positive topological entropy). To achieve our goal we will use recently developed concepts from the theory of parabolic flows on braid diagrams. We will especially need Conley index theory for such flows.

The first section of this paper contains a definition of twist maps and a proof that the trajectory of a point is fully determined by its first coordinates. Moreover we prove that a sequence of points has to satisfy certain relations to form an orbit of such a map. The second section presents the ideas from and a description of topological entropy and symbolic dynamics. It contains the investigation of a standard example of a chaotic system – the Smale Horseshoe map. We also prove that it is conjugate to the shift on two symbols. The section is concluded with a proof that the shift map has positive topological entropy. The third part of the paper contains the basic theory of braids and the fourth part contains a theory of the flows defined on them by recurrence relations which are also defined there. We have a first look at the Conley index for such flows. We also consider an interesting example of how a special flow on braid diagrams can lead to very complicated behavior. In the fifth section we deal with the problem of how to obtain a parabolic recurrence relation for orientation reversing twist maps. Since the results from section one do not suffice we will need to put some more effort in. Then in the next part of this section we translate our problem in terms of a parabolic flow. Then Conley index theory helps us to find many orbits of the iterated map. We finish with the proof that such a system is semi-conjugate to a shift on three symbols.

1. Twist Maps

In this paper we will use the concept of twist maps in the plane. To present some interesting properties we follow [1] (see also [2]). Let us begin with the definition:

Definition 1.1. Let φ be a smooth map such that

$$\varphi\colon \mathbb{R}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \varphi_1(x,y) \\ \varphi_2(x,y) \end{pmatrix} \in \mathbb{R}^2.$$

We will call φ a *monotone twist map* if $\frac{\partial \varphi_1}{\partial y} > 0$.

Remark 1.2. Classically twist maps are studied on an annulus or on a cylinder. The results from this paper can be applied to the lifts of such maps.

[1] See for example [3] or [8]

Remark 1.3. In this section we will assume that φ is orientation preserving. In later sections we will study orientation reversing twist maps. But the results for orientation preserving maps will prove to be useful.

Remark 1.4. One can also assume that $\frac{\partial \varphi_1}{\partial y} < 0$ (then we say that φ has a negative twist). This equivalent setting leads to the similar results as presented in this section. Note that monotonicity could also be assumed in φ_2 with respect to x.

Remark 1.5. For simplicity we will also assume that φ is invertible. Although this is not essential it simplifies the arguments.

The twist property has then astonishing effect. It can be shown that under this assumption, when iterating φ, we can retrieve the whole trajectory $\{(x_n, y_n)\}$[2] from just the sequence $\{x_n\}$. To show this let us start with the observation that the twist condition implies that there exists an open set U such that for any pair $x, x' \in U$ there exists a unique solution $Y(x, x')$ of the equation:

$$\varphi_1(x, Y(x, x')) = x'. \tag{1.1}$$

Again because $\frac{\partial \varphi_1(x,y)}{\partial y} > 0$ we get that Y is strictly increasing in x'. And by continuity of φ_1 it is continuous.

Now from the function Y we construct another one:

$$\tilde{Y}(x, x') := \varphi_2(x, Y(x, x')).$$

It is easily seen that this function is also continuous. Moreover it is strictly decreasing in x. To show that let us look at the image under the map φ of a line $x = x_0$, for fixed x_0. This image is given by the graph of the function $\varphi_2(x_0, Y(x_0, x'))$. This graph divides the plane into two parts (above the graph and below it). If we choose $x_1 > x_0$ then the image of line $x = x_1$ must be contained in one of these parts. Thus we have

$$\begin{array}{rcll} \varphi_2(x_0, Y(x_0, x')) & > & \varphi_2(x_1, Y(x_1, x')), & \text{or} \\ \varphi_2(x_0, Y(x_0, x')) & < & \varphi_2(x_1, Y(x_1, x')) & \end{array}$$

for all $x' \in \mathbb{R}$. Since φ is orientation preserving the first inequality holds. This proves that $\tilde{Y}$ is strictly decreasing in first variable (i.e. x).

Now we are ready to show that the orbit $\{x_n, y_n\}$ is fully determined by its first coordinates (i.e. x_n). To prove that let us consider a sequence $(x_k, y_k) \in U$ where $k \in \mathbb{Z}$. These points form an orbit of φ if and only if they satisfy the following equalities:

$$\begin{array}{rcl} x_{k+1} & = & \varphi_1(x_k, y_k) \\ y_k & = & \varphi_2(x_{k-1}, y_{k-1}) \end{array}$$

for all integers k. By the definition of Y we know that the first equality is equivalent to $y_k = Y(x_k, x_{k+1})$. By substituting this into the second equation and we obtain a system:

$$\begin{array}{rcl} y_k & = & Y(x_k, x_{k+1}) \\ y_k & = & \tilde{Y}(x_{k-1}, x_k) \end{array}$$

[2] $\binom{x_{n+1}}{y_{n+1}} = \varphi\binom{x_n}{y_n}$

Now we see that sequence (x_k, y_k) is an orbit of φ if and only if the x coordinates satisfy:

$$Y(x_k, x_{k+1}) - \tilde{Y}(x_{k-1}, x_k) = 0 \qquad \text{for all } k \in \mathbb{Z}. \tag{1.2}$$

Moreover the y coordinates are given by $y_k = Y(x_k, x_{k+1})$.

In the next sections we will define the *parabolic recurrence relations.* To put this in context, our main example of recurrence relation will be

$$\mathcal{R}_k(x_{k-1}, x_k, x_{k+1}) := Y(x_k, x_{k+1}) - \tilde{Y}(x_{k-1}, x_k). \tag{1.3}$$

From the properties of Y and $\tilde{Y}$ we get that $\mathcal{R}_k$ is continuous and it is increasing in x_{k-1} and x_{k+1}.

Remark 1.6. Here we defined $\mathcal{R}$ only on some region in $\mathbb{R}^3$. But later we are only interested in maps leading to $\mathcal{R}$ defined on whole $\mathbb{R}^3$. To ensure this we need to assume something more on φ. One of possible assumptions can be that φ satisfies the *infinite twist condition* which has form:

$$\lim_{y \to \pm\infty} \varphi_1(x, y) = \pm\infty. \tag{1.4}$$

This condition guaranties that both Y and $\tilde{Y}$ are defined on $\mathbb{R}^2$. From now on we will assume that φ is such that it produces a relation $\mathcal{R}$ defined on $\mathbb{R}^3$ (for example we can assume (1.4)).

We will finish this section with an example of how the mechanism presented above works in a specific case. Let us consider the well known *Hénon map.* It is a 2-dimensional invertible map given by formula:

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \beta y \\ 1 - \alpha y^2 + x \end{pmatrix}.$$

We observe that $\frac{\partial \beta y}{\partial y} = \beta$, and since we are free to choose the parameters α, β we can assume for simplicity that $\alpha = 1, \beta = 1$. For such a choice of parameters we have that the Hénon map is a twist map that satisfies the infinite twist condition. It is orientation reversing, but let us try to follow the procedure presented above nevertheless. First we need to construct function Y. Since $\beta = 1$ we have that Y is in fact the projection on the second coordinate (i.e. $Y(x, x') = x'$). Then $\tilde{Y}(x, x') = 1 - x'^2 + x$. So the sequence x_k forms a trajectory of F if it satisfies the relations

$$0 = Y(x_k, x_{k+1}) - \tilde{Y}(x_{k-1}, x_k) = x_{k+1} - 1 + {x_k}^2 - x_{k-1}$$

and we can retrieve the y coordinates from the simple equation

$$y_k = x_{k+1},$$

which is not surprising when one looks at the formula of the Hénon map. Since this map is orientation reversing we did not get monotonicity as presented above. We will deal with this problem in later sections.

2. Topological Entropy and Symbolic Dynamics

In this section we present some concepts closely related with the definition of chaos. This section is based mainly on material presented in the classical book [8].

Let X be a metric space and let $\Lambda \in X$ be a compact invariant set for a homeomorphism $F \colon X \to X$. Then for a positive integer n and $\varepsilon > 0$ we define an

(n, ε) *separated set* $S \subset \Lambda$ as a set which has the property: $x, y \in S$ and $x \neq y$ implies that there exists an $0 \leq i < n$ such that $d(F^i(x), F^i(y)) > \varepsilon$ (here $d(.,.)$ stands for the metric on X). By $s(n, \varepsilon)$ we denote the maximal cardinality of an (n, ε) separated subset of Λ. With this notation we give the following definition:

Definition 2.1. Define

$$h(F, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \ln s(n, \varepsilon),$$

and

$$h(F) = \lim_{\varepsilon \to 0} h(F, \varepsilon).$$

Then $h(F)$ is called the *topological entropy of* F.

Let us have a closer look at the definition of the topological entropy. We can fix ε as a criterion of how far the trajectories have to move away from each other in order that we could say that they are different in some sense (for example if we are studying some physical phenomenon we can think about ε as the precision of our measuring apparatus). Then topological entropy is a measure of the growth rate of the number of distinct trajectories in Λ as a function of the length of the trajectory (i.e. n).

We will also need the concept of *Symbolic Dynamics.* Let Σ_k denote the set of all bi-infinite sequences of k symbols. For example if $k = 2$ and the set of symbols is $\{1, 2\}$ then Σ_2 is the set of all sequences of the type $(\ldots, a_{-1}, a_0, a_1, \ldots)$ where $a_i \in \{1, 2\}$. We define a map

$$\sigma \colon \Sigma_k \to \Sigma_k$$

in a following way: $\sigma(a) = b$ where $b_i = a_{i+1}$. We call σ the *shift map* or *Bernoulli shift.* Symbolic dynamics is interesting on its own but it is very important because many chaotic dynamical systems admit it as a subsystem (in a sense that a locally chaotic system is topologically conjugate or more often semi-conjugate to a shift on an appropriate number of symbols). It is often a way of proving that certain systems are chaotic. Now let us equip Σ_k with a metric given by

$$d(a, b) = \sum_{i=-\infty}^{\infty} \delta_i 2^{-|i|},$$

where

$$\delta_i = \begin{cases} 0 & \text{if } a_i = b_i \\ 1 & \text{if } a_i \neq b_i. \end{cases}$$

The most classical application of this concept can be found in studies of the *Smale Horseshoe map.* Many complicated maps admit it as a subsystem which makes it very important. We begin its studies with $S = [0, 1] \times [0, 1]$ in the plane and a mapping $f \colon S \to \mathbb{R}^2$. The action of f can be describe in three steps (see figure 1).

