Bachelorproject Mathemetics:

The Generalised Riemann Integral

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Chapter 1

Introduction

Integration has been an important part of mathematics since the 17th century, when Gottfried Wilhelm Leibniz and Isaac Newton constructed the Fundamental Theorem of calculus, which says that area is essentially the same as taking an antiderative. In the 19th century, Cauchy investigated integrals of continuous functions. The integral that he used was later redefined by Bernhard Riemann, which he used to investigate integrals of discontinuous functions. His definition was far easier to understand and to teach than all of the previous ones, which resulted in the Riemann integral becoming the standard integral to be taught to students in a mathematics class.

The integral still had some setbacks that made it unsuitable for applications. Many important functions did not have a Riemann integral and the theory lacked strong convergence theorems for taking the integral of the limit of a sequence of functions. Swapping limit and integral would require uniform convergence, a rather heavy and restrictive requisite.

At the beginning of the 20th century, Henri Lebesque came up with a new kind of integral which would deal with many of these inconveniences. Many functions that the Riemann integral could not handle now became integrable, and the Monotone- and Dominated Convergence Theorems yielded significantly better results than the ones provided by Riemann theory. The Lebesque integral was based on measure theory and, when mastered, proved a very powerful tool in mathematical applications.

This measure theory made that the Lebesque integral was much harder to get familiar with, which is the main reason that teachers choose to educate the inferior Riemann integral with their students until they reach the final stages of their academic career.

In the following years, investigations into even more general forms of integration continued, mainly because there was lack of good methods to find primitives of functions. In 1957 the mathematicians Jaroslav Kurzweil and Ralph Henstock independently came up with a simple expression of an integral that was the result of beforementioned research.

This integral was based on the Riemann theory and just as easy to work with, but with powers to match and in certain cases even exceeds those of the Lebesque integral!

I find it quite surprising that an integral that is so much easier to master than the Lebesque integral, but with comparable results, has remained unknown even to most academic teachers so far. This was a reason for me to dig into this theory and learn more about it and report the main result in my bachelorthesis. It is my purpose to make it understandable for a wide audience, so students with some experience in set theory and using ϵ-δ-proofs should be able to read the first two chapters. Most of the proofs included contain a fair amount of details and may therefor only be interesting for teachers. These have been moved to the final chapter of this paper.

Chapter 2

Basic theory

We will start with the definition of the Riemann integral as we know it. We will then make some changes to this definition, which will then result in the superior integral mentioned in the preface. This will be followed in the next chapter by some theorems that will help us in determining whether a function has an integral and how it can be evaluated. It will also be shown how the Generalised version of the Riemann integral is an extension of the ordinary Riemann integral by showing that the ordinary version is a special case of its modern counterpart.

2.1 Riemann Integral

We will start our research with one-dimensional, bounded functions f over bounded intervals I. Because there are several ways in which the Riemann integral is brought to students, I will first explain the terms used in our definition of the integral.
A *division* of I is a finite collection of closed, nonoverlapping intervals whose union is I. A division is called a *tagged division* if every subinterval J has a *tag*, which is a point contained in that interval. So a tagged division is a finite collection $\mathcal{D}$ of ordered pairs, each consisting of a closed interval and its tag:

$$\mathcal{D} = \{([x_0, x_1], z_1), ([x_1, x_2], z_2), ..., ([x_{n-1}, x_n], z_n)\}, z_i \in [x_{i-1}, x_i], i = 1, 2, ..., n.$$

Let J be a such an interval in a tagged division and denote by $L(J)$ its (Euclidean) length. If $\mathcal{D}$ is a tagged division of I, then its *Riemann sum* is the sum of lengths of the intervals, each multiplied by the value that the considered function f takes in its tag:

$$fL(\mathcal{D}) := \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1}).$$

This is the familiar approximation of the area under the graph using rectangles. The idea for the Riemann integral is that this area can be approached arbitrarily close with a finite Riemann sum.

Definition: A number A is the *Riemann integral* of f over $I \Leftrightarrow \forall\epsilon\exists\delta$ such that for every tagged division $\mathcal{D}$ with $\max_{J\in\mathcal{D}} L(J) \leq \delta$: $|A - fL(\mathcal{D})| < \epsilon$.

This means that for a function to be integrable, the result of the Riemann sum cannot depend too much on the way I is broken up into intervals or on location of its tags. This is no problem when f is a continuous function or a function with only a finite amount of discontinuities. Too many gaps will cause trouble, though, as we shall see in the following familiar example.

Example 2.1: Observe the function $f := 1_{\mathbf{Q}}$ with $I := [0,1]$. Readers experienced in Lebesque theory will know that this function is 0 everywhere except on a set with Lebesque measure zero, and that therefor

$$\mathcal{L}\int_{[0,1]} 1_{\mathbf{Q}} = 0.$$

Since the Lebesque integral is an extension of the Riemann integral, the definition of Riemann integrability has hold for $A = 0$. So for given ϵ, will it be possible to find a δ such that every tagged division with all intervals no bigger than δ has Riemann sum smaller than ϵ? As many of us know already, the answer is negative. The puzzler is that every interval, no matter how small, contains both rational and irrational numbers, so to every subinterval in a division a rational tag can be assigned, which would then yield a Riemann sum of 1. Using just irrational tags we can also get a 0 there. This shows that f is NOT Riemann integrable, because a function can only have one intergral.

