Braids, Floer homology and forcing in two and three dimensional dynamics

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CHAPTER 1

Introduction

1.1. Low dimensional dynamics and topological forcing

A topological structure in 2 and 3 dimensional dynamics

Many problems in the natural sciences can be stated and studied in the mathematical language of dynamical systems. These systems often come with an underlying topological structure, which can be exploited in order to draw conclusions about their behavior. As an example of the aforementioned structure, let us consider the non-autonomous differential equation

$$x' = X(x,t),$$

where $x = (p,q) \in \mathbb{R}^2$, $t \in \mathbb{R}$ and $f : \mathbb{R}^2 \times S^1 \to \mathbb{R}$ is a sufficiently smooth function. In such a setting, let $x(t)$ and $y(t)$ be two periodic solutions of period one, that intersect in the (p,q)-plane. When lifted to the extended phase space $\mathbb{R}^3$ they *link* (see Figure 1.1). This link carries the topological information that we will exploit to study the evolution of the system, or to conclude existence of additional solutions. This also suggests another concept that is essential to our studies – forcing. The knowledge of the way that the two trajectories are intertwined may allow us to construct additional solutions.

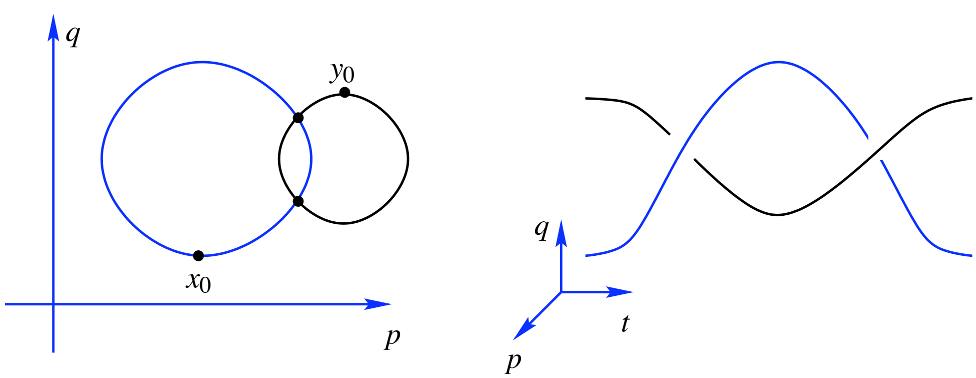

Figure 1.1: The trajectories of two periodic points x_0 and y_0, intersecting in the two dimensional phase space [left]. Their lifts link as the topological circles (end points are identified) [right].

A similar approach can be applied to autonomous systems in $\mathbb{R}^3$, that is, to equations of the form

$$x' = Y(x),$$

where $x \in \mathbb{R}^3$. This topological structure cannot be exploited in higher dimensional systems because linking is trivial in dimensions 4 and higher. Nevertheless, the above mentioned approach may be used to obtain results for a wide class of problems. For example, in the context of the non-autonomous Hamilton equations with one degree of freedom, or for systems generated by iterating certain orientation reversing diffeomorphisms of the plane.

REMARK 1.1. Throughout the introduction we only number the theorems proved in this thesis. The references to the results of other authors are given in the comments before or after their statements.

Dynamical order relations

A well known forcing result in dynamics is Sharkovskii's theorem proved in 1964 (see [**64**]). It concerns one dimensional discrete dynamical systems and can be formulated as follows. Let $f : I \to \mathbb{R}$ be a continuous map, where $I \subset \mathbb{R}$ is a bounded interval. The function f defines a dynamical system in a standard way ($x_{n+1} = f(x_n)$). Before stating the theorem we introduce an ordering on $\mathbb{N}$. Every positive integer n can be uniquely written in a form $2^r p$, where p is an odd number and r is such that 2^r is the highest power of 2 that divides n. Using this description we order the natural numbers in the following way

$$3 \succ 5 \succ 7 \ldots \succ 2\cdot 3 \succ 2\cdot 5 \ldots \succ 2^r \cdot 3 \succ 2^r \cdot 5 \succ \ldots \succ 2^r \succ \ldots \succ 2 \succ 1.$$

This is *Sharkovskii's ordering*. With this we can state the following.

THEOREM (Sharkovskii). *If f has a point of period n, then it necessary has at least one point of period n_1, provided $n \succ n_1$.*

In particular, if f has a period three point then it has periodic points of all periods. This explains why this theorem is sometimes referred to as "period three implies chaos" ([**44**]). The proof relies on studying carefully how the intervals bounded by the points of a given periodic orbit are mapped. In the simplest case one can use the intermediate value theorem to obtain forced solutions. This technique, while interesting and elementary, is restricted to dimension one.

Poincaré's geometric theorem

A well-known forcing result in dimension two, is Poincaré's geometric theorem, originally presented in 1912 ([**58**]) and proved by Birkhoff in 1917 ([**12**]). In its basic form it can be described as follows. Let A be an annulus of which the boundary consists of two circles $C(r_1)$ and $C(r_2)$, with radii r_1 and r_2 respectively ($r_1 > r_2 > 0$). Let $f : A \to A$ be a continuous area-preserving injective map rotating the outer circle in the counter clockwise and inner circle in the clockwise direction (cf. Figure 1.2). With this we can state the following.

THEOREM (Poincaré's geometric theorem). *Under the above assumptions, f has at least two fixed points in A.*

This theorem is an example, how general knowledge of a problem (area preservation) and some limited information on the behavior of the map (rotation at the boundary) allows

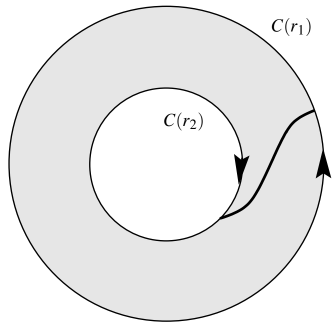

Figure 1.2: The annulus bounded by circles $C(r_1), C(r_2)$. The arrows indicate the directions in which the flow induced by f twists the boundary. Poincaré's theorem guarantees existence of at least two fixed points.

us to conclude the existence of stationary or periodic orbits. This kind of methodology will be central in our work.

Twist maps

Poincaré's geometric theorem above can be interpreted in the context of *twist maps*. We say that a continuously differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^2$ has the *twist property* if there exist global coordinates (x,y) such that $\frac{\partial x'}{\partial y} > 0$ ($\frac{\partial x'}{\partial y} < 0$), where $(x',y') = f(x,y)$. We call such a function f a *positive* (*negative*) *twist map*. As an example, consider the *Hénon map*. It is a invertible map of the plane given by

$$f : (x,y) = (-y, 1 - \alpha y^2 + x).$$

This is an orientation and area preserving negative twist map. It is well known for its strange attracting set, known as *Hénon attractor*, that exhibits Cantor-set like structures.

Let f be a twist map of the annulus in polar coordinates (r,θ). Then the graph of f can be schematically presented as in Figure 1.3. This guarantees that, in a sense, f rotates the outer circle faster than the inner circle. Poincaré's geometric theorem discussed above implies that f has at least two fixed points. Exploiting the fact that f is a twist map gives a simple proof to Poincaré's geometric theorem (see [**50**]). In general, let $(x',y') = f(x,y)$, then due to the implicit function theorem, there exist functions h_1, h_2 such that $y = h_1(x,x')$ and $y' = h_2(x,x')$. Moreover, area preservation shows that there exists a C^2 function $h(x,x')$ on $\mathbb{R}^2$ such that $h_1 = \partial_1 h$ and $h_2 = -\partial_2 h$ (see [**47**] or [**10**]). For fixed points of an annulus twist map the above yeilds a periodic function $h : S^1 \to \mathbb{R}$, and h has at least two critical points (maximum and minimum) which proves Poincaré's geometric. In general, for d-periodic points of f the trajectory $\{f^k(x,y)\}_{k=1}^d = \{(x_k,y_k)\}_{k=1}^d$, can be uniquely described using its x-coordinates. Moreover, a sequence $x = \{x_k\}$ yields an f-trajectory (first coordinates thereof) if and only if

$$\partial_i W(x) = 0 \qquad \text{(variational principle)},$$

where $W(x) \sum_{i=1}^d h(x_i, x_{i+1})$.

For an annulus twist map f, its trajectories can be fully determined using only the angle coordinates θ_i. Consider an d-periodic orbit starting in (r_0, θ_0). By the same token as above,

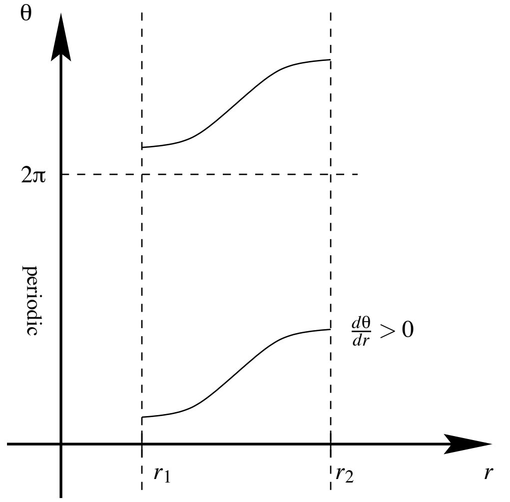

Figure 1.3: A twist map in the polar coordinates (r,θ). The radii r_1 and r_2 correspond to the boundary components of the annulus. The fact that $\frac{d\theta}{dr} > 0$ guarantees that the circles bounding the annulus have different rotation speeds under f.

there exists a function W such that

$$\partial_i W(\theta_0,\theta_1,\ldots,\theta_{n-1}) = 0 \qquad i = 1,\ldots,n, \tag{1.1}$$

where θ_i denotes the angular coordinates of the periodic orbit. Observe that W is defined on the n-dimensional torus. The question arises if the topological properties of the underlying manifold can provide us with any information about the critical points of W. Important contributions to the theory of twist maps were given by Moser [**56**], Mather [**47**], Aurby & Le Daeron [**10**], Angenent [**2**], and Boyland [**16**].

Morse theory

Morse theory, in its simplest form, can be described in the following way. Consider an n-dimensional smooth compact Riemannian manifold M. Let $h : M \to \mathbb{R}$ be a smooth function such that all its critical points (x is critical if $dh(x) = 0$) are non-degenerate (the Hessian of h at x, denoted by $H^2h(x)$, is invertible). Then to all points in

$$\mathrm{Crit}_h := \{x \in M \mid dh(x) = 0\}$$

one can assign the Morse index

$$\mu(x) = \dim \mathrm{Eig}^- H^2h(x),$$

where $\mathrm{Eig}^- H^2h(x)$ denotes the space spanned by eigenvectors corresponding to negative eigenvalues of $H^2h(x)$. Also

$$c_k := \#\mathrm{Crit}_h^k = \#\{x \in \mathrm{Crit}\, h \mid \mu(x) = k\},$$

is the number of critical points of index k.

Morse theory gives a relation between Crit_h and the topology of the underlying manifold M. To measure the topological properties of M one uses concepts from algebraic topology.

Recall that the Betti numbers of M are defined by

$$\beta_k = \dim H_k(M) \qquad k \in \{0, \dots, n\},$$

where $H_k(M)$ denotes the k-th singular homology group of M. We can now state the theorem about the *Morse inequalities*.

THEOREM (Morse inequalities). *For a smooth function $h : M \to \mathbb{R}$ with only non-degenerate critical points it holds that*

$$c_k - c_{k-1} + \dots \pm c_0 \geq \beta_k - \beta_{k-1} + \dots \pm \beta_0,$$

for all $k = 0, \dots, n-1$ and for $k = n$ we have equality

$$c_n - c_{n-1} + \dots \pm c_0 = \chi(M),$$

where $\chi(M) = \beta_k - \beta_{k-1} + \dots \pm \beta_0$ denotes the Euler characteristic of M.

If the condition of non-degeneracy of critical points is removed then the following estimate holds

$$\#\mathrm{Crit}_h \geq \mathrm{Cat}(M),$$

where the $\mathrm{Cat}(M)$ denotes (Ljusternik-Schnirelmann) category (minimal number of contractible subsets required to cover M). For example

$$\mathrm{Cat}(\mathbb{T}^n) = n + 1$$

In particular, for $n = 1$ we obtain that the number of critical points of a twist map on an annulus is greater than or equal to 2. That is why Morse theory can be considered a generalization of Poincaré's geometric theorem.

The theorem gives also a lower bound on the number of critical points of index k, i.e., $c_k \geq \beta_k$. The theory is due to Morse (see [**54**]) and was extended into several important directions by Thom [**67**], Smale [**65**], Milnor [**51**], Witten [**69**] and more recently, and most crucially for our work, by Floer (see Section 1.4).

Thurston-Nielsen theory

Let M be a compact, orientable two-manifold, possibly with boundary and let $f : M \to M$ be an orientation preserving homeomorphism. Iterations of f generate a dynamical system. We say that two homeomorphisms f_0, f_1 are isotopic (denoted $f_0 \simeq f_1$) if there exists an isotopy $f_t : M \times [0,1] \to M$ such that for all $t \in [0,1]$ the map $f_t(\cdot)$ is a homeomorphism. The set of all homeomorphisms of M can be divided into isotopy classes using the relation $\simeq$, and the collection of all isotopy classes together with composition forms a group, known as the *mapping class group*, denoted by $\mathrm{MCG}(M)$. If M has a boundary, one usually considers only homeomorphisms that are the identity on the boundary and isotopes that fix it pointwise, leading to $\mathrm{MCG}(M, \partial M)$. Additionally, it is very useful to consider isotopy classes relative to some finite set A, that is, we take into account only homeomorphisms leaving A invariant and isotopies fixing it. This yields $\mathrm{MCG}(M \text{ rel } A)$. Combining it with the boundary case, $\mathrm{MCG}(M \text{ rel } A, \partial M)$. In applications one should think of A as being a known periodic orbit.

The core of the Thurston-Nielsen classification theory is that every homeomorphism can be decomposed into components on which the dynamics is simple and components with complicated dynamics. To describe the chaotic part, recall the linear hyperbolic toral

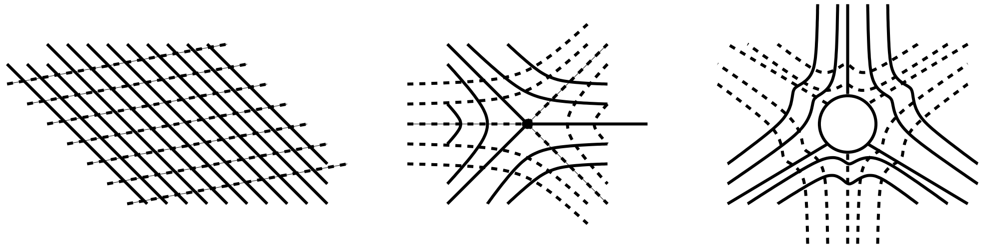

Figure 1.4: Local pictures of two transverse foliations (one depicted with solid lines and the other one with dashed lines) with singularities. The figure on the right presents a foliation near a boundary component.

automorphisms, namely smooth invertible maps of the torus that uniformly contract one direction and stretch the other. Those are standard examples of chaotic maps. It is possible to extend this concept to other surface maps. One needs to replace the orthogonal directions, as those may not exist in general. One uses *transverse foliations* with singularities, which we do not introduce formally here (for the intuitive picture cf. Figure 1.4). Using the concept of a *transverse measure* on foliations (measuring length of arcs transverse to the foliation), one can generalize the concept of stretching and contracting. We say that a map $f : M \to M$ is *pseudo-Anosov relative the finite set A* if there exist two transverse foliations with singularities only in points of A and corresponding measures such that the image of f along one of the measured foliations is stretched with constant $\lambda > 1$, whereas along the other it is contracted by a factor $\frac{1}{\lambda}$. We say that map is pseudo-Anosov (abbreviated pA map) if the set A is empty.

The dynamics of a pA map is complicated (chaotic). On the other side of the scale, we have maps that are fairly simple. We say that a map $g : M \to M$ is of *finite order* if there exists an $n > 0$ such that $g^n = \mathrm{id}$. Finally, a map f is *Thurston-Nielsen reducible*, if M can be decomposed (we do not make it precise here) into connected components that are dynamically separated by f and such that f restricted to each of them is either pA or finite order. The Thurston-Nielsen classification theorem can be summarized by saying that every isotopy class in $\mathrm{MCG}(M \text{ rel } A)$ contains a Thurston-Nielsen reducible representative (provided that $M \setminus A$ has negative Euler characteristic).

The philosophy is to find a periodic orbit (a set A) that may force chaotic, pA behavior. We should also mention that in general it is far from trivial to find a set A for which one can conclude that the given map is pA relative the set A. For an overview of the methods outlined here, one should consult [**17**]. The details of the proof of the classification theorem can be found in [**68**].

The above methods fall into two categories. Sharkovskii's theorem and Thurston-Nielsen theory allow one to prove the existence of periodic solutions using the knowledge of other periodic solutions, due to the dynamical forcing. On the other hand, results like Poicnaré's geometric theorem and Morse theory provide a lower bound on the number of critical points, using the topology of the underlying manifold. Combining the two above approaches, can lead to a new class of results that give insight into the dynamics of the low dimensional systems. This is the methodology that we will use in this thesis.

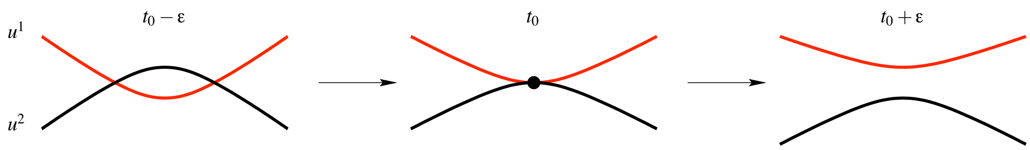

Figure 1.5: The local picture of the evolution of two solutions of a parabolic PDE that develop a tangency at t_0 [middle]. The number of intersections among the solutions drops by two in a neighborhood of t_0.

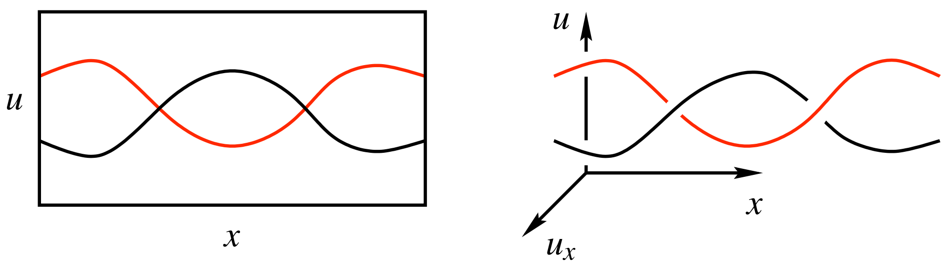

Figure 1.6: Two solutions of a parabolic equation [left]. Their lift to $(x,u,u_x,)$-space [right].

1.2. Braids and dynamics

Parabolic equations - braids enter the dynamics

The idea of using the topological structure of linked solutions (braids, knots) to obtain forcing results in dynamical systems was already used in several settings. In particular, we mention Thurston-Nielsen theory and the lap number techniques that we will discuss in this section.

An important motivation for using braid theory comes from the comparison principle. Consider a parabolic partial differential equation

$$u_t = f(x,u,u_x,u_{xx}) = u_{xx} + g(x,u,u_x), \quad x \in \mathbb{R}/\mathbb{Z},$$

where g is a smooth function. For two solutions of this equation, $u^1(x,t)$ and $u^2(x,t)$, define the number of crossings between them by

$$z_{u_1,u_2}(t) = \#\{x \mid u_1(x,t) = u_2(x,t)\}.$$

It turns out that z is a non-increasing function of time, and if u^1 and u^2 intersect non-transversally (topologically) then the intersection will be destroyed (cf. Figure 1.5). This was first observed by Sturm and later used and extended by many authors including Matano [**46**], Brunovsky and Fiedler [**19**], Angenent [**3**], [**6**], etc. Similar techniques were used in the context of curve-shortening to prove existence of geodesics on two dimensional manifolds (see [**6, 5**]).

Lifting u^1 and u^2 to the (x,u,u_x)-space, one recognizes a braid structure (cf. Figure 1.6) as two solutions wind around each other. The fact that z_{u_1,u_2} is non-increasing in t, translates into the language of the braid theory: along the evolution the complexity of the braid corresponding to those solutions cannot increase.

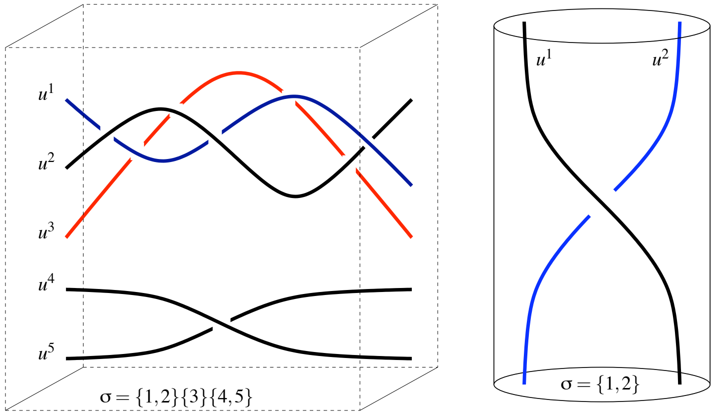

Figure 1.7: Two conventions for presenting braids. Horizontal – a braid on 5 strands consisting of two connected components [left]. Vertical – a braid on two strands (a single component) [right]. Both have the corresponding permutation indicated.

Braids and braid classes

Intuitively it is clear what a braid is, and the mathematical definition reflects this intuition. Roughly speaking, a braid consists of several strings that are intertwined. To be precise, we consider a braid to be a collection of n continuous curves $u^\alpha\,[0,1] \to \mathbb{R}^3$, called *strands*, that are transversal to all planes parallel to one of the directions. For example, we can assume $\partial u_1^\alpha > 0$ for all α, where $u^\alpha = (u_1^\alpha, u_2^\alpha, u_3^\alpha)$ (cf. Figure 1.7). Moreover, strands are assumed to have disjoint images (they do not intersect).

The braids considered in this thesis are closed. This does not necessary mean that all strands are periodic, but that there exists a permutation σ on n elements such that $u^\alpha(1) = u^{\sigma(\alpha)}(0)$ for all $\alpha \in \{1,\dots,n\}$ (cf. Figure 1.7). We will denote the braids using bold font, i.e., $\mathbf{u} = \{u^1,\dots,u^n\}$, omitting the corresponding permutation σ if it is clear from the context. Cycles of the permutation divide braids into *braid components* (see again Figure 1.7).

We say that, two braids are equivalent if one can be deformed into the other without creating any intersections along the path. The equivalence classes of this relation are called *braid classes*. For a schematic presentation of a braid class and examples of equivalent braids see Figure 1.8. Observe that two different braid classes on n strands are necessary separated by so called *singular braids*, i.e., collections of curves that *do* have intersections among the strands (cf. Figure 1.8).

Singular braids can have intersections of different degree of degeneracy ranging from two strands having a single isolated crossing to two (or more) strands collapsed onto each other. They can be viewed as the boundaries between braid classes. Observe that a continuous path between the representatives of two braid classes sharing a co-dimension one boundary component contains a singular braid that has one intersection among the strands (of course choosing a path through the aforementioned boundary components). At this point recall, the behavior of the parabolic flow, in which the transversal intersections of two solutions are destroyed along the evolution.

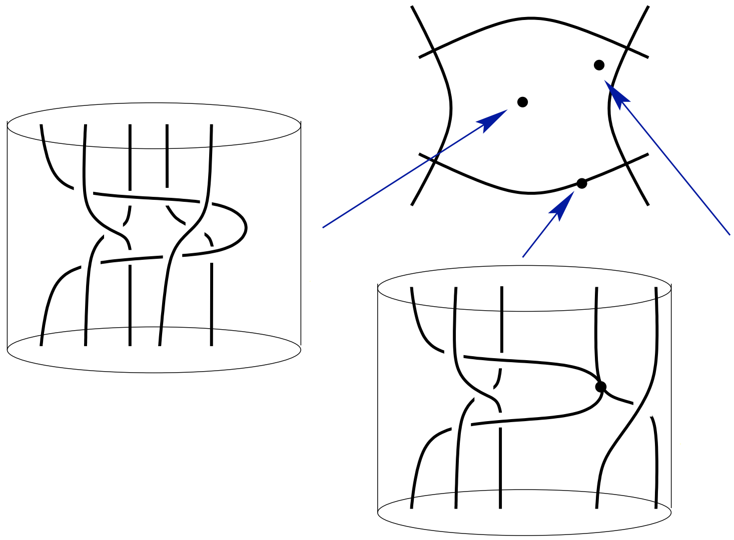

Figure 1.8: Schematic picture of a braid class with two representatives indicated [left] and [right] and a singular one corresponding to a point on the boundary of the class [middle].

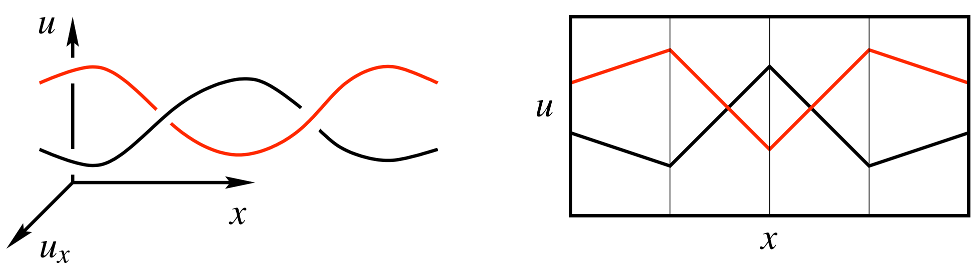

Figure 1.9: A braid on two strands [left]. The braid diagram (piecewise linear representant of the class of the two dimensional projection of the braid) [right].

The algebraic structure of braids

A *discretized braid diagram* is created from a braid by considering a (generic) two dimensional projection and then taking a piecewise linear approximation of it (cf. Figure 1.9). In this context, strands u^α are represented as k-tuples $u^\alpha := (u_1^\alpha, \dots, u_k^\alpha)$, and a braid as an unordered collection of such tuples. Here k denotes the number of discretization points. We also keep track of the type of crossings.

Braids carry the following algebraic structure. Let the configuration space be defined as $C_n := \{z \in \mathbb{C}^n \, z_j \neq z_i, i \neq j\}/\Sigma_n$ where Σ_n denotes the permutations on n symbols, i.e., C_n is a space of unordered n-tuples of distinct points in $\mathbb{C}$, or equivalently $\mathbb{R}^2$. Then the group of braids on n strands, denoted B_n, is defined as the fundamental group of C_n, that is, the space of equivalence classes of loops in C_n (i.e. $B_n = \pi(C_n)$). Observe that a loop $\mathbf{u}$ on this space can be viewed as a set of n unordered paths $u^\alpha : [0,1] \to \mathbb{R}^2$ such that $u^{\alpha_1}(t) \neq u^{\alpha_2}(t)$ for $\alpha_1 \neq \alpha_2$ and $u^\alpha(0) = u^{\sigma(\alpha)}(1)$ for some permutation $\sigma \in \Sigma_n$. By drawing graphs in $\mathbb{R}^3$ it is easy to see that we obtain n strands connecting the plane $(0,y,z)$ with $(1,y,z)$, which are

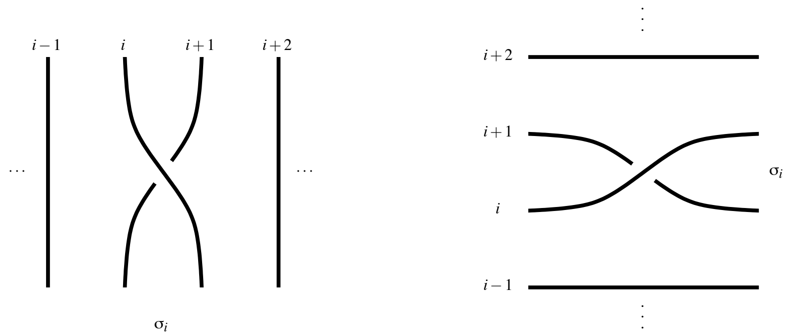

Figure 1.10: Standard representations of the generator σ_i.

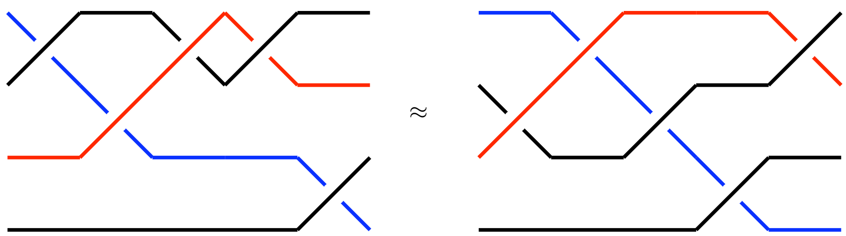

Figure 1.11: The figure presents two piecewise linear braids that have equivalent symbolic description: the one on the left $\sigma_3\sigma_2\sigma_3\sigma_3\sigma_1$ and the one on the right $\sigma_2\sigma_3\sigma_2\sigma_1\sigma_3$. Observe that these braids lie in the same braid class.

tangled. Concatenation of two such paths introduces the group structure. Again, two paths are homotopic if their stands can be homotoped without intersections along the path.

Artin in [**9**] showed that the braid group B_n can be given the following representation. Let σ_i be the braid that twists (exchanges) strands i and $i+1$ with i-th strand passing over $i+1$-st one (*positive crossing*) and leaving all other strands invariant (the standard representation of this generator is presented in Figure 1.10). Then the group B_n can be described using letters $\{\sigma_1,\ldots\sigma_{n-1}\}$ and their inverses (σ_i^{-1} twists the same strands as σ_i but i-th strand passes beneath the $i+1$-st; *negative crossing*) with two equivalence relation among the combinations of symbols, namely

$$\begin{cases} \sigma_i\sigma_j &= \sigma_j\sigma_i & \text{for } |i-j|>2,\ 1\leq i,j\leq n-1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1\leq i\leq n-2 \end{cases}$$

An element of this group is usually presented as a braid diagram i.e. a piecewise linear representative of the class (cf. Figure 1.11, which also shows the above equivalence relations).

It is worth to mention two problems related to the algebraic structure of braids. The first one is the word problem, that is, the question whether it is possible to decide algorithmically if the two given sequences of generators represent the same element in the braid group. The affirmative answer was already given by Artin in [**9**]. The second one is the conjugacy problem. We ask, given two braids $\mathbf{x}$ and $\mathbf{y}$, whether one can algorithmically find (and decide whether it exists) a braid $\boldsymbol{v}$ such that $\mathbf{y}=\boldsymbol{v}^{-1}\mathbf{x}\boldsymbol{v}$. This problem was first solved by

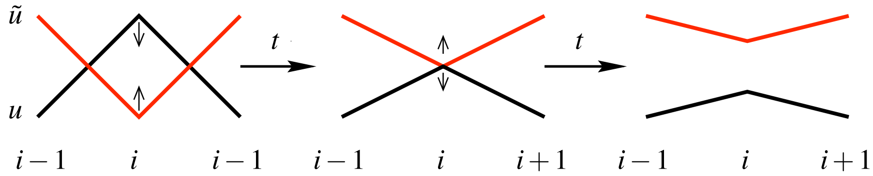

Figure 1.12: Along the evolution of the parabolic flow crossings can be destroyed, not created. Compare to Figure 1.5.

Garside in [**32**]. The results in this paper, in particular, the Garisde normal form of a braid will be also used in Chapter 3.

Parabolic flows and braid diagrams

Let us consider the equations of the form

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = 0, \tag{1.2}$$

where $\mathcal{R}_i : \mathbb{R}^3 \to \mathbb{R}$ are *increasing* in both the first and third variable. Moreover, assume that the sequence of $\mathcal{R}_i$ is periodic, i.e., $\mathcal{R}_{i+d} = \mathcal{R}_i$ for some $d \in \mathbb{N}$. Such a sequence $\mathcal{R}$ is called a *parabolic recurrence relation* (compare with monotone recurrence relations studied for example in [**4**] and [**37**]). We would like to point out that the properties of a parabolic recurrence relation resemble those obtained via generating function h (Section 1.1).

A periodic solution of Equation (1.2), $\{u_i\}$ with $u_{i+d} = u_i$, can be depicted in a diagram by connecting the points u_i with straight lines. We interpret a collection of such solutions (sequences) as a (discretized) braid diagram. Due to nature of our problem it is enough to restrict ourself to positive braids only, i.e., only positive generators. This superimposed structure becomes natural if we consider the *parabolic flow*

$$\frac{d}{dt} u_i = \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}).$$

The change of the position of the i-th point depends on its two nearest neighbors. One can deduce from the monotonicity properties of the parabolic recurrence relation that, along the flow, crossings among the strands can be destroyed but not created. If the evolution of the system develops an isolated tangency in a braid diagram (exactly two points of different strands collide, Figure 1.12), then the monotonicity conditions on $\mathcal{R}$ guarantee that

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) - \mathcal{R}_i(\tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}) = u_i' - \tilde{u}_i'$$

has a sign (negative in Figure 1.12). That is, the number of crossings before the collision will necessarily be greater than the number of crossings after. We already mentioned that the singular braids (possessing intersections among the strands) can be seen as boundary between two classes. With this we obtain the intuitive picture presented in Figure 1.13 (in analogy to the parabolic partial differential equation in Figure 1.5).