(1) First f_1 maps $(x, y) \in S$ into $(\lambda x, \mu y) \in \mathbb{R}^2$ where $0 < \lambda < 1$ and $\mu > 1$. So the whole S is mapped to a thin but long rectangle R.
(2) Secondly f_2 maps the rectangle R to a horseshoe shaped figure. This can be thought of as a folding of R in the vertical direction.
(3) And finally f_3 projects this figure onto S and truncates all parts which are not fitting in S.

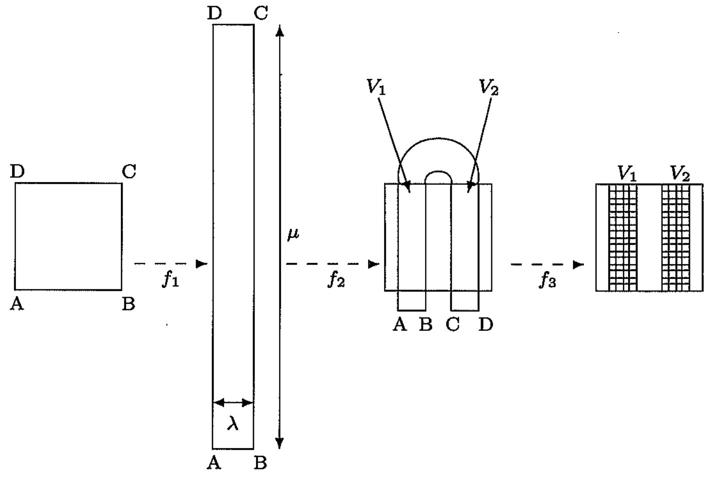

FIGURE 1. The Smale horseshoe map.

Now we define f as a composition $f_3 \circ f_2 \circ f_1$. As we can see from the construction of this function, it takes part of S and maps it into two vertical strips (V_1 and V_2) contained again in S. We can easily see that the next iterations of f will take those two strips into four strips and so on. In general after n iterations of f we will obtain 2^n vertical strips contained in S and each strip will have width equal to λ^n.

We can also see that one is able to define f^{-1} by inverting functions constructed in three steps (i.e. 1. expand and contract S by factors λ^{-1} and μ^{-1}; 2. fold resulting rectangle in a horizontal direction; 3. again truncate the parts which are outside of S;). So after one iteration of f^{-1} we will obtain two horizontal strips (H_1 and H_2) with height equal to μ^{-1}. And again further iterations will increase the number of those strips by a factor 2 and decrease their height by a factor μ^{-1}.

Define $\Lambda := \{x | f^i(x) \in S, i \in S\}$, where $f^0 = id$. We will try to describe the complicated topological structure of Λ, let us start with observing what is happening to V_1, V_2, H_1, H_2. We know that Λ has to be contained in the intersection $(V_1 \cup V_2) \cap (H_1 \cup H_2)$. It is easily seen that this intersection forms four small rectangles with dimensions λ and μ^{-1}. But after a further iteration of f and f^{-1} each strip is mapped again into two strips. For example f^2 have as its range $V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ and similarly for f^{-2} we have four horizontal strips (cf. Figure 2). And again Λ is contained in the intersection of these two sets of four strips which will lead to sixteen small rectangles. In general Λ will be contained in the intersection of 2^n vertical strips (the image of S under f^n) with 2^n horizontal strips (the image of S under f^{-n}). In fact one can show that Λ is the cartesian product of two Cantor sets.

These observations suggest that we can translate the problem into symbolic dynamic. Given a point $x_0 \in \Lambda$ we can follow its trajectory under iterations of f in the following way: we look at $f(x_0)$; if it is contained in V_1 we assign to x_0 the symbol 1 and when it is contained in V_2 we assign the symbol 2. Then we look at $f^2(x_0)$ if it is contained in V_{11} then we assign to x_0 symbol 11 and so on. We can also follow the trajectory of this point backwards in time and assign symbols to it, depending on in which H it is contained. In this manner we obtain a bi-infinite

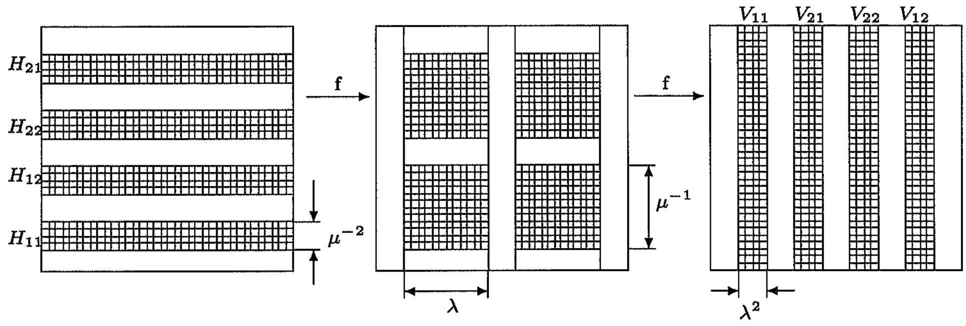

FIGURE 2. The Smale Horseshoe map. Iterations of f acting on H_{ij}.

sequence of symbols 1 and 2. If we look at the sequence $(\ldots, a_{-1}, a_0, a_1, \ldots)$ created in this way, symbol a_i tells us that $f^i(x_0)$ is contained in appropriate V for $i > 0$ or in appropriate H if $i < 0$.

Now one can show that the function ϕ which maps $x_0 \in \Lambda$ to the given sequence $a \in \Sigma_2$ is a one-to-one homeomorphism such that $\phi(f|_\Lambda) = \sigma(\phi)$ (i.e. ϕ is a topological conjugacy for $f|_\Lambda$ and σ). This allows us to study Σ_2 instead of the complicated Λ. It is quite easy to show that Σ_2 has an infinite number of periodic orbits (all sequences with property that $a_i = a_{i+k}$ for some k and for all i), aperiodic orbits and a dense orbit. We can thus state a theorem (for a proof see [8] chapter 5):

Theorem 2.2. *The Smale horseshoe map f has an invariant Cantor set Λ such that:*

(1) *Λ contains a countable set of periodic orbits of arbitrarily long periods.*
(2) *Λ contains an uncountable set of bounded nonperiodic motions.*
(3) *Λ contains a dense orbit.*

At the end of this section as an example we show that the topological entropy of $\sigma : \Sigma_k \to \Sigma_k$ is positive. For the Λ in definition 2.1 we will take Σ_k. Since we only intend to show the inequality let us first show that $s(n, \varepsilon) \geq k^n$. To accomplish this we will create an (n, ε) separated set with a cardinality greater or equal to k^n. Indeed for fixed $\varepsilon < 1$, we can take as (n, ε) separated set a set with sequences that vary on at least one position with index between 0 and $n - 1$. Since on each such a position we have k symbols to choose from we get that our set consist from k^n elements. We get then:

$$h(\sigma, \varepsilon) \geq \limsup_{n\to\infty} \frac{1}{n} \ln(k^n) = \ln(k)$$

and since the right hand side does not depend on ε we get:

$$h(\sigma) \geq \ln(k).$$

We conclude this paragraph with corollary that every dynamical system which contains a horseshoe map or Bernoulli shift as a subsystem[3] has positive topological entropy.

3. Theory of braids

In this section we define and describe the main tool which will allow us to prove that certain twist systems are chaotic. For a more detailed treatment see [6] and for basic definitions [4] or [9].

3.1. Definitions.

We start this section by describing some fundamental concepts.

Definition 3.1. As a *braid β on n strands* we will define a collection of embeddings $\{\beta^\alpha : [0,1] \to \mathbb{R}^3\}_{\alpha=1}^n$ with disjoint images satisfying the following properties

(a) $\beta^\alpha(0) = (0, \alpha, 0)$[4]
(b) $\beta^\alpha(1) = (1, \tau(\alpha), 0)$ where τ is a permutation of n elements.
(c) For each β^α its image is transverse to all planes $\{x = \text{const}\}$.

We can partition the set of braids into classes by saying that two braids are conjugate if we can continuously deform one braid to the other without any intersections among the strands. We call the classes created by dividing the space of braids by this relation *topological braid classes.*

For us it is especially interesting to consider closed braids. They arise from the ones defined above by identifying all points of a form $(0, y, z)$ with point $(1, y, z)$ (of course we still need conditions (a) and (b) from the definition of a braid in such identified points). Now a closed braid is a collection of disjoint embedded loops in $S^1 \times \mathbb{R}^2$ which are every where transverse to the $\mathbb{R}^2$ planes.

It is inconvenient to work in such a large space. Therefore we use a projection on the (x, y)-plane to decrease the dimension of the space in which we are considering our braids. This projection will be called a braid diagram. One can always perturb slightly (without changing the topological braid class) in such a way that projected strands are crossing transversally. We just need to set the convention how to represent two different types of crossings i.e. "top over bottom" and "bottom over top". We use the convention to mark the first ones $(-)$ and latter as $(+)$. From now on we choose all crossings to be of $(+)$ type.

It is also important to look at the algebraic interpretation of braids. We can add a group structure to this space by defining addition as concatenation of braids. Then one can represent each topological braid in terms of σ_i which interchange the i^{th} and $(i+1)^{\text{st}}$ strand. Now we can describe B_n (group of topological braids on n strands) using σ_i as generators and observing two relations among them i.e.

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for} \quad |i - j| > 1$$

(the order in which strands are interchanged is not important unless the interchanges are adjacent)

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \text{for} \quad i < n - 1$$

[3] In the sense that it is locally conjugate or semiconjugate to shift map. The proof that (semi)conjugacy preserves the property of having positive topological entropy can be founded in [5]

[4] One can think that assuming that $z = 0$ is artificial. But in fact since we are dealing with braids on finite number of strands we can always deform them in such way that end points are all on one line.

(since these sequences of interchanges lead to the same result as shown in figure 3).

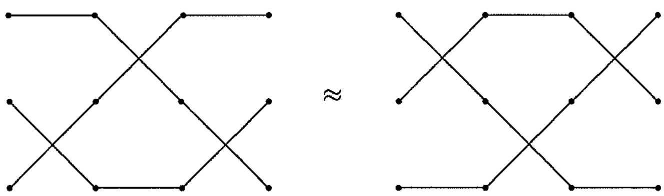

FIGURE 3. As we can see these two braids are equivalent in space of topological braids.

3.2. **Discretized braids.** Since we do not want to study a too complicated space we are working only with braids satisfying the following properties:

(a) They are positive — all crossing are of (+) type (we can thus reconstruct braid from its braid diagram).
(b) They are discretized — their diagrams are piecewise linear with constraints on the position of anchor points (see definition below).

Definition 3.2. The space of *discretized period d braids on n strands,* denoted $\mathcal{D}_d^n$, is the space of all pairs (U, τ) where:

(a) $\tau \in S_n$ is a permutation on n elements,
(b) U is an unordered collection of n strands $U = \{U^\alpha\}_{\alpha=1}^n$.