The problem here was that once ϵ is given there has to be a fixed δ for which every $\mathcal{D}$ with $L(J) \leq \delta$ for all J's behaves the way we want. This is a rather heavy requirement. In the case of the example, it would have been more convenient if the length of the interval were small in case of a rational tag, because it is these intervals for which the corresponding terms in the Riemann sum give poor approximations of the true area under the graph there, causing the total error to grow. If the tag is irrational on the other hand, the term in de Riemann sum perfectly describes the area under the graph, so we can choose the interval as big as we want. This is basically the idea behind this generalized version of the Riemann integral: to let the maximum length of the subintervals depend on how the function behaves in its tags.
This will be formalized in the next section.

2.2 Generalised Riemann Integral

We start by defining a special kind of function $\gamma : I \to \mathcal{O}$, with $\mathcal{O}$ being the collection of all open intervals in $\mathbf{R}$. This function produces for each possible location of a tag $z \in I$ an open interval around z, that limits the range for a subinterval J in the division $\mathcal{D}$ that has z as its tag. Such a function is called a *gauge*:

$$\gamma(z) := (z - c_z, z + d_z) \text{ for certain positive real numbers } c \text{ and } d.$$

A tagged division $\mathcal{D}$ is called γ-*fine* if $[x_{i-1}, x_i] \subset \gamma(z_i)$ for all i=1,2,...,n.
There's a theorem known as Cousin's Lemma that states that every gauge γ has at least one γ-fine tagged division. Its proof is included in the final chapter of this paper. It should be understandable for readers with some experience in set theory. It is based on the order property of the real line, and can therefor not be used for higher-dimensional spaces. More on that later.
We now hold the tools necessary to define the Generalised Riemann Integral.

Definition: A number A is integral of a function f over an interval $I \Leftrightarrow \forall\epsilon\exists\gamma$ such that $\forall\gamma$-fine tagged divisions $\mathcal{D} : |A - fL(\mathcal{D})| < \epsilon$.

This integral goes by many names, such as Henstock integral, Henstock-Kurzweil integral, gauge integral. I shall keep referring to it as the Generalised Riemann integral, to emphasize how close it stands to the ordinary Riemann integral.

It is clear from the definition that we're still approximating the area with rectangles, but that their allowed base lengths are now controlled by a function that is not (necessarily) constant. In order to show that a function is Generalised Riemann integrable, we need to construct a gauge that doesn't keeps the total error of a Riemann sum from getting bigger than ϵ whenever $\mathcal{D}$ is a γ-fine tagged division. We use the earlier example to illustrate this.

Example 2.2:
We revisit $f := 1_{\mathbf{Q}}$ with $I := [0, 1]$ from example 2.1. To show that this function has an integral, we need to find a gauge γ so that each γ-fine tagged division has Riemann sum smaller than ϵ, because we learnt from Lebesque that the integral must be 0. We mentioned before that when a tag is irrational there is no need to put any restrictions on a subinterval containing that tag. When $z \in \mathbf{Q}$ on the other hand, we want the containing subinterval to be as small as possible, because every term Jz with $z \in \mathbf{Q}$ will add $L(J)f(z) = L(J)$ to the accumulated error.

To ensure this, denote by $\{r_1, r_2, ...\}$ the (countable) collection of all the rationals in $[0, 1]$ and let n be the reference number for an element in this list. Define for given ϵ the gauge

$$\gamma(z) := \begin{cases} (z - \frac{\epsilon}{2^{n+1}}, z + \frac{\epsilon}{2^{n+1}}) & \text{if } z \in \mathbf{Q}; \\ (-1, 2) & \text{if } z \notin \mathbf{Q}. \end{cases}$$

First thing to note is that for reasonably small ϵ it is no longer possible to cover I with a finite amount of subintervals with only rational tags, because the combined length of all such intervals will never exceed ϵ. A suitable division $\mathcal{D}$ might have some rational tags, but because each of the corresponding terms zJ will only add $L(J) < \frac{2\epsilon}{2^{n+1}}$ to the Riemann sum, the total error will be bounded by ϵ:

$$fL(\mathcal{D}) = \sum_{i:z_i \in \mathbf{Q}} \frac{\epsilon}{2^i} < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

which proves that 0 is an integral for f.

Chapter 3

Basic properties

3.1 Uniqueness of the integral

This section will mention two important properties of the Generalised Riemann integral. The first is to show that if A_1 is an integral of a function f, that a number $A_2 \neq A_1$ cannot also be its integral. In other words, we want to prove that a function can only have up one integral.