Observe that in this setting a strand is just a sequence of points, but for the following two reasons we connect them piecewise linearly. First, visually it allows to deduce which points belong to one strand. Secondly, this formalism provides us with a way to encode the decreasing of intersections along the parabolic flow. The properties described above suggest that one should look at the boundary of a braid class and decide about the direction of the

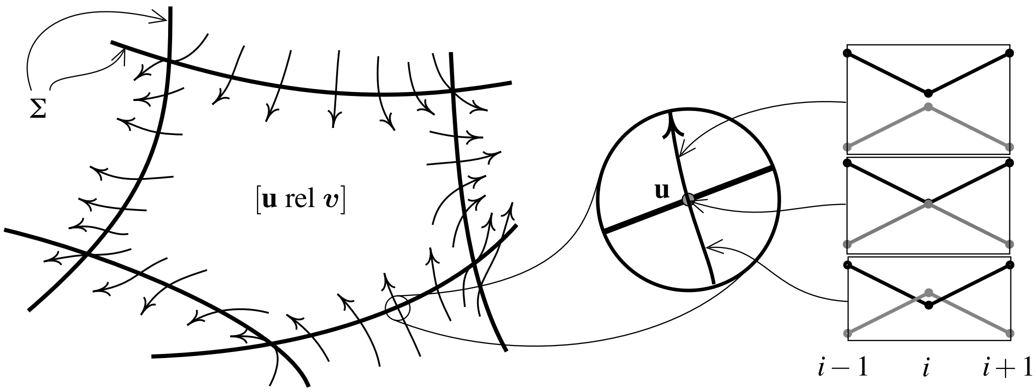

Figure 1.13: A schematic picture of a parabolic flow on a braid class. Σ denotes the set of singular braids (boundary of the class).

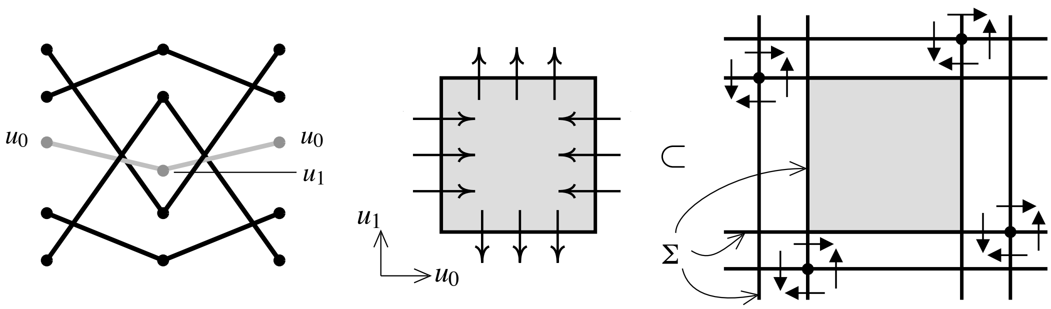

Figure 1.14: In the braid on the left, black strands denote the strands fixed by the flow and the gray one is allowed to move. In the middle its configuration space is shown and the direction of the parabolic flow on the boundary is indicated. On the right we see how the configuration space is positioned with respect to the stationary points of the flow (black strands), represented by the four dots.

flow on the singular braids by comparing the number of crossings in the adjacent classes. It turns out that this is indeed possible, but there are issues one needs to resolve to make this kind of argument precise (see chapter 2 and [**33**]).

To give some intuition of how one can use the properties mentioned above, let us discuss in more detail the example presented in Figure 1.14. Consider the braid depicted on the left in this figure. We deal with the braid diagram with two discretization points (points on the left are identified with those on the right). Coming back to the ideas of forcing, assume that we know four two-periodic solutions of the parabolic flow (depicted as black strands in the figure). We want to investigate the possible behavior of an additional grey strand under the evolution of the flow. Observe that the crossing property traps the point u_0 between the points of the black strands. Traversing any of them would increase the overall number of crossings of the braid. The situation for u_1 is different though, as once it touches any of the stationary strands it will be forced to move past them (the number of crossing drops in such a case). Drawing a schematic picture of the braid class in the (u_0,u_1)-plane and noting the direction of the flow on its boundary yields the picture in the middle of Figure 1.14. It suggests a hyperbolic stationary point, hence an additional (forced) solution.

To prove the existence of forced trajectories one uses the theory of the Conley index for braid diagrams. For now one should just think of it as an algebraic topology tool, that allows us to draw rigorous conclusions about the existence of additional solutions inside of given braid class. The details of the construction of the index are carried out in [**33**] and some additional explanation can be found in chapter 2. For the general Conley index theory we refer to [**20**] or [**53**].

1.3. Orientation reversing twist maps

The above ideas can be used to study orientation *reversing* twist maps of the plane.

DEFINITION 1.2. A map f is called an *orientation reversing twist map* of the plane if it is C^1-smooth and if there exist global coordinates $(x,y)\in\mathbb{R}^2$ such that f given by $(x',y') = f(x,y)$ satisfies the following two conditions: (i) $\det(df)<0$, and (ii) $\frac{dx'}{dy}>0$. It is called *an orientation reversing twist diffeomorphism* of the plane if in addition the function f is a diffeomorphism.

Property (ii) is referred to as a *positive twist property* or simply twist property. One of the questions that Chapter 2 deals with, is how an orientation reversing twist map (diffeomorphism) leads to a parabolic recurrence relation $\mathcal{R}$ (see [**33**]). In the case when f is an orientation reversing map the situation is more complicated. While it is still possible to express the second coordinates of the trajectory in terms of the first ones and obtain a recurrence relation $\tilde{\mathcal{R}}$ depending on two nearest neighbors, the monotonicity condition does not hold. That is, $\tilde{\mathcal{R}}$ is decreasing in the first and increasing in the third variable (details can be found in Chapter 2). We introduce the following modification:

$$\begin{aligned}
\mathcal{R}_0(x_{-1},x_0,x_1) &\stackrel{\text{def}}{=} \tilde{\mathcal{R}}(-x_{-1},-x_0,x_1)\\
\mathcal{R}_1(x_0,x_1,x_2) &\stackrel{\text{def}}{=} \tilde{\mathcal{R}}(-x_0,x_1,x_2)\\
\mathcal{R}_2(x_1,x_2,x_3) &\stackrel{\text{def}}{=} -\tilde{\mathcal{R}}(x_1,x_2,-x_3)\\
\mathcal{R}_3(x_2,x_3,x_4) &\stackrel{\text{def}}{=} -\tilde{\mathcal{R}}(x_2,-x_3,-x_4).
\end{aligned}$$

Using the monotonicity conditions of $\tilde{\mathcal{R}}$ we infer that $\mathcal{R}_i$ indeed forms a parabolic recurrence relation in the above (periodic) sense.

Another way to look at this, is observing that for an orientation reversing map, its *second* iterate f^2 can be decomposed as $f^2 = f_+\circ f_-$ with $f_+ = f\circ R_x$ and $f_- = R_x\circ f$, where R_x is a linear reflection in the y-axis. Observe that both f_+ and f_- are orientation preserving maps, and while f_+ has a positive twist, f_- has a negative twist. The *fourth* iterate of f can be written as a composition of four orientation preserving maps with positive twist:

$$f^4 = f_3\circ f_2\circ f_1\circ f_0,$$

where $f_0=-f_-$, $f_1=-f_+\circ(-\mathrm{id})$, $f_2=f_-\circ(-\mathrm{id})$ and $f_3=f_+$. Each of the maps defines an $\mathcal{R}_i$ and brings us back to the theory of parabolic recurrence relations.

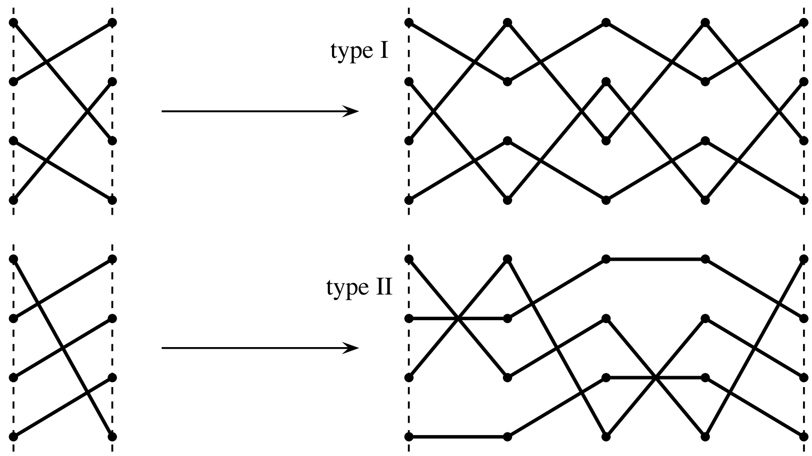

Figure 1.15: The flip transformation changes the braid diagram corresponding to a period four orbit point into a diagram of one of two types.

To introduce the braid diagrams in this setting, let z be a period four point of f and denote the points on its trajectory by $z_0 = z$, $z_1 = f(z)$, $z_2 = f^2(z)$ and $z_3 = f^3(z)$. Since the first coordinates x_i of the trajectory $z_i = (x_i, y_i)$ determine the trajectory fully, the corresponding braid diagram is obtained by connecting points (i, x_i) with piecewise linear graphs. Three additional strands are given by shifts of this trajectory, with respectively z_1, z_2 and z_3 as starting points. Now we need to incorporate the modifications applied to the recurrence relation to turn it into a parabolic recurrence relation. Analyzing the definition of $\mathcal{R}_i$, we see that we need to apply the following transformation to the x-coordinates of points of the trajectory:

$$(x_0, x_1, x_2, x_3) \to (-x_0, x_1, x_2, -x_3).$$

This will result in a sequence that satisfies $\mathcal{R}_i = 0$ if and only if the original $\{x_i\}$ satisfy $\tilde{\mathcal{R}} = 0$, hence form a trajectory of f. We call the above transformation a *flip*.

One also needs to investigate the effects of the flip on the braid diagrams constructed from a period four orbit. Initially those lead to six different diagrams depending on the permutation according to which the period four orbit visits its points. Applying the flip transformation to all six diagrams (by reversing the order of points on the flipped coordinates) surprisingly results in diagrams that can be divided into two types. To be more precise, the resulting diagrams lie in one of the two braid classes, as shown in Figure 1.15. If the resulting diagram of a given period-four trajectory belongs to the first of the classes we call it type I point and in the second case we call it type II. Our goal is to look for additional stationary solutions inside braid classes, with the strands corresponding to one of these two types being fixed. Hence we want to investigate the forcing properties (if any) of the period four points of type I and type II, see Figure 1.15.

The above mentioned type of braid diagram is a crucial element of the classification of period four orbits for an orientation reversing twist map. It turns out that we need a technical, but quite natural assumption, namely that f satisfies the *infinite twist condition*, i.e.,

$$\lim_{y\to\pm\infty} \pi_x f(x,y) = \pm\infty \quad \text{for all } x \in \mathbb{R},$$

where π_x denotes the projection onto the first coordinate. With this we can prove the following theorem:

THEOREM 1.3 (Chapter 2). *An orientation reversing twist map of the plane that satisfies the infinite twist condition and that has a type I period-4 point, is a chaotic system, i.e., the topological entropy of some bounded invariant set in $\mathbb{R}^2$ is positive. Conversely, there exists an orientation reversing twist map that satisfies the infinite twist condition, has a type II period-4 point, and that has zero entropy.*

Topological entropy is a measure of how complicated the dynamics of the system is and it essentially measures the growth of distinguishable periodic orbits as the period increases. If this growth is faster than exponential, the entropy is positive. We would like to stress here that under quite weak assumptions (especially no compactness is required) it is possible to obtain a fairly general result that classifies the period four orbits. In comparison, the Thurston-Nielsen theory does not give a bounded set on which a map is chaotic. Comapctifying $\mathbb{R}^2$ to the sphere allows applying the theory. Nevertheless, one cannot obtain a compact set on which the dynamics is chaotic after returning to $\mathbb{R}^2$ (more knowledge on the behavior near infinity is required).

What is crucial in the method is to use the diagram obtained through flipping from the period four orbit as fixed (for parabolic flow) strands of a braid. Through these strands we can thread an additional strand(s) and the braid class created like this will have a non-trivial Conley index. This can be seen by comparing Figures 1.14 and 1.15, since the braid of type I consist of two copies of the braid depicted on the left of Figure 1.14. The analysis at this point becomes similar to the one that was described when discussing the example in Figure 1.14. Adjusting it to the braid of type I allows the construction of many additional orbits leading to the construction of a semi-conjugacy to the shift on three symbols. This semi-conjugacy is used to obtain a lower bound on the topological entropy and to conclude the chaotic nature of the system having a period four orbit of type I.

To finish the classification of period four orbits, we need to construct an example of non-chaotic twist map having a period four orbit of type II. This is done by lifting a classical and thoroughly studied one dimensional logistic map to dimension two. Details can be found in Chapter 2. The rest of that chapter deals with a version of the above result that does not require the infinite twist condition, but instead requires map f to be a diffeomorphism.

THEOREM 1.4 (Chapter 2). *An orientation reversing twist diffeomorphism of the plane that has a type I period-4 point is a chaotic system, i.e., there exists a compact invariant subset $\Lambda \subset \mathbb{R}^2$ for which $f|_\Lambda$ has positive topological entropy. Conversely, there exists an orientation reversing twist diffeomorphism with a type II period-4 point that has zero entropy.*

As an example Let us come back to the *Hénon map*. Its general form is

$$f(x,y) = (\beta y, 1 - \alpha y^2 + x),$$

where α and β are real parameters. This map has the twist property for $\beta \neq 0$ and for $\beta > 0$, is orientation reversing, while for $\beta < 0$ it is orientation preserving. Observe that

$\frac{\partial x'}{\partial y} = \beta \neq 0$, and since

$$df(x,y) = \begin{pmatrix} 0 & \beta \\ 1 & -2\alpha y \end{pmatrix},$$

the Hénon map is area preserving if and only if $|\beta| = 1$. The results presented in Theorems 1.3 and 1.4 can be applied to this map (see Chapter 2.

1.4. Floer homology for relative braids

Area preserving maps and the Hamilton equations on the disc

The area preserving maps of the $\mathbb{D}^2$ can be seen as a generalization of twist maps of the disc (or $\mathbb{R}^2$). On the other hand, there exists a strong connection between the solutions of the Hamilton equations and area preserving maps. Consider compact 2-dimensional symplectic manifold (M,ω), i.e., ω is a non-degenerate closed 2-form on M. Let $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ be smooth, and let X_H be the Hamiltonian vector field generated by H, i.e.,

$$i_{X_H}\omega(\cdot,\cdot) = \omega(X_H,\cdot) = -dH.$$

Then the Hamilton equations are given by

$$\frac{dx}{dt} = X_H\big(t,x\big). \tag{1.3}$$

The solutions of this equation generate a family of maps $\psi_H^t : M \to M$ satisfying

$$\frac{d}{dt}\psi_H^t = X_H \circ \psi_H^t \qquad \psi_H^0 = \mathrm{Id}. \tag{1.4}$$

The link between the solutions of the Hamilton equations and area preserving maps can be observed by analyzing Equation (1.4) in the case $M = \mathbb{D}^2$. One can prove that the family ψ_H^t preserves the symplectic structure $\omega_0 = dp \wedge dq$, which is equivalent, in dimension two, to preserving the area and orientation. Given H, the Hamilton equations yield a time-1 map ψ_H^1 that is area preserving. On the other hand, for each orientation preserving map $f : \mathbb{D}^2 \to \mathbb{D}^2$ that preserves the area, there exists a hamiltonian H_f such that f is the time-1 map of the flow given by

$$\dot{x}(t) = X_{H_f}(t,x).$$

The above observations allow us to translate the results concerning orientation preserving maps of the disc to the results for the Hamilton equations, and vice versa. The information about the dynamics of time-1 maps gives insight into the flow defined by the Hamilton equations. As an example, fixed points of f correspond to 1-periodic orbits of the Hamiltonian flow. On the other hand, the periodic orbits of

$$\dot{x}(t) = X_H(t,x),$$

translate to the periodic orbits for the area and orientation preserving map (time-1 map of the flow).

Arnold Conjecture

The main motivation of Floer's original work was a long standing conjecture due to V.I. Arnold. Define the set of one periodic solutions of the Hamilton equations, which is equivalent to the set of fixed points of the time one map ψ_H^1 (see Equation (1.4))

$$\mathrm{Crit}_H := \left\{ x : \mathbb{R}/\mathbb{Z} \to M \;\middle|\; \frac{dx}{dt} = X_H(t,x) \right\}.$$

We use the notation of critical points analogous to Morse theory to stress the fact that the set of one periodic solutions corresponds to the critical points of an appropriate action functional (see below). In this context $x \in \mathrm{Crit}_H$ is called non-degenerate if

$$\det\Big(\mathrm{Id} - d\psi_H(1)\big(x(0)\big)\Big) \neq 0. \tag{1.5}$$

The Arnold conjecture was first proposed in [**7**] and can be stated as follows.

THEOREM (Arnold Conjecture). *Let (M,ω) and H be as above and assume that all $x \in$ $Crit_H$ are non-degenerate. Then*

$$\#\mathrm{Crit}_H \geq \sum_{i=0}^{2n} \dim H_i(M),$$

where $H_i(M)$ denotes the singular homology of M.

This theorem improves on the Lefschetz fixed point theorem proved in [**43**], which bounds the number of one-periodic points by the alternating sum of Betti numbers (dimensions of the homology groups). The result was also known previously for Hamiltonians that are t independent (autonomous systems), as in this case Crit_H coincides with critical points of H; it is then enough to show that the nondegeneracy condition from the theorem above implies that H is a Morse function, as we can apply standard Morse theory. The first progress in proving the conjecture in more general settings was obtained by Conley and Zehnder in [**21**], where they proved the conjecture for the n-dimensional torus using a variational principle on the loop space. Then Gromov [**35**] proved existence of at least one critical point using pseudo-holomorphic curves. Floer, in his series of papers [**27, 28, 29**], managed to combine the ideas of Conley and Zehnder with those of Gromov to develop an alternative for Morse theory that allowed him to prove the Arnold conjecture for a large class of symplectic manifolds. Further generalizations of the methods were done by Hofer and Salamon [**38**] and Ono [**57**]. Floer's proof of the Arnold conjecture has been extend to all compact symplectic manifolds by Fukaya and Ono in [**30**], Liu and Tian in [**45**].

The example of how Morse theory can be applied to an autonomous system suggests that it is possible to adapt the prove of the Arnold conjecture for time dependent Hamiltonians. On $\mathbb{D}^2$ (or $\mathbb{R}^2$) the Hamiltonian action functional on the space of loops is given by

$$f_H(x) = -\int_0^1 \alpha_0\big(x_t(t)\big) + \int_0^1 H(t,x(t))dt,$$

where $d\alpha_0 = \omega_0$ (i.e., $\alpha = pdq$ in the standard $\mathbb{R}^2$). In this setting, critical points of f_H are in fact solutions of the Hamilton equations.

When trying to follow the ideas of Morse theory, one is faced with the following obstructions. The first one is that the action functional is not bounded, neither from above nor from below. Moreover, fact that its spectrum is unbounded in both directions prevents us from defining a counterpart of the Morse index in any meaningful sense. Additionally, the L^2 gradient flow of f_H does not define a well posed initial value problem. Those are of the main reasons why the Arnold conjecture was open until the Floer's work.

As natural choices of gradient dynamics we may consider both the L^2 or the $W^{\frac{1}{2},2}$ gradient flow. The latter yields an ordinary differential equation with a well posed initial value problem and defines a flow. The L^2 gradient yields to an elliptic partial partial differential equation – the nonlinear Cauchy-Riemann system – for which the initial value problem is ill posed. In this case one needs to restrict to the set of bounded solutions. The advantage of using the L^2 gradient over the $W^{\frac{1}{2},2}$ gradient flow is that it satisfies the so called crossing principle, an analogue of the lap number techniques for parabolic equations, which is not satisfied by the $W^{\frac{1}{2},2}$ gradient flow. We will explain the implications of the crossing principle in the next section.

The Cauchy-Riemann equations in the setting of braids

To study n-periodic solutions, or a number of solutions of different integer periods, we can employ the structure of braids. Starting with an n-periodic solution of (1.3) we can describe the whole trajectory on the interval $[0,1]$ using its translates, i.e., define

$$x^1(t) = x(t),\ x^2(t) = x(t+1),\ \ldots,\ x^n(t) = x(t+n-1).$$

Each of the x^i forms a strand in a braid that corresponds to a given solutions x. Observe that if we want to consider the collection of above curves $x^i : [0,1] \to \mathbb{D}^2$ as a braid we need to show that their images are disjoint (we assume that n is a minimal period). This is, indeed the case due to the uniqueness of the initial value problem for the Hamilton equations. In fact, in this way we can superimpose several different solutions, possibly with different periods, as one braid, by considering the strands generated by the different solutions together.

To stress this braid oriented approach we use the notation $\mathbf{x}$ for the collection of strands $\{x^1,\ldots,x^n\}$. The associated permutation on n symbols σ is defined by

$$x^k(1) = x^{\sigma(k)}(0). \tag{1.6}$$

For a single solution the permutation σ of the corresponding braid is just a cyclic permutation, but for collections of solutions it is of course more complicated. The action functional for the Hamilton equations takes the form

$$f_H(x) = -\int_0^1 \alpha_0(x_t(t)) + \int_0^1 H(t,x(t))dt, \tag{1.7}$$

with $\alpha_0 = p\,dq$. Let $\mathbf{x}$ be a braid, then its action is defined by

$$\mathbf{f}_H(\mathbf{x}) := \sum_{k=1}^{n} f_H(x^k).$$

The Hamilton equations corresponding to this action functional can be viewed as a system of $2n$ equations on $(\mathbb{D}^2)^n$ with $\bar{H}(t,\mathbf{x}) = \sum_k H(t,x^k)$, coupled via the boundary conditions given in Equation (1.6).

Following the ideas of Morse theory, we want to analyze a 'negative gradient flow' of $\mathbf{f}_H$. Formally,

$$\begin{aligned} df_H(x)\delta x &= \int_0^1 \left[-p\,\delta q' - \delta p\, q' + H_p \delta p + H_q \delta q\right] dt \\ &= \int_0^1 \left[-q'\,\delta p + p'\,\delta q + H_p \delta p + H_q \delta q\right] dt \\ &= \int_0^1 \langle J_0 x' + \nabla H, \delta x\rangle dt, \end{aligned}$$

where

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We add a new variable time variable s to the system to define a negative gradient flow of $\mathbf{f}_H$, which reads

$$\frac{\partial u^k}{\partial s} + J_0 \frac{\partial u^k}{\partial t} + \nabla H(t, u^k) = 0. \tag{1.8}$$

Again, the equations for u^k are coupled through the boundary conditions in Equation (1.6). Here we use u to denote the solutions of the above equation to stress the dependence on s. The linear part of the above equations, $\frac{\partial u^k}{\partial s} + J_0 \frac{\partial u^k}{\partial t}$, is the reason why we call them the Cauchy-Riemann equations. Observe that in this setting, the periodic solutions of the Hamilton equations are the stationary braids of the Cauchy-Riemann equations (1.8). The equations do not define a well-posed initial value problem, as one can obtain a solution only if the Fourier coefficients of the initial condition decay sufficiently fast. Hence we restrict our attention to the set of bounded solutions, denoted $\mathcal{M}^{J,H}$. The next step is to prove that $\mathcal{M}^{J,H}$ is compact in the C^1_{loc} topology. This leads to a version of Floer's compactness theorem.

THEOREM 1.5 (Compactness, Chapter 3). *The set of bounded solutions (braids) $\mathcal{M}^{J,H}$ is compact in the topology defined by convergence on compact subsets of* $\mathbb{R} \times [0,1]$.

The proof of this theorem follows form interior regularity for the Cauchy-Riemann operator; details can be found in Chapter 3.

Crossing principle

The crucial interplay of the dynamics of the Cauchy-Riemann equations and the topological structure of braids is expressed in the *crossing principle*. The basis of this principle is the analysis in the case of two strands satisfying the Cauchy-Riemann. In the 'generic' case two strands intersect at a single point (Figure 1.16). This is the analogue of the property of parabolic flows on the braid diagrams. Observe that while two stands of a braid have disjoint images, they "overlap", i.e., intersect in projection. This overlap (corresponding to a generator in the braid word), is called a *crossing*, and we distinguished positive and negative types of crossings (cf. Figure 1.16). The crossing is *negative* if the strand 'coming from the left goes below the one from the right' and *positive* in the other case.

Consider the 'generic' local picture as in Figure 1.16. That is, two strands (solutions) u^1 and u^2 which intersect at an isolated point (s_0, t_0). The Cauchy-Riemann equations for

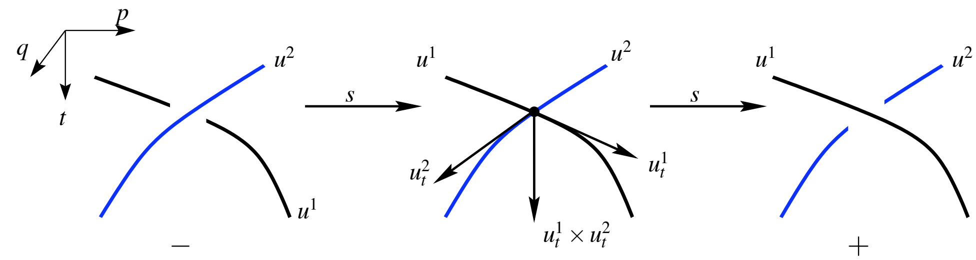

Figure 1.16: The labeling convention for crossing types: negative [left] and positive [right]. Two strands (solutions) x and y intersecting at a single point [center]. The flow (in s) changes negative into positive crossing.

$u^1 = (p^1, q^1)$ and $u^2 = (p^2, q^2)$ are

$$\begin{aligned} u^1_s + J_0 u^1_t + \nabla H(t, u^1) &= 0, \\ u^2_s + J_0 u^2_t + \nabla H(t, u^2) &= 0. \end{aligned}$$

Define the difference $z = u^1 - u^2$ and let us investigate how it changes along the evolution of the system. Observe that z satisfies

$$z_s + J_0 z_t + \nabla H(t, u^1) - \nabla H(t, u^2) = 0,$$

and at (s_0, t_0) it simplifies to

$$z_s + J_0 z_t = 0.$$

In the extended phase space, we have $u^1 = (p^1, q^1, t)$ and $u^2 = (p^2, q^2, t)$ which yields

$$u^1_t \times u^2_t = \begin{pmatrix} q^1_t - q^2_t \\ -(p^1_t - p^2_t) \\ p^1_t q^2_t - p^1_t q^2_t \end{pmatrix}.$$

On the other hand, in the p,q-plane, we have $z_t = (p^1_t - p^2_t, q^1_t - q^2_t)^T$. Hence we get

$$\pi_{p,q}(u^1_t \times u^2_t) = -J_0 z_t,$$

where $\pi_{p,q}$ denotes the projection on the p,q-plane. Consequently,

$$z_s \cdot \pi_{p,q}(u^1_t \times u^2_t) = -J_0 z_t \cdot \pi_{p,q}(u^1_t \times u^2_t) = \|J_0 z_t\|^2 > 0.$$

That is, z_s has the same direction as $\pi_{p,q}(u^1_t \times u^2_t)$, hence as s increases u^2 "moves downward faster" than u^1, as indicated in Figure 1.16.

We define the *crossing number* of a braid $\mathbf{x}$ by

$$\text{Cross}^*(\mathbf{x}) = \#\text{negative crossings} - \#\text{positive crossings}.$$

Observe that the crossing number is only defined for (not singular) braids. Along the evolution under the Cauchy-Riemann equations, $\mathbf{u} \in \mathcal{M}^{J,H}$ may develop an intersection at s_0. Then $\text{Cross}^*(\mathbf{u}(s_0))$ is not defined. We have the following.

LEMMA 1.6 (Crossing lemma, Chapter 3). *Where defined,* $\text{Cross}^*(\mathbf{u}(s))$ *is a non increasing function as a function of* s.

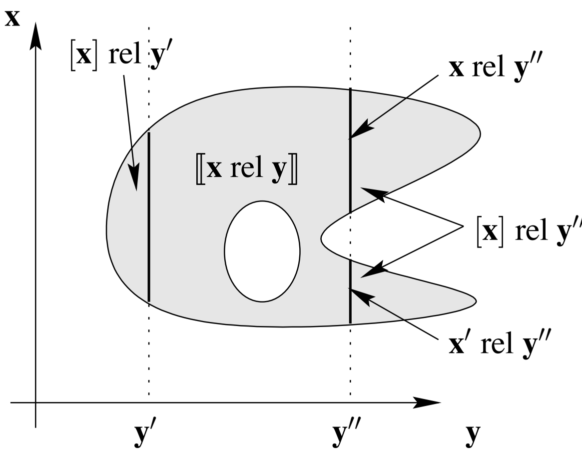

Figure 1.17: Sketch of a relative braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ and some representatives.

The proof (see Chapter 3) relies on the fact that the properties of solutions of the Cauchy-Riemann equations are similar to those of complex differentiable functions. In particular, two solutions are either equal for all (s,t) or their difference has only isolated zeros. This is due to the general similarity principle (see [**39**]).

Floer homology for relative braid classes

The idea of forcing leads to the concept of relative braid classes. Let Ω^n denote the set of braids on n strands. Define $\mathbf{x} \cup \mathbf{y}$ for $\mathbf{x}$ in Ω^n and $\mathbf{y}$ in Ω^m as the collection of strands of both braids. If none of the strands of $\mathbf{x} \cup \mathbf{y}$ intersect then $\mathbf{x} \cup \mathbf{y} \in \Omega^{n+m}$. The braid $\mathbf{y}$ will be called the *skeleton* and $\mathbf{x}$ the *free strands*. For fixed $\mathbf{y}$, define Ω^n rel $\mathbf{y}$, as the space consisting of all $\mathbf{x} \in \Omega^n$, such that $\mathbf{x} \cup \mathbf{y} \in \Omega^{n+m}$. We will call elements of this space *relative braids*.

In this setting $[\mathbf{x}]$ rel $\mathbf{y}$ denotes a path component of $\mathbf{x}$ rel $\mathbf{y}$ in Ω^n rel $\mathbf{y}$, that is, all equivalent relative braids with a fixed skeleton $\mathbf{y}$. Intuitively, $[\mathbf{x}]$ rel $\mathbf{y}$ describes a class of braids that can be deformed onto each other without creating any intersections along the homotopy while keeping the skeletal strands $\mathbf{y}$ fixed. The concept of the relative braid class allows us to obtain a finer decomposition of the space of braids (cf. Figure 1.17). Observe that one may consider a more general version of a relative braid class, namely let

$$\Omega^{n,m} = \left\{ (\mathbf{x}, \mathbf{y}) \in \Omega^n \times \Omega^m \mid \mathbf{x} \cup \mathbf{y} \in \Omega^{n+m} \right\},$$

and denote the path component of $\mathbf{x} \cup \mathbf{y} \in \Omega^{n+m}$ in $\Omega^{n,m}$ by $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ (cf. Figure 1.17).

A relative braid class is *proper* if its boundary does not contain a representative with a free strand that has entirely collapsed onto another one (be it free or fixed), or onto the boundary of the disc. Let $[\mathbf{x}]$ rel $\mathbf{y}$ be a proper relative braid class, then we define

$$\mathcal{M}^{J,H}([\mathbf{x}] \text{ rel } \mathbf{y}) := \{\mathbf{u} \in \mathcal{M}^{J,H} \mid \mathbf{u}(s,\cdot) \in [\mathbf{x}] \text{ rel } \mathbf{y} \text{ for all } s \in \mathbb{R}\},$$

a set of bounded solutions of the Cauchy-Riemann equations within a given braid class (cf. Figure 1.18). The crossing principle guarantees the isolation property for such proper classes, and with it the associated compactness properties.

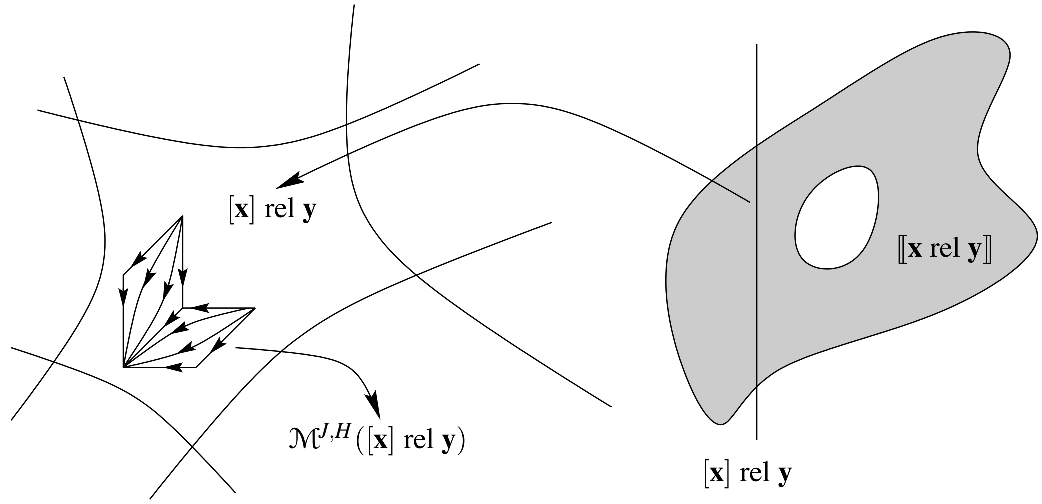

Figure 1.18: The set of bounded solutions $\mathcal{M}^{J,H}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big)$ restricted to a relative braid class $[\mathbf{x}]$ rel $\mathbf{y}$.