We require U to satisfy following properties:

(1) Each strand consist of $d+1$ anchor points: $U^\alpha = (u_0^\alpha, u_1^\alpha, \dots, u_d^\alpha) \in \mathbb{R}^{d+1}$.
(2) *periodicity* – For all $\alpha = (1, \dots, n)$, one has:

$$u_d^\alpha = u_0^{\tau(\alpha)}$$

(3) *transversality* – For any pair of distinct strands α and α' such that $u_i^\alpha = u_i^{\alpha'}$ for some i, we have:

$$(u_{i-1}^\alpha - u_{i-1}^{\alpha'})(u_{i+1}^\alpha - u_{i+1}^{\alpha'}) < 0 \tag{3.1}$$

We equip $\mathcal{D}_d^n$ with the standard topology of $\mathbb{R}^{n+1}$ on the strands and the discrete topology with respect to the permutation τ, modulo permutations which change the order of the strands (i.e. two pairs (U, τ) and $(\tilde{U}, \tilde{\tau})$ are close if and only if there exists a permutation $\sigma \in S_n$ such that $U^{\sigma(\alpha)}$ is close to $\tilde{U}^\alpha$ (as points in $\mathbb{R}^{n+1}$) for all α, with $\sigma \circ \tilde{\tau} = \tau \circ \sigma$).

Remark 3.3. In equation (3.1) and further in this paper we use the convention that all indices in coordinates u_i are considered *mod the permutation τ at d*; thus for all $j \in \mathbb{Z}$ we define recursively:

$$u_{d+j}^\alpha = u_j^{\tau(\alpha)}. \tag{3.2}$$

We will also usually denote a discretized braid as $\mathbf{u}$ instead of (U, τ) and $\mathbf{u}^\alpha$ as a strand of $\mathbf{u}$.

Now we see that with a braid $\mathbf{u} \in \mathcal{D}_d^n$ we can associate a braid diagram $\beta(u)$ defined with formula:

$$\beta^{\alpha}(s) := u^{\alpha}_{\lfloor d\cdot s\rfloor} + (d\cdot s - \lfloor d\cdot s\rfloor)(u^{\alpha}_{\lceil d\cdot s\rceil} - u^{\alpha}_{\lfloor d\cdot s\rfloor}), \tag{3.3}$$

for $s \in [0,1]$. Then we can superimpose such graphs for different strands of $\mathbf{u}$ (for an example of a braid diagram see fig.4). To make ideas given in this paragraph more

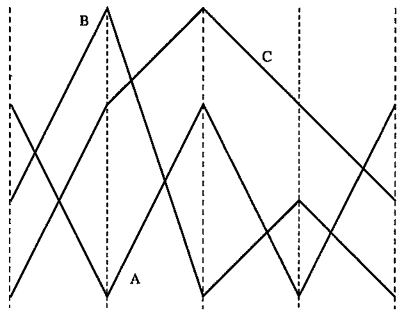

FIGURE 4. A braid diagram $\beta(\mathbf{u})$. Here we have $n = 3$ and $d = 4$. It is important to remember that end points of this diagram are identified.

clear let us consider fig.4 in a little more detail. First we see that its representation in B_n is $\sigma_1\sigma_2$ concatenated with $\sigma_1\sigma_2$, then with σ_2 and so on. We notice also the meaning of the transversality condition (3.1) in the definition of discretized braids: when it fails to hold for certain $\mathbf{u}'$ than it's diagram has a piecewise linear tangency in an anchor point in which this inequality fails to be true. One more thing worth noticing is that if we start on a strand B and follow it, then after four steps ($d = 4$) we switch to strand C and later we will return to strand B again. Starting on A we will stay on it. This of course follows from the fact that permutation τ can be written in terms of cycles as $((1),(2,3))$. This also suggest that we have, in a sense, two connected components of $\mathbf{u}$.

Of course many of the discretized braids are almost the same and since we are interested only on their key properties we give following definition:

Definition 3.4. We will say that two discretized braids $\mathbf{u}, \mathbf{u}' \in \mathcal{D}_d^n$ are of the same *discretized braid class* (denoted $[\mathbf{u}] = [\mathbf{u}']$) if and only if they are in the same path component of $\mathcal{D}_d^n$. The *topological braid class* denoted $\{\mathbf{u}\}$ is a path component of $\beta(\mathbf{u})$ in the space of positive topological braid diagrams.

Remark 3.5. One can proof (see for example [6]) that if $[\mathbf{u}] = [\mathbf{u}']$ than it follows that $\{\mathbf{u}\} = \{\mathbf{u}'\}$, but the converse is not always true.

Since generators of B_n can be seen as elements of $\mathcal{D}_1^n$ we notice that all essential properties of topological braids can be captured within the space of discretized braid diagrams.

3.3. **Singular braids.** As one can easily see $\mathcal{D}_d^n$ is not a vector space. Therefore we complete it with singular braids defined as follows:

Definition 3.6. Let $\bar{\mathcal{D}}_d^n$ denote the nd-dimensional vector space[5] of all discretized braids $\mathbf{u}$ satisfying properties (1) and (2) of definition (3.2), but they do not need to satisfy the transversality condition (i.e. (3.1)). Now $\Sigma_n^d := \bar{\mathcal{D}}_n^d \setminus \mathcal{D}_d^n$ is the set of *singular discretized braids.*

Hence Σ (we usually omit indices) consists of all $\mathbf{u} \in \bar{\mathcal{D}}_d^n$ such that there exist $i \in \{1 \dots d\}$ and $\alpha \neq \alpha'$ so that

$$u_i^\alpha = u_i^{\alpha'} \wedge (u_{i-1}^\alpha - u_{i-1}^{\alpha'})(u_{i+1}^\alpha - u_{i+1}^{\alpha'}) \geq 0$$

The number of distinct i for which the transversality condition fails to be satisfied is called the codimension of a singular braid. Now

$$\Sigma = \bigcup_{m=1}^{\infty} \Sigma[m]$$

where $\Sigma[m]$ denotes the singular braids with codimension m (see figure 5)

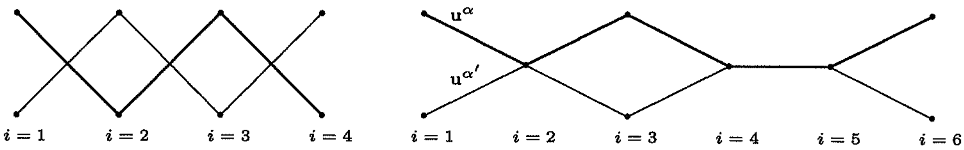

FIGURE 5. The meaning of the transversality condition in terms of braid diagrams. The picture on the left present a simple braid on two strands. The one on the right presents a singular braid on two strands. The transversality condition fails to be satisfied in three points thus this singular braid is in $\Sigma[3]$.

As suggested in the braid in figure 4, any closed (discretized or topological) braid can be partitioned into components by the permutation τ. We follow one strand and look into which others it will change until we go back to the starting one. This we call the first component. Then we select another strand which was not used before. And we continue in such a way. In fact components of braid are determined by cycles of τ. Here we are encountering one of the greatest obstacles in braid theory. As a moment of reflection shows: for a singular braids of sufficiently high codimension ($m \geq d$), different strands can be identical. This can lead to many difficulties, which is why we define *collapsed singularities* as follows:

$$\Sigma^- := \{\mathbf{u} \in \Sigma \mid u_i^\alpha = u_i^{\alpha'}, \forall i \in \mathbb{Z}, \text{ for some } \alpha \neq \alpha'\}.$$

Since in fact we have to reduce the number of strands of such singularity we have that: $\Sigma^- \subset \bigcup_{n'<n} \bar{\mathcal{D}}_d^{n'}$ (for $n = 1 : \Sigma^- = \varnothing$).

3.4. **Relative braids.** Usually when working with dynamical systems we know that some trajectories are fixed and in many cases systems are in some sense bounded. We want to have a possibility to ensure these properties also for systems defined on spaces of braids. That is why we are defining the following concepts.

Define $\mathbf{u} \cup \mathbf{v} \in \bar{\mathcal{D}}_d^{n+m}$ for $\mathbf{u} \in \bar{\mathcal{D}}_d^n$ and $\mathbf{v} \in \bar{\mathcal{D}}_d^m$ as the unordered union of strands. Then for given $\mathbf{v} \in \mathcal{D}_d^m$:

$$\mathcal{D}_d^n \text{ rel } \mathbf{v} := \{\mathbf{u} \in \mathcal{D}_d^n : \mathbf{u} \cup \mathbf{v} \in \mathcal{D}_d^{n+m}\}.$$

[5]In fact $\bar{\mathcal{D}}_d^n$ is not a vector of space but a union of vector spaces.***

It is important to remember that we are imposing transversality condition on $\mathbf{u} \cup \mathbf{v}$.

The path components of $\mathcal{D}_d^n$ rel $\mathbf{v}$ form *relative discrete braid classes* denoted by [$\mathbf{u}$ rel $\mathbf{v}$] (see figure 6 for intuition). Braid $\mathbf{v}$ is usually called the *skeleton*. Now it is easy to define relative versions of the concepts presented above i.e. Σ rel $\mathbf{v}$, Σ^- rel $\mathbf{v}$, $\bar{\mathcal{D}}_d^n$ rel $\mathbf{v}$ (as the sum $(\mathcal{D}_d^n$ rel $\mathbf{v}) \cup (\Sigma$ rel $\mathbf{v})$ or equivalently the closure of $(\mathcal{D}_d^n$ rel $\mathbf{v})$ in $\mathbb{R}^{nd}$) and $\{\mathbf{u}$ rel $\mathbf{v}\}$ (as *topological relative braid class*).

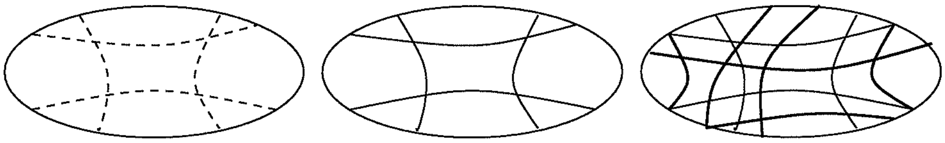

FIGURE 6. If we would think about $\mathcal{D}_d^n$ as a subset of $\mathbb{R}^2$ ([*left*]), then we can think of [*center*] as $\bar{\mathcal{D}}_d^n$ (walls are from Σ_n^d) and of [*right*] as $\bar{\mathcal{D}}_d^n$ rel $\mathbf{v}$ (thicker lines separates classes which arose from fixing the skeleton).