Uniqueness Theorem: If f is a function and A_1 and A_2 are both integrals of f over I, then $A_1 = A_2$.

Proof:
A_1 and A_2 are equal if and only if $|A_1 - A_2| = 0$.
Let ϵ be given. Then according to the definition, there is γ_1 such that $|A_1 - fL(\mathcal{D}_1)| < \epsilon$ and γ_2 such that $|A_2 - fL(\mathcal{D}_2)| < \epsilon$ whenever $\mathcal{D}_1$ is γ_1-fine and $\mathcal{D}_2$ is γ_2-fine.
Define a gauge γ by $\gamma(z) := \gamma_1(z) \cap \gamma_2(z)$. It is obvious from the definition that a tagged division $\mathcal{D}$ that is γ-fine is also γ_1-fine and γ_2-fine. The triangle inequality then tells us that

$$|A_1 - A_2| = |A_1 - fL(\mathcal{D}) + fL(\mathcal{D}) - A_2| \leq |A_1 - fL(\mathcal{D})| + |fL(\mathcal{D}) - A_2| < \epsilon + \epsilon = 2\epsilon.$$

Let ϵ run to 0 to see that what was claimed has now been proved. □

3.2 Expansion of the Riemann Integral

Example 2.2 showed that it may take some effort to construct a gauge and prove that a function is integrable. It would therefor be a reasonable demand that the generalised Riemann integral is at least an extension of the ordinary Riemann integral. This is indeed the case, and this major result is easy to prove. It will also be shown that the class of Riemann integrable functions is a special case of the Generalised Riemann integrable functions.

Theorem: A function $f : I \to \mathbf{R}$ that is Generalised Riemann integrable is also Riemann integrable.

Proof Suppose that f is Riemann integrable and let ϵ be given. Then there is δ such that $|\int_I f - fL(\mathcal{D})| < \epsilon$ for all tagged divisions with $L(J) < \delta$ for every $zJ \in \mathcal{D}$. Let R_δ be the collection of all such tagged divisions $\mathcal{D}$. Set $\gamma(z) := (z - \frac{\delta}{2}, z + \frac{\delta}{2})$ for all z in I. Then the collection $GR_{\frac{\delta}{2}} := \{\mathcal{D} : \mathcal{D} \text{ is } \gamma\text{-fine}\}$ contains intervals that are all smaller than δ and is therefor a subcollection of R_δ. So $|\int_I f - fL(\mathcal{D})| < \epsilon$ holds for all γ-fine $\mathcal{D}$. We conclude that f is Generalised Riemann integrable.

Assume now that f is Generalised Riemann integrable with the added property that $\gamma(z) := (z - \delta, z + \delta)$ is a gauge for which $|\int_I f - fL(\mathcal{D})| < \epsilon$ whenever $\mathcal{D}$ is γ-fine. Since all intervals in G_δ have length smaller than δ, a subinterval that has z as its tag will always be contained in an open interval of length 2δ centered around z. Consequently $R_{delta} \subset GR_\delta$, which tells us that f is Riemann integrable. □

3.3 Integration over unbounded intervals

So far we have sufficed with bounded functions over bounded intervals. From Calculus we know that in Riemann theory, we can use the limit of bounded integrals to determine the integrability of functions over infinitely long intervals, the so-called Improper Riemann Integral. We will show that the ordinary Generalised integral can also handle these, using the tools handed in the preceding sections.

What we first have to do is extend the real line by adding $\{-\infty\}$ and $\{\infty\}$. The result is usually denoted by $\overline{\mathbf{R}}$.
When taking a function f over an unbounded interval $I := [a, \infty]$ and try to find a gauge and divisions to prove that its integrable, one will face an obstruction. Namely that it is, of course, not possible to cover an unbounded interval with a finite amount of bounded subintervals. What we can do is cover a very large interval $[a, t]$ for some very big $t \in \mathbf{R}$ with a suitable tagged division $\mathcal{D}$ and then adjoining the interval $[t, \infty]$ to this, along with some tag (usually ∞). If we use the Euclidean length for this interval in the Riemann sum, we would always end up with ∞, so this is not the right approach here. Since an integrable function will have to be very close to zero for large values of x, a logical step would be to set

$$L([a, b]) := 0 \text{ if } a = \infty \text{ or } b = \infty,$$

because this term is expected to add little to the Riemann sum in the first place.

Example 2.4:
Again, the function $f := 1_{\mathbf{Q}}$ shall serve for clarification purposes, but this time we use

$I := [1, \infty]$. Assuming that this integral is still 0, our objective is to find a gauge on $[1, \infty]$ such that $fL(\mathcal{D}) < \epsilon$ whenever is γ-fine, using $f := 1_{\mathbf{Q}}$ for $x \in [1, \infty)$; $f(\infty)$ can be given any value we want. We can use the same gauge as we did in example 2.2, where this time $\{r_1, r_2, ...\}$ denotes the set of all rationals in $[1, \infty)$ (which is still countable) and add $\gamma(\infty) := (s, \infty)$ for some real number s. In this case, $\infty \in \gamma(z)$ only when $\gamma = \infty$.
It is easy to see that $fL(\mathcal{E}) < \epsilon$ for every γ-fine division $\mathcal{E}$ of $[1, t]$ for every real number $t > 1$. Because $L([t, \infty]) = 0$ by decree, we can conclude that $\int_1^\infty 1_Q$ exists and equals 0.