From this point on we follow the Floer's original work, although the braid structure forces us to replace some of the concepts with the counterparts that take into account the 'σ-periodicity' of braids. The first step is the definition of a relative index[1] for critical points of the action functional. Then we face the issue of transversality. Solving those two problems will finally allow us to define chain groups and the boundary operator. For the purpose of this introduction we only indicate the necessary adjustments and the reader is advised to go to Chapter 3 for the details.

The definition of a relative index is an adapted version of the theory presented in [**59**] and [**60**] that develops the Maslov index and Conley-Zehnder index for Lagrangian paths. For two stationary solutions $\mathbf{x}_-, \mathbf{x}_+ \in \mathrm{Crit}_H$ and an orbit $\mathbf{u}$ connecting them, the linearized Cauchy-Riemann operator can be written as

$$\bar{\partial}_{K,\Delta_\sigma} = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + K(s,t),$$

where $K(s,t)$ is a family of $2n \times 2n$ symmetric matrices. This operator acts on functions satisfying the boundary conditions given by Equation (1.6), and the family K satisfies a certain limit conditions dictated through $\mathbf{x}_-$ and $\mathbf{x}_+$. For non-degenerate (see Chapter 3) $\mathbf{x}_-$ and $\mathbf{x}_+$, the operator $\bar{\partial}_{K,\Delta_\sigma}$ is Fredholm on appropriate function spaces, that is, its kernel and co-kernel are both finite dimensional. Using this property one can obtain a relative index μ for stationary points so that

$$\mathrm{ind}\, \bar{\partial}_{K,\Delta_\sigma} = \dim\ker \bar{\partial}_{K,\Delta_\sigma} - \dim\mathrm{coker}\, \bar{\partial}_{K,\Delta_\sigma} = \mu(\mathbf{x}_-) - \mu(\mathbf{x}_+).$$

The relative index μ is defined using Maslov indices (see Chapter 3).

Roughly speaking, by choosing the Hamiltonians in a large enough class, bounded solutions generically have the property that $\bar{\partial}_{K,\Delta_\sigma}$ is Fredholm and surjective. This implies

[1]Compare with the fact that the Morse index is not well defined in this context as remarked in the 'Arnold Conjecture' section.

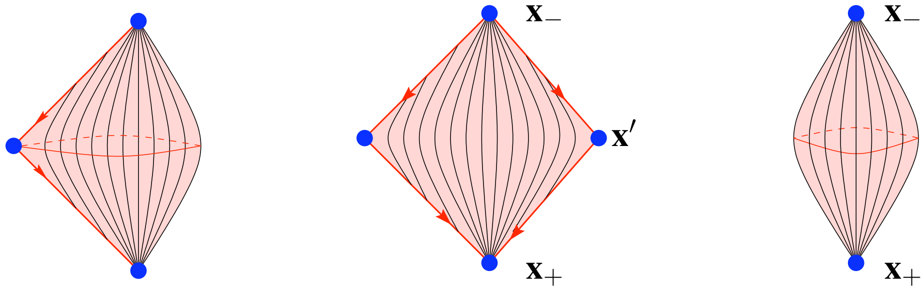

Figure 1.19: The possible boundary components of the manifold of connecting orbits between points with index difference two. In particular the number of connections via intermediate points is even.

via the implicit function theorem that the spaces $\mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}+} \subset \mathcal{M}^{J,H}$ of connecting orbits are smooth manifolds of dimension

$$\dim \mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}+} = \mu(\mathbf{x}_-) - \mu(\mathbf{x}_+). \tag{1.9}$$

Define the chain groups as free Abelian groups with coefficients in $\mathbb{Z}_2$ over the points of index k, i.e.,

$$C_k := \mathrm{span}_{\mathbb{Z}_2}\{\mathbf{x} \in \mathrm{Crit}_H \ : \ \mu(\mathbf{x}) = k\}.$$

Under genericity assumptions, Equation (1.9) guarantees that, modulo reparametrization of the trajectories, we have only finitely many connections between points with index difference one. This is crucial, as it allows us to define the boundary operator $\partial_k : C_k \to C_{k-1}$ as the linear map that for any $\mathbf{x} \in \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$ with $\mu(\mathbf{x}) = k$ is given by

$$\partial_k \mathbf{x} = \sum_{\mu(\mathbf{x}')=k-1} n(\mathbf{x},\mathbf{x}')\mathbf{x}',$$

where $n(\mathbf{x},\mathbf{x}') = \#\mathcal{M}^{J,H}_{\mathbf{x},\mathbf{x}'} \mod 2$.

We say that points $\mathbf{x}_-, \mathbf{x}_+$ with $\mu(\mathbf{x}_-) - \mu(\mathbf{x}_+) = 2$ are connected via a *broken trajectory* through $\mathbf{x}'$ (Figure 1.19), if $\mu(\mathbf{x}_-) - \mu(\mathbf{x}') = 1$, and $\mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}'}$ and $\mathcal{M}^{J,H}_{\mathbf{x}',\mathbf{x}_+}$ are both non-empty. In this setting the square of the boundary operator counts the number of broken trajectories connecting $\mathbf{x}_-, \mathbf{x}_+$ via all possible intermediate points. Further analysis of the Cauchy-Riemann equations reveals that the connected components of spaces $\mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}+}$ can only be homeomorphic to S^1 or $(0,1)$. The closure of $(0,1)$ in $\mathcal{M}^{J,H}$ is obtained by adding two 'ends' $0 \neq 1$, which corresponds to broken trajectories (gluing principle, cf. Figure 1.19). Hence the number of broken trajectories is always even and it holds that

$$\partial_k \circ \partial_{k+1} = 0.$$

We thus conclude that $\big(C_k, \partial_k\big)$ forms a chain complex. With this we can define the Floer homology of a proper relative braid class $[\mathbf{x}]$ rel $\mathbf{y}$ as

$$FH_k([\mathbf{x}] \text{ rel } \mathbf{y}, J, H) := \frac{\ker(\partial_k)}{\mathrm{im}(\partial_{k+1})}.$$

Floer homology satisfies the following invariance.

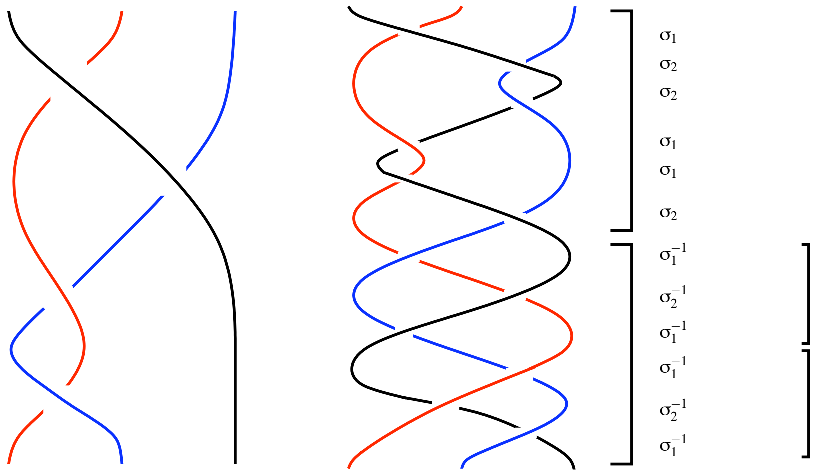

Figure 1.20: Braid with generators $\sigma_1\sigma_2\sigma_1^{-1}\sigma_1^{-1}$ [left] and the representative of the same braid class in a Garside normal form [right].

LEMMA 1.7 (Continuation). *For a proper relative braid class* $[\mathbf{x}]$ rel $\mathbf{y}$ *Floer homology does not depend on the choice of* J *or the Hamiltonian* H *fixing* $\mathbf{y}$.

Moreover, homotopies of the skeleton $\mathbf{y}$ (in an appropriate sense) do not change the Floer homology allowing us to define it for a proper braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ (cf. Figure 1.17).

THEOREM 1.8. *Let* $[\mathbf{x}]$ rel $\mathbf{y}$ *and* $[\mathbf{x}']$ rel $\mathbf{y}'$ *be in a proper relative braid class* $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$, *then it holds that*

$$FH_*([\mathbf{x}] \text{ rel } \mathbf{y}) = FH_*([\mathbf{x}'] \text{ rel } \mathbf{y}').$$

In particular, the Floer homology is an invariant of the class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$. *This justifies the notation* $FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$.

With this theorem we start investigating the properties of the Floer homology for the relative braid classes. Let H be a Hamiltonian fixing a given skeleton $\mathbf{y}$, then we have the following.

THEOREM 1.9 (Chapter 4). *Let* $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ *be a proper braid class. If*

$$FH_*\big([\mathbf{x}] \text{ rel } \mathbf{y}\big) \neq 0,$$

then $Crit_H([\mathbf{x}] \text{ rel } \mathbf{y}) \neq \varnothing$.

Additionally, we can obtain the following counterpart of the Morse inequalities. Let $\beta_k = \dim FH_k([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$ be the k-th Betti number of the Floer homology. Then, for a generic Hamiltonian H fixing $\mathbf{y}$, the number of critical points of index k in $[\mathbf{x}]$ rel $\mathbf{y}$ is bounded from below by β_k.

Every braid class has a representative that consist of a positive braid and a number of negative half twists. This corresponds to the *Garside normal form* of the braid (see [**32**]). As a corollary one can deduce that each braid class can be represented by a braid with a positive part concatenated with $l \in \mathbb{N}$ full negative twists. In Figure 1.20 we present an example of a braid and its normal form. Above guarantees that there exists a representant

$\beta(\mathbf{x} \text{ rel } \mathbf{y})$ of the class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ that can be written as

$$\beta(\mathbf{x} \text{ rel } \mathbf{y}) = \square^{-l} \cdot \beta(\mathbf{x}^+ \text{ rel } \mathbf{y}^+), \tag{1.10}$$

where $\beta(\mathbf{x}^+ \text{ rel } \mathbf{y}^+)$ contains only positive generators, $\square$ denotes a full positive twist and $l \in \mathbb{N}$. We have the following result

THEOREM 1.10 (Chapter 4). *For a proper class* $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ *on n strands (both skeletal and fixed). Then there exists a minimal natural number l satisfying Equation* (1.10) *such that*

$$FH_*(\mathbf{x} \text{ rel } \mathbf{y}) \cong FH_{*-2nl}(\mathbf{x}^+ \text{ rel } \mathbf{y}^+).$$

This allows us to restrict our attention to braids with positive generators only.

1.5. Extensions and future work

Connection with the Conley index for braids

Above we have defined the Floer homology of a relative braid class, suggesting that it allows us to obtain forcing results for the Hamilton equations. The problem that we face at this point is that a careful inspection of all the ingredients used to define the homology requires an almost complete knowledge of the system. Except for some special cases where one can continue the problem to integrable system. We need to know all periodic orbits of the Hamiltonian flow and all connections between them the Cauchy-Riemann flow. Such a detailed knowledge of the system defies the purpose of calculating it. Ideally, one would like to exploit topological information about the braid class to calculate the homology.

An ultimate goal for future work on this subject is to connect the Floer homology of a braid with the Conley index of a discretized braid diagram that corresponds to it, and then to use the latter to draw conclusions about the original system. We were able to prove one step in this direction. The first problem that we encounter in our search for a connection between the Floer homology and the Conley index for braids, is the fact that the latter theory deals only with positive braids, while the former allows also negative crossings. The Garside normal form of a braid and the shift Theorem 1.10 eliminates this problem.

Connection with Thurston-Nielsen theory

In the case of the disc, the mapping class group relative to a finite set (periodic orbit) A is $\mathrm{MCG}(\mathbb{D}^2 \text{ rel } A, \partial\mathbb{D}^2)$. One can identify this group with Artin's braid group on n-strands B_n divided by its center, where n is number of points in A (or period in case of a periodic orbit). This correspondence is best explained via an example. Let us consider the case $A = \{a_1, a_2, a_3\}$. Words in the braid group B_3 are generated by letters $\sigma_1^{\pm 1}$ and $\sigma_2^{\pm 1}$, with the relations described earlier in this introduction. Besides, let $\phi_1 \in \mathrm{MCG}(\mathbb{D}^2 \text{ rel } A, \partial\mathbb{D}^2)$ denote a homeomorphism that interchanges a_1 with a_2 in a counterclockwise direction (its inverse does so in the opposite direction), and ϕ_2 interchanges in an analogous way a_2 and a_3. Observe that

It turns out that the map $G : B_3 \to \mathrm{MCG}(\mathbb{D}^2 \text{ rel } A, \partial\mathbb{D}^2)$ that sends the generator σ_i to an isotopy class $[\phi_i]_\simeq$ is a group isomorphism, providing the required correspondence (cf. Figure 1.21). In this sense one can think of an (isotopy class of) homeomorphism as being represented by a braid (class).

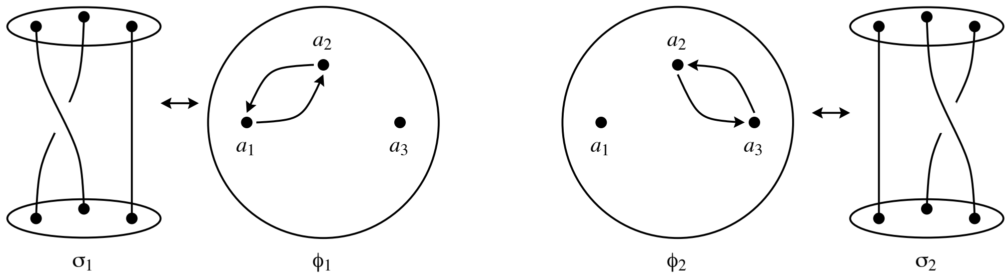

Figure 1.21: The schematic picture of the correspondence between the representatives ϕ_1 and ϕ_2 of $\mathrm{MCG}(\mathbb{D}^2 \text{ rel } A, \partial\mathbb{D}^2)$, and the generators σ_1 and σ_2 of the braid group B_3.

To explain this intuitively one needs the concepts of the suspension manifold and the suspension flow. The *suspension manifold* corresponding to a homeomorphism is defined as the quotient

$$M_f := \frac{M \times [0,1]}{(x,0) \sim (f(x),1)}.$$

In the case of the disc it can be thought of as a torus created from a cylinder with the ends glued along f. The *suspension flow* ψ_f^t defined on M_f is just a unit speed flow on the second coordinate of $M \times [0,1]$ projected onto M_f. In this context, a braid is created from an element of the isotopy class by extracting orbits of the points of A in the suspension flow. Observe that all representants of a given isotopy class yield braids in the same braid class.

So far we have only mentioned the application of the braids in Thurston-Nielsen context as a tool to analyze the group $\mathrm{MCG}(\mathbb{D}^2 \text{ rel } A, \partial\mathbb{D}^2)$. We now present some results to which the use of braids may lead. Recall that the periodic orbit of a homeomorphism leads to a braid in the suspension flow and conversely. One hopes for a kind of Sharkovskii order among braids, that ideally would lead to a (partial) ordering on the braid classes in such a way that the existence of braid β as a periodic orbit of a given map, forces existence of all periodic orbits corresponding to the braids that are lower in the ordering than β. Of course, in many cases this is too much to ask for, and only partial results can be obtained. Below we will try to give some examples following [**17**].

We recall that $\mathrm{MCG}(\mathbb{D}^2 \text{ rel } A, \partial\mathbb{D}^2)$ can be identified with the braid group on n strings (where $n = \#A$). In this context, a braid β is pseudo-Anosov if its image under the group morphism $G : B_3 \to \mathrm{MCG}(\mathbb{D}^2 \text{ rel } A, \partial\mathbb{D}^2)$ introduced above, is a pseudo-Anosov map (or its isotopy class contains a representant with a component that is pseudo-Anosov, see Section 1.1).

We finally introduce the concept of the *exponential sum* of a braid, defined as the sum of exponents of its generators in the word describing the braid. For example, for the braid $\beta = \sigma_1^{-1}\sigma_2^1\sigma_1^2$ its exponential sum, denoted $es(\beta)$, is 2. Let $R_{n/m} : \mathbb{D}^2 \to \mathbb{D}^2$ denote the rotation map of the disc, which in polar coordinates can be expressed as $R_{n/m}(r,\theta) = (r, \theta + \frac{n}{m})$. The following classification of homeomorphisms of the disc can be obtained.

THEOREM (Theorem 8.3 in [**17**]). *If f is an orientation-preserving homeomorphism of the disk that satisfies $f^m = id$, than f is topologically conjugate to $R_{n/m}$ for some* $0 \le n \le m$.

It is not too hard to see that for a braid $\beta_{n/m}$ corresponding to the map $R_{n/m}$, we have that $es(\beta_{n/m}) = n(m-1)$. On the other hand we have the following:

THEOREM (Proposition 9.4 in [**17**]). *If a braid* β *on m strands, where m is prime, does not correspond to any of the maps* $R_{n/m}$*, then the map containing* β *is pseudo-Anosov.*

In particular, combining these results, we obtain that if $es(\beta) \neq n(m-1)$ for any $0 < n < m$, then β forces the pA property on the map containing it.

Let us come back for a moment to the case of braids on three strands, since in such a setting one is able to obtain a much stronger result in the spirit of Sharkovskii's theorem. It turns out that any braid on three strands can be described using the generators σ_1 and σ_2^{-1} only, and with this we obtain the following.

THEOREM (Theorem 9.4 in [**17**]). (a) *A braid* β *on three stands leads to a pA map if and only if its minimal word in the* σ_1, σ_2^{-1} *description contains both generators.*

(b) *Let* β_1, β_2 *be two pA braid classes. Moreover, assume that the minimal braid word in the* σ_1, σ_2^{-1} *description of* β_2 *is contained as a sub-word in the* σ_1, σ_2^{-1} *word of* β_1*. Then, if a homeomorphism contains the periodic orbit corresponding to* β_1*, it also has the one corresponding to* β_2.

The partial orders on braid types were studied by several authors, including Matsuoka [**49**], Benardete e.a. [**11**], de Carvalho and Hall [**23**]. Despite the fact that the comparison of the Thurston-Nielsen classification theory and our methods still requires more elaborate studies, we would like to point out some of the similarities and distinctions between the theories.

Orientation reversing twist maps. To explain the differences between our application of the Conley index to an orientation reversing twist maps of the plane and the Thurston-Nielsen theory we refer to Chapter 2. The differences between the two approaches are that we require the map to have the twist property, and we essentially exploit the variational structure that comes with it, while the Thurston-Nielsen classification is also suitable for non-twist maps. On the other hand, Thurston-Nielsen theory requires the underlying manifold to be compact, whereas our maps are defined on the whole plane. In general, to be able to restrict the dynamics of such maps onto a disc, one needs control of the behavior at infinity, which is irrelevant in our case (for twist diffeomorphisms). This makes the methods somewhat complementary as they are applicable in different settings.

Floer homology. A comparison of the Floer homology for relative braids and the Thurston-Nielsen theory requires more in-depth studies. Both theories require the underlying manifold to be compact and two-dimensional. As already mentioned above, both methods use a braid theoretic approach. This makes both theories very similar in spirit. Thurston-Nielsen theory uses a braid algorithm (train tracks) to study the classification of the dynamics forced by a braid corresponding to a given periodic orbit. In the case of Floer homology additional studies are required to make it computable, so that relevant information about the system can be extracted. Nevertheless, the approach is intrinsically different and interesting in its own right. Moreover, it is much more analytic in nature, which may have certain advantages, such as being able to deal with singularities, as well as the possibility to study non-equilibrium solutions of the Cauchy-Riemann equations.

CHAPTER 2

Orientation reversing twist maps of the plane

2.1. Introduction

Orientation preserving twist maps have been studied by many authors over the past decades. In particular we mention the important contributions by Moser [**55, 56**], Mather [**48**], Aubry & Le Daeron [**10**], Angenent [**2, 4**], Boyland [**17**] and Le Calvez [**42**]. Most of these works consider area and orientation preserving twist maps and make use of the variational principle that comes with it. This is a powerful tool for studying periodic points, in particular when the domain of the map is an annulus.

In this paper we are interested in dynamical systems generated by *orientation reversing* twist maps that do not necessarily preserve area and that are defined on the whole plane. Specifically, we are interested in periodic orbits and the minimal dynamics they force. We postpone a discussion of related work to the end of this introductory section. First, we give the necessary definitions and introduce a topological principle for such orientation reversing maps.

A well-known example of a two dimensional orientation reversing twist map is the family of Hénon maps. The discrete time dynamics that are obtained by iterating such maps have emerged as models from various applications in the physical sciences. Orientation *preserving* (twist) maps of (sub-regions of) the plane are often obtained as time-1 maps in non-autonomous Hamiltonian systems in the plane, or as first return maps to a Poincaré section in three dimensional dynamical systems. On the other hand, maps that reverse orientation do not occur as such section maps.

In this paper we are mainly concerned with *diffeomorphisms* of the plane, i.e. bijective C^1 maps.

DEFINITION 2.1. A diffeomorphism $f:\ \mathbb{R}^2 \to \mathbb{R}^2$ is called an *orientation reversing twist diffeomorphism* of the plane if there exist global coordinates $(x,y) \in \mathbb{R}^2$ such that f is given by $(x',y') = f(x,y)$ and satisfies the assumptions: (i) $\det(df) < 0$, and (ii) $\frac{\partial x'}{\partial y} > 0$. Due to the latter condition, which we will refer to as the *twist property*, f is said to have *positive* twist. If the bijectivity assumption is dropped but f is still C^1 and satisfies properties (i) and (ii) then f is called an orientation reversing twist *map*.

In the following, to indicate the coordinate functions x' and y' of f, we use the composition with the orthogonal projections π_x and π_y onto the x- and y-coordinate respectively, i.e. $x' = \pi_x f(x,y)$ and $y' = \pi_y f(x,y)$.

As will be explained in Section 2.3 (see also [**33**]) (compositions of) orientation preserving (positive) twist maps have a natural topological structure, which is less straightforward in the orientation reversing case. There exists an easy procedure to circumvent this obstacle and find a useful topological tool for orientation reversing twist maps. Note that even powers of f are orientation preserving maps, but compositions of twist maps are in general not twist maps. The second composite iterate f^2 can be written as a composition of two orientation *preserving* twist maps as follows: $f^2 = f_+ \circ f_-$, with $f_+ = f \circ R_x$, and $f_- = R_x \circ f$, where R_x is a linear reflection in the y-axis. The drawback is that f_+ is a positive twist map and f_- a negative twist map. If we consider the fourth iterate f^4 we have the decomposition

$$f^4 = f_3 \circ f_2 \circ f_1 \circ f_0, \tag{2.1}$$

where the maps f_i are defined as follows:

$$f_0 = -f_-, \quad f_1 = -f_+ \circ (-\mathrm{id}), \quad f_2 = f_- \circ (-\mathrm{id}), \quad \text{and} \quad f_3 = f_+.$$

One can easily verify that all four maps are orientation preserving maps with *positive* twist. The theory of parabolic recurrence relations in [**33**] (summarized in Section 2.3) is now applicable since it applies to compositions of orientation preserving positive twist maps. Using this formulation we can study periodic points of period $n = 4k$ (for other periods symmetry requirements could be imposed, but we will not pursue this issue here).

Recall that a point $z = (x, y)$ is a period-n point for f if $f^n(z) = z$, where f^n denotes the n-th iterate of f. The period n is assumed to be *minimal*, i.e. $f^k(z) \neq z$ for all $0 < k < n$. Instead of describing a period-4 point in terms of the images of f, i.e. $(z, f(z), f^2(z), f^3(z))$, a natural way to describe orbits is to do so in accordance to the decomposition given by (2.1). We write an orbit as $\{z_i\}_{i=0}^3$, with $z_i = f_i(z_{i-1})$. This applies to period-$4k$ points, by defining f_i via $f_{i+4} = f_i$, for all $i \in \mathbb{Z}$. The theory of parabolic recurrence relations in [**33**] now dictates that orbits $\{z_i\}$ should be represented as braid diagrams, which we will explain next.

Let z be a period-4 point of f. By choosing the points z, $f(z)$, $f^2(z)$, and $f^3(z)$ as different initial points we obtain four different orbits for the composition $f_3 \circ f_2 \circ f_1 \circ f_0$, namely the orbits defined by $z_i = f_i(z_{i-1})$ while setting $z_0 = f^k(z)$ for $k = 0, 1, 2, 3$. For each orbit we connect the consecutive points (i, z_i) via piecewise linear functions. This yields a piecewise linear closed braid consisting of four strands. By projecting the braid on the x-coordinates one obtains a closed braid diagram. Braid diagrams are discussed in more detail in Section 2.3. Figure 2.1 depicts the braid diagrams which result from this construction starting from two different period-4 orbits. Since the braid diagram is only concerned with the x-coordinates the construction of the braid diagram is, for all practical purposes, equivalent to the following: let (x^0, x^1, x^2, x^3) be the x-coordinates of a period-4 point orbit $\{f^k(z)\}_{k=0}^3$, i.e. $x^i = \pi_x f^i(z)$, then perform a flip on these coordinates to obtain $(x_0, x_1, x_2, x_3) = (-x^0, x^1, x^2, -x^3)$, and finally connect the points (i, x_i) in the plane by line segments. This gives one strand and the total braid diagram is obtained by performing this transformation to all shifts of the orbit through z.

Notice that period-4 points can occur in a variety of six different "permutations" (of the x-coordinates, see also Section 2.4). However, permutations do not have topological meaning with respect to parabolic recurrence relations and permutations are thus not suitable for classifying period-4 points. On the other hand, via the above construction each permutation yields a unique braid class that has a topological meaning. It follows that period-4

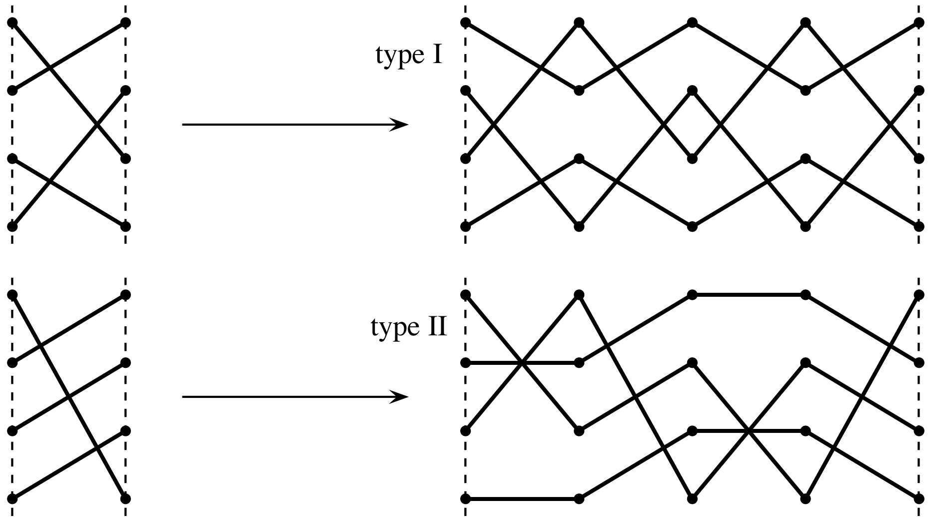

Figure 2.1: Period-4 points lead to *two* possible braid classes. In the braid diagrams on the right one may think of all the crossings as being positive, i.e. the strand with the larger slope going on top.

points give rise to exactly two types of braid classes. Figure 2.1 shows the two possible braid classes: type I and type II. In other words, *any* period-4 orbits is either of type I or of type II, according to the braid class that results from the above transformations. More details on this classification are supplied in Section 2.4. Period-4 points of type I imply chaos, while those of type II do not, as is stated in our main theorem.

THEOREM 2.2. *An orientation reversing twist diffeomorphism of the plane that has a type I period-4 point is a chaotic system, i.e., there exists a compact invariant subset* $\Lambda \subset \mathbb{R}^2$ *for which* $f|_\Lambda$ *has positive topological entropy. Conversely, there exists an orientation reversing twist diffeomorphism with a type II period-4 point that has zero entropy.*

We want to point out that the theorem is stated under quite weak assumptions; in particular, there are *no* compactness assumptions (the twist property in a way compensates this lack of compactness). The bijectivity assumption in the theorem is certainly stronger than strictly necessary. In fact, instead, for twist maps it is more natural to assume the infinite twist condition: a twist map is said to satisfy the *infinite twist* condition if

$$\lim_{y\to\pm\infty} \pi_x f(x,y) = \pm\infty \qquad \text{for all } x \in \mathbb{R}. \tag{2.2}$$

For twist maps on the plane this condition in some sense means that the map has positive twist at infinity (not an infinite amount of twist). Under the infinite twist condition we have the same result as for diffeomorphisms.

THEOREM 2.3. *An orientation reversing twist map of the plane that satisfies the infinite twist condition and that has a type I period-4 point, is a chaotic system — chaotic as explained in Theorem 2.2. Conversely, there exists an orientation reversing twist map that satisfies the infinite twist condition, has a type II period-4 point, and that has zero entropy.*

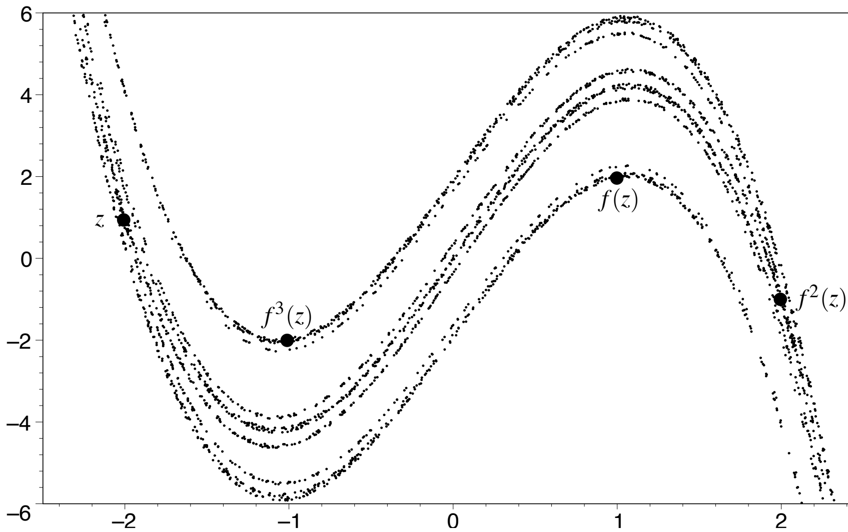

Figure 2.2: Orbits of the map $f(x,y) = (y, x + \frac{17}{3}y - \frac{5}{3}y^3)$. A period-4 orbit of type I is indicated by the large dots.

We can also give a lower bounds on the entropy for Theorems 2.2 and 2.3. Namely the entropy satisfies $h(f) \geq \frac{1}{2}\ln(1+\sqrt{2})$ and $h(f) \geq \frac{1}{2}\ln 3$ in Theorems 2.2 and 2.3 respectively.

The infinite twist condition makes the topological/variational principle we use easier to apply and the proof less technical. This is strongly related to the fact that the infinite twist condition is a more natural assumption in the context of twist maps than bijectivity. We will therefore explain all the details by proving Theorem 2.3. In Section 2.7 we make the necessary technical adaptations to the method in order to prove Theorem 2.2.

The method discussed in this paper makes extensive use of the twist property. On the other hand, we stress that it needs no compactness conditions, nor information about the asymptotics of f near infinity. It allows us to study periodic solutions of orientation reversing twist maps, in particular those of which the period is a multiple of four (but other periods can be dealt with as well). Theorems 2.2 and 2.3 are representative for the kind of results that can be obtained, but the method is much more general. We note that there is an additional variational structure that can be exploited in this setting if the (absolute value of the) area is preserved (see Remarks 2.5 and 2.15).

Of course the theorem does not detect all occurrences of chaos. An important example of orientation reversing twist maps is the Hénon map $f(x,y) = (\beta y, 1 - \alpha y^2 + x)$, where $\alpha \in \mathbb{R}$ and $\beta > 0$ are parameters. It is well known that for various parameter choices the system is chaotic, while a type I period-4 point is hard/impossible to find. Nevertheless, concerning the practical aspects of the above theorem we note that to establish chaos one can search for a type I period-4 point with the help of a computer. This can be done in a mathematically rigorous manner, for example with the help of a software package like GAIO, see [**25, 22**]. Furthermore, in the family of generalized Hénon maps $f(x,y) = (y, x + ay - by^3)$, which are orientation reversing twist maps, a period-4 orbit of type I can be found analytically (exploiting the symmetry) for $a > 4\sqrt{2}$ and any $b > 0$. In Figure 2.2 a period-4 orbit of type I is indicated and the chaotic nature of the dynamics is apparent.

To obtain an example of a non-chaotic map with a period-4 orbit of type II we return to the classical Hénon map, for convenience rescaled to read $f(x,y) = (y, \varepsilon x + \lambda[y - y^2])$. For $\varepsilon = 0$ this is a one dimensional map and for λ not too large it is non-chaotic. For small positive ε the 1-dimensional map perturbs to a 2-dimensional map, which for appropriately chosen λ has a period-4 point of type II and which remains non-chaotic. The details of the construction are given in Section 2.6. This provides a proof of second statements in Theorems 2.2 and 2.3.

We like to point out the similarity of the above theorem and the famous Sharkovskii theorem [**64, 44**], which states that a one dimensional system having a period-3 point necessarily has periodic points of all periods. In our case chaos is forced by certain period-4 points. In a one dimensional system the Sharkovskii ordering has little implications for a map containing a period-4 point. Nevertheless, also in the one dimensional case certain types of period-4 orbits (depending on the permutation of the points) force chaos (proved via the usual one dimensional techniques).