It is now also possible that two classes [$\mathbf{u}$ rel $\mathbf{v}$] and [$\mathbf{u}'$ rel $\mathbf{v}'$] are in fact the same. That is why we consider:

$$\mathcal{D} := \{(\mathbf{u}, \mathbf{v}) \in \mathcal{D}_d^n \times \mathcal{D}_d^m \mid \mathbf{u} \cup \mathbf{v} \in \mathcal{D}_d^{n+m}\}.$$

Path components of $(\mathbf{u}, \mathbf{v})$ in $\mathcal{D}$ will be denoted by $[\mathbf{u} \text{ rel } [\mathbf{v}]]$. So our new partition of the space of relative braid classes is given by the relation:

$$[\mathbf{u} \text{ rel } \mathbf{v}] \sim [\mathbf{u}' \text{ rel } \mathbf{v}'] \Leftrightarrow [\mathbf{u} \text{ rel } [\mathbf{v}]] = [\mathbf{u}' \text{ rel } [\mathbf{v}']] .$$

In the same manner $\{\mathbf{u} \text{ rel } \{\mathbf{v}\}\}$ is the set of equivalent *topological* relative braid classes defined by relation $\{\mathbf{u} \text{ rel } \mathbf{v}\} \sim \{\mathbf{u}' \text{ rel } \mathbf{v}'\}$ if and only if there exist a continuous family of topological (positive, closed) braid diagram pairs deforming $(\mathbf{u}, \mathbf{v})$ to $(\mathbf{u}', \mathbf{v}')$

4. Parabolic flows on braid diagrams

4.1. **Recurrence relations.** As was mentioned before we will consider dynamical systems in the space of relative braid diagram classes. For this we need to adopt the previously known concept of recurrence relation. Let us start with the definition:

Definition 4.1. A sequence of real-valued $\mathcal{C}^1$ functions $\mathcal{R} = (\mathcal{R}_i)_{i \in \mathbb{Z}}$ from $\mathbb{R}^3$ satisfying following properties:

(A1) *monotonicity* – $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_1 \geq 0$ for all $i \in \mathbb{Z}$,
(A2) *periodicity* – For some $d \in \mathbb{N}$ we have: $\mathcal{R}_{i+d} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$,

is called a *parabolic recurrence relation* $\mathcal{R}$ on $\mathbf{X}$ (where $\mathbf{X}$ denotes $\mathbb{R}^{\mathbb{Z}}$).

4.2. **Induced flow.** Now we need to define a flow using the recurrence relation $\mathcal{R}$. The way to do it is to write the differential equation:

$$\frac{d}{dt} u_i = \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) \quad \text{where } \mathbf{u}(t) \in \mathbf{X} \text{ and } t \in \mathbb{R}.$$

It is known that such an equation defines a (local) $\mathcal{C}^1$ flow ψ^t on $\mathbf{X}$ under periodic boundary conditions provided they are of period nd. We call it a *parabolic flow* on

X. As the reader could already notice, it is easy to "move" this flow onto the space $\bar{\mathcal{D}}^n_d$, by considering the equation:

$$\frac{d}{dt}u_i^\alpha = \mathcal{R}_i(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha) \quad \text{where } \mathbf{u} \in \bar{\mathcal{D}}^n_d. \tag{4.1}$$

Because $\mathbf{u}$ is of period d and so is $\mathcal{R}$, it follows that the flow given by this equation is well defined. We also see that the change of position of the i^{th} anchor point under ψ^t depends on its current position and on the positions of the two nearest neighbors. The flow defined by equation (4.1) is called *parabolic flow on discretized braids.* As it was suggested before we will be especially interested on flows for braids from $\bar{\mathcal{D}}^n_d$ rel $\mathbf{v}$ for some fixed skeleton $\mathbf{v}$. Then we restrict our attention only to those flows on $\bar{\mathcal{D}}^{n+m}_d$ which are fixing all anchor points of the skeleton. We will denote it by writing $\psi^t(\mathbf{v}) = \mathbf{v}$. As we remember from the description of the group B_n, we do not have a unique representation of $\mathbf{u} \in \mathcal{D}^n_d$, but the word length of such representation is unique. The number of generators σ_i required to represent a braid is in fact a word metric from geometric group theory. By the way, the geometric meaning of this value is quite obvious, it is the number of pairwise crossings in braid diagram $\beta(\mathbf{u})$, corresponding to a braid $\mathbf{u}$ for which we want to calculate the word metric. Moreover one is able to show ([6]) that this metric acts as discrete Lyapunov function for any parabolic flow on $\mathcal{D}^n_d$, in fact one can show that

Theorem 4.2. *Let ψ^t be parabolic flow on $\bar{\mathcal{D}}^n_d$.*

(a) *For each point $\mathbf{u} \in \Sigma - \Sigma^-$, the local orbit $\{\psi^t(\mathbf{u}) : t \in [-\varepsilon, \varepsilon]\}$ intersects Σ uniquely at $\mathbf{u}$ for all ε sufficiently small.*
(b) *For any such $\mathbf{u}$, the length of braid diagram $\psi^t(\mathbf{u})$ for $t > 0$ in the word metric is strictly less then that of the diagram $\psi^t(\mathbf{u})$ for $t < 0$.*

As we see from this theorem every parabolic flow evolve in such way that when we are moving from one braid class to another crossing through Σ (but not Σ^-!) we only decrease the complexity of the braid class (in the sense that we have less crossings). So typically such events can be imagined as shown in figure 7.

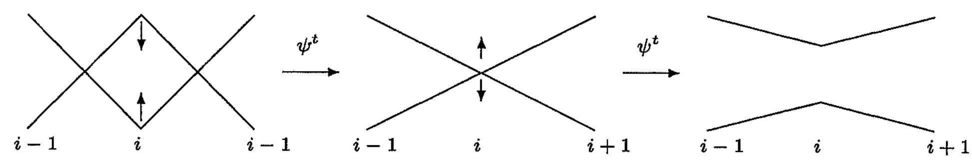

FIGURE 7. The evolution of a parabolic flow on a simple braid diagram. Two local crossings [*left*]. Singular braid from $\Sigma \setminus \Sigma^-$ [*center*]. No local crossing [*right*]. The word metric decreased by 2.

4.3. **Proper and bounded braid classes.** Theorem (4.2) shows also that we are "in danger" when our system evolves near to Σ^-. Because of this we restrict our interest only to proper braids of which the definition is given below.

Definition 4.3. We call a topological braid class $\{\mathbf{u} \text{ rel } \mathbf{v}\}$ *proper* if it is impossible to find a continuous path of braid diagrams $\mathbf{u}(t)$ rel $\mathbf{v}$ for $t \in [0, 1]$ such that:

(1) $\mathbf{u}(0) = \mathbf{u}$

(2) For all $t \in [0,1)$: $\mathbf{u}(t)$ rel $\mathbf{v}$ defines a braid (non-singular!)
(3) $\mathbf{u}(1)$ rel $\mathbf{v}$ is a diagram where an entire component of the closed braid has collapsed onto itself or onto other component of $\mathbf{u}$ or $\mathbf{v}$.

A discretized relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is called proper if the associated topological braid class is proper, otherwise it is *improper.*

For the examples of proper and improper braids look at figure 8. It is important to remember that the fact whether one braid is proper or not highly depends on which skeleton we are fixing. So the same braid can be improper for a certain skeleton and be proper for another one. Also as was mentioned before we are

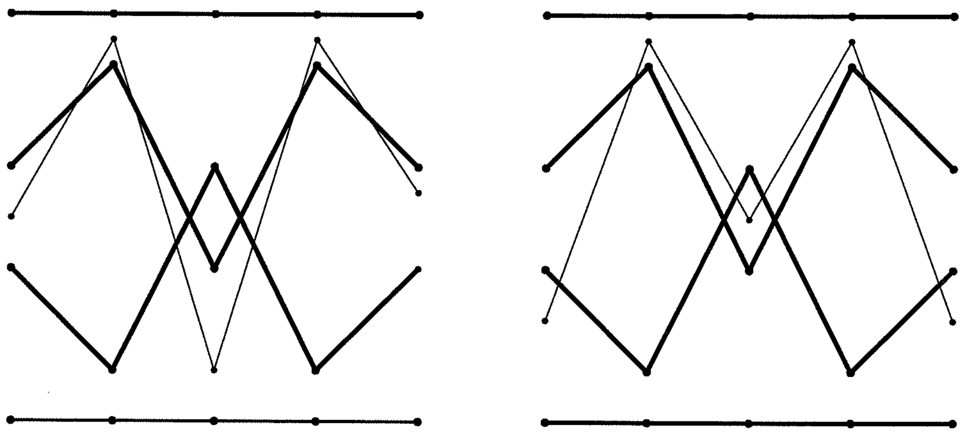

FIGURE 8. Two different braids on the same skeleton (free strand is marked by thinner line). The one on the left is improper (we can deform it to one of the skeleton's strands), the one on the right is proper.

considering mainly bounded systems, that motivates defining:

Definition 4.4. A topological relative braid class $\{\mathbf{u} \text{ rel } \mathbf{v}\}$ is called bounded if there exist a uniform bound on all representatives $\mathbf{u}$ of the equivalence class, i.e. on the strands $\beta(\mathbf{u})$ (in $C^0([0,1])$). A discrete relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is called bounded if the set $[\mathbf{u} \text{ rel } \mathbf{v}]$ is bounded.

As the reader may easily observe these concepts are well defined for properties of equivalence classes of braids.

4.4. **Conley index for braids .** This very powerful tool of showing how complicated certain parts of flows are, will be much used in this paper. For now we will just apply part of terminology from its theory which may help more advanced readers to observe general ideas hidden by our studies. The Conley index for braid classes is defined in [6] (for more detail in a general setting see [10]) where the reader can also find theorems showing that it is well defined, as well as it's fundamental properties.

In general the Conley index is defined for isolating neighborhoods. Let X be a locally compact metric space. A compact set $N \subset X$ is an *isolating neighborhood* for a flow ψ^t on X if the maximal invariant set

$$\mathrm{INV}(N) := \{x \in N | \mathrm{cl}\{\psi^t\}_{t \in \mathbb{R}} \subset N\}$$

is contained in the interior of N. The set $\mathrm{INV}(N)$ is then called *compact isolated invariant set* for ψ^t. One can show that such set admits pair (N, N^-) satisfying following properties:

(i) $\mathrm{INV}(N) = \mathrm{INV}(\mathrm{cl}(N - N^-))$ where $N - N^-$ forms a neighborhood of $\mathrm{INV}(N)$;
(ii) N^- is positively invariant in N;
(iii) N^- is an *exit set* for N (i.e. given $x \in N$ and $t_1 > 0$ such that $\psi^{t_1}(x) \notin N$, then there exist a $t_0 \in [0, t_1]$ for which $\{\psi^t(x) \ : \ t \in [0, t_0]\} \subset N$ and $\psi^{t_0}(x) \in N^-$.)