This example also shows that there is no need for improper integrals in this theory.

3.4 Fundamental Theorem

One of the most powerful features of the Riemann Integral is that it provides us with an easy way to actually determine the value of an integral. In Calculus we learn of the Fundamental Theorem that assures us that when f is integrable and there exists a continuous function F for which $F'(x) = f(x)$ for all $x \in [a, b]$, possibly with the exception of a countable and nowhere dense set C - where *nowhere dense* means that every member of C can be contained in an open interval that does not hold any other members of C - that

$$\int_a^b f = F(b) - F(a).$$

One can wonder whether we can apply this useful tool for the Generalised Riemann integral. Fortunately, it does. Of course it goes whenever f was already Riemann integrable, but we can loosen the requirements a little so that it works for a wider collection of functions. Looking back at example, we can see that the value of the integral doesn't change if a function trails from its expected path, as long as the set where it does so is countable.
This gives us the idea that $F'(x) = f(x)$ only has to hold everywhere except on a countable set. This is easy to see because we can construct a gauge that limits the total error caused by terms in a Riemann sum that have tags z for which $F'(z) \neq f(z)$ as much as we want by the same method used in example 2.2. We still need the continuity of F to assure the existence of an integral over unbounded functions/intervals, which we will see later. An F that holds on to these two requirements is called a *primitive* of f.

Fundamental Theorem: If $f : [a, b] \subset \overline{\mathbf{R}} \to \mathbf{R}$ has a primitive F, then f is integrable and $\int_a^b f \mathrm{d}x = F(b) - F(a)$.

The proof is quite long and contains a number of details and is therefor moved to the end of this paper.

Example 2.3 As an example we consider $f = 1_{\mathbf{Q}}$ on $[0, 1]$ again. Here we see that the function $F \equiv 1$ is a primitive of f, since $F'(x) = 0 = f(x)$ everywhere except for the countable set $\mathbf{Q} \cap [0,1]$. Since f has a primitive, the fundamental theorem assures us that it's integrable and has integral $F(1) - F(0) = 1 - 1 = 0$

3.5 Convergence Theorems

One of the main reasons behind the Lebesque integral's popularity is the convergence theorems that apply to it. Quite surprisingly, the adaptions made to the definition of the Riemann integral yields a space that also has these remarkable properties. The Riemann theory has a theorem that allows us to claim that $\int_I f = \lim_{n\to\infty} \int_I f_n$ if (f_n) converges to f uniformly. The following two theorems allow us to make the same conclusion, but with much weaker requirements. The (long) proof of both theorems is again postponed until the final chapter.

Monotone Convergence Theorem: Let (f_n) be a monotone sequence of functions with $\lim_{n\to\infty} f_n(x) = f(x)$ for all x in I. Then

$$\int_I f \text{ exists } \leftrightarrow \lim_{n\to\infty} \int_I f_n < \infty. \text{ Moreover } \int_I f = \lim_{n\to\infty} \int_I f_n.$$

Dominated Convergence Theorem: Suppose f_n and h are integrable on I and $|f_n| < h$ for all and n and that $\lim_{n\to\infty} f_n(x) = f(x)$ for all x in I.
Then f is integrable and

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Chapter 4

Proofs

4.1 Cousin's compatibility theorem

Suppose γ is a gauge on an interval $I \in \overline{\mathbf{R}}$. Then there exists a tagged division that is γ-fine.

Proof:
When $I = [a, b]$ and $E := \{x : x \in (a, b]$ and $\exists\gamma$-fine tagged division of $[a, x]\}$. This set is not empty, because if $x \in \gamma(a) \cap [a, b]$ then $\{(a, [a, x])\}$ is a γ-fine tagged division of $[a, x]$. Set $y := \sup_x E$. Then there is $x < y \in \gamma(y)$. Addjoin to the tagged division of $[a, x]$ the interval $[x, y]$ tagged with y. The result is a γ-fine division of $[a, y]$, which we will call $\mathcal{D}$ for convenience.
Assume now that $y < b$ and let $w \in \gamma(y) \cap (y, b)$. Now adjoin to $\mathcal{D}$ the term $(y, [y, w])$. The result is a γ-fine partition of $[a, w]$, so w has to be in E.
This contradicts our earlier assignment of y as the least upper bound of E. □

It is obvious that this proof cannot be used to prove compatibility for higher dimensions, because they don't have the special order-property that the real line possesses. While the theorem is also true for gauges in higher dimensions, additional theory is necessary to prove this and will therefor be left out of this paper.