On compact surfaces of genus G (with or without boundary) the results in [**36**] and [**14**] show that if an orientation reversing diffeomorphism has at least $G+2$ periodic points of distinct odd periods, then there exist periodic points for infinitely many different periods, and in particular the topological entropy of the map is positive. The maps in this paper are maps on $\mathbb{R}^2$ and therefore the above result does not immediately apply. However, in the special circumstance that an orientation reversing map on $\mathbb{R}^2$ allows extension to S^2 with a fixed point at infinity, then the existence of a period-3 point, or any other odd period for that matter, implies, by the above mentioned result, that the map has positive topological entropy. To translate this result back to the context of the original map on $\mathbb{R}^2$ one needs (detailed) information about the local behavior near the point at infinity (the asymptotics of the map). In contrast, Theorems 2.2 and 2.3 are applicable without prior knowledge of asymptotic behavior. Moreover, our result gives insight in what happens when we have information about period-4 points, which complements the results on periodic orbits with odd periods in [**14, 36**].

The relation to Thurston's theory

Once again, the method of proof in this paper strongly relies on the fact that we consider (compositions of) twist maps, which allows an elementary construction of infinitely many periodic points and a semi-conjugacy to a (sub-)shift on 3 symbols. This draws strongly on the elegant topological principle for twist maps. A different approach would be to employ Thurston's classification theorem of surface diffeomorphisms [**68**]. Thurston's result does not restrict to twist maps, however compactness is required (we come back to this point in a moment).

Since the results for arbitrary maps on compact surfaces via Thurston's theory are complementary to those for twist maps on the (non-compact) plane in the present paper, let us explain how our results relate to Thurston's theory. For sake of simplicity, let us assume that the maps can be extended to homeomorphisms on for example D^2. In that case the classification theorem is applicable. In order to follow the approach using Thurston's classification theorem we first need to decide what distinguishes period-4 points. In our approach there is a natural distinction into two types of period-4 points via discrete four strand braids. In

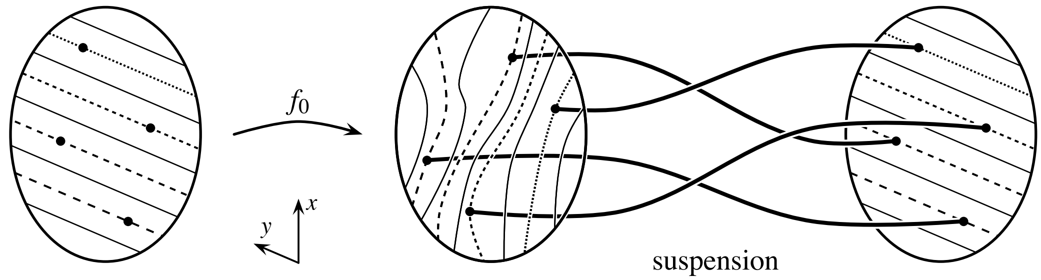

Figure 2.3: The map f_0 is orientation preserving and has the twist property, hence the suspension looks like a distorted rotation, which leads to a positive braid.

the approach using Thurston's result the braids are used to determine the isotopy class of a map in question, see e.g. [**17**].

It is easier to visualize this for orientation preserving maps, so we consider $g = f^4$, which is an orientation preserving map and which can be written as a composition of four orientation preserving positive twist maps $g = f_3 \circ f_2 \circ f_1 \circ f_0$. In the case of a period-4 orbit $P = \{f^i(z)\}_{i=0}^3$ for f, the map g has four fixed points P. Therefore one considers the mapping class group $\mathrm{MCG}(D^2 \text{ rel } P)$, where the maps are orientation preserving and fix P (as a set) and ∂D^2 (a homeomorphism of the boundary). Using the results in [**13**] it can be shown easily that $\mathrm{MCG}(D^2 \text{ rel } P) \simeq B_4/\text{center}$, where B_4 is Artin's braid group on four strands, and the center of the braid group B_4 is the infinite cyclic subgroup generated by $(\sigma_1\sigma_2\sigma_3)^4$, the full twists.

In general it is quite hard to determine the mapping class of a map, but for twist maps this is a little easier. In fact, identifying the mapping class with the braid group, the mapping class for f^4 is exactly the positive braid we have constructed above. We illustrate this for the first of the composite maps f_0 for a type I period-4 orbit in Figure 2.3. Besides the permutation of the (x-coordinates of the) points in P, the twist property gives global information about the map, so that the suspension can be understood (note that f_0 does not fix P, but this does not lead to undue complications). The other three maps are similar and the total braid is obtained by the natural addition in the braid group. We refer to [**15**] for a further discussion on the application of Thurston's theory to twist maps on an annulus. As a final point, the same construction can be carried out for $\tilde{g} = f^2 = f_+ \circ f_-$. One needs to take into account that f_- has negative twist and thus leads to a braid with negative generators. Of course, repeating the braid for $\tilde{g}$ twice leads to a braid that is equivalent to the one for g.

Using Thurston's classification the braid of type I is pseudo-Anosov, and thus the corresponding mapping class is also pseudo-Anosov, hence chaotic. In order to draw conclusions for the original map on $\mathbb{R}^2$ one needs to find a compact invariant set in the *interior* of D^2 on which the entropy is positive. This requires detailed information about the behavior near ∂D^2, and thus about the asymptotic behavior of the original map on $\mathbb{R}^2$. This is *not* needed in our results however. The braid of type II is reducible and contains only components of finite type (and thus no pseudo-Anosov component, in fact the braid is a cable of cabled braids), hence the corresponding map is not necessarily chaotic. We point out that our construction of a non-chaotic map with a type II period-4 point confirms the latter conclusion. However, Thurston's classification theorem does not provide a non-chaotic map within the class of twist maps as required here. See also [**18**] for details on pseudo-Anosov maps and mapping classes.

The organization of the paper is as follows. In Section 2.2 we recall some facts about twist maps and for orientation reversing maps we introduce a transformation that associates a parabolic recurrence relation to such maps. In Section 2.3 we summarize the concepts we need from braid theory and parabolic flows, which were thoroughly studied in [**33**]. In Section 2.4 the focus shifts to period-4 orbits and their classification in types I and II. We combine these concepts in Section 2.5 to prove the first assertion in Theorem 2.3 by constructing a semi-conjugacy to the shift on three symbols. In Section 2.6 we show an example of a non-chaotic map with a period-4 orbit of type II, which establishes the second part of the theorem. Finally, Section 2.7 is devoted to extending the techniques to bijective maps and proving Theorem 2.2.

Acknowledgement

The authors wish to thank R.W. Ghrist for a number of fruitful discussions on this subject.

2.2. Twist Maps

We collect some facts about both orientation preserving and reversing twist maps.

Recurrence relations for twist maps

A C^1 map from $\mathbb{R}^2$ to $\mathbb{R}^2$, denoted by $f(x,y) = (\pi_x f, \pi_y f)$, is a (positive) twist map if $\frac{\partial \pi_x f}{\partial y} > 0$. It is orientation preserving if $\det(df) > 0$ and orientation reversing if $\det(df) < 0$. Of course, one could also consider $\frac{\partial \pi_y f}{\partial x} > 0$ and/or negative twist, but a change of coordinates reduces these cases to $\frac{\partial \pi_x f}{\partial y} > 0$.

Note that iterates f^k of a twist map are not necessarily twist maps, but the crucial property of twist maps is that they allow us to retrieve whole trajectories $\{(x_k, y_k)\} = \{f^k(x_0, y_0)\}$ from just the sequence $\{x_k\}$. To show this we follow [**4**] (see also [**2**]). Let us start with the observation that the twist property implies that there exists an open set U such that for any pair $x, x' \in U$ there exists a unique solution $Y(x,x')$ of the equation

$$\pi_x f(x, Y(x,x')) = x'.$$

It also follows from the twist property that Y is monotone in x':

$$\frac{\partial Y}{\partial x'} > 0.$$

From the function Y we construct yet another function:

$$\widetilde{Y}(x,x') \stackrel{\text{def}}{=} \pi_y f(x, Y(x,x')).$$

This second function $\widetilde{Y}$ also has a monotonicity property that follows directly from the inverse function theorem. The map f is locally invertible and the derivative of its inverse

f^{-1} is given by $\partial_2(\pi_x f^{-1}) = -(\det(df))^{-1}\partial_2(\pi_x f)\circ f^{-1}$, hence

$$\partial_2(\pi_x f^{-1}) < 0 \quad \text{and} \quad \frac{\partial \widetilde{Y}}{\partial x} < 0 \qquad \text{if } f \text{ is orientation preserving,}$$
$$\partial_2(\pi_x f^{-1}) > 0 \quad \text{and} \quad \frac{\partial \widetilde{Y}}{\partial x} > 0 \qquad \text{if } f \text{ is orientation reversing.}$$

Obviously the reason for these definitions is that if $(x_{k+1}, y_{k+1}) = f(x_k, y_k)$ then

$$y_k = Y(x_k, x_{k+1}) \qquad \text{and} \qquad y_{k+1} = \widetilde{Y}(x_k, x_{k+1}).$$

That is, the functions Y and $\widetilde{Y}$ can be used to retrieve the whole trajectory $\{(x_k, y_k)\}$ from the sequence $\{x_k\}$. It easily follows that a sequence $\{(x_k, y_k)\}$ forms an orbit of f if and only if the x-coordinates satisfy

$$Y(x_k, x_{k+1}) - \widetilde{Y}(x_{k-1}, x_k) = 0 \qquad \text{for all } k \in \mathbb{Z}.$$

We therefore introduce the notation

$$\mathcal{R}(x_{k-1}, x_k, x_{k+1}) \stackrel{\text{def}}{=} Y(x_k, x_{k+1}) - \widetilde{Y}(x_{k-1}, x_k). \tag{2.3}$$

Solutions $\{x_k\}$ of the recurrence relation $\mathcal{R}(x_{k-1}, x_k, x_{k+1}) = 0$ thus correspond to trajectories of the map f. From the properties of Y and $\widetilde{Y}$ we see that $\mathcal{R}$ is increasing in x_{k+1}, and if f is orientation preserving then $\mathcal{R}$ is also increasing in x_{k-1}. In this case $\mathcal{R}$ will be referred to as a *parabolic recurrence relation*. When f is orientation reversing then $\mathcal{R}$ is not increasing, but decreasing in x_{k-1}. In Section 2.2 we explain how we can, nevertheless, associate a parabolic recurrence relation to an orientation reversing map.

The function Y (and similarly $\widetilde{Y}$) has a domain of the form

$$D = \{(x, x') \,|\, x \in \mathbb{R}, g(x) < x' < h(x)\},$$

where the functions $g, h : \mathbb{R} \to [-\infty, \infty]$ are upper/lower semi-continuous with $g(x) < h(x)$, see Section 2.7 for more details. A way to ensure that the domain D is the whole plane, is to assume the *infinite twist* condition (2.2). To simplify the exposition in the following sections we assume that $D = \mathbb{R}^2$. In Section 2.7 we show how to extend our results to maps that are bijective to $\mathbb{R}^2$ (i.e. diffeomorphisms of the plane). Note that bijectivity does not imply the infinite twist condition, nor does it guarantee that $D = \mathbb{R}^2$.

REMARK 2.4. Any twist map that satisfies the infinite twist condition is injective. Namely, let $f(x_0, y_0) = f(x_1, y_1) = (x', y')$. If $x_0 = x_1$ then it follows from the twist property that $y_0 = y_1$. Suppose $x_0 \neq x_1$, say $x_0 < x_1$, then the infinite twist condition implies that for any $x \in [x_0, x_1]$ there is a (unique) $y(x)$ such that $\pi_x f(x, y(x)) = x'$, with $y(x_0) = y_0$ and $y(x_1) = y_1$. Since $\frac{\partial \widetilde{Y}(x,x')}{\partial x} \leqslant 0$ we have $\frac{d\pi_y f(x,y(x))}{dx} \leqslant 0$, contradiction the fact that $\pi_y f(x_0, y(x_0)) = \pi_y f(x_1, y(x_1)) = y'$.

REMARK 2.5. When f is an orientation and *area* preserving twist map there exists an additional structure, namely generating functions (see e.g. [**7**]). A smooth function $S : \mathbb{R}^2 \to \mathbb{R}$ exists with the property that if $(x', y') = f(x, y)$, then $y = \partial_1 S(x, x')$, and $y' = -\partial_2 S(x, x')$. This generating function S allows one to formulate the existence of periodic points in terms of critical points of an action function. A period-n point corresponds to a critical point of

$$W(x_0, x_1, \ldots, x_{n-1}) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} S(x_i, x_{i+1}), \qquad \text{with } x_n = x_0.$$

The parabolic recurrence relation is then given by the gradient of W: $\mathcal{R}(x_{i-1},x_i,x_{i+1}) = \frac{\partial W}{\partial x_i}$. For orientation reversing area preserving maps a similar variational structure exists. The difference is that the relations between the generating function S and the y coordinates are $y = \partial_1 S(x,x')$ and $y' = \partial_2 S(x,x')$, i.e. with the same sign. A period-$2m$ point corresponds to a critical point of

$$W(x_0,x_1,\dots,x_{2m-1}) \stackrel{\text{def}}{=} \sum_{i=0}^{2m-1}(-1)^i S(x_i,x_{i+1}), \qquad \text{with } x_{2m} = x_0.$$

The recurrence relation is not quite given by the gradient, but by $\frac{\partial W}{\partial x_i} = (-1)^i \mathcal{R}(x_{i-1},x_i,x_{i+1})$, so there is still a correspondence between critical points of W and solutions of $\mathcal{R}$. However, it is more convenient to deal with such a situation through the (flip) transformation described in Section 2.2 below.

We finish this section with an example.

EXAMPLE 2.6. Let us consider the well known *Hénon map*. The Hénon map is a two-dimensional invertible map given by formula:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \beta y \\ 1 - \alpha y^2 + x \end{pmatrix}.$$

It is an orientation reversing twist map for all $\beta > 0$ and $\alpha \in \mathbb{R}$. It is bijective and also satisfies the infinite twist condition (2.2). It is not difficult to construct the recurrence relation:

$$\mathcal{R}(x_{k-1},x_k,x_{k+1}) = -1 - x_{k-1} + \alpha\beta^{-2}x_k^2 + \beta^{-1}x_{k+1}.$$

Parabolic recurrence relations for orientation reversing twist maps

Consider the case that f is an orientation *reversing* twist map. From the previous subsection it then follows that the trajectory of a periodic point can be retrieved from the sequence $\{x_k\}$ satisfying the recurrence relation

$$\widetilde{\mathcal{R}}(x_{k-1},x_k,x_{k+1}) = 0,$$

where $\widetilde{\mathcal{R}}$ is defined by (2.3), with $\partial_1\widetilde{\mathcal{R}} < 0$ and $\partial_3\widetilde{\mathcal{R}} > 0$. Since the theory of braid flows (see Section 2.3) is defined using parabolic recurrence relations (i.e. $\partial_1\mathcal{R} > 0$ and $\partial_3\mathcal{R} > 0$), we need to make a modification. In Section 2.1 we explained that f^4 can be written as a composition of four orientation preserving positive twist maps f_i. For each f_i we can derive the recurrence function $\mathcal{R}_i$, which has the properties that

$$\partial_1\mathcal{R}_i > 0 \qquad \text{and} \qquad \partial_3\mathcal{R}_i > 0.$$

This is equivalent to defining the functions $\mathcal{R}_i$ as follows

$$\begin{aligned}
\mathcal{R}_0(x_{-1},x_0,x_1) &\stackrel{\text{def}}{=} \widetilde{\mathcal{R}}(-x_{-1},-x_0,x_1)\\
\mathcal{R}_1(x_0,x_1,x_2) &\stackrel{\text{def}}{=} \widetilde{\mathcal{R}}(-x_0,x_1,x_2)\\
\mathcal{R}_2(x_1,x_2,x_3) &\stackrel{\text{def}}{=} -\widetilde{\mathcal{R}}(x_1,x_2,-x_3)\\
\mathcal{R}_3(x_2,x_3,x_4) &\stackrel{\text{def}}{=} -\widetilde{\mathcal{R}}(x_2,-x_3,-x_4).
\end{aligned}$$

It is easily verified that the recurrence functions are indeed parabolic and we define the sequence $\{\mathcal{R}_i\}$ periodically: $\mathcal{R}_{i+4} = \mathcal{R}_i$. This change of coordinates naturally also effects the trajectory $x = \{x_k\}$. To make this precise we define the transformation

$$\lambda(x)_k = \begin{cases} -x_k & \text{for } k = 0,3 \bmod 4 \\ x_k & \text{for } k = 1,2 \bmod 4 \end{cases} \tag{2.4}$$

We call the transformation λ on sequences a *flip*. Clearly $\lambda^2 = \mathrm{id}$ and it commutes with σ^4, where σ is the shift map $\sigma(x)_k = x_{k+1}$. Now $x = \{x_k\}$ solves $\widetilde{\mathcal{R}} = 0$ if and only if $\lambda(x)$ solves $\mathcal{R}_i = 0$.

LEMMA 2.7. *Every solution $x = \{x_k\}$ of $\mathcal{R}_i = 0$ yields a solution $\lambda(x)$ of $\widetilde{\mathcal{R}} = 0$, and thus corresponds to a trajectory of f, namely $\{(\lambda(x)_k, Y(\lambda(x)_k, \lambda(x)_{k+1}))\}$.*

2.3. Braid diagrams and the Conley index

Discretized braids and braid diagrams

In this section we define and describe the main topological structure which is used in the proofs of Theorems 2.2 and 2.3. As pointed out in Section 2.1 the way we deal with sequences is to consider them as piecewise linear functions by connecting the consecutive points via linear interpolation.

DEFINITION 2.8 ([**33**]). The space of *discretized period d braids on n strands,* denoted $\mathcal{D}^n_d$, is the space of all pairs $(\mathbf{u}, \tau)$, where $\tau \in S_n$ is a permutation on n elements, and $\mathbf{u}$ is an unordered collection of n strands $\mathbf{u} = \{\mathbf{u}^\alpha\}_{\alpha=1}^n$, which satisfy the following properties:

(a) Each strand consist of $d+1$ anchor points: $\mathbf{u}^\alpha = (u_0^\alpha, u_1^\alpha, \ldots, u_d^\alpha) \in \mathbb{R}^{d+1}$.

(b) *periodicity* – For all $\alpha = 1, \ldots, n$, one has: $u_d^\alpha = u_0^{\tau(\alpha)}$.

(c) *transversality* – For any pair of distinct strands α and α' such that $u_i^\alpha = u_i^{\alpha'}$ for some i, we have:

$$(u_{i-1}^\alpha - u_{i-1}^{\alpha'})(u_{i+1}^\alpha - u_{i+1}^{\alpha'}) < 0. \tag{2.5}$$

We equip $\mathcal{D}^n_d$ with the standard topology of $\mathbb{R}^{nd}$ on the strands, and the discrete topology with respect to the permutation τ, modulo permutations which change the order of the strands (i.e., two pairs $(\mathbf{u}, \tau)$ and $(\tilde{\mathbf{u}}, \tilde{\tau})$ are close if there exists a permutation $\sigma \in S_n$ with $\sigma \circ \tilde{\tau} = \tau \circ \sigma$), such that $\mathbf{u}^{\sigma(\alpha)}$ is close to $\tilde{\mathbf{u}}^\alpha$ (as points in $\mathbb{R}^{nd}$) for all α.

We will say that two discretized braids $\mathbf{u}, \mathbf{u}' \in \mathcal{D}^n_d$ are of the same *discretized braid class* (denoted $[\mathbf{u}] = [\mathbf{u}']$) if they are in the same path component of $\mathcal{D}^n_d$. The discrete topology on the permutations leads to the following useful interpretation. Consider a continuous family of braids and pick one of the permutations in the equivalence class (subsequently dropped from the notation). These discretized braids of period d on n strands are then completely determined by their coordinates $\{u_i^\alpha\}_{i=1\ldots d}^{\alpha=1\ldots n}$, i.e., every discretized braid corresponds to a point in the configuration space $\mathbb{R}^{nd}$. We come back to this point of view later.

Let us now compare the notion of a discretized braid with that of a topological braid. In topology a braid β on n strands is a collection of embeddings $\{\beta^\alpha : [0,1] \to \mathbb{R}^3\}_{\alpha=1}^n$ with disjoint images such that (a) $\beta^\alpha(0) = (0, \alpha, 0)$, (b) $\beta^\alpha(1) = (1, \tau(\alpha), 0)$ for some permutation $\tau \in S_n$, and (c) the image of each β^α is transverse to all the planes $\{x = \text{constant}\}$.

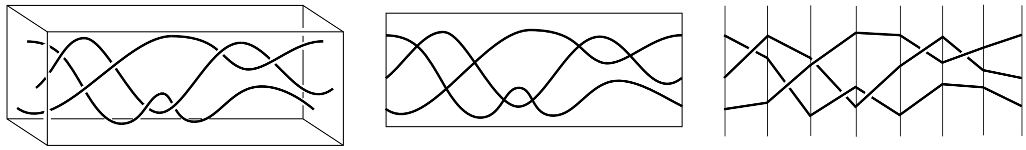

Figure 2.4: Example of a braid on three strands. [left] A braid with all crossings positive (bottom over top), [middle] its 2-d projection, and [right] the associated piecewise linear braid diagram, a discretized braid. Its braid word is $\sigma_2\sigma_1\sigma_2\sigma_1^2\sigma_2^2$.

The projection of a topological braid onto an appropriate plane, e.g. the (x,y)-plane, is called a *braid diagram* if all crossings of strands are transversal in this projection. In this braid diagram a marking $(+)$ indicates a crossing which is "bottom over top", whereas a marking $(-)$ indicates a crossing "top over bottom". A positive $(+)$ crossing of the i-th and $(i+1)$-st strands corresponds to a generator σ_i, while a negative crossing corresponds to σ_i^{-1}. The use of these generators σ_i leads to a natural group structure (see e.g. [**13**] for more background). The sequence of generators ("reading" the braid from left to right) is called the braid word.

The link between discretized braids and topological braids is the following. Any discretized braid $\mathbf{u}$ can be interpreted as the braid diagram of a topological braid when we use linear interpolation between the points $(i,u_i^\alpha)\in\mathbb{R}^2$, where u_i^α are the anchor points of strand α. Here we choose the convention that all crossings in this discretized braid diagram are *positive*. The resulting positive piecewise linear braid diagram is denoted by $\beta(\mathbf{u})$. It is also useful to consider braid diagrams that are not piecewise linear. A (positive, closed) topological braid diagram is a collection of strands $\{\beta^\alpha\in C([0,1])\}_{\alpha=1}^n$ such that (a) $\beta^\alpha(1)=\beta^{\tau(\alpha)}(0)$ for some permutation $\tau\in S_n$, and (b) all intersections among pairs of strands are isolated and topologically transverse. The *topological braid class* $\{\mathbf{u}\}$ is a path component of $\beta(\mathbf{u})$ in the space of positive topological braid diagrams. Figure 2.4 depicts a braid in its various appearances. Since for positive braids the braid word consists of positive generators only, it follows that the number of generators in the braid word, the *braid word length*, is an invariant of a discretized braid class, and even of a topological braid class. For a more detailed account we refer to [**33**].

Since discretized braids are periodic we extend all strands periodically:

$$u_{i+d}^\alpha = u_i^{\tau(\alpha)} \qquad \text{for all } i\in\mathbb{Z}, \quad \alpha=1,\ldots,n.$$

As explained above, $\mathcal{D}_d^n$ is a subset of a collection of copies of $\mathbb{R}^{nd}$ (one for each equivalence class of permutations). Fixing an appropriate permutation, we may identify a discretized braid class with a subset of $\mathbb{R}^{nd}$, its configuration space. The connected components of $\mathcal{D}_d^n$, i.e. the discretized braid classes, are separated by co-dimension-1 varieties in $\mathbb{R}^{nd}$, called the singular braids:

DEFINITION 2.9. Let $\overline{\mathcal{D}}_d^n$ denote the collection of nd-dimensional vector spaces of all discretized braid diagrams $\mathbf{u}$ satisfying properties (1) and (2) of Definition 2.8. Now $\Sigma \stackrel{\text{def}}{=} \overline{\mathcal{D}}_d^n\setminus\mathcal{D}_d^n$ is the set of *singular discretized braids*.

The set $\overline{\mathcal{D}}_d^n$ is the closure of $\mathcal{D}_d^n$, hence its elements do not necessarily satisfy the transversality condition (2.5). The braids in Σ are said to have a *tangency*. A moments reflection shows that in singular braids of sufficiently high co-dimension $(m \geq d)$, different strands can collapse onto each other. This set of specific singularities plays an important role later on and is defined as

$$\Sigma^- \stackrel{\text{def}}{=} \{\mathbf{u} \in \Sigma \,|\, u_i^\alpha = u_i^{\alpha'}, \forall i \in \mathbb{Z}, \text{ for some } \alpha \neq \alpha'\}.$$

If one wants to braid a strand, one needs something to braid it through. This leads us to the introduction of a so-called *skeleton* braid through which we can braid so-called *free* strands. Define $\mathbf{u} \cup \mathbf{v} \in \overline{\mathcal{D}}_d^{n+m}$, with $\mathbf{u} \in \overline{\mathcal{D}}_d^n$ and $\mathbf{v} \in \overline{\mathcal{D}}_d^m$ as the (unordered) union of strands. Then for given a $\mathbf{v} \in \mathcal{D}_d^m$ we define

$$\mathcal{D}_d^n \text{ rel } \mathbf{v} \stackrel{\text{def}}{=} \{\mathbf{u} \in \mathcal{D}_d^n \,|\, \mathbf{u} \cup \mathbf{v} \in \mathcal{D}_d^{n+m}\}.$$

It is important to remember that the transversality condition (2.5) is imposed on the strands in $\mathbf{u} \cup \mathbf{v}$.

The path components of $\mathcal{D}_d^n$ rel $\mathbf{v}$ form *relative discretized braid classes*, denoted by $[\mathbf{u} \text{ rel } \mathbf{v}]$. The braid $\mathbf{v}$ is usually called the *skeleton*, and $\mathbf{u}$ are called the *free strands*. Now it is easy to define relative versions of the concepts presented above, i.e. Σ rel $\mathbf{v}$, Σ^- rel $\mathbf{v}$, $\overline{\mathcal{D}}_d^n$ rel $\mathbf{v}$, and $\{\mathbf{u} \text{ rel } \mathbf{v}\}$ (as *topological relative braid class*).

It is also possible that two classes $[\mathbf{u} \text{ rel } \mathbf{v}]$ and $[\mathbf{u}' \text{ rel } \mathbf{v}']$ are topologically the same. The set of equivalent topological relative braid classes $\{\mathbf{u} \text{ rel } \{\mathbf{v}\}\}$ is defined by the relation $\{\mathbf{u} \text{ rel } \mathbf{v}\} \sim \{\mathbf{u}' \text{ rel } \mathbf{v}'\}$ if and only if there exist a continuous family of topological (positive, closed) braid diagram pairs deforming $(\mathbf{u}, \mathbf{v})$ to $(\mathbf{u}', \mathbf{v}')$. See [**33**] for more details.

Parabolic flows on braid diagrams

In [**33**] the topology of discretized braids is used to find solutions of parabolic recurrence relations. This is done by embedding the problem into an appropriate dynamical setting. Before briefly explaining the ideas we recall the definition of parabolic recurrence relations.

DEFINITION 2.10 ([**33**]). A sequence of functions $\mathcal{R} = (\mathcal{R}_i)_{i \in \mathbb{Z}}$, with $\mathcal{R}_i \in C^1(\mathbb{R}^3, \mathbb{R})$, satisfying

(i) $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i \geq 0$ for all $i \in \mathbb{Z}$,
(ii) for some $d \in \mathbb{N}$ we have $\mathcal{R}_{i+d} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$,

is called a *parabolic recurrence relation*.

Here we only consider parabolic recurrence relations defined on $\mathbb{R}^3$, although one can also study parabolic recurrence relations on more general domains, see Section 2.7.

Let $\mathcal{R}$ be a parabolic recurrence relation and consider the differential equation

$$\frac{du_i}{dt} = \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) \quad \text{where } \mathbf{u}(t) \in \mathbf{X} = \mathbb{R}^{\mathbb{Z}} \text{ and } t \in \mathbb{R}.$$

It is straightforward to show that such an equation defines a (local) C^1-flow ψ^t on $\mathbf{X}$ under periodic boundary conditions, provided they are of period nd. We call such a flow, generated by a parabolic recurrence relation, a *parabolic flow* on $\mathbf{X}$. Notice that it is easy to regard this flow as a flow on the space $\overline{\mathcal{D}}_d^n$ by considering the equation

$$\frac{du_i^\alpha}{dt} = \mathcal{R}_i(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha), \quad \text{where } \mathbf{u} \in \overline{\mathcal{D}}_d^n. \tag{2.6}$$

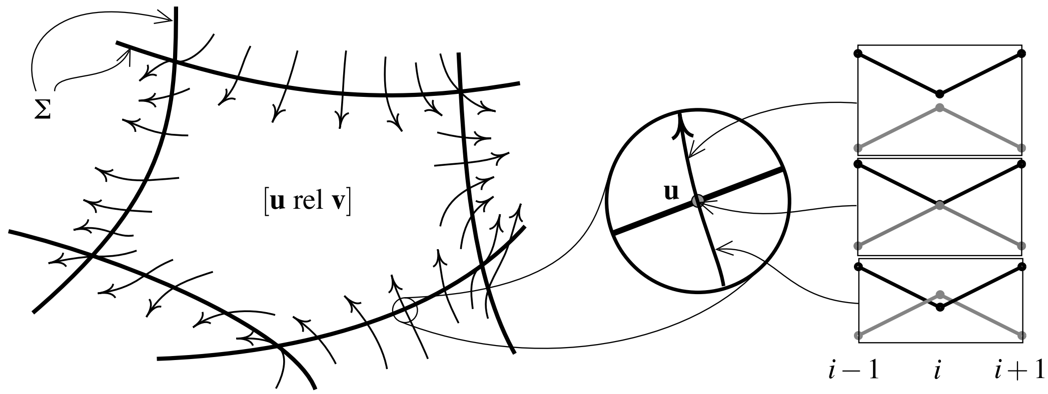

Figure 2.5: A schematic picture of a parabolic flow on a (bounded and proper) braid class.

This equation is well-defined by the periodicity requirement in Definition 2.10. The next-neighbor coupling and the monotonicity of a parabolic recurrence relation have far reaching consequences for the corresponding parabolic flow. Namely, along flow lines the total number of intersections in a braid, i.e. the braid word length, can only decrease in time (as indicated in Figure 2.5). The following proposition is a precise statement of this property.

PROPOSITION 2.11 ([**33**]). *Let* ψ^t *be a parabolic flow on* $\overline{\mathcal{D}}^n_d$.

(a) *For each point* $\mathbf{u} \in \Sigma \setminus \Sigma^-$, *the local orbit* $\{\psi^t(\mathbf{u}) \mid t \in [-\varepsilon, \varepsilon]\}$ *intersects* Σ *uniquely at* $\mathbf{u}$ *for all* ε *sufficiently small.*

(b) *For any such* $\mathbf{u}$, *the braid word length of the braid diagram* $\psi^t(\mathbf{u})$ *for* $t > 0$ *is strictly less then that of the braid diagram* $\psi^t(\mathbf{u})$ *for* $t < 0$.

As a direct consequence of this proposition flow lines cannot re-enter a braid class after leaving it. In other words, the dynamics of (2.6) obeys the natural co-orientation of the braid classes, i.e., if we co-orient the boundary $\Sigma \setminus \Sigma^-$ in the direction of decreasing intersection number, then the vector field, and thus the flow, is co-oriented in the same way.

In Section 2.3 we will define the Conley index of a braid class, hence we need the braid class to be isolating, i.e., the flow at the boundary should have no internal tangencies. Proposition 2.11 shows that we are "in danger" when our system evolves near to Σ^-, since a parabolic flow displays invariant behavior in Σ^-. For this reason, a discretized relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is called *proper* if its boundary (which is a subset of Σ rel $\mathbf{v}$) does not intersect Σ^- rel $\mathbf{v}$. Figure 2.6 gives a simple examples of a proper and an improper braid class. Besides properness we also need the braid classes to be compact. A discretized relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is called *bounded* if the set $[\mathbf{u} \text{ rel } \mathbf{v}] \subset \mathbb{R}^{(n+m)d}$ is bounded.

Conley index for braids

The Conley index is a powerful tool for studying the complexity of dynamical systems. For braid classes the Conley index is defined in [**33**] and we refer to that paper for all details, proofs and much additional information. For more details about the general setting of the Conley index, see [**20, 52**]. Proposition 2.11 implies that $\mathrm{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$ is isolating for the

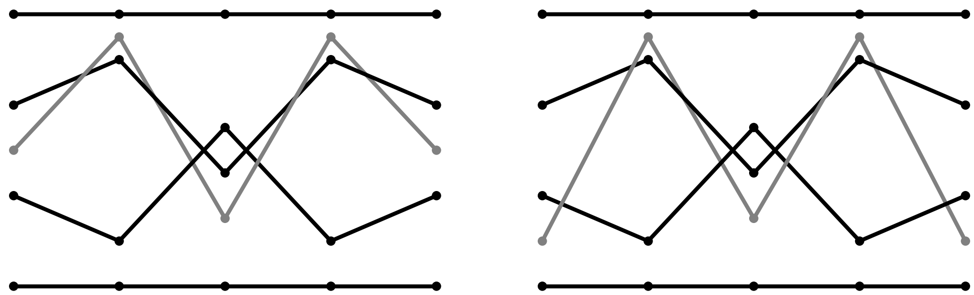

Figure 2.6: Two bounded braids with the same skeleton (black lines); the free strand is the gray line. The braid on the left is improper (one can deform the free strand to one of the strands of the skeleton), the one on the right is proper.

flow generated by a parabolic recurrence relation, provided the braid class is proper and bounded. Let N denote $\mathrm{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$, and let $N^- \subset \partial N$ be the exit set for a parabolic flow ψ^t. Then the Conley index $h(\mathbf{u} \text{ rel } \mathbf{v})$ is the homotopy type of the pointed space $(N/N^-, [N^-])$, denoted by $[N/N_-]$. Note that N^- can also be characterized purely in terms of braids by using the co-orientation of $\Sigma \setminus \Sigma^-$.