We call such a pair an *index pair* for $\mathrm{INV}(N)$. Then we define the Conley index as the homotopy type of the pointed space $(N/N^-, [N^-])$, usually denoted by $[N/N^-]$. Since generally it is difficult to compute the homotopy type it is usually convenient to use singular homologies instead.

In fact with a little effort one can prove that we can use a different definition of this concept in our special case. For a proper relative bounded class $[\mathbf{u} \text{ rel } \mathbf{v}]$ we define its Conley index in a following way: First denote by N the set $\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}]$ within $\bar{\mathcal{D}}^n_d$. Since we are only taking into account proper bounded braids we know that N is compact and $\partial N \cap \Sigma^- = \emptyset$. Then consider any point $\mathbf{w}$ on $\partial N \subset \Sigma$. There exists a small neighborhood W of $\mathbf{w}$ in $\bar{\mathcal{D}}^n_d$ for which the subset $W - \Sigma$ consist of a finite number of connected components $\{W_j\}$. Assume that $W_0 := W \cap N$, then:

$$N^- := \mathrm{cl}\{\mathbf{w} \in \partial N : |W_0|_{\text{word}} \geq |W_j|_{\text{word}} \text{ for all } j > 0\}$$

i.e. N^- consists of such $\mathbf{w}$ for which word metric is locally maximal on W_0. One can show then that such a pair (N, N^-) is an index pair for any parabolic flow fixing the skeleton $\mathbf{v}$ and that $[N/N^-]$ gives the Conley index.

4.5. **Example** . Let us try to calculate the homotopy index for a simple period two braid shown in figure 9. Let us start with the braid shown on the left part of

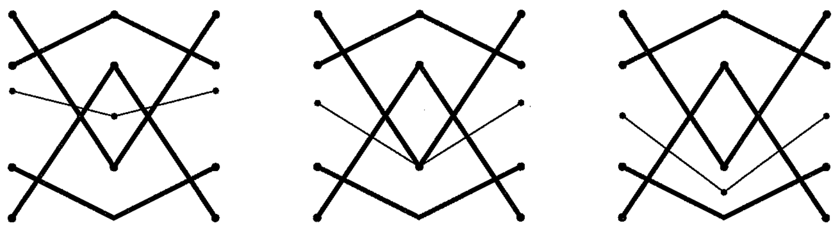

FIGURE 9. Evolution a of braid flow. As usual thicker lines denote the skeleton and thinner lines the free strand.

the picture. It is of period two and it is proper (but observe that if one deletes any of skeleton's strands then we get an improper braid). We can move the free strand by changing one of the two anchor points, let us call first one u_0 and second one u_1 (it is important to remember that this braid is closed so the end points are identified). We know that such a braid can evolve only in certain ways. The anchor point u_0 cannot move up or down to cross the anchor points of skeleton since this would led to an increased number of crossing. So u_0 is "trapped" between anchor points of the skeleton. The middle point of the braid is allowed to move up and down and moreover it can cross the nearest anchor points since this decreases the number of crossings (middle part of the picture). Of course then our braid is sent to an improper class (right part of figure). Let's now try to calculate the Conley index of such braid. We can think of braid class as a product of two intervals (since moving anchor points between the skeleton does not change the braid class). So $N = I \times I$, now we need to find exit set. By definition N^- is contained in ∂N (in this case the boundary of N is a square). But as mentioned above u_0 cannot evolve out of interval bounded by anchor points of skeleton. In contrast u_1 is allowed to do that. So the exit set in fact consists of two singular braid classes (one of which is shown in the middle part of the picture and the other has the middle point on the upper anchor point). Hence the exit set is formed from those points for which u_1 is on boundary. Thus we see that the homotopy index of $[N/N^-] \simeq S^1$ (the end points of the second interval are identified).

4.6. **Folding of recurrence relation and braid diagrams.** This basic idea behind theorems forcing existence of periodic points follows from the observation that we are not only interested in periodic points but even more in trajectories evolving close to them. And since in general these are not periodic, we have to look at periodic orbits not just until they return to starting points but further as well.

As one can easily see from definition 4.1, every recurrence relation of period d is also of period kd for all integers $k > 0$. In the same way we can fold a given braid diagram of period d, instead of "drawing" it only once we can draw several segments, one after another. For an example of folded braid diagram consider figure 10.

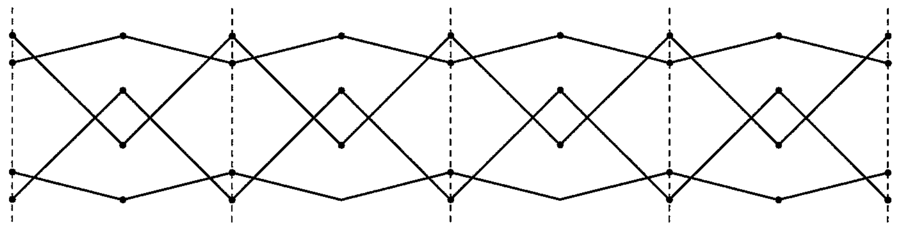

FIGURE 10. Folding of a braid. This is a 4-fold cover of the skeleton braid diagram presented in the previous example.

To make our study a bit more concrete let us consider parabolic recurrence relation fixing the skeleton from example 4.5. As presented above we have a non-trivial Conley index for the free strand having its middle point in the inner loop. Now consider a $2n$-fold cover of this recurrence relation for some natural number $n \geq 1$ (it fixes a $2n$-folded skeleton as presented on pictures). Then we are interested in all possible proper braids having the property that every even i anchor point of the free strand is between the skeleton's anchor points and every odd one is placed in any of three positions: above inner loop, below it or in it (as suggested in figure 11). Since we are dealing with topological braid classes, in fact it is not important what exact positions of the anchor points are we chosen (as long as we do not change the intervals bounded by the skeleton's anchor points). In this sense we have only one position for every even i and three position for every odd one. The reader can observe a similarity with the horseshoe map and the way we were assigning symbols to points from its domain. We can use this scheme here as well. Given a free strand

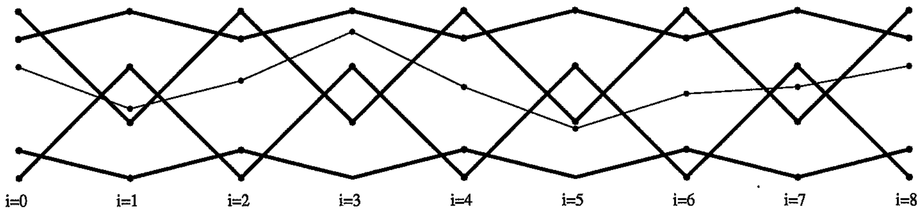

FIGURE 11. One of the possible choices of the free strand for the folded braid.

of braid diagram class we look at its odd anchor points starting form $i = 1$. If it is above inner the loop we assign symbol $+1$ to this braid, if it is in the middle then symbol 0 and if it is below -1. Then we move to $i = 3$ and we add to the previously chosen symbol a new one in the same way. Lets try to follow this procedure for the braid presented on the picture 11. For $i = 1$ we get symbol 0 since anchor point is in the inner loop, for $i = 3$ we get $+1$, and further -1 and again 0. So in total we get sequence $\{0, +1, -1, 0\}$. Lets assume now that the end points of this braid are positioned in the same place (with respect to the vertical coordinate), then our braid be can thought of as closed of period $2n$. And for every symbol sequence of length n consisting of tree symbols there exists a braid for which we can assign it. Well, as the attentive reader could discover, this is not fully true. Two of this sequences are forming improper braids i.e.

$$\underbrace{\{+1, \ldots, +1\}}_{\text{n times}} \text{ and } \underbrace{\{-1, \ldots, -1\}}_{\text{n times}}.$$

So for this braid we have $3^n - 2$ possible different sequences which corresponds to proper braid classes.

Again this gives us the chance to move the problem of how complicated the given system is into symbolic dynamics. Into space Σ_3 - all sequences of three symbols. But for now it is not obvious how to do it since our flow does not act in the same way as the shift map on the space of sequences. In fact we are not really interested in flows on braid diagrams but more on the iterates of twist map. Then it will be possible to move our studies into symbolic dynamics.

5. PROBLEM OF PERIODIC POINTS FOR TWIST MAPS

Finally we are ready to connect the ideas given throughout the paper into one theory. The concept of a twist maps with parabolic recurrence relations, then with flows on them and symbolic dynamics.

5.1. **Parabolic recurrence relation for twist maps.** From the section about the twist maps we know that the whole trajectory of a point can be retrieved from the sequence x_n which has to satisfy:

$$\mathcal{R}_k(x_{k-1}, x_k, x_{k+1}) = 0,$$

but since

$$\mathcal{R}_k(x_{k-1}, x_k, x_{k+1}) = Y(x_k, x_{k+1}) - \tilde{Y}(x_{k-1}, x_k)$$

the functions on the right hand side do not depend on k, hence

$$\mathcal{R}_k = \mathcal{R} \qquad \text{for all integers } k.$$

This means that sequence $(\mathcal{R}_k)_{k\in\mathbb{Z}}$ satisfies the periodicity condition (with $d = 1$) from definition (4.1). And from the paragraph about twist maps we have that this sequence is monotone in the sense of definition 4.1. So we have that $(\mathcal{R}_k)_{k\in\mathbb{Z}}$ forms a parabolic recurrence relation.

Remark 5.1. As was mentioned in the first section (remark 1.6) we don't have to assume that the twist map satisfies the infinite twist condition but since we want to work with maps defining recurrence relation on $\mathbb{R}^3$ we need to assume some condition guaranteeing it. The infinite twist condition is an example of such a condition.

5.2. **Parabolic recurrence relation for orientation reversing twist map.** Now we will assume that:

$$T\colon \mathbb{R}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \in \mathbb{R}^2$$

is a orientation reversing, invertible twist map (i.e. $\frac{\partial f_1}{\partial y} > 0$) satisfying the infinite twist condition (or other condition guaranteeing that the relation that we will get is defined on $\mathbb{R}^3$).