4.2 Fundamental Theorem

If $f : [a, b] \subset \overline{\mathbf{R}} \to \mathbf{R}$ has a primitive F, then f is integrable and $\int_a^b f \mathrm{d}x = F(b) - F(a)$.

Proof:
We'll be using the definition of the Generalised Riemann integral to show that $F(b)-F(a)$ is the integral of f, so our goal is to find a gauge that selects divisions whose Riemann sums approximate $F(b) - F(a)$ arbitrarily close.

For convenience, set $\Delta F(J) := F(v) - F(u)$ when $J = [u, v]$. Then

$$F(b) - F(a) = \sum_{zJ \in \mathcal{D}} \Delta F(J)$$

when $\mathcal{D}$ is a division of $[a, b]$, because the sum telescopes. Consequently,

$$|F(b) - F(a) - RS(\mathcal{D})| = \left| \sum_{zJ \in \mathcal{D}} \Delta F(J) - \sum_{zJ \in \mathcal{D}} f(z)L(J) \right| \leq \sum_{zJ \in \mathcal{D}} |\Delta F(J) - f(z)L(J)|$$

using the triangle inequality.

Because of the definition of primitive, there are two types of points to be considered when we try to construct a gauge: points where $F'(z) = f(z)$ and points where $F'(z) \neq f(z)$.

Let C :={ $c_1, c_2, ...$} be an infinitely countable set for which $F'(z) = f(z)$ when $z \in [a, b] \cap C^c$. If there's only a finite amount of points where $F'(z) \neq f(z)$, we can always add points with $F'(z) = f(z)$ to make it infinite. Because $f(x)$ is bounded on the closed interval $[a, b]$, it is possible to make a gauge γ such that $|f(c_n)L(J)| < \frac{\epsilon}{2^{n+3}}$ and $|\Delta F(J)| < \frac{\epsilon}{2^{n+3}}$ whenever $c_n \in J$ and $J \subset \gamma(c_n)$. For such a pair (c_n, J), we have $|\Delta F(J) - f(c_n)L(J)| < \frac{\epsilon}{2^{n+2}}$, again using the triangle inequality.

For the complementary set, we call upon an auxiliary function $G : \overline{\mathbf{R}} \to \mathbf{R}$, defined by

$$G(x) := \frac{x}{2 + |x|}$$

when $x \in \mathbf{R}$, $= -\frac{1}{2}$ when $x = -\infty$ and $= \frac{1}{2}$ when $x = \infty$. This function is continuous and has $G'(x) > 0$ for all $x \in \mathbf{R}$.
We will now introduce two analytical tools which we will use to bind the effect of terms in the Riemann sum with tags in $[a, b] \cap C^c$, the Straddle Lemma and a corollary:

Straddle Lemma: Supose $F : [a, b] \to \mathbf{R}$ is differentiable at z. Then $\forall \epsilon$ $\exists \delta$ such that

$$|F(v) - F(u) - F'(z)(v - u)| < \epsilon(v - u)$$

whenever $z \in [u, v]$ and $[u, v] \subset (z - \delta, z + \delta) \cap [a, b]$.

Corollary:
Suppose $G'(z) > 0$. Then there exists δ such that

$$G'(z)(v - u) < 2(G(v) - G(u))$$

whenever $z \in [u, v]$ and $[u, v] \subset (z - \delta, z + \delta)$.

Now assume that $z \in [a,b] \cap C^c$. In first applying the Straddle Lemma, using $\frac{\epsilon}{4}G'(z)$ instead of ϵ, and then its corollary, we have that

$$|F(v) - F(u) - f(z)(v-u)| < \frac{\epsilon}{4}G'(z)(v-u) < \frac{\epsilon}{2}(G(v) - G(u))$$

whenever $z \in [u,v]$ and $[u,v] \subset (z-\delta, z+\delta) \cap [a,b]$. Then set $\gamma(z) = (z-\delta, z+\delta)$ for $z \in [a,b] \cap C^c$. This δ may have different value for each z, since ϵ used in the Straddle Lemma depends on z.

Now let $\mathcal{D}$ be a γ-fine division of $[a,b]$ and let $\mathcal{E}$ and $\mathcal{F}$ be subsets of the collection $\mathcal{D}$, consisting of the zJ with tags in C and $[a,b] \cap C^c$, respectively.