PROPOSITION 2.12 ([**33**]). *Suppose* $[\mathbf{u} \text{ rel } \mathbf{v}]$ *is a bounded proper relative discretized braid class and* ψ^t *is a parabolic flow that fixes the skeleton* $\mathbf{v}$. *Then*

(1) $\mathrm{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$ *is an isolating neighborhood for* ψ^t*, which yields a well-defined Conley index* $h(\mathbf{u} \text{ rel } \mathbf{v}, \psi^t)$.
(2) *The index* $h(\mathbf{u} \text{ rel } \mathbf{v}, \psi^t)$ *is independent of the choice of the parabolic flow* ψ^t *as long as* $\psi^t(\mathbf{v}) = \mathbf{v}$. *Therefore the index is denoted by* $h(\mathbf{u} \text{ rel } \mathbf{v})$.

REMARK 2.13. The Conley index is in fact an invariant of the topological relative braid class $\{\mathbf{u} \text{ rel } \mathbf{v}\}$, provided one slightly generalizes the definitions. First, the definitions of proper and bounded are extended in a straightforward manner to $\{\mathbf{u} \text{ rel } \mathbf{v}\}$. Furthermore, an equivalence class of topological relative braids $\{\mathbf{u} \text{ rel } \{\mathbf{v}\}\}$ is proper/bounded if for all $\mathbf{v}' \in \{\mathbf{v}\}$ any class $\{\mathbf{u}' \text{ rel } \mathbf{v}'\} \in \{\mathbf{u} \text{ rel } \{\mathbf{v}\}\}$ is proper/bounded.

Second, several discretized braid classes may be part of equivalent topological braid classes. For fixed period d, let $[\mathbf{u}(0) \text{ rel } \mathbf{v}']$ be a discretized braid class such that on the topological level $\{\mathbf{u}(0) \text{ rel } \mathbf{v}'\} \in \big\{\mathbf{u} \text{ rel } \{\mathbf{v}\}\big\}$. Let $[\mathbf{u}(j) \text{ rel } \mathbf{v}']$, $j = 0, \ldots, m$ denote *all* the different discretized braid classes relative to $\mathbf{v}'$ such that $\{\mathbf{u}(j) \text{ rel } \mathbf{v}'\} \in \big\{\mathbf{u} \text{ rel } \{\mathbf{v}\}\big\}$. The set $\widetilde{N} = \bigcup_{j=0}^{m} \mathrm{cl}([\mathbf{u}(j) \text{ rel } \mathbf{v}'])$ is isolating for any parabolic flow fixing $\mathbf{v}'$, and the exit set is denoted by $\widetilde{N}^-$. The Conley index $\mathbf{H}(\mathbf{u} \text{ rel } \mathbf{v}')$ of the topological relative braid class $\{\mathbf{u} \text{ rel } \mathbf{v}'\}$ is the homotopy type of the pointed space $(\widetilde{N}/\widetilde{N}^-, [\widetilde{N}^-])$. It does not depend on the period d, the choice of $\mathbf{v}'$ or the parabolic flow. The Conley index $\mathbf{H}(\mathbf{u} \text{ rel } \mathbf{v}')$ is an invariant of $\big\{\mathbf{u} \text{ rel } \{\mathbf{v}\}\big\}$.

The homotopy index is usually not very convenient to work with and therefore we use the *homological* Conley index

$$CH_*(\mathbf{u} \text{ rel } \mathbf{v}) \stackrel{\text{def}}{=} H_*(N, N^-)$$

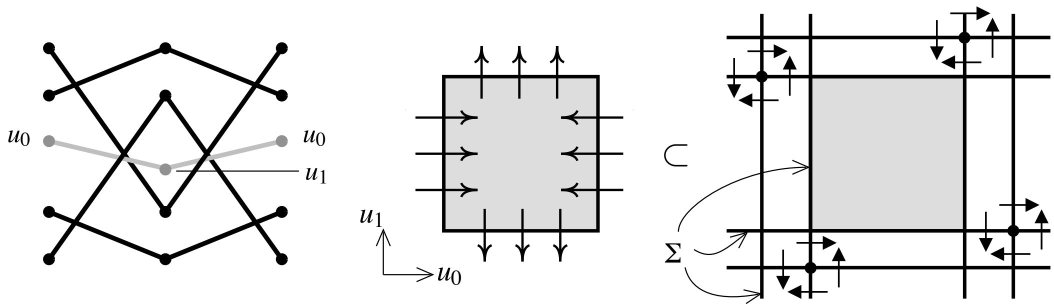

Figure 2.7: In the relative braid on the left black lines denote the skeleton and gray lines the free strand. In the middle its configuration space is shown and the direction of the parabolic flow on the boundary is indicated. On the right we see how the configuration space is positioned with respect to the stationary points of the skeleton, represented by the four dots.

where $N = \mathrm{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$, N^- is its exit set, and H_* is the relative homology of the pair (N, N^-). One can assign to such an index a characteristic polynomial

$$CP_t(\mathbf{u} \text{ rel } \mathbf{v}) \stackrel{\text{def}}{=} \sum_{k \geq 0} \beta_k t^k,$$

where β_k is a free rank of $CH_k(\mathbf{u} \text{ rel } \mathbf{v})$. For the parabolic flows under consideration Morse inequality can be used to draw conclusions from the characteristic polynomial about fixed points and periodic orbits (see Section 7 of [**33**]). In this paper we use the only the simplest consequence:

LEMMA 2.14. *Let* $[\mathbf{u} \text{ rel } \mathbf{v}]$ *be a discretized relative braid class that is bounded and proper. If* $CP_{-1}(\mathbf{u} \text{ rel } \mathbf{v})$ *is nonzero, then there is at least one stationary point in* $[\mathbf{u} \text{ rel } \mathbf{v}]$ *for any parabolic flow* ψ^t *that leaves* $\mathbf{v}$ *invariant.*

REMARK 2.15. A special situation occurs when the recurrence relation is exact, i.e., when there exists a d-periodic sequence of $C^2(\mathbb{R}^2)$ functions S_i such that

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_{i-1}(u_{i-1}, u_i) + \partial_1 S_i(u_i, u_{i+1}) \qquad \text{for all } i \in \mathbb{Z}.$$

Note that a recurrence relation is exact if it originates from a composition of area preserving twist maps, see Remark 2.5. The main example in our context is when the orientation reversing twist map f is area preserving. Setting $W(\mathbf{u}) = \sum_{i=1}^{d} S_i(u_i, u_{i+1})$ the corresponding parabolic flow is a gradient flow: $\frac{d\mathbf{u}}{dt} = \nabla W$. This implies that invariant sets consists of fixed points and connecting orbits only. The second order character of the recurrence relation leads to the following strong result (see [**33**, section 7]): for an exact parabolic flow on a bounded proper relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$, the number of fixed points is bounded below by the number of distinct nonzero monomials in the characteristic polynomial $CP_t(\mathbf{u} \text{ rel } \mathbf{v})$.

EXAMPLE 2.16. We calculate the homotopy index of the braid shown at the left in Figure 2.7. It is of period two and it is proper and bounded. A braid can evolve only in such a way as to decrease the number of intersections (cf. Proposition 2.11 and Figure 2.5). Hence along the flow the free anchor point u_0 cannot cross the anchor points of skeleton since this would lead to an increased number of crossing, i.e., u_0 is “trapped” between anchor points of the skeleton. On the other hand, the middle point u_1 of the free strand can evolve in such a way that it crosses the nearest anchor points, since this decreases the number of crossings. Of course, on crossing the anchor point of the skeleton, the free strand leaves the braid class. The configuration space and the flow on the boundary are shown in the middle in Figure 2.7. The exit set N^- consists of the top and bottom boundaries. The homotopy index of this braid class is $[N/N^-] \simeq (S^1, \mathrm{pt})$, hence $CP_t = t$ and any parabolic flow leaving $\mathbf{v}$ invariant has at least one fixed point inside the braid class.

2.4. Period-4 points for orientation reversing twist maps

We now apply the theory of braids and parabolic flows to orientation reversing twist maps. Let f be an orientation reversing twist map. As explained in the introduction and Section 2.2 we can write it as the composition of four orientation preserving twist maps. This leads to a parabolic recurrence relation $\mathcal{R} = (\mathcal{R}_i)_{i\in\mathbb{Z}}$ which is 4-periodic: $\mathcal{R}_{i+4} = \mathcal{R}_i$. Lemma 2.7 gives the correspondence between trajectories of f and solutions of the recurrence relation via the flip transformation (2.4).

Suppose now that $\{(x^i, y^i)\}_{i=1}^4$ is a period-4 orbit of f, i.e., its minimal period is four. Let $x = \{x^i\}_{i\in\mathbb{Z}}$, then the flipped sequence $\lambda(x)$ is a solution of the recurrence relation $\mathcal{R} = 0$. Obviously, any shift $\sigma^\alpha(x)$ of the sequence x corresponds to the same period-4 orbit of f. Hence $\lambda(\sigma^\alpha(x))$ for $\alpha = 1,2,3,4$ are four solutions of the parabolic recurrence relation $\mathcal{R} = 0$, labeled $\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4$ respectively, and they thus form the four stationary strands of a closed discretized braid diagram $\mathbf{v} = \{\mathbf{v}^\alpha\} \in \mathcal{D}_4^4$. A priori $\mathbf{v}$ is only in $\overline{\mathcal{D}}_4^4$, but if $\mathbf{v} \in \Sigma$, then necessarily $\mathbf{v} \in \Sigma^-$, since Proposition 2.11 implies there are no stationary points of a parabolic flow on $\Sigma \setminus \Sigma^-$. On the other hand, if $\mathbf{v} \in \Sigma^-$, then at least two of the strands $\lambda(\sigma^\alpha(x))$ coincide, hence the minimal period is smaller than four. However, we are assuming that the initial orbit is a true period-4 orbit and hence the corresponding braid diagram $\mathbf{v}$ is a discretized braid in $\mathcal{D}_4^4$.

The next question is: which braid classes do these period-4 orbits represent? Because we need to make sure that we consider all possible cases, we start simply from the quadruple (x^0, x^1, x^2, x^3). Assume, without loss of generality, that $x^0 = \min\{x^i\}$. There are six non-degenerate orderings (degenerate ones are discussed below), namely

$$\begin{array}{lll} x^0 < x^1 < x^2 < x^3, & x^0 < x^1 < x^3 < x^2, & x^0 < x^2 < x^1 < x^3, \\ x^0 < x^2 < x^3 < x^1, & x^0 < x^3 < x^1 < x^2, & x^0 < x^3 < x^2 < x^1. \end{array} \tag{2.7}$$

For each of these six possibilities the procedure described above leads to a closed discretized braid diagram. The easiest way to do this is depicted in Figure 2.8. Namely, one draws the four iterates of the four shifts of the periodic solution. Then one inverts the order of the points at the zeroth and third coordinates to obtain a *braid diagram*. It is perhaps good to point out that the picture in the middle of Figure 2.8, i.e. before the flip, is *not* interpreted as a braid diagram, since it is not related to a parabolic flow. For the six possible orderings the resulting braid diagrams are shown shown in Figure 2.9.

The six discretized braid diagrams can be grouped in two distinct topological braid classes, type I and type II, see Figure 2.9. We note that they are in four distinct discretized braid classes in $\mathcal{D}_4^4$, but on the topologically level these reduce to two classes. Type I has (periodic) braid word $\sigma_2^2\sigma_1^2\sigma_3^2\sigma_2^2\sigma_1^2\sigma_3^2$ and corresponds to orderings $x^0 < x^3 < x^1 < x^2$ and $x^0 < x^1 < x^3 < x^2$, while the (periodic) braid word of type II is $\sigma_1^2\sigma_2\sigma_1^2\sigma_2\sigma_3^2\sigma_2\sigma_3^2\sigma_2$.

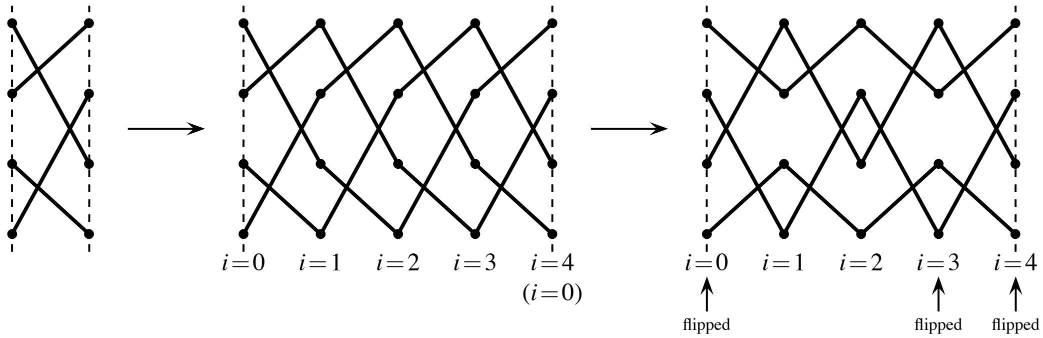

Figure 2.8: Starting from an ordering of the points (x^0, x^1, x^2, x^3) on the left (f permutes the ordered points), one uses four iterates (middle) and then applies the flip (i.e. inverting the order at the zeroth and third coordinates) to obtain the *braid diagram* on the right.

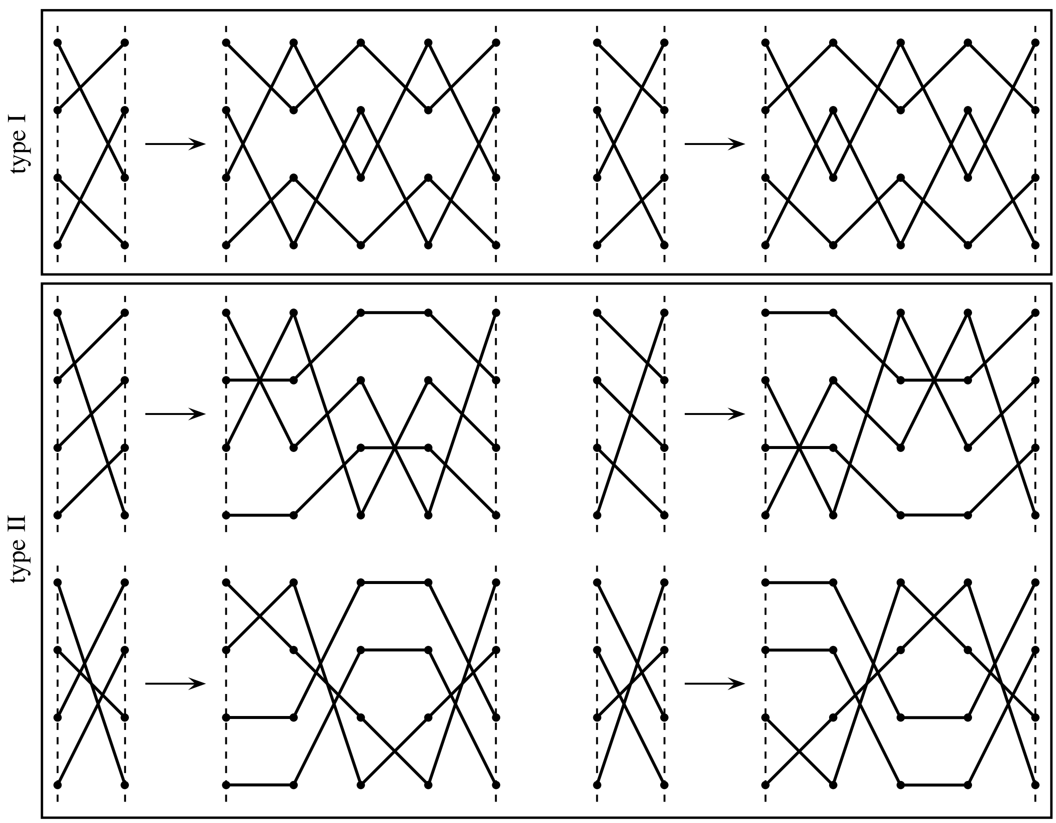

Figure 2.9: The six period-4 orbits and their corresponding braid diagrams.

As discussed above, since the braid consists of stationary solutions of a parabolic flow, the braid cannot have tangencies. Of course, anchor points can nevertheless coincide, which corresponds to a degenerate case in the ordering of the quadruple x^0,x^1,x^2,x^3. That is, some of the inequalities in (2.7) are replaced by equalities. Since tangencies in the braid diagram are excluded and since we start from a true period-4 orbit, the only possible degenerate cases turn out to be

$$x^0 < x^1 = x^2 < x^3 \qquad \text{and} \qquad x^0 < x^2 = x^3 < x^1,$$

which both lead to a braid of type II.

REMARK 2.17. The fact that we have four different discretized braid diagrams but only two topological braid classes may lead to notational difficulties that we clarify here while we are at it. The two discretized braid classes within one topological braid class are related by a shift σ or a double shift σ^2. We can thus go back and forth between the two by applying shifts to both $\mathbf{v}$ and $\mathcal{R}$. When we obtain results for a parabolic flow generated by $\mathcal{R}$ that has stationary braid $\mathbf{v}$, then these results carry over to $\sigma(\mathcal{R})$ and $\sigma(\mathbf{v})$, since $\sigma(\mathcal{R})$ is a parabolic recurrence relation that fixes $\sigma(\mathbf{v})$. We may thus restrict our attention to just one of the discretized braid classes in each topological braid class.

2.5. Positive topological entropy

We are ready to assemble the machinery previously presented in order to prove that a twist map with a period-4 point of type I is chaotic. Throughout this section we assume the infinite twist condition (2.2), which leads to a proof of Theorem 2.3, see Section 2.7 for the case of a diffeomorphism. We will show that for f there exists a compact invariant set $\Lambda \subset \mathbb{R}^2$ on which f has positive topological entropy.

Our strategy is to first consider the second iterate f^2 and to show that there is a compact set $\Lambda_1 \subset \mathbb{R}^2$, invariant under f^2, on which it is semi-conjugate to the shift map on three symbols, which has positive entropy. Standard results about the entropy then imply that the map f also has positive entropy on $\Lambda = \Lambda_1 \cup f(\Lambda_1)$. The set of all sequences on three symbols is denoted by $\Sigma_3 = \{-1,0,+1\}^{\mathbb{Z}}$, and $\sigma\colon \Sigma_3 \to \Sigma_3$ maps $\{a_n\}_{n\in\mathbb{Z}}$ to the shifted sequence $\{a_{n+1}\}_{n\in\mathbb{Z}}$.

Let $\bar{z}$ be a period-4 point of type I. According to Section 2.4 this means that we may assume that the x-coordinates of its orbit, denoted by $x^i = \pi_x f^i(\bar{z})$, are ordered in a certain way. In particular, in view of Remark 2.17 and considering an iterate of $\bar{z}$ if necessary, we may without loss of generality assume that

$$x^0 < x^3 < x^1 < x^2.$$

Let $S \subset \mathbb{R}^2$ be the set of all complete orbits of f and define

$$\Lambda_1 \stackrel{\text{def}}{=} \{ z \in S \mid \pi_x f^{2i}(z) \in [x^0,x^2] \text{ and } \pi_x f^{2i+1}(z) \in [x^3,x^1] \text{ for all } i \in \mathbb{Z} \}. \tag{2.8}$$

Remark 2.4 shows that f^{-1} is well-defined (at least on the image of f). We note that $\bar{z}$ and $f^2(\bar{z})$ are elements of Λ_1. The set Λ_1 is invariant under f^2 and it is bounded. By definition the x-coordinates are uniformly bounded on Λ_1, while boundedness of the y-coordinate follows from the fact that the functions $Y(x,x')$ and $\widetilde{Y}(x,x')$ from Section 2.2 are continuous on $\mathbb{R}^2$ and thus bounded on bounded sets. Furthermore, since f and f^{-1} are continuous (differentiable) functions it is not hard to see that Λ_1 is compact.

Let $\varphi : \Lambda_1 \to \Sigma_3$ be the function that assigns a symbol sequence to each point in Λ_1 as follows:

$$\varphi(z) = \{a_n\}_{n\in\mathbb{Z}} \iff \begin{cases} a_n = +1 & \text{if } \pi_x f^{2n}(z) \in (x^1, x^2], \\ a_n = 0 & \text{if } \pi_x f^{2n}(z) \in [x^3, x^1], \\ a_n = -1 & \text{if } \pi_x f^{2n}(z) \in [x^0, x^3). \end{cases} \tag{2.9}$$

The sequence $\{a_n\}_{n\in\mathbb{Z}}$ will be called the symbolic description of a point (trajectory) in Λ_1. We note that

$$\varphi(\bar{z}) = \{(-1)^{n-1}\}_{n\in\mathbb{Z}} \qquad \text{and} \qquad \varphi(f^2(\bar{z})) = \{(-1)^n\}_{n\in\mathbb{Z}}. \tag{2.10}$$

Our goal is to show that φ is a semi-conjugacy. It follows from the construction that $\varphi \circ f^2(z) = \sigma \circ \varphi(z)$ for all $z \in \Lambda_1$. We still need to show that φ is surjective and continuous. Continuity is proved in Lemma 2.21, while surjectivity follows from Lemma 2.20. Leading up to that we first state and prove the crucial lemma, which uses the concepts of the flip transformation, braid diagrams and their Conley index.

LEMMA 2.18. *For any periodic symbol sequence* $\{a_n\}_{n\in\mathbb{Z}} \in \Sigma_3$ *there exists a point in* Λ_1 *that has* $\{a_n\}_{n\in\mathbb{Z}}$ *as its symbolic description.*

PROOF. Let p be the minimal period of the sequence $\{a_n\}_{n\in\mathbb{Z}}$, and let $4q$ denote the smallest common multiple of $2p$ and 4.

Step 1. Construction of relative braid classes.

In Section 2.4 we explained in detail how a period-4 point yields a braid $\mathbf{v} \in \mathcal{D}_4^4$ that is stationary for the parabolic flow associated to the recurrence relation $\mathcal{R} = (\mathcal{R}_i)_{i\in\mathbb{Z}}$. In this section $\mathbf{v}$ is assumed to be a type I braid. By concatenating $\mathbf{v}$ (just repeating it) we obtain more stationary skeletons. To be precise, define $\#_q\mathbf{v}$ to be the q-concatenation of $\mathbf{v}$. Clearly $\#_q\mathbf{v} \in \mathcal{D}_{4q}^4$, and it is a stationary skeleton for $\mathcal{R}$ (cf. Figure 2.10).

Using the skeletons $\#_q\mathbf{v}$ we can now construct numerous relative braid classes by weaving in a free strand with the skeletal strands. Given a periodic symbol sequence $\{a_n\}$, a free strand $\mathbf{u} = (u_i)_{i=0}^{4q-1}$ can be characterized as follows:

(i) For i odd, $u_i \in (x^3, x^1)$ when $i = 1 \bmod 4$, and $u_i \in (-x^1, -x^3)$ when $i = 3 \bmod 4$.

(ii) The position of the even anchors is determined by the sequence $\{a_n\}_{n=0}^{2q-1}$:

$$\begin{array}{lllll} \text{if} & a_n = +1 & \text{then } u_{2n} \in (x^1, x^2) \text{ for } n \text{ odd, and } u_{2n} \in (-x^2, -x^1) \text{ for } n \text{ even;} \\ \text{if} & a_n = 0 & \text{then } u_{2n} \in (x^3, x^1) \text{ for } n \text{ odd, and } u_{2n} \in (-x^1, -x^3) \text{ for } n \text{ even;} \\ \text{if} & a_n = -1 & \text{then } u_{2n} \in (x^0, x^3) \text{ for } n \text{ odd, and } u_{2n} \in (-x^3, -x^0) \text{ for } n \text{ even.} \end{array}$$

Moreover, let $u_{4q} = u_0$. The subdivision of the range of n (basically $n = 0, 3 \bmod 4$ and $n = 1, 2 \bmod 4$) is needed since we are working with the (flipped) coordinates for parabolic recurrence relations. Figure 2.10 shows an example of a relative braid class obtained in this way. Denote the equivalence class of the relative braids described above by $[\mathbf{u} \text{ rel } \#_q\mathbf{v}]$. If $a_n \not\equiv \pm(-1)^n$, then the these braid classes are bounded and proper. For the sequences $a_n \equiv \pm(-1)^n$ the corresponding points in Λ_1 are given by (2.10), and we will exclude these special sequences from our considerations.

Step 2. Non-triviality of the Conley index.

We now calculate the Conley index for the braid classes described in step 1. Since each coordinate u_i can only move in the designated intervals as described above, the configuration space $N = \mathrm{cl}([\mathbf{u} \text{ rel } \#_q\mathbf{v}])$ is a cartesian product of intervals, i.e. $N \simeq I^{4q}$, a $4q$-dimensional

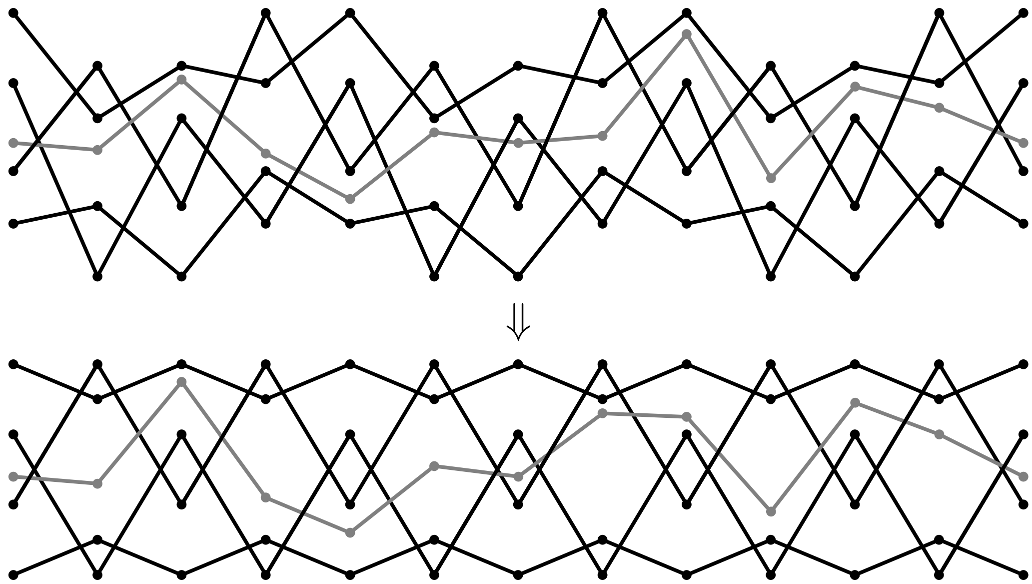

Figure 2.10: Braid diagrams corresponding to $\{\dots,a_0,a_1,a_2,a_3,a_4,a_5,a_6,\dots\} = \{\dots,0,+1,+1,0,-1,+1,0,\dots\}$. At the top is a generic non-symmetric situation, while at the bottom the skeleton is deformed into a symmetric one, which has the same topological information and has the advantage that it is a lot easier to survey. The homotopy type of this braid class is the pointed space (S^2,pt).

hypercube. We now proceed by determining N^-, the exit set. As in Example 2.16 the flow can only decrease the total number of intersections if $u_{2n} \in (x^3,x^1)$ or $u_{2n} \in (-x^1,-x^3)$. Then the number of intersections decreases when u_{2n} moves through the boundary of these intervals. The number of anchor points for which this is possible is equal to the number of zeroes in $\{a_n\}_{n=0}^{2q-1}$. Denote this number by k. This way N^- consists only of opposite faces. Therefore, $h = [N/N^-] \simeq (S^k,\mathrm{pt})$. A standard result from homology theory then shows that

$$H_*(N,N^-) = H_*((S^k,\mathrm{pt})) = \begin{cases} \mathbb{R} & \text{if } * = k, \\ 0 & \text{otherwise,} \end{cases}$$

and $CP_t(h) = t^k$, proving that the Conley index is non-trivial for any periodic symbol sequence $\{a_n\}$ with $a_n \not\equiv \pm(-1)^n$. Such symbol sequences will be earmarked as non-trivial.

Step 3. Existence of periodic points.
From the previous step we have that $CP_{-1}(h) = (-1)^k \neq 0$. Lemma 2.14 then proves that there exists at least one stationary point, i.e. a solution of $\mathcal{R} = 0$, in the relative braid class $[\mathbf{u} \text{ rel } \#_q\mathbf{v}]$ that is associated to each of the non-trivial periodic symbol sequences $\{a_n\}$. The considerations in Section 2.2, in particular Lemma 2.7, imply that the stationary solution $\mathbf{u}$ constructed this way corresponds to a periodic point of f. Hence it corresponds to a $2q$ periodic orbit of f^2 and the construction of the braid classes ensures that this periodic orbit is in Λ_1 and has symbolic description $\{a_n\}$. □

The proof of Lemma 2.18 does not show that every periodic symbol sequences of minimal period p corresponds to a periodic trajectory with period p of f^2 (only when p is even this is clear). Nor do we obtain uniqueness of points in Λ_1 that have a particular periodic symbolic description. However, since we are only building a *semi*-conjugacy, neither of these points matter.

REMARK 2.19. For any $z \in \Lambda_1$ the x-coordinates of the even iterates cannot be on the boundary of the intervals distinguishing the different symbolic descriptions, i.e. $\pi_x f^{2n}(z) \neq x^1, x^3$. Namely, suppose $\pi_x f^{2n}(z) = x^1$ or x^3, then after applying the flip transformation and interpreting the flipped trajectory of z as a strand in the braid diagram (see Figure 2.10), this strand is stationary and has a tangency at anchor point $2n$ with one of the strands of the skeleton. This is impossible, as stated in Proposition 2.11. An alternative is to compare the trajectories of z and $\bar{z}$ and to use the twist property of the orientation reversing twist map f to obtain a contradiction directly.

The periodic symbol sequences from Lemma 2.18 allow us to deal with the general case.

LEMMA 2.20. *For any sequence $\{a_n\}_{n\in\mathbb{Z}} \in \Sigma_3$ there exist a point in Λ_1 that has $\{a_n\}_{n\in\mathbb{Z}}$ as its symbolic description.*

PROOF. Let $a = \{a_n\}_{n\in\mathbb{Z}}$ be any sequence in Σ_3. We can approach a by periodic sequences $a^k \in \Sigma_3$, where $a^k_n = a_n$ for $|n| \leq k$ with periodic extension $a^k_n = a^k_{n-2k-1}$ for all n. Clearly $a^k \to a$ as $k \to \infty$, with a^k being periodic (the metric is given explicitly in the proof of the next lemma). Lemma 2.18 shows that there exist points $z_k \in \Lambda_1$ such that $\varphi(z_k) = a^k$. Since Λ_1 is compact, there exists a convergent subsequence $z_{k_m} \to z \in \Lambda_1$ as $m \to \infty$. Let $\varphi(z) = b \in \Sigma_3$, then we claim that $b = a$. For any fixed $n \in \mathbb{Z}$, $\pi_x f^{2n}(z)$ is either in $[x^0, x^3)$, (x^3, x^1) or $(x^1, x^2]$, because the values x^1 and x^3 are excluded by Remark 2.19. Hence it follows that for m sufficiently large $\pi_x f^{2n}(z_{k_m})$ is in the same of these intervals as $\pi_x f^{2n}(z)$. Since the intervals encode the symbolic description, this implies $b_n = a^{k_m}_n$ for sufficiently large m, and thus indeed $b = a$. □

LEMMA 2.21. *The map φ defined in* (2.9) *is continuous.*

PROOF. The arguments resemble the ones used in the previous proof. We use the metric $d(a,b) = 2^{-\max\{m | a_n = b_n \text{ for } |n| < m\}}$ on Σ_3. Let z_k be any convergent sequence in Λ_1, $z_k \to z \in \Lambda_1$. Let $\varphi(z) = b \in \Sigma_3$ and $\varphi(z_k) = b^k$. For any fixed $n \in \mathbb{Z}$, $\pi_x f^{2n}(z)$ is either in $[x^0, x^3)$, (x^3, x^1) or $(x^1, x^2]$, because the values x^1 sand x^3 are excluded by Remark 2.19. Hence it follows that for k sufficiently large $\pi_x f^{2n}(z_k)$ is in the same of these intervals as $\pi_x f^{2n}(z)$, which implies $b_n = b^k_n$ for sufficiently large k. In particular, for any (large) $m \in \mathbb{N}$ there exists a $K(m) \in \mathbb{N}$ such that $b_n = b^k_n$ for all $|n| \leq m$ and $k \geq K$. In other words, $|\varphi(z) - \varphi(z_k)| \leq 2^{-m-1}$ for $k \geq K$, which establishes continuity. □

From the previous lemmas we conclude that φ as defined by (2.9) is a semi-conjugacy from $f^2|_{\Lambda_1}$ to $\sigma|_{\Sigma_3}$. To carry over this information to the map f we define

$$\Lambda \stackrel{\text{def}}{=} \Lambda_1 \cup f(\Lambda_1),$$

which is invariant under f, and the entropy of f on Λ can be estimated in terms of the entropy of the shift on three symbols.