From the section 1 we know that for such map we can get a relation $\mathcal{R}$ given by

$$\mathcal{R}_T(x_{k-1}, x_k, x_{k+1}) = Y(x_k, x_{k+1}) - \tilde{Y}(x_{k-1}, x_k)$$

but because it is an orientation reversing map we we will not get the monotonicity condition from the definition 4.1. In fact we have that $\partial_1\mathcal{R}_T < 0$ and $\partial_3\mathcal{R}_T > 0$. Since the theory of braid flows is defined using the parabolic recurrence relation we need to modify the one we have just obtained. We do it defining

$$\begin{aligned}
\mathcal{R}_{\pm_0}(u_3, u_0, u_1) &:= \mathcal{R}_T(-u_3, -u_0, u_1) \\
\mathcal{R}_{\pm_1}(u_0, u_1, u_2) &:= \mathcal{R}_T(-u_0, u_1, u_2) \\
\mathcal{R}_{\pm_2}(u_1, u_2, u_3) &:= -\mathcal{R}_T(u_1, u_2, -u_3) \\
\mathcal{R}_{\pm_3}(u_2, u_3, u_0) &:= -\mathcal{R}_T(u_2, -u_3, -u_0).
\end{aligned}$$

Now appropriate derivatives are positive and we have a parabolic recurrence relation of period four. But we are left with the question: what effect does this change have on a given trajectory $\{x_n\}$. As one can see it is:

$$\{...x_0, x_1, x_2, x_3, x_4, x_5, x_6...\} \to \{... - x_0, x_1, x_2, -x_3, -x_4, x_5, x_6...\}.$$

This transformation is of course invertible and it is, in a sense, of period four. Every sequence satisfying $\mathcal{R}_\pm$ satisfies also $\mathcal{R}_T$ (we need to reverse transformation). To distinguish between trajectories of T and the ones after modification we will use the symbol u to denote the latter. This shows that the following lemma is true:

Lemma 5.2. *Every solution of $\mathcal{R}_\pm$ leads to a solution of $\mathcal{R}_T$ so in fact to a trajectory of T.*

Remark 5.3. It is worth to notice that transformation defined above preserves orbits of period four.

5.3. **Braid diagrams for T.** As one can suspect we will try to express such solution in terms of braid diagrams.

We need to find a way to construct a braid diagram corresponding to a given period four solution. We do it in the following way. We take an initial point and we mark it in $\mathbb{R}^2 \times \{0\}$ then next point in section $\mathbb{R}^2 \times \{1\}$ and so on until $\mathbb{R}^2 \times \{4\}$ where we are marking the initial point again (the orbit is of period four). In fact we just created a discretized braid. Now as suggested in the section about braids an appropriate projection creates a braid diagram. We connect points according to the trajectory of T. Since as the initial point we can choose each of the four points, we create in this way a braid diagram on four strands (each one is in fact a shifted copy cf. figure 12). To such a diagram we are adding a permutation type of a trajectory. We do it in following way – first label points with symbols 1,2,3 and 4 (highest point in projection gets 1, second 2 and so on). Then we add a permutation to such a diagram by following the strand starting in 1. We move along this strand and we look in what order it is "visiting" points. In this way we get one of the following six permutations:

$$(1,2,4,3) \quad (1,2,3,4) \quad (1,3,2,4) \quad (1,2,4,2) \quad (1,4,2,3) \quad (1,4,3,2).$$

For an example look at the figure 12.

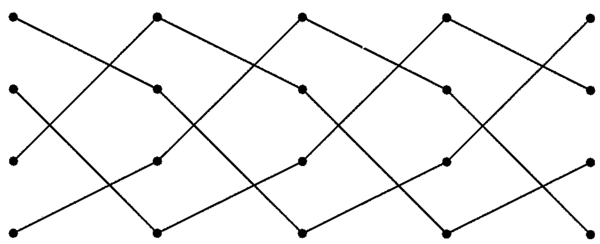

FIGURE 12. Trajectory with a braid diagram with a permutation type $(1,2,4,3)$.

For other examples look at the left part of picture 14. Now we know how to obtain a braid diagram for a given (period four) trajectory of T. But we do not have a parabolic recurrence relation for this function. We have one only for the modified sequences. How to express this modification in terms of braid diagrams? It can be easily accomplished when one observes that the changing of a sign on a trajectory reverses the order of the points on them, 1 becomes 4, 2 becomes 3 and so on. So for example if we want to modify sequence on 0^{th} position we just exchange anchors in a manner presented above. It is important to notice that we are changing the sequence (so the braid as well) only on positions 0 or 3 modulo 4(cf. formulas for $\mathcal{R}_{\pm}$). This leads to a modification of the braid from picture 12 to the one presented on figure 13. We will call this modification of the braid diagram and trajectory a flip. In this way we have also obtained picture 14 for other types of permutations. This figure shows that a period four solution corresponding to permutation $(1,2,4,3)$ or $(1,3,4,2)$ leads to the braid diagram from example 4.5 (one is shifted but as one may suspect this does not change much).

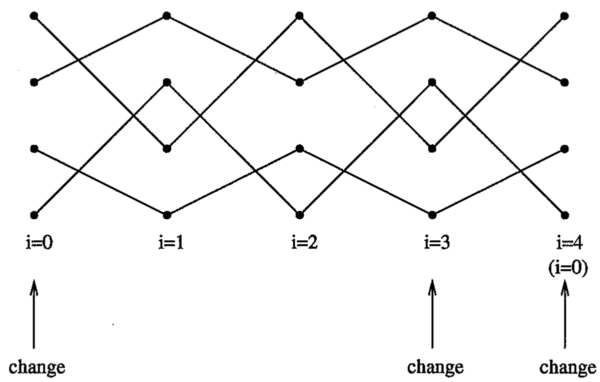

FIGURE 13. Modification of braid diagram for the trajectory corresponding to permutation type $(1,2,4,3)$.

5.4. **Braid flow for $\mathcal{R}_{\pm}$.** Like in section 4 we define a parabolic flow on braid diagrams using $\mathcal{R}_{\pm}$. It is given by the equation:

$$\frac{d}{dt}u_i^\alpha = \mathcal{R}_{\pm i}(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha) \tag{5.1}$$

As mentioned above, example 4.5 suggests that a skeleton like the one for the first two permutation types could lead to very complicated behavior of this flow. That is why we assume from now on that map T has a period four point with corresponding permutation type $(1,2,4,3)$[6]. Now from previous sections we get that such a trajectory leads to a solution of $\mathcal{R}_{\pm}$ corresponding to the braid diagram from picture 13 (we will call it type (I) and the one for $(1,3,4,2)$ – type (II)). Since this is a solution we have that $\mathcal{R}_{\pm} = 0$ on this sequence. This proves that the parabolic flow defined by (5.1) fixes braid (I). So we can consider our flow on the space $\bar{\mathcal{D}}_d^n$ rel $\mathbf{v}$, where $\mathbf{v}$ denotes braid diagram (I).

Now the reader may be wondering what is the connection between the evolution of the parabolic flow on $\bar{\mathcal{D}}_d^n$ rel $\mathbf{v}$ and trajectories of T. It is not obvious at first sight that there is any. To show it we present the following lemma.

Lemma 5.4. *Every connected component of a fixed point* $\mathbf{u}$ *rel* $\mathbf{v}$ *for the parabolic flow* ψ^t *defined by* $\mathcal{R}$*, is solution of this relation.*

Proof: Take $\mathbf{u}_i^\alpha$ a connected component of $\mathbf{u}$ rel $\mathbf{v}$ (remember that $u_{i+d}^\alpha = u_i^{\tau(\alpha)}$). Since by assumption $\mathbf{u}$ rel $\mathbf{v}$ is a fixed point for the flow we get:

$$\frac{d}{dt}\mathbf{u}_i^\alpha = 0.$$

Then also $\mathcal{R}(u_{i-1}, u_i, u_{i+1}) = 0$ by the definition of the induced flow. So $\mathbf{u}_i^\alpha \in \mathbf{X}$ is a solution of $\mathcal{R}$. Now we can follow the same argument for every component of $\{\mathbf{u}$ rel $\mathbf{v}\}$. This concludes the proof. ■

Then we get as an immediate consequence:

Corollary 5.5. *Every connected component of a fixed point of* ψ^t *defined by (5.1) is a solution of* $\mathcal{R}_{\pm}$*. Therefore it leads to a trajectory of* T .

[6]Reasoning for the permutation type $(1,3,4,2)$ is analogous (cf. remark 6.8).

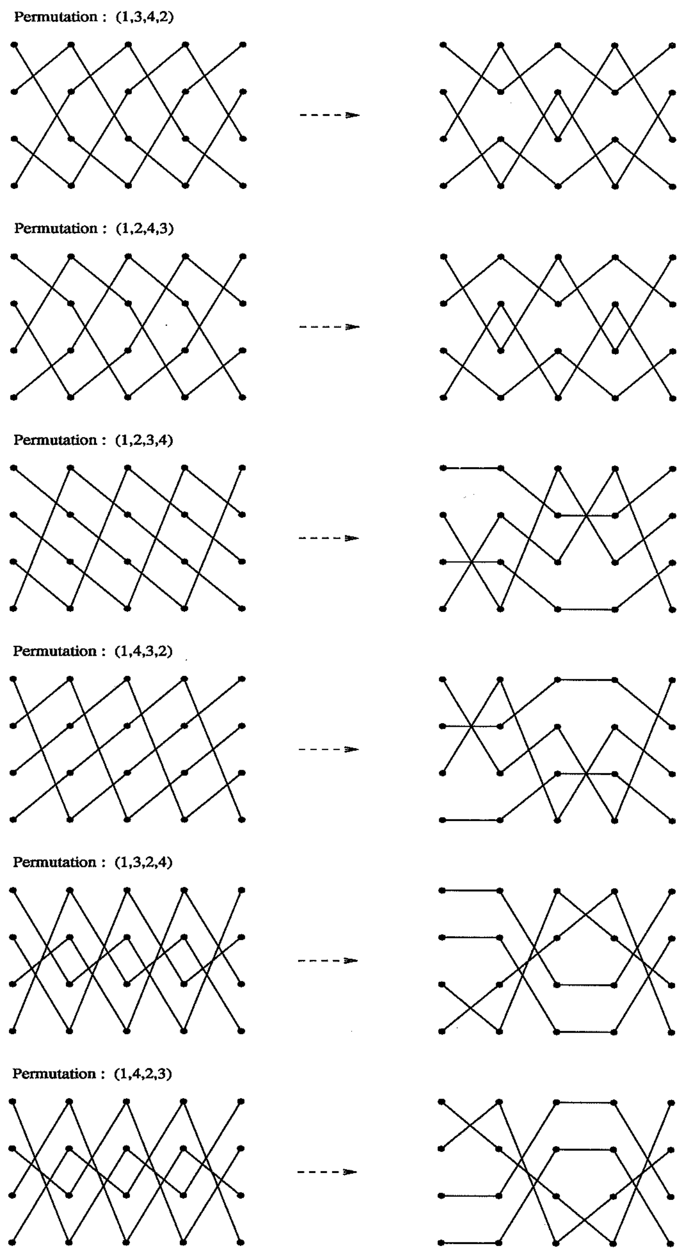

FIGURE 14. Flips. The left part of the picture presents the braid diagram corresponding to a particular permutation. The right part presents the flipped diagram.