When $zJ \in \mathcal{E}$, we had $|\Delta F(J) - f(c_n)L(J)| < \frac{\epsilon}{2^{n+2}}$. Each c_n can be the tag of at most two subintervals, so that

$$\sum_{zJ \in \mathcal{E}} |\Delta F(J) - f(z)L(J)| < 2\sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+2}} = \frac{\epsilon}{2}.$$

For $zJ \in \mathcal{F}$, we know that

$$\sum_{zJ \in \mathcal{F}} |\Delta F(J) - f(z)L(J)| \leq \sum_{zJ \in \mathcal{F}} \frac{\epsilon}{2}\Delta G(J)$$

$$\leq \sum_{zJ \in \mathcal{D}} \frac{\epsilon}{2}\Delta G(J) \leq \frac{\epsilon}{2}(G(b) - G(a)) \leq \frac{\epsilon}{2}(G(\infty) - G(-\infty)) = \frac{\epsilon}{2}$$

This leads to the final result

$$|F(b) - F(a) - RS(\mathcal{D})| \leq \sum_{zJ \in \mathcal{D}} |\Delta F(J) - f(z)L(J)| =$$

$$\sum_{zJ \in \mathcal{E}} |\Delta F(J) - f(z)L(J)| + \sum_{zJ \in \mathcal{F}} |\Delta F(J) - f(z)L(J)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Which conclude that f is integrable with integral $F(b) - F(a)$. □

4.3 Monotone and dominated convergence theorems

MCT:
Let (f_n) be a monotone sequence of functions with $\lim_{n\to\infty} f_n(x) = f(x)$ for all x in I. Then

$$\int_I f \text{ exists } \Leftrightarrow \lim_{n\to\infty} \int_I f_n < \infty. \text{ Moreover } \int_I f = \lim_{n\to\infty} \int_I f_n.$$

DCT:
Suppose f_n and h are integrable on I and $|f_n| < h$ for all and n and that $\lim_{n\to\infty} f_n(x) = f(x)$ for all x in I.
Then f is integrable and

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Proof:
It is easy to see that

$$\int_I f < \int_I g \text{ whenever } f < g$$

and both functions are integrable. Just consider a gauge that selects the right tagged division for both functions (e.g. $\gamma(z) := \gamma_f(z) \cap \gamma_g(z)$, where γ_f and γ_g select divisions required to make f and g integrable, respectively). Because in case of a monotone sequence $f_n < f$ or $f_n > f$ for all n (depending on whether (f_n) is an increasing or decreasing sequence) this means that the $\Rightarrow$-implication in the MCS is always true.
It remains to be shown that the integral of f exists and equals $\lim_{n\to\infty} \int_I f$ in both cases.
The **Iterated Limits theorem** tells us that

$$\lim_{n\to\infty}(\lim_{\mathcal{D}} f_n M(\mathcal{D})) = \lim_{\mathcal{D}}(\lim_{n\to\infty} f_n M(\mathcal{D}))$$

whenever there's a gauge γ such that $|\int_I f_n - f_n M(\mathcal{D})| < \epsilon$ for all γ-fine tagged divisions $\mathcal{D}$ (**) for all n greater than a fixed number $N \in \mathbf{N}$. A number A is the limit of f according to $\mathcal{D}$ provided that A is the (unique) integral of f over I.
The proof that such a gauge exists is quite long and will therefor be broken up in a number of segments.

a) First, consider a function $G : \overline{\mathbf{R}} \to \mathbf{R}$ that is continuous and differentiable, and has $G'(x) > 0$ for all $x \in \mathbf{R}$, a second function with $g(x) := G'(x)$ and a function $\tau(J) := 2\Delta G(J)$, where J is a closed interval.

Calling upon the corollary of the Straddle Lemma used in the proof of the Fundamental Theorem, there is $\delta > 0$ such that $G'(z)(v-u) < 2(G(v) - G(u))$ whenever $z \in [u, v]$ and $[u, v] \subset (z - \delta, z + \delta) =: \gamma_g(z)$. This means that $g(z)M(J) < \tau(J)$ if $z \in J$ and $J \subset \gamma_g(z)$.
Next, we can make G so that $\tau(\overline{\mathbf{R}}) = 1$. Observe for example an arctan-function divided by π and that has values $-\frac{1}{2}$ and $\frac{1}{2}$ at the end-points of the extended real line. Because τ is an additive function, it is obvious that

$$gM(\mathcal{D}) = \sum_{zJ\in\mathcal{D}} g(z)M(J) < \sum_{zJ\in\mathcal{D}} \tau(J) \leq 1.$$

b) Let γ_n be a gauge such that $|\int_I f_n - f_n M(\mathcal{D}| < \frac{\epsilon}{2^n}$ whenever $\mathcal{D}$ is γ_n-fine.
Let

$$F_n := \bigcap_{k=n}^{\infty} \left\{ z : |f_k(z) - f(z)| \leq \frac{\epsilon}{8}\, g(z) \right\}.$$

(F_n) is an increasing sequence since we intersect over less sets as n increases. Furthermore, its limit is I because $lim_{n\to\infty} |f_n(z) - f(z)| = 0$ for all $z \in I$.
Now fix $N \in \mathbf{N}$ and define

$$E_N := F_N \text{ and } E_n := F_n \cap F_{n-1}^c \text{ for } n \geq N.$$

Then $\{E_n\}_{n=N}^{\infty}$ is a partition of I since $F_{n-1} \subset F_n$. For $z \in E_n$, define

$$\gamma(z) := \gamma_g(z) \cap \gamma_1(z) \cap ... \cap \gamma_n(z).$$

We will show that this is the gauge for which (**) holds.