THEOREM 2.22. *An orientation reversing twist map of the plane that satisfies the infinite twist condition and that has a type I period-4 point, has positive topological entropy restricted to the compact invariant set Λ.*

PROOF. We use the semi-conjugacy φ to estimates the entropy $h(f|_\Lambda)$ of f on Λ. Standard properties of the entropy (e.g. see [**24**]) give the estimates

$$h(f|_\Lambda) = \frac{1}{2}h(f^2|_\Lambda) \geq \frac{1}{2}h(f^2|_{\Lambda_1}) \geq \frac{1}{2}h(\sigma|_{\Sigma_3}) = \frac{1}{2}\ln(3).$$

□

REMARK 2.23. As an alternative strategy one can consider the fourth iterate of f instead of the second one. This is perhaps more natural in view of the decomposition of f^4 in terms of orientation preserving twist maps, as discussed in the introduction. On the other hand, the notation becomes a bit more involved. Anyway, it is not difficult to see that arguments analogous to the ones used for the second iterate lead to a semi-conjugacy of $f^4|_{\Lambda_1}$ to the shift on the space Σ_9 of sequences on nine symbols. This approach gives exactly the same lower bound for the topological entropy of f:

$$h(f|_\Lambda) \geq \frac{1}{4}h(f^4|_{\Lambda_1}) \geq \frac{1}{4}h(\sigma|_{\Sigma_9}) = \frac{1}{4}\ln(9) = \frac{1}{2}\ln(3).$$

2.6. Type II periodic points

In the previous section we have proved that a period-4 orbit of type I forces orientation reversing twist maps to be chaotic. Now we will show that the theorem is "sharp" in the sense that we construct an example of a map with a period-4 orbit of type II that has zero topological entropy, i.e., the entropy of the dynamics restricted to any bounded invariant set is zero.

We start with the well known quadratic family of one dimensional maps

$$x_{k+1} = \lambda x_k(1-x_k),$$

where λ is a parameter. This map is a good starting point since it has a simple formula and its dynamics has been studied extensively. The property of most interest to us is that for λ slightly larger than

$$\lambda^* = 1+\sqrt{6},$$

the system has a period-4 orbit which is stable (the period-2 orbit undergoes a period doubling bifurcation at $\lambda = \lambda^*$). Moreover, the topological entropy of the map on the maximal bounded invariant set is zero.

We want to embed this system into $\mathbb{R}^2$ and turn it into an orientation reversing twist map. To accomplish this we use the family of maps

$$f_\varepsilon : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ \varepsilon x + \lambda y(1-y) \end{pmatrix},$$

which are orientation reversing twist diffeomorphisms for all $\varepsilon > 0$, while for $\varepsilon = 0$ we retrieve the quadratic family in disguise (f_0 is *not* a diffeomorphism). Notice that for $\varepsilon = 0$, and λ slightly larger than λ^*, the period-4 orbit is of type II (cf. Section 2.4). Intuition suggests that for small $\varepsilon > 0$ the perturbation εx will not change the dynamics much (in particular, the entropy remains zero). The remainder of this section is spent on making this precise.

Since our aim is to show that the maps for $\varepsilon > 0$ have zero topological entropy we prove that their non-wandering sets are all "the same", and in a sense "copies" of the non-wandering set at $\varepsilon = 0$, i.e., we will prove a version of Ω-stability for this particular situation. Let S^ε be the set of all all points in $\mathbb{R}^2$ through which there is a complete bounded orbit of f_ε, and let Ω^ε be the set of non-wandering points of f_ε. We start with proving that all interesting dynamics is contained in the compact set $N \stackrel{\text{def}}{=} [-1,2] \times [-1,2]$.

LEMMA 2.24. *For* $\varepsilon \in [0,1/2)$ *and* $\lambda \in [1,4]$ *it holds that* $\Omega^\varepsilon \subset S^\varepsilon \subset \operatorname{int}(N)$.

PROOF. The case $\varepsilon = 0$ corresponds to the one-dimensional quadratic map and the statements are easily seen to hold. We turn to the case $\varepsilon \in (0,1/2)$, for which f_ε is invertible. First we show that $S^\varepsilon \subset N$. Let us start with the bound $x_n, y_n < 2$. By contradiction, assume that $x_0 \geq 2$, then since $\lambda y_n(1-y_n) \leq 1$ we have

$$x_{n-2} \geq \frac{x_n - 1}{\varepsilon}.$$

Hence the sequence $x_{-2k} \to \infty$, as $k \to \infty$. This contradicts the fact that trajectory is bounded, so indeed $x_n < 2$ for all n. Since $y_n = x_{n+1}$ we then also have $y_n < 2$.

Next we prove that $x_n, y_n > -1$. If $y_0 \leq -1$ then the inequality

$$y_{n+1} < \lambda y_n(1-y_n) + 1$$

implies that $y_k \to -\infty$ as $k \to \infty$. Therefore $y_n > -1$ and again the same holds for x_n.

We now show that also $\Omega^\varepsilon \subset N$. From the previous argument we see that if this is not the case then there has to be some point $(x_0, y_0) \in \Omega^\varepsilon$ for which $x_0 \geq 2$. It then follows that $x_{-2k} \to \infty$, and since x_0 is non-wandering x_{-2m+1} has to be arbitrarily close to $x_0 \geq 2$ for some $m \in \mathbb{N}$. The same reasoning as before then shows that $x_{-2m+1-2k} \to \infty$ as $k \to \infty$, contradicting the fact that $(x_0, y_0) \in \Omega^\varepsilon$. We have thus established that $\Omega^\varepsilon \subset N$. Finally, if $z \in \Omega^\varepsilon$, then $f_\varepsilon(z) \in \Omega^\varepsilon$ and $f_\varepsilon^{-1}(z) \in \Omega^\varepsilon$, hence $\Omega^\varepsilon \subset S^\varepsilon$. □

In Ω-stability theory the concept of axiom A maps and the no-cycle property are usually essential (see for example [**61**]). Let us recall their standard definitions. For a compact manifold M, we say that a map $f : M \to M$ satisfies *axiom A* if the set $\Omega(f)$ is hyperbolic and the periodic points are dense in $\Omega(f)$. When f satisfies axiom A then the non-wandering set $\Omega(f)$ can be written as a finite disjoint union $\Omega = \Omega_0 \cup \cdots \cup \Omega_k$ of closed invariant sets on which f is topologically transitive (the spectral decomposition theorem, cf. [**61**]). The sets Ω_i are called basic sets. We say that $\Omega_i \leq \Omega_j$ if $(W^s(\Omega_i) \setminus \Omega_i) \cap (W^u(\Omega_j) \setminus \Omega_j) \neq \varnothing$, where the stable and unstable sets are given by

$$W^s(\Omega_i) = \{x \in M \mid f^n(x) \to \Omega_i \text{ as } n \to \infty\}$$
$$W^u(\Omega_i) = \{x \in M \mid f^{-n}(x) \to \Omega_i \text{ as } n \to \infty\}.$$

A map f satisfying axiom A has the no-cycle property if for every choice of distinct indices $\{i_k\}_{k=1}^n$, $n \geq 1$ it is impossible to have the inequalities

$$\Omega_{i_1} \leq \Omega_{i_2} \leq \ldots \leq \Omega_{i_n} \leq \Omega_{i_1}.$$

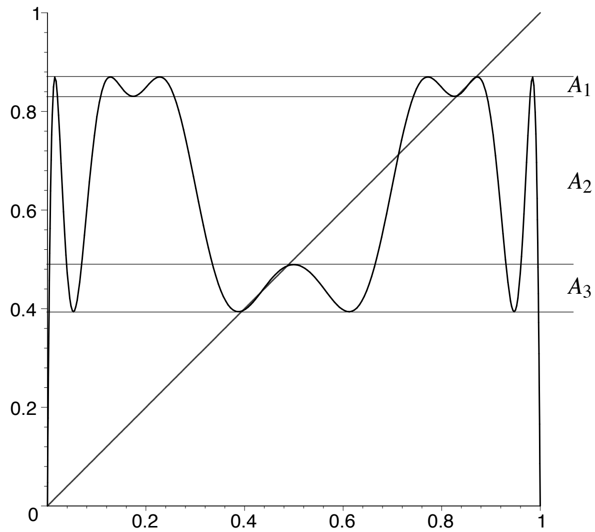

Figure 2.11: The graph of the fourth power of the quadratic map. The parameter λ is set to $\lambda^* + 0.04$. The intervals A_1, A_2 and A_3 (bounded by the extrema) are indicated.

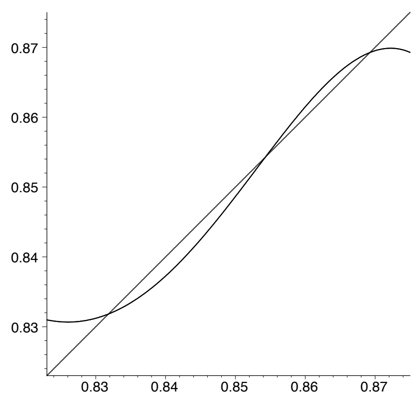

Figure 2.12: One of the two trapping regions of the period-4 orbit. We choose λ so close to λ^* that the function is monotone between the three fixed points of F_λ^4 in the picture.

Since our case does not fit in the usual setting of diffeomorphisms on a compact manifold we will now adapt these concepts to the family f_ε. For f_0 the invariant set is

$$S^0 = \{(x,y) \,|\, x \in [0, \lambda/4] \text{ and } y = \lambda x(1-x)\}.$$

For values of λ slightly larger than λ^* there are two unstable fixed points, an unstable period-2 orbit and stable period-4 orbit. For simplicity we write

- Ω_1 – period-4 orbit;
- Ω_2 – period-2 orbit;
- Ω_3 – non-trivial fixed point;
- Ω_4 – fixed point $(0,0)$.

We would like to show that these are the only non-wandering points. To analyze the dynamics we observe that for the fourth power F_λ^4 of quadratic map $F_\lambda(x) = \lambda x(1-x)$ eventually maps any point $x_0 \in (0,1)$ into the interval $A = [F_\lambda(\lambda/4), \lambda/4]$ (cf. Figure 2.11). On the other hand, in A we can distinguish three intervals $A_1 = [F_\lambda(\lambda/4), F_\lambda^3(\lambda/4)]$, $A_2 = (F_\lambda^3(\lambda/4), F_\lambda^2(\lambda/4))$ and $A_3 = [F_\lambda^2(\lambda/4), \lambda/4]$. Monotonicity of F_λ^4 on $A_2 \cap (F_\lambda^4)^{-1}(A_2)$ guarantees that any point in A, with the exception of the fixed point Ω_2, will eventually enter A_1 or A_3 under iterates of F_λ^4. Apart from the period-2 orbit any point in A_1 and A_3 approaches the period-4 orbit due to monotonicity of F_λ^4 on these intervals (cf. Figure 2.12). Our choice of λ is sufficiently close to λ^* so that the function F_λ^4 is monotone between the three fixed points of F_λ^4 in A_1 and A_3. Since f_0 mimics the dynamics of F_λ, it follows that any point in S^0 that is not eventually periodic has Ω_3 as its ω-limit set. Moreover, we have proved that there are no other non-wandering points then the orbits contained in Ω_i for $i = 1,2,3,4$, i.e.

$$\Omega^0 = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4.$$

One can easily see that the eigenvalues of df_0 in a point in Ω^0 are $\alpha_1 = 0$ and α_2, which is equal to the eigenvalue of the corresponding point of F_λ. Again, since for λ sufficiently close to λ^*, and $\lambda > \lambda^*$, we have $F_\lambda' \neq \pm 1$ at the fixed points, the period-2 and the period-4 orbit. Hence, they are all hyperbolic, and Ω_i is hyperbolic for $i = 1,2,3,4$.

The reasoning above shows that f_0 has a hyperbolic non-wandering set which only consists of periodic orbits. Moreover, we have identified the basic sets to be Ω_i with $i = 1,2,3,4$. Now we turn to the no-cycle property. To simplify the notation we write $\widetilde{W}^s(\Omega_i) = W^s(\Omega_i) \setminus \Omega_i$ and $\widetilde{W}^u(\Omega_i) = W^u(\Omega_i) \setminus \Omega_i$. To exclude the existence of a cycle let us start with the observation that $\widetilde{W}^s(\Omega_4) \cap S^0 = \varnothing$. This ensures that $\Omega_i \not\lesssim \Omega_4$ for $i = 1,2,3,4$. On the other hand $\widetilde{W}^u(\Omega_1) = \varnothing$, so $\Omega_1 \not\lesssim \Omega_i$ for $i = 1,2,3,4$. From the arguments above (illustrated in Figures 2.11 and 2.12) it follows that $\widetilde{W}^u(\Omega_3) \subset W^s(\Omega_1) \cup W^s(\Omega_2)$ and $\widetilde{W}^u(\Omega_2) \subset W^s(\Omega_1)$. Combining these observation we see that there are no cycles among $\{\Omega_i\}_{i=1}^4$. This reasoning shows that

LEMMA 2.25. *For λ slightly larger than λ^* the map f_0 has a finite hyperbolic non-wandering set, and there are no cycles among the basic sets.*

We are now in a position to prove Ω-stability and in particular

LEMMA 2.26. *There exists an ε^* such that for all $\varepsilon \in [0, \varepsilon^*]$ the set Ω^ε is finite.*

PROOF. We will mimic the proof of Ω-stability theorem for diffeomorphisms on a compact set in [**61**]. From the lemmas above we know that $\Omega^0 = \mathrm{per}(f_0) = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$. It is well know that for f_0 there exist a Lyapunov function (see [**40**]). So there exists a function $V : N \to \mathbb{R}$ satisfying the following conditions. It is decreasing along trajectories of f_0, except on Ω_i, $i = 1,2,3,4$, where V is constant. Furthermore, because of the no-cycle property we may assume that $V(\Omega_i) \neq V(\Omega_j)$ for $i \neq j$. Also, we can rescale V so that $V : N \to (\frac{1}{2}, 4\frac{1}{2}]$ and $V(\Omega_i) = i$. We define the (compact) sets

$$M_j \stackrel{\text{def}}{=} V^{-1}((-\infty, j + 1/2]) \cap N.$$

The sets M_j have the properties of a filtration:

(1) $N = M_4 \supset M_3 \supset M_2 \supset M_1 \supset M_0 = \varnothing$;

(2) $f_0(M_j) \subset \operatorname{int}(M_j)$;
(3) $\Omega_j \subset \operatorname{int}(M_j \setminus M_{j-1})$;
(4) $\Omega_j = \bigcap_{k=-\infty}^{\infty} f_0^k(M_j \setminus M_{j-1})$;

where f_0^{-k} denotes the k-th pre-image. These properties follow from the definition of M_j and the structure of Ω^0. For simplicity denote

$$U_j \overset{\text{def}}{=} M_j \setminus M_{j-1}.$$

By the continuity of the family f_ε and the compactness of N we can choose ε_1 so small that property (2) holds for all $\varepsilon \leq \varepsilon_1$, i.e. $f_\varepsilon(M_j) \subset \operatorname{int}(M_j)$ for all j.

Since Ω_i consists of a hyperbolic periodic orbit, Ω_i continues under perturbations. The perturbed periodic orbit, denoted by Ω_i^ε, is again hyperbolic for ε sufficiently small, say $\varepsilon \leq \varepsilon_2 \leq \varepsilon_1$. Clearly $\Omega_i^\varepsilon \subset \Omega^\varepsilon$ for all i. To conclude the proof we show the other inclusion $\Omega^\varepsilon \subset \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Omega_3^\varepsilon \cup \Omega_4^\varepsilon$.

We will prove the two following claims. Firstly, for ε sufficiently small, $\Omega_j^\varepsilon = S^\varepsilon(U_j)$, where $S^\varepsilon(U_j)$ is the set of all points in $\mathbb{R}^2$ whose complete orbits lie entirely in U_j. Secondly, if $z \in \Omega^\varepsilon \cap U_j$ for some $\varepsilon \leq \varepsilon_2$ and some j, then $f_\varepsilon^i(z) \in U_j$ for all $i \in \mathbb{Z}$. Let us assume for the moment that the claims are true for $\varepsilon \leq \varepsilon^* \leq \varepsilon_2$. Let $z_0 \in \Omega^\varepsilon$ for some $\varepsilon \in (0, \varepsilon^*]$. By property (1) of the sets M_j the point z_0 has to be in some U_{j_0}. By the second claim the whole trajectory of z_0 is contained in U_{j_0}. Then the first claim shows that $z_0 \in \Omega_j^\varepsilon$. This proves that the non-wandering set for f_ε consists entirely of the perturbation of the non-wandering set of f_0. In particular, Ω^ε is finite. Now we return to the proof of the claims.

Claim 1: $\Omega_j^\varepsilon = S^\varepsilon(U_j)$ for all j and all ε sufficiently small. Because of property (3) of the set M_j we get $\Omega_j^\varepsilon \subset S^\varepsilon(U_j)$ for ε sufficiently small. For the other inclusion we argue by contradiction. Setting $\varepsilon = 1/n$ we assume that for all $n \geq n_0 \in \mathbb{N}$ there is a $z_n \in S^{1/n}(U_j)$ such that $z_n \notin \Omega_j^{1/n}$. By the hyperbolicity of Ω_j (and Ω_j^ε) there is a $\delta > 0$ such that $f_{1/n}^{k(n)}(z_n)$ is not in a δ-ball $B_\delta(\Omega_j)$ around Ω_j for some $k(n) \in \mathbb{Z}$. Set $w_n = f_{1/n}^{k(n)}(z_n)$, then $w_n \in S^{1/n}(U_j) \setminus B_\delta(\Omega_j)$. By compactness of N there exists a subsequence $m_0(n)$ so that $w_{m_0(n)} \to v_0 \in \overline{U_j} \setminus B_\delta(\Omega_j)$. We want to show that $v_0 \in U_j$ and that there is an complete orbit in U_j through v_0. First we prove that $f_0^i(v_0) \in U_j$ for all $i \geq 0$. If this would not be the case then $f_0^i(v_0) \in M_{j-1}$ for some $i \geq 0$. From the property (2) of the sets M_j we get $f_0^{i+1}(v_0) \in \operatorname{int}(M_{j-1})$, and from the continuity of the family f_ε and the continuity of the map it follows that $f_{1/m_0(n)}^{i+1}(w_{m_0(n)}) \in \operatorname{int}(M_{j-1})$ for n large, which contradicts the assumption that $w_{m_0(n)} \in U_j$. We thus have that $f_0^i(v_0) \in U_j$ for all $i \geq 0$. To get the same for pre-images of v_0 we need to extract further subsequences.

From the sequence $m_0(n)$ we extract yet another subsequence $m_1(n)$ such that $f_{1/m_1(n)}^{-1}(w_{m_1(n)})$ converges to, say, v_{-1}. It easily follows that $f_0(v_{-1}) = v_0$. Similarly, from the sequence $m_1(n)$ we can extract a subsequence $m_2(n)$ such that $f_{1/m_2(n)}^{-2}(w_{m_2(n)}) \to v_{-2}$, and $f_0(v_{-2}) = v_{-1}$. We can repeat this procedure inductively and we end up with a sequence $\{v_k\}_{k=-\infty}^{0} \subset \overline{U_j}$ and $f_0(v_{-k}) = v_{-k+1}$. In fact, $v_{-k} \in U_j$ $(k \in \mathbb{N})$, because if $v_{-k} \in \overline{U_j} \setminus U_j$, then $v_{-k+1} = f(v_{-k}) \in \operatorname{int}(M_{j-1})$, a contradiction.

We have now constructed a whole trajectory $\{v_k\}_{k=-\infty}^{0} \cup \{f_0^k(v_0)\}_{k=0}^{\infty}$ of f_0 contained in $S^0(U_j)$. By property (4) of the sets M_j this trajectory has to be contained in Ω_j, but since $v_0 \notin B_\delta(\Omega_j)$ we get a contradiction, which concludes the proof of the claim 1.

Claim 2: For all $z \in \Omega^\varepsilon$ with $\varepsilon \in (0, \varepsilon_2]$ it holds that if $z \in U_j$, then $f_\varepsilon^i(z) \in U_j$ for all $i \in \mathbb{Z}$. It is worth recalling that $\varepsilon \le \varepsilon_2$ implies that $f_\varepsilon(M_j) \subset \mathrm{int}(M_j)$. Assume that $z \in \Omega^\varepsilon \cap U_j$ for some j.

Firstly, we show that $f_\varepsilon^i(z) \in U_j$ for all $i \ge 0$. Since z is in M_j we know that $f_\varepsilon^i(z)$ is in the interior of M_j for every positive i. Next, $f_\varepsilon^i(z) \notin M_{j-1}$ for all $i > 0$. Namely, if $f_\varepsilon^i(z) \in M_{j-1}$ for some $i > 0$, then the next iterate is in the interior of M_{j-1}. The continuity of the map guarantees that $f_\varepsilon^{i+1}(B_\delta(z)) \subset \mathrm{int}(M_{j-1})$, for δ sufficiently small, which also implies $f_\varepsilon^{i+1+k}(B_\delta(z)) \subset \mathrm{int}(M_{j-1})$ for all $k \in \mathbb{N}$. On the other hand, $B_\delta(z) \cap M_{j-1} = \varnothing$, for sufficiently small δ. Hence, $f_\varepsilon^{i+1+k}(B_\delta(z)) \cap B_\delta(z) = \varnothing$ for all positive k and δ sufficiently small, which contradicts $z \in \Omega^\varepsilon$.

Secondly, we show that also the negative iterates $f_\varepsilon^{-i}(z) \in U_j$, for all $i > 0$. We have to show that $f_\varepsilon^{-i}(z) \notin M_{j-1}$ and that $f_\varepsilon^{-i}(z) \notin U_{j+m}$, where $i, m > 0$. As above it follows that if $f_\varepsilon^{-i}(z) \in M_{j-1}$, then $z = f_\varepsilon^{-i+i}(z) \in \mathrm{int}(M_{j-1})$, whereas $z \in U_j$, which shows that $f_\varepsilon^{-i}(z) \notin M_{j-1}$. To prove that $f_\varepsilon^{-i}(z) \notin U_{j+m}$, $m > 0$, we observe that the non-wandering set Ω^ε is invariant under f_ε and f_ε^{-1}. If we would have that $\tilde{z} = f_\varepsilon^{-i}(z) \in U_{j+m}$ for some $i > 0$ and some $m > 0$, then $\tilde{z} \in \Omega^\varepsilon$ and $f_\varepsilon^k(\tilde{z}) \in U_{j+m}$, for all $k \ge 0$, by the result on positive iterates established above. This contradicts the fact that $f_\varepsilon^i(\tilde{z}) = z \in U_j$, concluding the proof of claim 2 and therefore the lemma. □

We have thus found our counterexample.

LEMMA 2.27. *The orientation reversing twist maps f_ε, with λ slightly larger than λ^* and ε sufficiently small, which have a period-4 orbit of type II, have zero topological entropy (as explained at the beginning of this section).*

PROOF. Lemma 2.26 proves that the non-wandering set Ω^ε of f_ε is finite. Standard results on the topological entropy show that the entropy of f_ε on S^ε is equal to the entropy on Ω^ε, and the entropy of the map on a finite set is zero (e.g. see [**61**]). □

2.7. Twist diffeomorphisms of the plane

We now extend our results to situations where the parabolic recurrence relation is not defined on the whole of $\mathbb{R}^3$. Since this requires some careful analysis, this section is substantially more technical than the previous ones. We first introduce the necessary frame work and in Section 2.7 we apply it to period-4 orbits of orientation reversing twist diffeomorphisms and we prove Theorem 2.2.

The domain of parabolic recurrence relations

We are interested in *bijective* orientation reversing twist maps. In the introduction it has been explained that the fourth iterate can be decomposed in four orientation preserving positive twist maps, to which we can apply the theory of parabolic flows. We thus restrict our attention here to orientation *preserving* twist diffeomorphisms, and compositions thereof. We assume f is an orientation preserving twist diffeomorphism, i.e. f is bijective to $\mathbb{R}^2$,

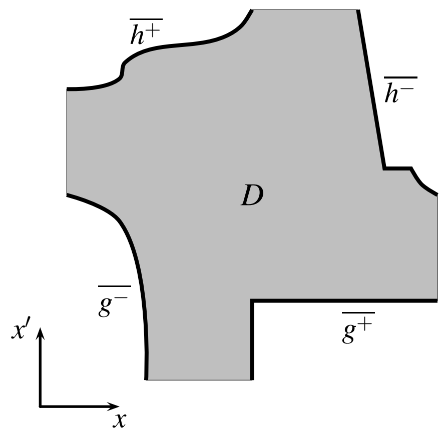

Figure 2.13: The domain D of the functions Y_f and $\widetilde{Y}_f$ for a twist diffeomorphism f.

$df \neq 0$, and $\partial_2(\pi_x f) > 0$. By definition the function f^{-1} is defined on $\mathbb{R}^2$, it is differentiable by the inverse function theorem, and $\partial_2(\pi_x f^{-1}) < 0$, i.e. f^{-1} has negative twist.

We recall and refine some notation from Section 2.2. Let $(x',y') = f(x,y)$, then there are differentiable functions Y_f and $\widetilde{Y}_f$ with $\partial_2 Y_f > 0$ and $\partial_1 \widetilde{Y}_f < 0$, such that

$$y = Y_f(x,x') \qquad \text{and} \qquad y' = \widetilde{Y}_f(x,x'). \tag{2.11}$$

Since f^{-1} has negative twist, the same reasoning as in Section 2.2 gives differentiable functions $Y_{f^{-1}}$ and $\widetilde{Y}_{f^{-1}}$ with $\partial_2 Y_{f^{-1}} < 0$ and $\partial_1 \widetilde{Y}_{f^{-1}} > 0$, such that $y' = Y_{f^{-1}}(x',x)$ and $y = \widetilde{Y}_{f^{-1}}(x',x)$. Obviously

$$Y_f(x,x') = \widetilde{Y}_{f^{-1}}(x',x) \qquad \text{and} \qquad \widetilde{Y}_f(x,x') = Y_{f^{-1}}(x',x).$$

Let us consider the domain D of Y_f and $\widetilde{Y}_f$, and define

$$g(x) \stackrel{\text{def}}{=} \lim_{y\to-\infty} \pi_x f(x,y) \qquad \text{and} \qquad h(x) \stackrel{\text{def}}{=} \lim_{y\to\infty} \pi_x f(x,y). \tag{2.12}$$

These are functions from $\mathbb{R}$ to $[-\infty,\infty]$. Since they are limits of monotone sequences of continuous functions, g is upper semi-continuous and h lower semi-continuous, and $g(x) < h(x)$ for all $x \in \mathbb{R}$. The domain of Y_f and $\widetilde{Y}_f$ is the open set given by

$$D = \{(x,x') \,|\, x \in \mathbb{R},\, g(x) < x' < h(x)\}.$$

When f is invertible we can use the same arguments for f^{-1}. We define $G(x) = \lim_{y\to\infty} \pi_x f^{-1}(x,y)$ and $H(x) = \lim_{y\to-\infty} \pi_x f^{-1}(x,y)$. The domain of $Y_{f^{-1}}$ and $\widetilde{Y}_{f^{-1}}$ is given by $\widetilde{D} = \{(x',x) \,|\, x' \in \mathbb{R},\, G(x') < x < H(x')\}$. Obviously $(x,x') \in D$ if and only if $(x',x) \in \widetilde{D}$, i.e. $\widetilde{D} = D^{-1}$. This gives us a lot of information on g and h. In fact, the boundary ∂D of D consists of at most four pieces, each of which is a monotone graph. This is depicted in Figure 2.13.

It takes some notation to make this precise. The function $h : \mathbb{R} \to (-\infty,\infty]$ is lower semi-continuous; there is a point $x_h \in [-\infty,\infty]$ such that $h(x_h) = \infty$ and h is non-decreasing for $x < x_h$, and non-increasing for $x > x_h$. This means that h consists of at most two pieces of real-valued functions on (semi-)infinite intervals, a non-decreasing function h^+ and a non-increasing one h^-. Since h and/or x_h can be infinite, h^- and/or h^+ may be nonexistent. The associated graphs are

$$\overline{h^{\pm}} = \mathrm{gr}(h^{\pm}) = \partial\{(x,x') \in \mathbb{R}^2 \,|\, x \in \mathrm{dom}(h^{\pm}), x' < h^{\pm}(x)\} \subset \mathbb{R}^2.$$

In Figure 2.13 $\overline{h^+}$ is the northwest boundary and $\overline{h^-}$ is the northeast boundary. A similar description is valid for g, with g^+ being non-decreasing (the southeast) and g^- being non-increasing (the southwest). The boundary of $D \subset \mathbb{R}^2$ thus consists of the (at most four) graphs $\overline{g^{\pm}}$ and $\overline{h^{\pm}}$.

Since the parabolic recurrence relation, and hence the parabolic flow, is not defined on the boundary ∂D, we need to define (preferably smooth) approximations to it. We construct here the smooth approximations to the northwest boundary $\overline{h^+}$. The other boundaries are dealt with similarly. Let $h^{+\varepsilon}$ be a cutoff/extension function of h^+: $h^{+\varepsilon}(x) = \min\{\varepsilon^{-1}, h^+(x)\}$ for $x \in \mathrm{dom}(h^+)$ and $h^{+\varepsilon}(x) = \varepsilon^{-1}$ for $x \notin \mathrm{dom}(h^+)$. We make it smooth by using a one-sided mollification as follows. Let $z(x)$ be a nonnegative function with support in $[0,1]$ and integral $\int_{\mathbb{R}} z = 1$; let $z_\varepsilon(x) = \varepsilon^{-1} z(x/\varepsilon)$. Define

$$h^+_\varepsilon(x) = \varepsilon\,\frac{\mathrm{arctanh}(x) - 1}{2} + \int_{\mathbb{R}} z_\varepsilon(y) h^{+\varepsilon}(x-y)\,dy.$$

The ε-approximation h^+_ε of $\overline{h^+}$ is smooth on $\mathbb{R}$. Because of the one-sided mollification and the addition of a small increasing term, h^+_ε is increasing, hence $h^{+\prime}_\varepsilon > 0$, and

$$h^+(x-\varepsilon) - \varepsilon < h^+_\varepsilon(x) < h^+(x) \tag{2.13}$$

provided $x - \varepsilon \in \mathrm{dom}(h^+)$. This means that $\mathrm{gr}(h^+_\varepsilon)$ is $\sqrt{2}\varepsilon$-close to $\overline{h^+}$ on the piece where $h^+_\varepsilon < \varepsilon^{-1}$. The cut-off along the x' coordinate will cause no problems since we will only be interested in bounded braid classes, i.e. bounded subset of $\mathbb{R}^2$. Furthermore, h^+_ε is strictly increasing in ε. It follows from (2.11), (2.12) and (2.13) that

$$\lim_{\varepsilon\to 0} Y(x, h^+_\varepsilon(x)) = \infty \qquad \text{for all } x \in \mathrm{dom}(h^+), \tag{2.14a}$$

$$\lim_{\varepsilon\to 0} \widetilde{Y}((h^+_\varepsilon)^{-1}(x'), x') = \infty \qquad \text{for all } x' \in \mathrm{range}(\overline{h^+}). \tag{2.14b}$$

Since we are interested in compositions $f_{d-1} \circ f_{d-2} \circ \cdots \circ f_1 \circ f_0$ of orientation preserving twist maps, we index the corresponding g and h accordingly. The ε-approximation of the domain D_i of Y_i is thus

$$D_{i,\varepsilon} = \{(x_i, x_{i+1}) \,|\, g^{\pm}_{i,\varepsilon}(x_i) \le x_{i+1} \le h^{\pm}_{i,\varepsilon}(x_i)\}.$$

Restricted braid classes

The spaces of *restricted* braid diagrams are defined as (cf. [**33**])

$$\overline{\mathcal{E}}^n_d \stackrel{\mathrm{def}}{=} \overline{\mathcal{D}}^n_d \cap \{\mathbf{u} \,|\, (u^\alpha_i, u^\alpha_{i+1}) \in D_i \text{ for } i = 0\ldots d-1 \text{ and } \alpha = 1\ldots n\},$$
$$\mathcal{E}^n_d \stackrel{\mathrm{def}}{=} \mathcal{D}^n_d \cap \{\mathbf{u} \,|\, (u^\alpha_i, u^\alpha_{i+1}) \in D_i \text{ for } i = 0\ldots d-1 \text{ and } \alpha = 1\ldots n\},$$
$$\Sigma_{\mathcal{E}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{E}}^n_d \setminus \mathcal{E}^n_d.$$

For $\mathbf{u} \in \mathcal{E}^n_d$ the restricted braid class $[\mathbf{u}]_{\mathcal{E}}$ is defined as $[\mathbf{u}] \cap \mathcal{E}$. For $\mathbf{v} \in \mathcal{E}^m_d$ and $\mathbf{u} \cup \mathbf{v} \in \mathcal{D}^{n+m}_d$ the restricted relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ is $[\mathbf{u} \text{ rel } \mathbf{v}] \cap \{\mathbf{u} \in \mathcal{E}^n_d\}$.