Proof: The first part is just a special case of the above theorem. The second part follows directly from lemma 5.2. ■

This corollary draws our attention to fixed points of the parabolic flow.

5.5. **Fixed points for ψ^t.** To prove the existence of fixed points for a parabolic flow we will combine two ideas presented already in this paper. The idea of folding will allow us to obtain numerous examples of braids. Later we will show that they have non-trivial Conley index. Then the theory behind the Conley index will prove existence of fixed points for ψ^t.

5.5.1. *Folding.* We start with following folding lemma:

Lemma 5.6. *Assume that* $\mathbf{u}$ *is a solution of parabolic recurrence relation* $\mathcal{R}$ *on* $\mathcal{D}^n_d$. *Then for every integer* $N > 1$, *there exists a lifted parabolic recurrence relation* $\tilde{\mathcal{R}}$ *on* $\mathcal{D}^n_{Nd}$ *for which the lift of* $\mathbf{u}$ *is a solution. Furthermore, any solution to the lifted dynamics projects to some period-d braid diagram.*

Proof: For the first part it is enough to observe that every solution of period d is also a solution of period Nd for $N > 1$. So in fact for we can take

$$\tilde{\mathcal{R}}_i = \mathcal{R}_j \qquad i = j \bmod d.$$

Now since $\mathbf{u}$ is a solution of $\mathcal{R}$, for every strand α and every i we have:

$$\mathcal{R}_i(u^\alpha_{i-1}, u^\alpha_i, u^\alpha_{i+1}) = 0$$

Since of course every strand α' of the lifted $\mathbf{u}$ consist locally of points from some strand α of $\mathbf{u}$. So we get:

$$\tilde{\mathcal{R}}_i(u^{\alpha'}_{i-1}, u^{\alpha'}_i, u^{\alpha'}_{i+1}) = \mathcal{R}_i(u^\alpha_{i-1}, u^\alpha_i, u^\alpha_{i+1}) = 0.$$

Here we have used our convention of indexing periodically and the fact that different strands for lifted braid are in fact concatenations of the ones for $\mathbf{u}$. This proves that the lifted solution of $\mathcal{R}$ is a solution of $\tilde{\mathcal{R}}$. The second part is obvious (cf. figure 15). Note that the projection has the same number of connected components, but it may have a larger number of strands. ■

We will usually use folded braids with one free strand. Then we need to remember that the projection in general will have more then one strand. The lemma above will alow us to construct many braids diagrams having non-trivial Conley index. Then we will be able to find solutions of $\mathcal{R}_\pm$.

5.5.2. *Conley index.* To present one of the main results from the Morse theory for braids we need to extend the notation connected with the Conley index. In section 4.4 we have defined it as a homotopy type of $[N, N^-]$ and suggested that this is not really convenient. That is why we use the *homological Conley index*

$$CH_*(N) := H_*(N, N^-)$$

where N, N^- are like before and H_* is the singular homology. We can always assign to such an index a characteristic polynomial

$$CP_t(N) := \sum_{k \geq 0} \beta_k t^k$$

where β_k is a free rank of $CH_k(N)$. Having in mind that we will usually use $N = cl([\mathbf{u} \text{ rel } \mathbf{v}])$ we can simplify notation writing

$$CP_t(h) := CP_t(cl([\mathbf{u} \text{ rel } \mathbf{v}])$$

(h usually denotes the Conley index).

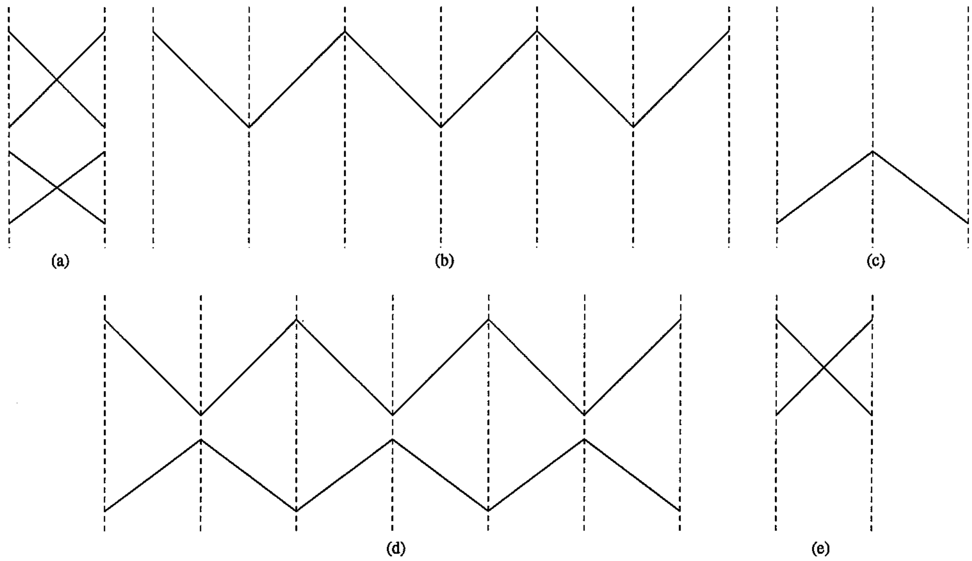

FIGURE 15. Part (a) - a given period 1 braid diagram. Part (b) and (d) - two different foldings of braid from (a) in this case $N = 6$. Part (c) - This time $N = 2$. Part (e) – projection of braid (b) on period 1 braid diagrams.

Before presenting the results which we will use to locate fixed points of a parabolic flows it is maybe useful to present some intuition behind it. For this let us review part of the theory of planar flows. For those we have the Poincaré-Bendixon theorem showing that α-limit and β-limit sets can in fact consist only of fixed points and closed orbits (periodic trajectory or equilibrium connections). If we could find a point for which we would know that its α-limit set cannot contain closed orbits then we have just "found" fixed point. One of the approaches to this idea is the concept of an index for curves. Given a planar flow we draw a simple closed curve C not passing through any equilibrium points. Then we consider the orientation of vector field at some $p = (x, y) \in C$. Letting p travel along C anticlockwise, the vector field rotates continuously and upon returning to the starting position it must have rotated through an angle $2\pi k$ for some integer k (we take the convention that we count angle anticlockwise). We call k the index of C and it can be shown ([8]) that this index depends only on the character of the fixed points encircled by C. Moreover inside any closed orbit or curve having non-trivial index we can find at least one fixed point.

For a parabolic flow on braid diagrams the Poincaré-Bendixon theorem also holds (see [7]). In fact one can show that such a flow is of almost Morse-Smale type i.e.

(1) The number of fixed points and periodic orbits is finite (here we do not have the condition that they are hyperbolic).
(2) All stable and unstable manifolds intersect transversally (or do not intersect at all).
(3) The nonwandering set consist only of fixed points and periodic orbits.

Very important for us is that it is also possible to show a *Morse inequality* saying that each fixed point within N contributes t^k to the Conley index and that each closed orbit contributes $(1+t)t^k$ for some natural number k. Now the analogy with planar flows makes us to suspect that non-trivial Conley index (which can be thought of as a generalization of the one for planar flows) can force the existence of a fixed point provided that we can exclude somehow periodic orbits. It is true as the lemma below show.

Lemma 5.7. *An arbitrary parabolic flow on a bounded relative braid class is forced to have a fixed point if* $\chi(h) := CP_{-1}(h)$ *is nonzero.*

The proof of this lemma can be founded in [6]. Condition $CP_{-1} \neq 0$ usually means that we have $\mathbf{u}$ rel $\mathbf{v}$ having nontrivial index. Moreover characteristic polynomial does not contain $(1+t)$ as a factor which by Morse inequalities excludes the existence of only periodic orbits and no fixed points inside $N = \mathrm{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$.

6. Positive topological entropy for twist maps

Now we are ready to assemble the machinery previously presented to prove that a twist map having a period four point of a certain type is chaotic. For simplicity we will denote by Λ_1 all the points $(x,y) \in \mathbb{R}^2$ for which flipped trajectory under iterations of T is between the anchor points of the skeleton[7]. So intuitively we are considering only the points which flipped first coordinates stay in the interesting region for all iterates of T.

The correspondence between points from this set and sequences of three symbols will be built by means of braid diagrams exactly as suggested in example 4.5. So for a given point $(x,y) \in \Lambda_1$ it is done in the following steps:

(1) We iterate T on (x,y) and we obtain the sequence $\{x_n\}$.
(2) To such a sequence we apply the flip transformation i.e. we change sign of all x_i with i equal to 0 or 3 modulo 4.
(3) We look at the odd anchor point and check in which out of three possible position it is placed. And we add one of symbols $\{+1, 0, -1\}$ as presented in the example 4.5[8].

The map assigning to a point in Λ_1 its symbolic description in Σ_3 we will call h. Since one can see that the description is constructed in such a way that it is concentrated only on odd anchors we can capture all the interesting behavior with T^2. That is why our first aim is to prove that h is semiconjugacy of T^2 restricted to Λ_1 and σ. To accomplish this let us start with the following lemma.

Lemma 6.1. *Assume that* $d > 1$. *Then for any simple d-periodic*[9] *sequence* $\{a_n\}_{n=-\infty}^{+\infty}$ *of three symbols there exists a point in* Λ_1 *having* $\{a_n\}$ *as its symbolic description. Moreover trajectory of this point is uniformly bounded.*

Proof: Denote by q the smallest common multiple of d and 4.
Step 1. Construction of braid diagram.

[7] Strictly speaking for even i – x_i has to be placed between the inner anchor points of the skeleton and for odd i – x_i has to be placed between the outer skeleton's anchors.

[8] In the case when our trajectory is not periodic we write down its sequence by considering diagrams with finite but increasing length. Then the limit sequence will be its symbolic description

[9] Then the assumption $d > 1$ guaranties that we avoid the "dangerous" cases i.e. sequences leading to the improper braids $(\{\ldots, 1, 1, 1, \ldots\}$ and $\{\ldots, -1, -1, -1, \ldots\})$.