c) For a tagged division $\mathcal{D}$ of I, define the collections

$$\mathcal{D}_i := \{ zJ \in \mathcal{D} : z \in E_i \},\ \mathcal{E}_n := \bigcup_{i=N}^{n-1} \mathcal{D}_i \text{ and } \mathcal{F}_n := \bigcup_{i=n}^{\infty} \mathcal{D}_i.$$

When $\mathcal{D}$ is γ-fine, then $\mathcal{D}_i$ is $\gamma_1 \cap ... \cap \gamma_1$-fine, so $\mathcal{F}_n$ is γ_n-fine, but $\mathcal{E}_n$ is not. So for $n > N$ we have, with the additivity of the integral and the triangle inequality, that

$$\left| \int_I f_n - f_n M(\mathcal{D}) \right| = \left| \sum_{zJ\in\mathcal{D}} \int_J f_n - f_n M(\mathcal{E}_n) - f_n M(\mathcal{F}_n) \right|$$

$$\leq \left| \sum_{zJ\in\mathcal{E}_n} \int_J f_n - f_n M(\mathcal{E}_n) \right| + \left| \sum_{zJ\in\mathcal{F}_n} \int_J f_n - f_n M(\mathcal{F}_n) \right| \leq \left| \sum_{zJ\in\mathcal{E}_n} \int_J f_n - f_n M(\mathcal{E}_n) \right| + \frac{\epsilon}{2^n}$$

$$\leq \left| \sum_{zJ\in\mathcal{E}_n} \int_J f_n - f_n M(\mathcal{E}_n) \right| + \frac{\epsilon}{4} \text{ since } n \geq 2.$$

In the second last inequality we made use of *Henstock's Lemma*, which asserts that a gauge selects Riemann sums as well on subintervals of I as it does on the whole interval I.
Formally:
Henstock's Lemma: If k is a function on I and γ_k is a gauge such that $|\int_I k - kM(\mathcal{D})| < \epsilon$ when $\mathcal{D}$ is γ_k-fine. Let $\mathcal{E}$ be a subcollection of $\mathcal{D}$. Then

$$\left| \sum_{zJ\in\mathcal{E}} \left[\int_J k - kM(zJ) \right] \right| \leq \epsilon.$$

Back to the proof of the convergence theorems. For the collection $\mathcal{E}_n$ we have

$$\left|\sum_{zJ\in\mathcal{E}_n}\int_J f_n - f_n M(\mathcal{E}_n)\right| = \left|\sum_{i=N}^{n-1}\sum_{zJ\in\mathcal{D}_i}\int_J f_n - \sum_{i=N}^{n-1} f_n M(\mathcal{D}_i)\right| \leq$$

$$\sum_{i=N}^{n-1}\left|\sum_{zJ\in\mathcal{D}_i} f_n - \sum_{zJ\in\mathcal{D}_i} f_i\right|$$

$$+\sum_{i=N}^{n-1}\left|\sum_{zJ\in\mathcal{D}_i}\int_J f_i - f_i M(\mathcal{D}_i)\right|$$

$$+\sum_{i=N}^{n-1}\left|f_i M(\mathcal{D}_i) - f_n M(\mathcal{D}_i)\right|.$$

The second sum is smaller than $\sum_{i=N}^{n-1}\frac{\epsilon}{2^i} < \frac{\epsilon}{4}$ because each of the terms here are smaller than $\frac{\epsilon}{2^i}$. The third sum can also be bounded:

$$\sum_{i=N}^{n-1}\left|f_i M(\mathcal{D}_i) - f_n M(\mathcal{D}_i)\right| = \sum_{i=N}^{n-1}\left|\sum_{zJ\in\mathcal{D}_i}(f_i(z) - f_n(z))M(J)\right| \leq \sum_{i=N}^{n-1}\left|\sum_{zJ\in\mathcal{D}_i}\frac{2\epsilon}{8}g(z)M(J)\right|$$

$$= \sum_{i=N}^{n-1}\frac{\epsilon}{4}gM(\mathcal{D}_i) = \frac{\epsilon}{4}gM(\mathcal{E}_n) \leq \frac{\epsilon}{4},$$

because $\mathcal{E}_n$ is γ_g-fine and therefor with the result in (a) is $gM(\mathcal{E}_n) < 1$.
So far we have

$$\left|\int_I f_n - f_n M(\mathcal{D})\right| = \sum_{i=N}^{n-1}\left|\sum_{zJ\in\mathcal{D}_i} f_n - \sum_{zJ\in\mathcal{D}_i} f_i\right| + \frac{3\epsilon}{4},$$

so it remains to be shown that $\sum_{i=N}^{n-1}|\sum_{zJ\in\mathcal{D}_i} f_n - \sum_{zJ\in\mathcal{D}_i} f_i| < \frac{\epsilon}{4}$