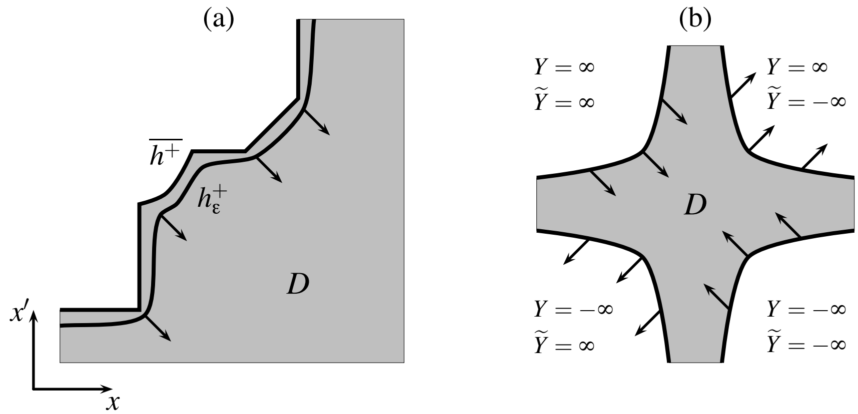

Figure 2.14: (a) The northwest corner and its ε-approximation with the direction of the flow. (b) Cartoon of the four pieces of boundary of the domain.

The boundary of a restricted relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ consists of two parts, namely the singular braids in $\partial[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}} \cap \Sigma_{\mathcal{E}}$, and the braids that violate the restriction in $\partial[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}} \setminus \Sigma_{\mathcal{E}}$. The parabolic flow is well-defined on $\partial[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}} \cap \Sigma_{\mathcal{E}}$ but not on $\partial[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}} \setminus \Sigma_{\mathcal{E}}$. To overcome this difficulty we may of course use the ε-approximations of Section 2.7:

$$[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}^{\varepsilon} = [\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}} \cap \{\mathbf{u} \,|\, (u_i^\alpha, u_{i+1}^\alpha) \in D_{i,\varepsilon} \text{ for } i = 0 \dots d-1 \text{ and } \alpha = 1 \dots n\}.$$

Now the flow is well-defined on the whole boundary $\partial[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}^{\varepsilon}$.

As an example, let us suppose (u_k, u_{k+1}) is close to the northwest boundary $\overline{h^+}$ of D_k, say $u_{k+1} = h_\varepsilon^+(u_k)$ and $\varepsilon \to 0$. If all other pairs of coordinates (u_i, u_{i+1}), $i \neq k$ are not close to the boundary, then, by (2.14), we have for sufficiently small ε that

$$\begin{aligned} \frac{du_k}{dt} &= Y_k(u_k, u_{k+1}) - \widetilde{Y}_{k-1}(u_{k-1}, u_k) > 0, \qquad (2.15) \\ \frac{du_{k+1}}{dt} &= Y_{k+1}(u_{k+1}, u_{k+2}) - \widetilde{Y}_k(u_k, u_{k+1}) < 0. \end{aligned}$$

Since $h'_\varepsilon > 0$ the flow is thus directed inwards at this point, see Figure 2.14a. On the other boundaries similar arguments hold, which leads to the (mental) picture in Figure 2.14b.

However, we may have a problem when for example (u_{k-1}, u_k) also approaches a boundary. If it approaches the southeast or northeast boundary then there is no problem, since then $\widetilde{Y}_{k-1}(u_{k-1}, u_k) < 0$ and (2.15) still holds. On the other hand, if (u_{k-1}, u_k) approaches the northwest or southwest boundary then the two terms in (2.15) do not cooperate and we can draw no conclusion about the sign. In that case we are unable to conclude that $[\mathbf{u} \text{ rel } \mathbf{v}]_E^\varepsilon$ is isolating for the parabolic flow. We therefore need to introduce the notion of cooperation.

DEFINITION 2.28. A restricted relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ is *cooperating*, if for any braid $\mathbf{u}$ in the boundary piece $\partial[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}} \setminus \Sigma_{\mathcal{E}}$, the following holds:

(1) if $(u_i^\alpha, u_{i+1}^\alpha) \in \overline{h_i^\pm}$, then $(u_{i-1}^\alpha, u_i^\alpha) \notin \overline{h_{i-1}^+} \cup \overline{g_{i-1}^-}$;

(2) if $(u_i^\alpha, u_{i+1}^\alpha) \in \overline{g_i^\pm}$, then $(u_{i-1}^\alpha, u_i^\alpha) \notin \overline{h_{i-1}^-} \cup \overline{g_{i-1}^+}$.

We want to link the index of the restricted braid class to that of the unrestricted braid class. For that purpose we need a stronger assumption, that also takes points in $[\mathbf{u} \text{ rel } \mathbf{v}] \setminus [\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ into account.

DEFINITION 2.29. A restricted relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ is *strongly* cooperating if for any $\mathbf{u} \in \text{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$ the following holds:

(1) if $u^\alpha_{i+1} \geq h^\pm_i(u^\alpha_i)$, then $u^\alpha_i < h^+_{i-1}(u^\alpha_{i-1})$ and $u^\alpha_i > g^-_{i-1}(u^\alpha_{i-1})$;
(2) if $u^\alpha_{i+1} \leq g^\pm_i(u^\alpha_i)$, then $u^\alpha_i < h^-_{i-1}(u^\alpha_{i-1})$ and $u^\alpha_i > g^+_{i-1}(u^\alpha_{i-1})$.

We are now ready to state the main result. The statement and proof are similar to Section 8.3 in [**33**], where the restrictions on the domain were simpler and cooperation was automatic.

THEOREM 2.30. *Let $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ be a cooperating restricted braid class and let the unrestricted braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ be bounded and proper.*

(a) *Then the ε-approximation $N_\varepsilon = \text{cl}([\mathbf{u} \text{ rel } \mathbf{v}]^\varepsilon_{\mathcal{E}})$ is an isolating neighborhood for the parabolic flow for all sufficiently small ε, which yields a well-defined Conley index, denoted by $h(\mathbf{u} \text{ rel } \mathbf{v}, \mathcal{E})$.*
(b) *Moreover, if $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ is strongly cooperating, then the index of the restricted braid class is the same as that of the unrestricted braid class: $h(\mathbf{u} \text{ rel } \mathbf{v}, \mathcal{E}) = h(\mathbf{u} \text{ rel } \mathbf{v})$.*

PROOF. Denote by $\widetilde{\mathcal{R}}_i$ the parabolic recurrence relation under consideration, with parabolic flow $\widetilde{\psi}^t$ defined on N_ε for all small ε. We first need to show that N_ε is isolating for sufficiently small ε. For any point $\mathbf{u} \in \partial N_\varepsilon \cap \Sigma_{\mathcal{E}}$ the flow $\widetilde{\psi}^t$ leaves N_ε in forward or backward time by Proposition 2.11. For any point $\mathbf{u} \in \partial N_\varepsilon \setminus \Sigma_{\mathcal{E}}$ the flow $\widetilde{\psi}^t$ leaves N_ε in forward or backward direction by the definition of a cooperating braid class and the arguments that lead up to its Definition 2.29. We thus conclude that N_ε is isolating, hence its Conley index is well-defined and is independent of (sufficiently small) ε.

Next consider the unrestricted braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$. There exists a parabolic flow that fixes $\mathbf{v}$ (see Appendix of [**33**]), given by a recurrence relation $\mathcal{R}^0$ defined on $\mathbb{R}^3$. We are going to change the recurrence relation so that it still fixes $\mathbf{v}$, while the invariant set is guaranteed to be in the smaller set $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$. Clearly $\mathbf{v} \in \mathcal{E}$ and also $\mathbf{v} \in \mathcal{E}_{2\varepsilon}$ for sufficiently small ε. Let $\eta \in C^\infty(\mathbb{R})$ such that $\eta(x) = 0$ for $x \leq 0$ and $\eta(x) = Ke^{-1/x}$ for $x > 0$, with large K to be chosen later. We construct a nonnegative function $\zeta_i(x,x')$ that is 0 on $D_{i,2\varepsilon}$ and that is large in some sense (see below) on the complement of the slightly larger $D_{i,\varepsilon}$. Namely, we define

$$\zeta_i(x,x') = \eta\big(x' - h^+_{i,2\varepsilon}(x)\big) + \eta\big(x' - h^-_{i,2\varepsilon}(x)\big) - \eta\big(g^+_{i,2\varepsilon}(x) - x'\big) - \eta\big(g^-_{i,2\varepsilon}(x) - x'\big);$$
$$\xi_i(x,x') = \eta\big(x' - h^+_{i,2\varepsilon}(x)\big) - \eta\big(x' - h^-_{i,2\varepsilon}(x)\big) - \eta\big(g^+_{i,2\varepsilon}(x) - x'\big) + \eta\big(g^-_{i,2\varepsilon}(x) - x'\big).$$

It is important that $\partial_2\zeta_i \geq 0$ and $\partial_1\xi_i \leq 0$ (they mirror the behavior of Y_i and $\widetilde{Y}_i$). For any large square N in $\mathbb{R}^2$ we can choose ε sufficiently small, so that the four terms have disjoint support in N. Additionally, for any $C > 0$ there is (by a straightforward compactness argument) a sufficiently large K so that

$$\zeta_i(x,x') = \eta(x' - h^{\pm}_{i,2\varepsilon}(x)) > C \qquad \text{on } N \cap \{x' \geq h^{\pm}_{i,\varepsilon}(x)\}; \tag{2.16a}$$

$$\zeta_i(x,x') = -\eta(g^{\pm}_{i,2\varepsilon}(x) - x') < -C \qquad \text{on } N \cap \{x' \leq g^{\pm}_{i,\varepsilon}(x)\}; \tag{2.16b}$$

$$\xi_i(x,x') = \pm\eta(x' - h^{\pm}_{i,2\varepsilon}(x)) \gtrless \pm C \qquad \text{on } N \cap \{x' \geq h^{\pm}_{i,\varepsilon}(x)\}; \tag{2.16c}$$

$$\xi_i(x,x') = \mp\eta(g^{\pm}_{i,2\varepsilon}(x) - x') \lessgtr \mp C \qquad \text{on } N \cap \{x' \leq g^{\pm}_{i,\varepsilon}(x)\}. \tag{2.16d}$$

We define for $s \in [0,1]$

$$\mathcal{R}^s_i(x_{i-1},x_i,x_{i+1}) = \mathcal{R}^0_i(x_{i-1},x_i,x_{i+1}) + s\left[\zeta_i(x_i,x_{i+1}) - \xi_{i-1}(x_{i-1},x_i)\right].$$

Let ψ^t_s be the flow generated by $\mathcal{R}^s$. By computing $\partial_1\mathcal{R}^s_i$ and $\partial_3\mathcal{R}^s_i$, it is not difficult to check that $\mathcal{R}^s$ is a parabolic recurrence relation and ψ^t_s a parabolic flow for all $s \in [0,1]$. Since $\mathcal{R}^s = \mathcal{R}^0$ on $\mathcal{E}_{2\varepsilon}$ the whole family fixes $\mathbf{v}$. Since $[\mathbf{u} \text{ rel } \mathbf{v}]$ is bounded it is contained in a large cube, say $(u^\alpha_i, u^\alpha_{i+1}) \in N$ for all i and α. Using the strongly cooperating property of $[\mathbf{u} \text{ rel } \mathbf{v}]$ we can deduce from (2.16) that for sufficiently large K the recurrence relations $\mathcal{R}^1_i$ and $\mathcal{R}^1_{i+1}$ have fixed sign whenever $(x_i,x_{i+1}) \in N \setminus D_{i,\varepsilon}$, for example, $\mathcal{R}^1_i > 0$ and $\mathcal{R}^1_{i+1} < 0$ when $x_{i+1} \geq h^+_{i,\varepsilon}(x_i)$. This implies that in forward or backward time the orbit through such a point leaves N. Therefore the invariant set for ψ^t_1 in $[\mathbf{u} \text{ rel } \mathbf{v}]$ is completely contained in $[\mathbf{u} \text{ rel } \mathbf{v}]^\varepsilon_{\mathcal{E}}$.

We now use the fact that the Conley index is a property not only of an isolating neighborhood, but also of an invariant set. Let S be the invariant set of $[\mathbf{u} \text{ rel } \mathbf{v}]$ under the flow ψ^t_1. Since $[\mathbf{u} \text{ rel } \mathbf{v}]^\varepsilon_{\mathcal{E}}$ is also an isolating neighborhood of S for ψ^t_1, we see that the Conley indexes of $[\mathbf{u} \text{ rel } \mathbf{v}]$ and $[\mathbf{u} \text{ rel } \mathbf{v}]^\varepsilon_{\mathcal{E}}$ are the same, namely the index of S.

Finally, consider the flows given by the interpolating parabolic recurrence relations $(1-\lambda)\widetilde{\mathcal{R}} + \lambda\mathcal{R}^1$, $\lambda \in [0,1]$, with parabolic flow $\widetilde{\psi}^t_\lambda$. Note that $\widetilde{\psi}^t_0 = \widetilde{\psi}^t$ and $\widetilde{\psi}^t_1 = \psi^t_1$, and the whole family fixes $\mathbf{v}$. Furthermore, $[\mathbf{u} \text{ rel } \mathbf{v}]^\varepsilon_{\mathcal{E}}$ is an isolating neighborhood for $\widetilde{\psi}^t_\lambda$ for any $\lambda \in [0,1]$, since the signs of $\mathcal{R}^1_i$ and $\widetilde{\mathcal{R}}_i$ on the restricting boundaries are the same. Hence the Conley index does not change along the continuation from $\widetilde{\psi}^t$ to ψ^t_1:

$$h(\mathbf{u} \text{ rel } \mathbf{v}, \mathcal{E}) = h([\mathbf{u} \text{ rel } \mathbf{v}]^\varepsilon_{\mathcal{E}}, \widetilde{\psi}^t) = h([\mathbf{u} \text{ rel } \mathbf{v}]^\varepsilon_{\mathcal{E}}, \psi^t_1) = h([\mathbf{u} \text{ rel } \mathbf{v}], \psi^t_1) = h(\mathbf{u} \text{ rel } \mathbf{v}).$$

This finishes the proof. □

Positive entropy for bijective twist diffeomorphisms

Let us apply the theory developed in the previous section to prove Theorem 2.2. We can try to emulate Section 2.5 up to the point where we need to calculate the Conley index, which is now replaced by the restricted index $h([\mathbf{u} \text{ rel } \mathbf{v}], \mathcal{E})$. We need to be sure that the restricted index is well-defined. The braid classes under consideration are bounded and proper, but they might not all be cooperating.

Let us look at the shape of the domains D_i. Since the skeleton $\mathbf{v}$ (cf. Figures 2.7 (right) and 2.10) consists of stationary points, it must be that $(v^\alpha_i, v^\alpha_{i+1}) \in D_i$ for $\alpha = 1,2,3,4$. These points are shown in Figure 2.15 for even and odd i. For each i the projection of the unrestricted braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ under consideration onto the (u_i, u_{i+1})-plane is one of the three blocks indicated in Figure 2.15. As a consequence of the fact that $(v^\alpha_i, v^\alpha_{i+1}) \in D_i$ and of our knowledge about the shape of the boundary ∂D_i, we see that the northeast and

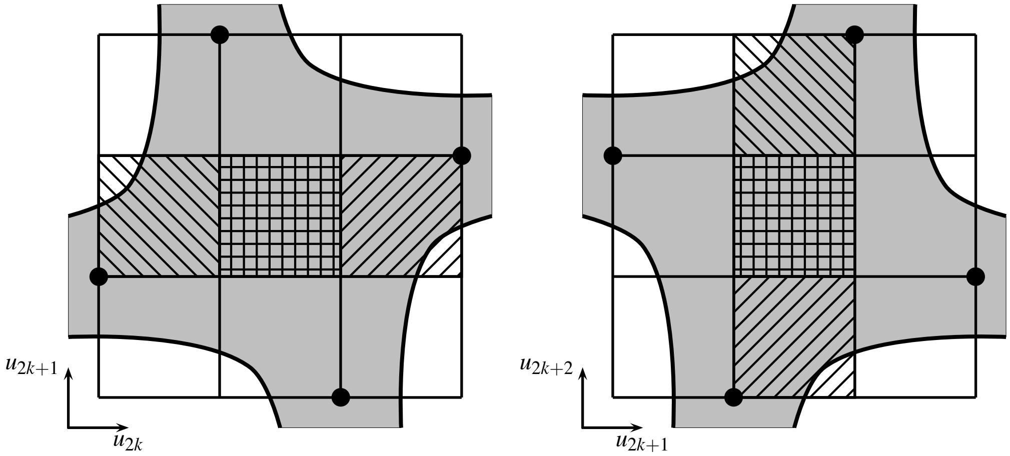

Figure 2.15: The dots represent the points $(v_i^\alpha, v_{i+1}^\alpha) \in D_i$, $\alpha = 1,2,3,4$ for even i (left) and odd i (right). The domain D_i is indicated in gray. The projections of the unrestricted braid classes onto the (u_i, u_{i+1})-plane are hatched. Of the four boundaries of D_i only two can intersect the unrestricted braid classes.

southwest boundary never come into play for any of the braid classes under consideration, see Figure 2.15 again.

According to the definition of cooperating braid classes we need to prevent that (u_{i-1}, u_i) and (u_i, u_{i+1}) can be both on the northeast or both on the southwest boundary. When one retraces the steps, in particular the application of a flip in the proof of Lemma 2.18, one sees that this can only happen if in the associated symbol sequence -1 is adjacent to -1 or $+1$ is adjacent to $+1$, and we thus need to exclude these possibilities. To ensure the braid class is cooperating we therefore go back a step and replace the (full) shift on three symbols by a subshift with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

In words, only sequences in which -1 is followed by 0 or $+1$, and $+1$ is followed by -1 or 0, are allowed. The corresponding braid classes are now all cooperating, and even strongly cooperating, so the remainder of the proof follows the path described in Section 2.5, using Theorem 2.30 to compute the restricted Conley index.

There is one more issue to deal with, namely compactness. The set Λ_1 as defined in (2.8) is not necessarily bounded, since, as should be clear at this point, it is harder to control the y-coordinates Y and $\widetilde{Y}$ for diffeomorphisms than it is for maps with the infinite twist condition. To resolve this problem, consider the set Λ_2 of periodic orbits of f^2 that is "constructed" in the same way as in Lemma 2.18 with the restriction on the symbol sequences due to the cooperating braid classes described above. To be more precise, for every symbol sequence in the subshift defined by A the proof of Lemma 2.18 gives a corresponding periodic point/orbit of f^2, and the collection of these orbits we call Λ_2.

Since Λ_2 consists of orbits it is invariant under f^2 and we claim that it is also bounded. Clearly the x-coordinates are uniformly bounded. The parameter ε, that is used to regularize in Section 2.7, can be chosen in a uniform manner, since there are only four different maps and four different domains $D_{i,\varepsilon}$ to consider. In the ε-approximations $[\mathbf{u} \text{ rel } \mathbf{v}]^{\varepsilon}_{\mathcal{E}}$ of the braid classes considered in Lemma 2.18, the pairs (x_i, x_{i+1}) are in a bounded subset of $D_{i,\varepsilon}$, and on these sets the continuous functions Y and $\widetilde{Y}$ are bounded. Hence also the y-coordinates of the points in Λ_2 are uniformly bounded.

The set Λ_1 in Section 2.5 is now replaced by the (smaller) compact set $\widetilde{\Lambda}_1 = \text{cl}(\Lambda_2)$, which is invariant under f^2. Clearly this set $\widetilde{\Lambda}_1$ also suffices in Lemma 2.20, because that lemma essentially consists of taking the closure of the periodic trajectories. Replacing Λ_1 by $\widetilde{\Lambda}_1$ does not change any of the other arguments in Section 2.5. The resulting lower bound on the entropy of the bijective twist map is half of the entropy of the subshift, which is log of $1+\sqrt{2}$, the largest eigenvalue of the matrix A (cf. [**41**]).

CHAPTER 3

Floer homology for relative braid classes

3.1. Hamiltonian systems on the 2-disc

Let $\mathbb{D}^2 \subset \mathbb{R}^2$ be the 2-disc given by $\mathbb{D}^2 = \{x = (p,q) \in \mathbb{R}^2 \mid |x|^2 \leq 1\}$, and let $\omega_0 = dp \wedge dq$ be the standard area form on $\mathbb{D}^2$. The pair $(\mathbb{D}^2, \omega_0)$ is a 2-dimensional symplectic manifold. We consider a class of Hamiltonian functions $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ that satisfy the following hypotheses (with respect to $\mathbb{D}^2$):

(h1) $H \in C^2(\mathbb{R} \times \mathbb{R}^2; \mathbb{R})$;
(h2) $H(t+1,x) = H(t,x)$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^2$;
(h3) $H(t,x) = 0$ for all $x \in \partial\mathbb{D}^2$ and all $t \in \mathbb{R}$.

We denote this class of Hamiltonians by $\mathcal{H}(\mathbb{D}^2)$, or $\mathcal{H}$ for short if there is no ambiguity about the symplectic manifold.

For a given Hamiltonian $H \in \mathcal{H}$ we define the time-dependent Hamilton vector field X_H by

$$i_{X_H}\omega_0 = -dH.$$

In other words, $X_H(t,x) = J_0 \nabla H(t,x) = -H_q(t,x)\frac{\partial}{\partial p} + H_p(t,x)\frac{\partial}{\partial q}$, and

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the standard symplectic matrix, which is defined via the relation $\langle \cdot,\cdot \rangle = \omega_0(\cdot, J_0 \cdot)$. If $\mathbb{R}^2$ is regarded as the complex plane $\mathbb{C}$ then J_0 corresponds to complex multiplication with i. The vector field X_H is a C^1-function from $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$ to $\mathbb{R}^2$. We restrict X_H to $\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$.

Since the Hamiltonian H is 1-periodic in time (Hypothesis (h2)) closed characteristics or trajectories of X_H in $\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$ correspond to n-periodic ($n \in \mathbb{N}$) solutions $\big(x(t+n) = x(t),\ \forall t\big)$ of the Hamilton differential equations

$$x_t = X_H(t,x), \quad (t,x) \in \mathbb{R}/\mathbb{Z} \times \mathbb{D}^2. \tag{3.1}$$

This system has a variational structure. In particular, the 1-periodic solutions $x(t)$ are critical points of the Hamilton action functional

$$f_H(x) = -\int_0^1 \alpha_0(x_t(t))dt + \int_0^1 H(t,x(t))dt, \tag{3.2}$$

where $\alpha_0 = pdq$. The first variation is given by

$$\begin{aligned} df_H(x)\xi &= \int_0^1 \omega_0\big(x_t(t),\xi(t)\big)dt + \int_0^1 dH(t,x(t))\xi(t)dt \\ &= \int_0^1 \omega_0\Big(x_t(t) - X_H(t,x(t)),\xi(t)\Big)dt. \end{aligned}$$

REMARK 3.1. Hypothesis (h3) reflects that $\mathbb{D}^2$ is invariant for the Hamiltonian flow, and so is $\partial\mathbb{D}^2$. In that setting, hypothesis (h3) is not a restriction on H. Namely, in order for $\mathbb{D}^2$ to be invariant for the Hamilton equations (3.1) we need that $H(t,\cdot)|_{\partial\mathbb{D}^2} =$ constant for each $t \in \mathbb{R}$. For such Hamiltonians H, the same flow is obtained from the "normalized" Hamiltonian $\tilde{H}(t,x) = H(t,x) - H(t,\cdot)|_{\partial\mathbb{D}^2}$, which satisfies (h3). Therefore, we can assume (h3) without loss of generality.

The initial value problem for Equation (3.1) defines the solution operator, or time-τ map $\Psi = \Psi^\tau$ as follows: $\Psi(x_0) \stackrel{\text{def}}{=} x(\tau;x_0)$, $x \in \mathbb{D}^2$. By compactness of $\mathbb{D}^2$ the initial value problem yields solutions for $\tau \in [-\tau_0,\tau_0]$ for some $\tau_0 > 0$. Since the boundary $\partial\mathbb{D}^2 \cong S^1$ is invariant for Equation (3.1) solutions $x(t;x_0)$, $x_0 \in \text{int}(\mathbb{D}^2)$, cannot intersect the boundary in finite time and therefore $|x(t;x_0)| < 1$. This implies that $\tau_0 = \infty$ and the time-τ map is defined for all $\tau \in \mathbb{R}$. The time-τ map Ψ is area preserving

$$\Psi^*\omega_0 = \omega_0.$$

At each $x \in \mathbb{D}^2$ it holds that $\det(d\Psi(x)) = 1$ and at each $x \in \mathbb{D}^2$, $d\Psi(x)$ is a symplectic matrix with respect to ω_0. Recall that symplectic 2×2 matrices S are defined by the relation

$$S^T J_0 S = J_0,$$

which is derived from the condition $S^*\omega_0 = \omega_0$, i.e. $\omega_0(S\xi,S\eta) = \omega_0(\xi,\eta)$. In particular, for 2×2 matrices this is equivalent to $\det(S) = 1$. The symplectic 2×2 matrices form a group which is denoted by $\text{Sp}(2,\mathbb{R})$, which (in dimension two) is equal to the special linear group $\text{SL}(2,\mathbb{R})$.

A 1-periodic solution of Equation (3.1) is equivalent to a fixed point of the time-1 map Ψ^1. Let $x(t+1) = x(t)$ be a solution of Equation (3.1), then $\Psi^1(x(t)) = x(t)$ for any $t \in \mathbb{R}$, and vice versa, if $\Psi^1(x_0) = x_0$ then $x(t;x_0)$ is a 1-periodic solution of Equation (3.1). For n-periodic solutions the same holds; n-periodic solutions are equivalent to n-periodic points of the time-1 map Ψ^1, i.e. $x(t+n) = x(t)$ implies that $\Psi^n(x(t)) = x(t)$ for all $t \in \mathbb{R}$ and if $\Psi^n(x_0) = x_0$ then $x(t;x_0)$ is an n-periodic solution.

3.2. Closed braids

In order to study n-periodic solutions of Equation (3.1) we exploit a topological structure that is available only in 2-dimensional Hamiltonian systems. Let x be a periodic solution of Equation (3.1) of integer period n. By restricting t to the interval $[0,1]$ we can describe x via its translates

$$x^1(t) = x(t),\ x^2(t) = x(t+1),\ \ldots,\ x^n(t) = x(t+n-1),$$

see also Figure 3.1. Since $x^1,\ldots,x^n$ are solutions of Equation (3.1), the uniqueness of the initial value problem implies that, if the period n is minimal, the n 'curves', or *strands*, never intersect, i.e. $x^k(t) \neq x^{k'}(t)$, for any $k \neq k'$ and any $t \in [0,1]$. If the period is not minimal

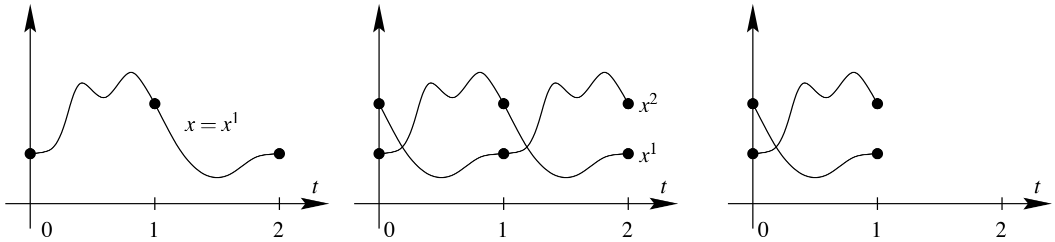

Figure 3.1: A periodic function with minimal period 2; a periodic function and its integer translates; the restriction to the fundamental interval $t \in [0,1]$.

then some strands coincide. Since the collection of curves or strands is constructed from an n-periodic solution x it holds that

$$\{x^1(1),\dots,x^n(1)\} = \{x^1(0),\dots,x^n(0)\}$$

as (unordered) sets. Of course, if we consider a single periodic solution with minimal period n, the values of $x^k(1)$ and $x^k(0)$ do not match pointwise in k. As a matter of fact given a family of solutions curves (not 1-periodic) $\{x^k(t)\}_{k=1}^n$ for which the end points are cyclicly permuted, is equivalent to having an n-periodic solution x by concatenating the strands. As such, appropriate collections of functions can be regarded as solutions of Equation (3.1). This lead to the following definition.

DEFINITION 3.2. The space of closed braids on n strands, denoted Ω^n, consists of pairs $(\mathbf{x},\sigma)$, where $\sigma \in S_n$ is a permutation, and $\mathbf{x}$ an unordered collection of C^0-functions $\mathbf{x} = \{x^k\}$, $k = 1,\dots,n$, with $x^k\colon [0,1] \to \mathbb{D}^2$, called strands, which satisfy the following properties;

(i) for all $k = 1,\dots,n$ it holds that $x^k(1) = x^{\sigma(k)}(0)$;[1]
(ii) for any pair of strands $k \neq k'$, it holds that $x^k(t) \neq x^{k'}(t)$ for all $t \in [0,1]$.

On Ω^n we consider the standard (strong) metric topology of $\left(C^0([0,1];\mathbb{D}^2)\right)^n$, and the discrete topology with respect to permutations, modulo permutations that change the ordering of strands. To be more specific, two braids $(\mathbf{x},\sigma)$ and $(\tilde{\mathbf{x}},\tilde{\sigma})$ are close in the topology of Ω^n if and only if for some permutation $\theta \in S_n$ it holds that $x^{\theta(k)}$ and $\tilde{x}^k$ are close in C^0, for all k, and $\tilde{\sigma} = \theta^{-1}\sigma\theta$.

We remark that identification of braids which are equivalent under permutation of the strands is not necessary to develop the theory. On the one hand, it is elegant topologically, but on the other hand, it is a bit cumbersome computationally. In most of the arguments we shall (silently) order the strands, and it will be implicit that all arguments are invariant under the choice of this ordering (i.e. choosing a representative). A collection $\mathbf{x}$ satisfying Condition (i) in fact defines a permutation σ. Therefore, we often omit σ from the notation if there is no ambiguity about the permutation

[1] The action of σ is defined by $\sigma\big((1,\dots,n)\big) = (\sigma(1),\dots,\sigma(n))$.

In the spirit of this definition n-periodic solutions yield closed braids which can be interpreted as a solutions of the product system of Equation (3.1).[2] In general a closed n-braid is a collection of periodic solution of Equation (3.1) if all its strands satisfy Equation (3.1). The sets of strands that correspond to the same cycle of the permutation σ are called the *components* of the braid. The number of components determines the number of periodic solutions that is represented by the braid. In this context it is natural to extend any strand $x^k(t), t \in [0,1]$ to all $t \in \mathbb{R}$ by requiring that

$$x^k(t+1) = x^{\sigma(k)}(t) \quad \text{for all } t \in \mathbb{R} \text{ and } k = 1, \dots, n. \tag{3.3}$$

When $n = 1$, then $\Omega^1 = \overline{\Omega}^1 = C^0([0,1]; \mathbb{D}^2)$. Condition (ii) makes Ω^n disconnected for all $n \geq 2$, and we are especially interested in the *path components* of Ω^n.

DEFINITION 3.3. Two closed braids $\mathbf{x}$ and $\mathbf{x}'$ in Ω^n are in the same *braid class*, denoted $[\mathbf{x}] = [\mathbf{x}']$, if and only if there exists a continuous path $\mathbf{x}(s) \in \Omega^n$, for all $s \in [0,1]$, such that $\mathbf{x}(0) = \mathbf{x}$ and $\mathbf{x}(1) = \mathbf{x}'$.

The (C^0-)closure of Ω^n, denoted $\overline{\Omega}^n$, can be characterized as the set of all maps $\mathbf{x} = \{x^k\}$ satisfying Condition (i) of the Definition 3.2 but not necessarily Condition (ii). The braids failing to satisfy the Condition (ii) of Definition 3.2 are called singular braids.

DEFINITION 3.4. $\Sigma^n = \overline{\Omega}^n \setminus \Omega^n$ is the set of *singular* braids.

It is easy to see that, generically, a singular braid is a collection of strands $\mathbf{x}$ for which $x^k(t_0) = x^{k'}(t_0)$ for exactly one pair $k \neq k'$ and exactly one $t_0 \in [0,1]$. We will say that such a collection of strands is a codimension 1 singularity and denote the set consisting of such braids as $\Sigma^n_1 \subset \Sigma^n$. We can also define $\Sigma^n_k \subset \Sigma^n$ as the set of codimension k singularities, which consists of singular braids $\mathbf{x}$ with at most 'k intersections'. The set Σ^n_1 serves as the 'walls' between path components of Ω^n.

At the other extreme, an important subset of singular braids are the so-called *collapsed singularities* $\Sigma^n_- \subset \Sigma^n$. This may happen in two slightly different ways; either a component collapses into a braid with fewer strands, or two different components coalesce into one component. To be more precise

$$\Sigma^n_- \stackrel{\text{def}}{=} \left\{ \mathbf{x} \in \Sigma^n \,|\, x^k(t) = x^{k'}(t), \text{ for all } t \in \mathbb{R}, \text{ for some } k \neq k' \right\}.$$

Note that Σ^n_- can be identified with the spaces $\overline{\Omega}^{n'}$, $n' < n$, under the appropriate identification of collapsed strands. When $n = 1$, then $\Sigma^1 = \Sigma^1_- = \varnothing$.