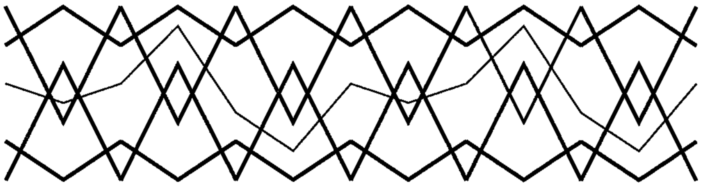

FIGURE 16. Braid diagram corresponding to the sequence $\{\ldots 0, +1, -1, 0, +1, -1, \ldots\}$. Here we have $d = 3$, $q = 12$, $k = 2$. The homotopy type of this braid class is S^2.

First we are folding the skeleton (I) to period q (in fact we are folding $\mathcal{R}_{\pm}$ but as lemma 5.6 shows we are then also folding its solution forming the skeleton). For such a lifted skeleton we add free strand in following way:

(1) Every even anchor point is between the inner anchors of the skeleton.
(2) The position of the odd anchors is determined by a sequence $\{a_n\}$

if	$a_i = +1$	then	anchor $2i + 1$ is placed above the inner loop
if	$a_i = 0$	then	anchor $2i + 1$ is placed inside the inner loop
if	$a_i = -1$	then	anchor $2i + 1$ is placed below the inner loop

(cf. example 4.5 and figure 16). We construct this braid in such a way that the obtained braid is closed (i.e. $u_0 = u_q$). The fact that $d > 1$ guarantees that this braid is proper. We denote this constructed braid by $\mathbf{u}$ rel $\mathbf{v}$.
Step 2. Non-triviality of the Conley index.
Now for the braid created in step 1 we will calculate its index. Consider the anchor points of the free strand. Since we are allowed to move each one independently from the others and only between the anchors of the skeleton, the configuration space for a parabolic flow on this braid is in fact the cartesian product of q intervals. So N is I^q, a q dimensional hypercube. Now we need to determine N^-, the exit set. As argued in example 4.5, the flow can change braid class only for the odd anchor points placed inside the inner loop. We will denote the number of this points by k ($0 \leq k \leq q$, k is equal to number of zeros in $\{a_0, \ldots, a_q\}$). Since opposite faces of N always have the same direction of the flow we get that $N/N^- \simeq S^k$. The standard result from the theory of homology shows that

$$H_*(N, N^-) = H_*(S^k) = \begin{cases} \mathbb{R} & * = k \\ 0 & \text{otherwise} \end{cases}$$

and then $CP_t(h) = t^k$.
Step 3. Existence of a required point.
Since from the previous step we have $\chi(h) = CP_{-1}(h) = (-1)^k \neq 0$, lemma 5.7 proves that there exists a fixed point of the parabolic flow in the interior of $N = cl([\mathbf{u} \text{ rel } \mathbf{v}])$. By corollary 5.5 this fixed point leads to a trajectory of T so also to its initial point (x_0, y_0). It is now obvious that this point has $\{a_n\}$ as its symbolic description. This concludes the proof since the fact that such a trajectory is bounded follows easily from the observation that fixed points constructed in this way are bounded by values of the skeleton anchors (because all elements of N are bounded in this way). This shows also that (x_0, y_0) is in Λ_1. ∎

Remark 6.2. A moment of reflection shows that sequences

$$\{\ldots, 1, 1, 1, \ldots\} \quad \text{and} \quad \{\ldots, -1, -1, -1, \ldots\}$$

lead to improper braids, so the construction is invalid for them (in fact this is why we have assumed that $d > 1$). However, for a sequence $\{\ldots, 0, 0, 0, \ldots\}$ the corresponding braid is proper and we still can apply the lemma. We will use this fact in the proof of next lemma.

Remark 6.3. It is worth to notice that the proof presented above do not shows that periodic symbol sequences are corresponding to periodic trajectories of T. We have that only for sequences with the period equal to a multiple od four. But since we are only building a semiconjugacy it is not a problem.

Now with the trajectories from this lemma we are able to construct others.

Lemma 6.4. *For any sequence $\{a_n\}_{n=-\infty}^{+\infty}$ consisting of three symbols, there exist a point $(x', y') \in \Lambda_1$ for which $\{a_n\}$ is a symbolic description.*

Proof: We shall construct a sequence $\{\tau_n\}$ of points converging to the desired one. From lemma 6.1 we have that there exists a point (x_0, y_0) having as its symbolic description the sequence

$$\{\ldots, a_0, 0, a_0, 0, a_0, 0, \ldots\}^{10} \qquad \text{(this sequence is 2 periodic).}$$

We will call this point τ_0. Then in general we define τ_n as a point having description

$$\{\ldots, a_n, 0, a_0, a_1, \ldots, a_{n-1}, a_n, 0, a_0 \ldots\} \qquad \text{(it is } n+1 \text{ periodic).}$$

Again we know that such τ_n exist from lemma 6.1. Since from this lemma we have also a uniform bound on $\{\tau_n\}$ then we can choose a subsequence converging to some point which we call (x', y'). The same argument shows that (x', y') is in Λ_1. Moreover since T is continuous and since all points form τ_n are generated by trajectories passing within certain intervals given by positions on the skeleton anchor points, it follows that (x', y') has $\{a_n\}$ as its symbolic description. ∎

The above lemma gives us an infinite number of points but it does not guarantee their uniqueness (in a sense that only one point has a given symbolic description). This is not a problem but maybe it is good to give formal definitions of the two closely related concepts topological conjugacy and topological semi-conjugacy.

Definition 6.5. Let $f\colon A \to A$ and $g\colon B \to B$ be two maps. The functions f and g are said to be *topologically conjugate* if there exist a homeomorphism $\phi\colon A \to B$ such that $\phi \circ f = g \circ \phi$. This homeomorphism is called a *topological conjugacy.*

This definition states that in fact the systems defined by f and g are identical (there is one-to-one correspondence between trajectories). In many cases this condition is too strong, which is why we define *semi-conjugacy* by the above definition but where we do not require ϕ to be one-to-one but just onto. For us a semi-conjugacy suffices since it shows that the topological entropy of the first system is greater or equal to that of the second one (see for example [5]).

First let us make simple observation.

Lemma 6.6. *The set Λ_1 is invariant under action of T^2.*

[10] The symbol 0 after a_0 is added to ensure that we do not get sequences leading to an improper braid cf. remark 6.2

Proof: From the definition of Λ_1 it easily follows that if (x, y) is in Λ_1 then also $T^2(x, y) \in \Lambda_1$. ∎

Now we can state theorem that will allow us to prove the main result.

Theorem 6.7. *There exist a surjective continuous correspondence h between Λ_1 and Σ_3 such that*

$$h \circ (T^2|_{\Lambda_1})(x) = \sigma \circ h(x) \quad \textit{for all } x \in \Lambda_1$$

i.e. $T|_{\Lambda_1}$ and σ are topologically semi-conjugate.

Proof: As mentioned above

$$h : \Lambda_1 \to \Sigma_3$$

will be defined as a map assigning to a point in Λ_1 its symbolic description. The fact that every point has a unique symbolic description (otherwise it would mean that there exists a point having some x_n in two disjoint intervals defined by the anchors of the skeleton) we get that h is well defined function. Moreover from the lemma 6.4 we get that h is onto.

To prove continuity of this map let us take $\varepsilon > 0$. The fact that two sequences $\{a_n\}, \{b_n\} \in \Sigma_3$ are such that $d(\{a_n\}, \{b_n\}) < \varepsilon$ means by the definition of metric on Σ_3 that these sequences are equal on a certain central block i.e. $a_k = b_k$ for all $|k| < i$ where i is such that $2^{-i} > \varepsilon$. Now for given $(x, y) \in \Lambda$ we look at the distances of sequence $\{x_k\}_{k=-2i}^{2i}$ from the anchor points of a skeleton. We choose the smallest of them and denote it by ξ. Using the twist property, it is not too difficult to show that $\xi > 0$. By the continuity of maps T and T^{-1} we can find δ such that first coordinates of trajectories of all points in Λ_1 starting not further than δ from (x, y) will not move from its trajectory for more than ξ. Now for this δ we know that all points starting δ-close to (x, y) have the symbolic description which is equal to the one of (x, y) up to the i^{th} and down to the $-i^{\text{th}}$ position. So their symbolic description will be ε-close to the (x, y) one. This proves the continuity of h.

Now take arbitrary $(x_0, y_0) \in \Lambda_1$. It is obvious that if (x_0, y_0) has $\{a_n\}$ as its symbolic description then $T^2(x_0, y_0)$ is described by $\{a_{n+1}\}$. This shows that

$$h(T^2|_{\Lambda_1}(x_0, y_0)) = \{a_{n+1}\}$$

and of course

$$\sigma(h(x_0, y_0)) = \{a_{n+1}\}.$$

So

$$h \circ (T^2|_{\Lambda_1})(x, y) = \sigma \circ h(x, y) \qquad \text{for all } (x, y) \in \Lambda_1,$$

which concludes the proof. ∎

Remark 6.8. We can follow almost exactly the same reasoning as presented above to show that a point of period four with permutation type $(1, 3, 4, 2)$ leads to complicated behavior of T^2. The only difference is in the construction of symbolic description of a point (we need to change role of even and odd anchors). The rest of reasoning can be just repeated.

We need to define one more set before stating final theorem. Let

$$\Lambda := \Lambda_1 \cup T(\Lambda_1).$$

It is obvious that Λ is invariant under action of T. With this definition we can form a result announced in the introduction:

Theorem 6.9. *An orientation reversing twist homeomorphism having a four periodic point of type* $(1,2,4,3)$ *or* $(1,3,4,1)$ *has positive topological entropy.*

Proof: By remark 6.8 it is enough to consider a permutation type $(1,2,4,3)$. Since the whole system has trivially greater or equal topological entropy than the restricted one (we are taking maximal separated sets) then

$$h(T) \geq h(T|_\Lambda) \qquad \text{for all } \varepsilon > 0.$$

The standard result shows that (see [5])

$$h(T|_\Lambda) = \frac{1}{2}h(T^2|_\Lambda) \geq \frac{1}{2}h(T^2|_{\Lambda_1})$$

Now the theorem above show that:

$$h(T^2|_{\Lambda_1}) \geq h(\sigma_3).$$

In section 2 we proved that the topological entropy of the shift on three symbols is positive, so we get:

$$h(T|_\Lambda) \geq \frac{1}{2}h(\sigma_3) \geq \frac{1}{2}\ln(3) > 0.$$

This concludes the proof. ■

Remark 6.10. It is also possible to use a different strategy to construct semiconjugacy. We could use T^4 instead of T^2. Then we would need to use the shift on nine symbols instead of three. This approach gives exactly the same lower bound on topological entropy since

$$h(T) \geq \frac{1}{4}h(T^4|_{\tilde{\Lambda}})) \geq \frac{1}{4}h(\sigma_9) \geq \frac{1}{4}\ln(9) = \frac{1}{2}\ln(3)$$

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