(d) To that end we will now use the convergence properties of the sequence of functions. First let (f_n) be monotonous. Then all the terms $\int_J f_n - \int_J f_i$ have the same sign for all i. Also, because $N \leq i < n$, we have

$$\left|\int_J (f_n - f_i)\right| \leq \left|\int_J (f_n - f_N)\right|$$

so that

$$\sum_{i=N}^{n-1}\left|\sum_{zJ\in\mathcal{D}_i}(f_n - f_i)\right| \leq \sum_{i=N}^{n-1}\sum_{zJ\in\mathcal{D}_i}\left|\int_J (f_n - f_i)\right| = \sum_{zJ\in\mathcal{E}_n}\left|\int_J (f_n - f_i)\right| \leq \sum_{zJ\in\mathcal{E}_n}\left|\int_J (f_n - f_N)\right|$$

$$= \left| \sum_{zJ \in \mathcal{E}_n} \int_J (f_n - f_N) \right| \leq \left| \sum_{zJ \in \mathcal{D}} \int_J (f_n - f_N) \right| = \left| \int_I (f_n - f_N) \right|.$$

We were trying to prove the if-statement in the MCT, so we're assuming that $\int_I f_n$ converges. This means we can choose N so big that $|\int_I f_n - \int_I f_N| < \frac{\epsilon}{4}$ when $n > N$ which concludes the proof of the Monotone Convergence Theorem.

e) Now assume that $|f_n - f_m| < h$ for all n, m and that h, f_m and f_n are integrable. This is the same assumption as was made in the DCT, since h integrable implies that $2h$ is integrable. Then (it is easy to see that) for j and k with $j \leq n < m \leq k$, $\max_{j \leq n < m \leq k} |f_n - f_m|$ is an integrable function that is $\leq h$ and increasing as k grows. Define

$$t_j := \lim_{k \to \infty} \left[\max_{j \leq n < m \leq k} |f_n - f_m| \right].$$

This function is bounded by h and decreases to 0 if j goes to infinity, so that the limit of $(t_j)_j$ is integrable on account of the MCT. Therefor we know that

$$\lim_{j \to \infty} \int_I t_j = 0.$$

This means we can choose N so that $\int_I t_N < \frac{\epsilon}{4}$. $|f_n - f_i| \leq t_N$ for $N \leq i < n$, which leads to the conclusion that

$$\sum_{i=N}^{n-1} \left| \sum_{zJ \in \mathcal{D}_i} f_n - \sum_{zJ \in \mathcal{D}_i} f_i \right| \leq \sum_{zJ \in \mathcal{E}_n} \left| \int_J (f_n - f_i) \right| \leq \sum_{zJ \in \mathcal{E}_n} t_N \leq \int_I t_N < \frac{\epsilon}{4}.$$

$\square$

Chapter 5

Conclusion

5.1 Comparison to Lebesque

Those who have enjoyed education in Lebesque theory may be interested in how this Generalized Riemann integral compares to the Lebesque integral. I will suffice with giving some of the results. Proofs of these can be found in *A modern theory of integration* by Robert Gardner Bartle.

-Suppose f is a bounded function. Then the following properties are equivalent:

-f is Lebesque integrable,
-f is Generalised Riemann integrable,
-f is Lebesque measurable.

-For general f it's a fact that f is Lebesque integrable if and only if both f and $|f|$ are Generalised Riemann integrable.
-If f is differentiable everywhere on I, then its derative is Generalised Riemann integrable, *but not necessarily Lebesque integrable.*

5.2 Final words

To those readers is will also be clear that this integral is, as long as we limit ourselves to one dimension, far easier to master than the Lebesque version. No understanding of σ-algebra's, measurable sets and measurable functions is required. Although proofs of the heavier statements such as the convergence theorems in chapter 3 tend to get long and unsuitable for undergraduate students, showing that a function is integrable is (usually) an easy to perform task.

A downside are the higher dimensions. While the Lebesque integral can easily be applied to higher dimensions once understanding of the basic theory has been acquired, this is a much more difficult issue in Henstock's theory. The Lebesque integral may therefor be more preferable for applications, such as stochastical processes. Also, the kind of functions that are Generalised Riemann, but not Lebesque integral are only valuable when trying to come up with counterexamples rather than practical purposes.
An example of such a function is $x^2 \cos(\frac{1}{x^2})$.

I would therefor refrain from going as far as suggesting to teach this integral instead of the Lebesque one, but giving students the tools to deal with more irregular functions, such as the one used in several examples throughout the thesis, during an introduction to analysis-class near the end of the first year would certainly add to their capabilities later on.

I hope to have provided a clear overview of this integral. More (detailed) information can be found in the articles mentioned in the reference section.

Chapter 6

Reference

The Generalized Riemann Integral Robert M. McLeod
A modern theory of integration Robert Gardner Bartle
http://www.math.vanderbilt.edu/ schectex/ccc/gauge/