In the case $n = 1$ the variational principle for Equation (3.1) is given by the action functional in (3.2). For closed n-braids we can adjust the variational principle as follows. Given any $\mathbf{x} \in \overline{\Omega}^n$ define its action by

$$\mathbf{f}_H(\mathbf{x}) = \sum_{k=1}^{n} f_H(x^k), \tag{3.4}$$

where f_H is defined by (3.2). We may also regard the action as a Hamiltonian action for a Hamiltonian system on $(\mathbb{D}^2)^n$, with $\overline{\omega}_0 = \omega_0 \times \cdots \times \omega_0$, and $\overline{H}(t, \mathbf{x}) = \sum_k H(t, x^k)$, i.e. an

[2] The product system is obtained by constructing an uncoupled system on $(\mathbb{D}^2)^n$ by repeating the equations n times. The only coupling is in the boundary conditions as given by Condition (i) of Definition 3.2.

uncoupled system with coupled boundary conditions. The functional $\mathbf{f}_H$ is well-defined on $\overline{\Omega}^n \cap C^1$. The stationary, or critical closed braids, including singular braids, are denoted by

$$\mathrm{Crit}_H(\overline{\Omega}^n) = \{\mathbf{x} \in \overline{\Omega}^n \cap C^1 \mid d\mathbf{f}_H(\mathbf{x}) = 0\}.$$

Since the boundary conditions are given by Condition (i) in Definition 3.2, the first variation of the action yields that the individual strands satisfy Equation (3.1). We can establish the following property with respect to critical points of $\mathbf{f}_H$ on $\overline{\Omega}^n$.

LEMMA 3.5. *The set* $\mathrm{Crit}_H(\overline{\Omega}^n)$ *is compact in* $\overline{\Omega}^n$. *As a matter of fact* $\mathrm{Crit}_H(\overline{\Omega}^n)$ *is compact in the* C^2*-topology.*

PROOF. From Equation (3.1) we derive that $|x_t^k| = |\nabla H(t, x^k)| \le C$, by the assumptions on H. Since $|x^k| \le 1$, for all k, we obtain the a priori estimate

$$\|\mathbf{x}\|_{W^{1,\infty}} \le C,$$

which holds for all $\mathbf{x} \in \mathrm{Crit}_H(\overline{\Omega}^n)$. Via compact embeddings (Arzela-Ascoli) we have that a sequence $\mathbf{x}_n \in \mathrm{Crit}_H$ converges in C^0, along a subsequence, to a limit $\mathbf{x} \in \overline{\Omega}^n$. Using the equation we obtain the convergence in C^1 and $\mathbf{x}$ satisfies the equation with the boundary conditions given by Condition (i) of Definition 3.2. Therefore $\mathbf{x} \in \mathrm{Crit}_H(\overline{\Omega}^n)$, thereby establishing the compactness of $\mathrm{Crit}_H(\overline{\Omega}^n)$ in $\overline{\Omega}^n$. The C^2-convergence is achieved by differentiating Equation (3.1) once. This concludes the compactness of $\mathrm{Crit}_H(\overline{\Omega}^n)$ in C^2. □

REMARK 3.6. The critical braids in $\mathrm{Crit}_H(\overline{\Omega}^n)$ have one additional property that plays an important role. For the strands x^k of $\mathbf{x} \in \mathrm{Crit}_H(\overline{\Omega}^n)$ it holds that either $|x^k(t)| = 1$, for all t, or $|x^k(t)| < 1$, for all t. This is a consequence of the uniqueness of solutions for the initial value problem for (3.1). We say that a braid $\mathbf{x}$ in supported in $\mathrm{int}(\mathbb{D}^2)$ if $|x^k(t)| < 1$, for all t and for all k.

In the same spirit, we can define the subset of stationary braids restricted to a braid class $[\mathbf{x}]$, notation $\mathrm{Crit}_H([\mathbf{x}])$. Due to the fact that strands can coalesce, this space is not necessarily compact in $\overline{\Omega}^n$. By the same token, the union of all braid classes, i.e. $\mathrm{Crit}_H(\Omega^n)$, is not necessarily compact and $\mathrm{Crit}_H(\Omega^n) \subset \mathrm{Crit}_H(\overline{\Omega}^n)$. The following proposition gives a refined compactness statement for set of stationary braids. Before stating this result let us first explain the conjugacy classes of permutations of closed braids. Let $I_n = \{1, \dots, n\}$, and S_n is the group of permutations on I_n. A permutation $\sigma' \in S_l$, for some $l \le n$, is said to *coarsen* a permutation $\sigma \in S_n$, if there exist disjoint sets $A_i \subset I_n$, $i = 1, \dots, l$ satisfying $\bigcup_i A_i = I_n$, such that

$$A_{\sigma'(i)} = \sigma(A_i) \stackrel{\text{def}}{=} \{\sigma(a) \mid a \in A_i\},$$

for all $i = 1, \dots, l$. In terms of braids, the coarsening means that the strands of one or more components of the braid are identified.

PROPOSITION 3.7. *The compact space* $\mathrm{Crit}_H(\overline{\Omega}^n)$ *can be decomposed as follows:*

$$\mathrm{Crit}_H(\overline{\Omega}^n) = \bigcup_{l=1}^{n} \mathrm{Crit}_H(\Omega^l).$$

Elements in $\mathrm{Crit}_H(\Omega^l)$*,* $l < n$*, can occur as limits of sequences in* $\mathrm{Crit}_H(\Omega^n)$ *when strands are counted with multiplicity. The compactness in* $\mathrm{Crit}_H(\overline{\Omega}^n)$ *can, in view of the decomposition above, be described as follows; let* $[\mathbf{x}]$ *be a braid class in* Ω^n*, then for any sequence*

$\{(\mathbf{x}_m,\sigma_m)\}\subset \mathrm{Crit}_H([\mathbf{x}])$, $\sigma_m\in[\sigma]$, *there exists a subsequence* $\{(\mathbf{x}_{m_j},\sigma')\}\subset \mathrm{Crit}_H([\mathbf{x}])$ *(i.e.* $\sigma'\in[\sigma]$*), and a braid* $(\tilde{\mathbf{x}},\tilde\sigma)\in \mathrm{Crit}_H(\Omega^l)$*, for some* $l\le n$*, such that* $\tilde\sigma$ *coarsens* σ' *with respect to a decomposition* $\{A_i\}$ *of* I_n*, and*

$$x^{k_i}_{m_j}\xrightarrow{C^0}\tilde{x}^i,\quad m_j\to\infty,$$

for all $k_i\in A_i$ *and for all* $i=1,\dots,l$*. For the action it holds that*

$$\mathbf{f}_H(\mathbf{x}_{m_j})\longrightarrow\sum_{i=1}^{l}|A_i|\,f_H(\tilde{x}^i),\quad m_j\to\infty,$$

where $|A_i|$ *is the number of elements in* A_i*.*

PROOF. Compactness follows from Lemma 3.5, and the uniqueness of the initial value problem for Equation (3.1) implies that limits are necessarily closed braids, possibly with strands coalescing. Simple inspection of limits reveals how closed braids may collapse and how the action of the limits relates the action along a sequence. □

3.3. The Cauchy-Riemann equations

The matrix J_0 defines a compatible almost complex structure on $\mathbb{D}^2$. In general, a constant compatible almost complex structure on $\mathbb{D}^2$ is a matrix $J:T_x\mathbb{D}^2\cong\mathbb{R}^2\to T_x\mathbb{D}^2$ satisfying

$$J^2=-\mathrm{Id},$$

and such that the quadratic form $g(\cdot,\cdot)=\omega_0(\cdot,J\cdot)$ defines a Riemannian metric, or inner product g on $\mathbb{R}^2$.

The set of constant almost complex structures is denoted by $\mathcal{J}^+$, and it follows that J is a symplectic matrix (for the constructions in this paper we only need J to be constant matrix, i.e. independent of t or x). Indeed, let

$$J=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\ \text{ then }\ J^2=\begin{pmatrix} a^2+bc & b(a+d)\\ c(a+d) & d^2+bc\end{pmatrix}=\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix},$$

which implies that $d=-a$ and $\det(J)=-a^2-bc=1$. The set of all matrices J for which $J^2=-\mathrm{Id}$ is denoted by

$$\mathcal{J}=\left\{J=\begin{pmatrix} a & b\\ c & -a\end{pmatrix}\;\middle|\; a^2+bc=-1,\ a,b,c\in\mathbb{R}\right\},$$

which is a smooth 2-dimensional submanifold of $\mathrm{Sp}(2,\mathbb{R})$ with two connected components $\mathcal{J}^+=\{J\in\mathcal{J}\mid c>0\}$ and $\mathcal{J}^-=\{J\in\mathcal{J}\mid c<0\}$, of which $\mathcal{J}^+$ are the constant almost complex structures. For instance $J_0\in\mathcal{J}^+$ corresponds to complex multiplication.

In terms of the standard inner product $\langle\cdot,\cdot\rangle$ the metric g is given by $g(\xi,\eta)=\langle -J_0J\xi,\eta\rangle$, where $-J_0J$ is positive definite symmetric matrix when $c>0$. With respect to the metric g it holds that $J\nabla_gH=X_H$. The extended class of almost complex structures on $\mathbb{D}^2$ is not needed in the present work, but it may be exploited in future applications (e.g., when considering the connection between Cauchy-Riemann equations and parabolic heat flows).

In order to study 1-periodic solutions of Equation (3.1) the variational method due to Floer and Gromov explores the perturbed (non-linear) Cauchy-Riemann equations

$$\bar\partial_{J,H}(u)\stackrel{\text{def}}{=}u_s+J\big[u_t-X_H(t,u)\big]=0. \tag{3.5}$$

for functions $u:\mathbb{R}\times\mathbb{R}\to\mathbb{D}^2$. In the case of 1-periodic solutions we invoke the boundary conditions $u(s,t+1)=u(s,t)$. It is immediate that stationary solutions, i.e. $u(s,t)=x(t)$, are 1-periodic solutions of the Hamilton equations (3.1). The parameters in the equation are $H\in\mathcal{H}$, and $J\in\mathcal{J}^+$. The latter yields another means of writing the Cauchy-Riemann equations:

$$u_s+Ju_t+\nabla_g H(t,u)=0, \tag{3.6}$$

where $\nabla_g H=-JX_H$.

In order to find closed braids as critical points of $\mathbf{f}_H$ on $\overline{\Omega}^n$ we invoke the Cauchy-Riemann equations. The bounded 'flowlines' or solutions of the Cauchy-Riemann equations are used to devise a Morse type theory for critical points of $\mathbf{f}_H$ in the spirit of Floer's construction [**29**]. A collection of C^1-functions $\mathbf{u}(s,t)=\{u^k(s,t)\}$ is said to satisfy the Cauchy-Riemann equations if its components u^k satisfy Equation (3.5) for all k and the boundary conditions in t, given by Condition (i) of Definition 3.2, are satisfied for all s. The Cauchy-Riemann equations for the collection $\mathbf{u}$ can be given the structure of Cauchy-Riemann equations in the symplectic product $\big((\mathbb{D}^2)^n,\overline{\omega}_0\big)$, where $\overline{\omega}_0=\omega_0\times\cdots\times\omega_0$. For a given almost complex matrix $J\in\mathcal{J}^+$ the induced almost complex matrix $\overline{J}\in\mathrm{Sp}(2n,\mathbb{R})$ is defined by the relation

$$\overline{g}(\cdot,\cdot)=\overline{\omega}_0(\cdot,\overline{J}\cdot),$$

where $\overline{g}=g\times\cdots\times g$. The equations

$$\bar\partial_{\overline{J},\overline{H}}(\mathbf{u})=\mathbf{u}_s+\overline{J}\mathbf{u}_t+\nabla_{\overline{g}}\overline{H}(t,\mathbf{u})=0, \tag{3.7}$$

are Cauchy-Riemann equations in $\big((\mathbb{D}^2)^n,\overline{\omega}_0\big)$ with almost complex matrix $\overline{J}$ and Hamiltonian $\overline{H}(t,\mathbf{u})=\sum_k H(t,u^k)$. These form an uncoupled system of n identical equations coupled only via the boundary conditions of Definition 3.2. In Section 3.7 we show that for generic Hamiltonians H the set of stationary solutions is non-degenerate. By embedding Equation (3.7) into Hamiltonian perturbations $\mathfrak{h}$ on $\mathbb{R}/\mathbb{Z}\times(\mathbb{D}^2)^n$ we can put the equations in general position with respect to generic properties for connecting orbits, see Section 3.7. In Floer's original article [**29**] the Equations (3.7) are studied on 'isolating neighborhoods'. In the subsequent sections we adopt this philosophy by considering proper braid classes as isolating neighborhoods. In this case the Equations (3.7) are put in general positions as Cauchy-Riemann equations on $(\mathbb{D}^2)^n$.

Define the set of entire solutions as

$$\mathcal{F}^n_{J,H}=\Big\{\mathbf{u}=\{u^k\}_{k=1}^n\;\Big|\;u^k\in C^1(\mathbb{R}\times[0,1];\mathbb{D}^2),\,\bar\partial_{\overline{J},\overline{H}}(\mathbf{u})=0\Big\}.$$

We still need to incorporate the "periodicity" condition

$$\{u^1(1),\ldots,u^n(1)\}=\{u^1(0),\ldots,u^n(0)\} \tag{3.8}$$

in our notion of solution. This requirement is fulfilled precisely by braids $\mathbf{u}(s,\cdot)\in\overline{\Omega}^n$ for all s. We therefore define the space of bounded solutions of (3.5) or (3.7) by

$$\mathcal{M}^{J,H}=\mathcal{M}^{J,H}(\overline{\Omega}^n)=\big\{\mathbf{u}\in\mathcal{F}^n_{J,H},\sigma\in S_n\;\big|\;(\mathbf{u}(s,\cdot),\sigma)\in\overline{\Omega}^n\big\}$$

As before, we will drop the permutation σ from our notation. Note that solutions in $\mathcal{M}^{J,H}$ extend to C^1-functions $u^k:\mathbb{R}\times\mathbb{R}\to\mathbb{D}^2$, by periodic extension in t of $\mathbf{u}=\{u^k\}$ (see (3.3)). If there is no ambiguity about the dependence on J and H we abbreviate notation by writing $\mathcal{F}$ and $\mathcal{M}$.

Compactness

The following statement is in essence Floer's compactness theorem adjusted to the present situation. We will give a self-contained proof here.

PROPOSITION 3.8. *The space $\mathcal{M}^{J,H}$ is compact in the topology of uniform convergence on compact sets in $(s,t) \in \mathbb{R}^2$, with derivatives up to order 1. Moreover, $\mathbf{f}_H$ is uniformly bounded along trajectories $\mathbf{u} \in \mathcal{M}$, and*

$$\lim_{s\to\pm\infty} |\mathbf{f}_H(\mathbf{u}(s,\cdot))| = |c_\pm(\mathbf{u})| \le C(J,H),$$
$$\int_{\mathbb{R}}\int_0^1 |\mathbf{u}_s|_g^2 dtds = \sum_{k=1}^n \int_{\mathbb{R}}\int_0^1 |u_s^k|_g^2 dtds \le C'(J,H),$$

for all $\mathbf{u} \in \mathcal{M}^{J,H}$, and some constants $c_\pm(\mathbf{u})$, and C, C' depending on J and H only.

PROOF. In this proof, the constant C changes from line to line. Define the operators

$$\partial_J = \frac{\partial}{\partial s} - J\frac{\partial}{\partial t}, \quad \bar{\partial}_J = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t}.$$

Equation (3.6) can now be written as

$$\bar{\partial}_J u^k = -\nabla_g H(t,u^k) \stackrel{\text{def}}{=} f^k(s,t), \qquad \text{for all } k.$$

By the hypotheses on H and the fact that $|u^k| \le 1$ for all k we have that $f^k(s,t) \in L^\infty(\mathbb{R}^2)$. The latter follows from the fact that the solutions u^k can be regarded as functions on $\mathbb{R}^2$ via periodic extension in t of the collection $\mathbf{u} = \{u^k\}$. From the interior regularity estimates due to Douglis and Nirenberg for elliptic systems [**26**], and in particular the operators ∂_J and $\bar{\partial}_J$ we have the following L^p-estimates for functions $u \in W_0^{k+1,p}(B_1(0))$, $1 < p < \infty$, and $0 \le k$:

$$\|u\|_{W_0^{k+1,p}} \le C(p,J)\|\partial_J u\|_{W^{k,p}}, \quad \|u\|_{W_0^{k+1,p}} \le C(p,J)\|\bar{\partial}_J u\|_{W^{k,p}},$$

which also follow from the Calderon-Zygmund inequality for the Laplacian $\Delta = \bar{\partial}_J\partial_J = \partial_J\bar{\partial}_J$, see [**34**]. Using a partition of unity we derive the standard interior regularity estimates for the Cauchy-Riemann operator from the above interior estimates, e.g. [**62**]. Let $K \subset\subset G \subset \mathbb{R}^2$, with K, G compact domains, then

$$\|u\|_{W^{1,p}(K)} \le C(p,J,K,G)\Big(\|\bar{\partial}_J u\|_{L^p(G)} + \|u\|_{L^p(G)}\Big), \tag{3.9}$$

for $1 < p < \infty$. Indeed, let $\varepsilon < \text{dist}(K, \partial G)$, then the compact set K can be covered by balls $B_{\varepsilon/2}(x_0)$ for finitely many $x_0 \in K$. Furthermore, let $\{\omega_{\varepsilon,x_0}\}_{x_0}$ be a partition of unity of $\bigcup_{x_0} B_{\varepsilon/2}(x_0) \supset K$ subordinate to $\{B_\varepsilon(x_0)\}_{x_0}$. Recall that $\|fg\|_{L^p} \le \|f\|_{L^\infty}\|g\|_{L^p}$, which yields the estimate

$$\begin{aligned}
\|\omega_{\varepsilon,x_0} u\|_{W^{1,p}} &\le C\|\omega_{\varepsilon,x_0} u\|_{W_0^{1,p}} \\
&\le C\|\bar{\partial}_J(\omega_{\varepsilon,x_0} u)\|_{L^p} \\
&\le C\|\omega_{\varepsilon,x_0}\bar{\partial}_J u\|_{L^p} + C\|u\bar{\partial}_J\omega_{\varepsilon,x_0}\|_{L^p} \\
&\le C\|\bar{\partial}_J u\|_{L^p(G)} + C\|u\|_{L^p(G)}.
\end{aligned}$$

Since $\{\omega_{\varepsilon,x_0}\}_{x_0}$ is a partition of unity it follows that

$$\|u\|_{W^{1,p}(K)} = \Big\|\sum_{x_0}\omega_{\varepsilon,x_0}u\Big\|_{W^{1,p}(K)} \leq \sum_{x_0}\|\omega_{\varepsilon,x_0}u\|_{W^{1,p}(B_\varepsilon(x_0))},$$

which proves (3.9).

We apply these basic regularity estimates to the non-linear Cauchy-Riemann equation (3.5) to obtain a prirori estimates on the Hölder norm $\|u\|_{C^{1,\lambda}(\mathbb{R}^2)}$. Define the nested sets

$$G_T^j = [T-j, T+j+1]\times[-j, j+1], \quad j = 0,1,2.$$

Note that T just represents a shift in the (periodic) variable t. Hence, although T is arbitrary, the estimates will be independent of T. Choose $K = G_T^1 \subset\subset G_T^2 = G$. It holds that $\|f^k\|_{L^p(G_T^2)} \leq C_0(p,H)$, since $f^k \in L^\infty$. Similarly, $\|u^k\|_{L^p(G_T^2)} \leq C_0'(p)$ by the assumption that $|u^k| \leq 1$. Therefore,

$$\begin{aligned}
\|u^k\|_{W^{1,p}(G_T^1)} &\leq C(p,J)\Big(\|\bar\partial_J u^k\|_{L^p(G_T^2)} + \|u^k\|_{L^p(G_T^2)}\Big)\\
&= C(p,J)\Big(\|f^k\|_{L^p(G_T^2)} + \|u^k\|_{L^p(G_T^2)}\Big) \leq C_1(p,J,H) < \infty.
\end{aligned}$$

In order to further bootstrap the regularity of solutions we argue as follows. Recall that $\|fg\|_{W^{1,p}} \leq C\big(\|f\|_{W^{1,p}}\|g\|_{L^\infty} + \|g\|_{W^{1,p}}\|f\|_{L^\infty}\big)$. As before, on balls $B_\varepsilon(x_0)$ we obtain

$$\begin{aligned}
\|\omega_{\varepsilon,x_0}u\|_{W^{2,p}} &= \|\omega_{\varepsilon,x_0}u\|_{W_0^{2,p}} \leq C\|\bar\partial_J(\omega_{\varepsilon,x_0}u)\|_{W^{1,p}}\\
&\leq C\|\omega_{\varepsilon,x_0}\bar\partial_J u\|_{W^{1,p}} + C\|u\bar\partial_J\omega_{\varepsilon,x_0}\|_{W^{1,p}}\\
&\leq C\|\bar\partial_J u\|_{L^\infty(G)} + C\|\bar\partial_J u\|_{W^{1,p}(G)} + C\|u\|_{L^\infty(G)} + C\|u\|_{W^{1,p}(G)}.
\end{aligned}$$

Since $\{\omega_{\varepsilon,x_0}\}_{x_0}$ is a partition of unity we obtain the estimate

$$\|u\|_{W^{2,p}(K)} \leq C'(p,J,K,G)\Big(\|\bar\partial_J u\|_{W^{1,p}(G)} + \|u\|_{W^{1,p}(G)} + \|\bar\partial_J u\|_{L^\infty(G)} + \|u\|_{L^\infty(G)}\Big),$$

for compact domains $K \subset\subset G$, and $1 < p < \infty$. Now choose $K = G_T^0 \subset\subset G_T^1 = G$. In order to estimate the term $\|\bar\partial_J u^k\|_{W^{1,p}(G_T^1)}$ in the above inequality, observe that $\bar\partial_J u^k = -\nabla_g H(t,u^k)$. Then, by the $W^{1,p}$-interior estimates, the components

$$\begin{aligned}
g_1^k(s,t) &\stackrel{\text{def}}{=} \partial_s(-\nabla_g H(t,u^k)) = -d_{t,u}\nabla_g H(t,u^k)(0,u_s^k),\\
g_2^k(s,t) &\stackrel{\text{def}}{=} \partial_t(-\nabla_g H(t,u^k)) = -d_{t,u}\nabla_g H(t,u^k)(1,u_t^k),
\end{aligned}$$

both lie in $L^p(G_T^1)$, and thus also $g^k = (g_1^k, g_2^k)$ lies in $L^p(G_T^1)$. From the $W^{2,p}$-interior estimates it then follows that

$$\begin{aligned}
&\|u\|_{W^{2,p}(G_T^0)} \leq C'(p,J)\Big(\|g^k\|_{L^p(G_T^1)}\\
&\quad + \|u^k\|_{W^{1,p}(G_T^1)} + \|f^k\|_{L^\infty(G_T^1)} + \|u^1\|_{L^\infty(G_T^1)}\Big) \leq C_2(p,J,H) < \infty.
\end{aligned}$$

Additional regularity is obtained from the Sobolev embeddings [**1**, Theorem 4.12], $W^{2,p}(G_T^0) \to C^{1,\lambda}(\overline{G}_T^0)$, for $0 < \lambda \leq 1 - 2/p$. This yields the a priori estimate

$$\|\mathbf{u}\|_{C^{1,\lambda}(\mathbb{R}^2)} \leq C'(J,H). \tag{3.10}$$

The estimate for $\mathbb{R}^2$ makes use of the following fact. From the Sobolev embedding we derive that $\|\mathbf{u}\|_{C^{1,\lambda}([T,T+1]\times[d,d+1])} \le C'(J,H)$, and thus $\|\mathbf{u}\|_{C^{1,\lambda}(B_1(z))} \le C'(J,H)$, for any $z \in \mathbb{R}^2$. For functions ψ we have that

$$\begin{aligned}
&\sup_{z,z'\in\mathbb{R}^2} \frac{|\psi(z)-\psi(z')|}{|z-z'|^\lambda}\\
&\quad = \max\left\{ \sup_{|z-z'|\le 1} \frac{|\psi(z)-\psi(z')|}{|z-z'|^\lambda}, \sup_{|z-z'|\ge 1} \frac{|\psi(z)-\psi(z')|}{|z-z'|^\lambda} \right\}\\
&\quad \le \sup_{|z-z'|\le 1} \frac{|\psi(z)-\psi(z')|}{|z-z'|^\lambda} + 2\sup_{z\in\mathbb{R}^2} |\psi(z)|.
\end{aligned}$$

Consequently,

$$\|\mathbf{u}\|_{C^{1,\lambda}(\mathbb{R}^2)} \le \|\mathbf{u}\|_{C^{1,\lambda}(B_1(z))} + 2\|\nabla\mathbf{u}\|_{L^\infty(\mathbb{R}\times[0,1])} \le 3C'(J,H).$$

With these a priori estimates in place, we tackle the compactness assertion in the proposition. Let $\mathbf{u}_n$ be a sequence in $\mathcal{M}^{J,H}$. In view of the compactness of the embedding $C^{1,\lambda}(\mathbb{R}^2) \hookrightarrow C^{1,\lambda'}(K)$, for any compact domain $K \subset \mathbb{R}^2$, and $0 \le \lambda' < \lambda$, there exists a subsequence, again denoted by $\mathbf{u}_n$, and a function $\widehat{\mathbf{u}} \in C^{1,\lambda'}$, such that.

$$\mathbf{u}_n \longrightarrow \widehat{\mathbf{u}}, \qquad \text{in} \quad C^{1,\lambda'}(K), \quad \text{as} \quad n\to\infty.$$

The limit function $\widehat{\mathbf{u}}$ satisfies the equation $\bar{\partial}_{J,H}(\widehat{\mathbf{u}}) = 0$ and the periodicity condition (3.8), and therefore $\widehat{\mathbf{u}} \in \mathcal{M}^{J,H}$, which proves the compactness of the space of bounded trajectories $\mathcal{M}^{J,H}$.

Due to the a priori bound in $C^{1,\lambda}$ it holds for the action $\mathbf{f}_H$ that $|\mathbf{f}_H(\mathbf{u}(s,\cdot))| \le C(J,H)$. Since

$$\frac{d}{ds}\mathbf{f}_H(\mathbf{u}(s,\cdot)) = -\int_0^1 |\mathbf{u}_s|_g^2 dt \le 0,$$

it follows then that the limits $\lim_{s\to\pm\infty} \mathbf{f}_H(\mathbf{u}(s,\cdot)) = c_\pm$ exist and are a priori bounded by the same $C(J,H)$. Finally, for any $T_1, T_2 > 0$

$$\int_{-T_1}^{T_2}\int_0^1 |\mathbf{u}_s|_g^2 dtds = \sum_{k=1}^n \int_{-T_1}^{T_2}\int_0^1 |u_s^k|_g^2 dtds = \mathbf{f}_H(\mathbf{u}(T_2,\cdot)) - \mathbf{f}_H(\mathbf{u}(-T_1,\cdot)).$$

By the uniform boundedness of the action along all orbits $\mathbf{u} \in \mathcal{M}_{J,H}$ we obtain the estimate

$$\int_{\mathbb{R}}\int_0^1 |\mathbf{u}_s|_g^2 dtds \le C(J,H),$$

which completes our proof. □

REMARK 3.9. In the compactness proof in [**62**] a condition on the symplectic manifolds M is that

$$\int_{S^2} h^*\omega_0 = 0,$$

for any smooth mapping $h : S^2 \to M$. This property is used in the compactness proof for more general symplectic manifolds M. It implies that holomorphic maps $h : S^2 \to M$ must be constant due to the identity $\int_{S^2} h^*\omega_0 = \frac{1}{2}\int_{S^2} |\nabla h|^2$. This property is used in the blow-up analysis to obtain a priori bounds on $\nabla\mathbf{u}$. For comparison, if $M = \mathbb{D}^2$, we have that $\omega_0 = d\alpha_0$, and thus by Stokes Theorem

$$\int_{S^2} h^*\omega_0 = \int_{S^2} h^* d\alpha_0 = \int_{S^2} d(h^*\alpha_0) = 0.$$

Therefore, the topological condition on $\mathbb{D}^2$ is automatically satisfied.

REMARK 3.10. If we reverse time $s \mapsto -s$ and consider the conjugate action

$$\bar{f}_H(x) = \int_0^1 \alpha_0(x_t(t))dt - \int_0^1 H(t,x(t))dt = -f_H(x),$$

then the conjugate Cauchy-Riemann equations become

$$\partial_{J,H}(u) = u_s - Ju_t - \nabla_g H(t,u) = 0,$$

which is again a negative gradient flow equation, i.e. $\frac{d}{ds}\bar{f}_H(u(s,\cdot)) \leq 0$. We still assume that $J \in \mathcal{J}^+$. All results discussed here also hold for the conjugate equation, except for an occasional minus sign. We will point out these differences as we go along.

Additional compactness

Consider the non-autonomous Cauchy-Riemann equations

$$u_s + J(s)u_t + \nabla_g H(s,t,u) = 0, \tag{3.11}$$

where $s \mapsto J(s)$ is a smooth path in $\mathcal{J}^+$ and $s \mapsto H(s,\cdot,\cdot)$ is a smooth path in $\mathcal{H}$. Assume that $|H_s| \leq \kappa(s) \to 0$ as $s \to \pm\infty$ uniformly in $(t,x) \in \mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$, with $\kappa \in L^1(\mathbb{R})$. Moreover, both paths have the property that the limits $s \to \pm\infty$ exists. Choose the change of variable $u = \Phi(s)v$, where the path $s \mapsto \Phi(s)$ in $\mathrm{GL}(2,\mathbb{R})$ satisfies the identity $J(s)\Phi(s) = \Phi(s)J_0$. Such a smooth path in $\mathrm{GL}(2,\mathbb{R})$ can be chosen by solving the above matrix equation. Indeed, the almost complex structure $J(s)$ is given by

$$J(s) = \begin{pmatrix} a(s) & b(s) \\ c(s) & -a(s) \end{pmatrix}, \quad \text{with} \quad a^2 + bc = -1.$$

The space of matrices A satisfying the above matrix equation is given by

$$\Phi = \begin{pmatrix} \lambda a(s) + \mu & -\lambda + \mu a(s) \\ \lambda c(s) & \mu c(s) \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R},$$

and $\det(\Phi) = (\lambda^2 + \mu^2)\, c(s)$. By choosing λ and μ constant we obtain a path $s \mapsto \Phi(s)$ with $\det(\Phi(s)) > 0$, since $J(s)$ is a path in $\mathcal{J}^+$ and therefore $c(s) > 0$. The positivity of $\det(\Phi(s))$ will be used in Section 3.4. As matter of fact, since $J(s)$ has limits it holds that $0 < c_- \leq c(s) \leq c_+$, and the functions $b(s)$ and $a(s)$ are bounded as well.

The Cauchy-Riemann equations now transform to

$$v_s + J_0 v_t - \Phi^{-1}(s) J J_0 (\Phi^{-1}(s))^T \nabla \widehat{H}(s,t,v) = 0,$$

where $\widehat{H}(s,t,v) = H(s,t,\Phi(s)v)$. This equation is again of the form $\bar{\partial}_{J_0} v = f(s,t)$. In the new coordinates v is again a priori bounded in L^∞ and the compactness for Equation (3.11) follows from Proposition 3.8, using the bounds on $a(s)$, $b(s)$ and $c(s)$. Define $\mathbf{f}_H(s,x)$ as the action with Hamiltonian $H(s,\cdot,\cdot)$. The first variation with respect to s can be computed as in Section 3.1:

$$\begin{aligned}\frac{d}{ds}\mathbf{f}_H(s,\mathbf{u}(s,\cdot)) &= \frac{\partial \mathbf{f}_H}{\partial s} + \sum_{k=1}^{n}\int_0^1 \omega_0\big(u_t^k - X_H(t,u^k), u_s^k\big)dt \\ &= \frac{\partial \mathbf{f}_H}{\partial s} - \sum_{k=1}^{n}\int_0^1 |u_t^k - X_H(t,u^k)|_g^2 dt \\ &= \frac{\partial \mathbf{f}_H}{\partial s} - \int_0^1 |\mathbf{u}_s|_g^2 dt.\end{aligned}$$

The partial derivative with respect to s is given by

$$\frac{\partial \mathbf{f}_H}{\partial s} = \sum_{k=1}^{n}\int_0^1 \frac{\partial H}{\partial s}\Big(s,t,u^k(s,t)\Big)\,dt,$$

and $\left|\frac{\partial \mathbf{f}_H}{\partial s}\right| \le C\kappa(s) \to 0$ as $s \to \pm\infty$. For a non-stationary solution $\mathbf{u}$ it holds that $\int_0^1 |\mathbf{u}_s|_g^2 dt > 0$, and thus for $|s|$ sufficiently large $\frac{d}{ds}\mathbf{f}_H(s,\mathbf{u}(s,\cdot)) < 0$ which proves that the limits $\lim_{s\to\pm\infty}\mathbf{f}_H(s,\mathbf{u}(s,\cdot)) = c_\pm$ exist. Since $\kappa \in L^1(\mathbb{R})$ we also obtain the integral $\int_{\mathbb{R}}\int_0^1 |\mathbf{u}_s|_g^2 dtds \le C(J,H)$.

This non-autonomous version of the Cauchy-Riemann equations will be used in Section 3.8 to establish continuation for Floer homology.

3.4. Crossing numbers and a priori estimates

We start with an important property of the Cauchy-Riemann equations in dimension two. We consider Equation (3.5), or more generally Equation (3.6), and local solutions of the form $u : G \subset \mathbb{R}^2 \to \mathbb{R}^2$, where $G = [\sigma,\sigma'] \times [\tau,\tau']$. For two local solutions $u,u' : G \to \mathbb{R}^2$ of (3.5) assume that

$$u(s,t) \neq u'(s,t), \quad \text{for all } (s,t) \in \partial G.$$

Intersections of u and u', i.e. $u(s_0,t_0) = u'(s_0,t_0)$ for some $(s_0,t_0) \in G$, have special properties. Consider the difference function $w(s,t) = u(s,t) - u'(s,t)$, then by the assumptions on u and u' we have that $w|_{\partial G} \neq 0$, and intersections are given by $w(s_0,t_0) = 0$. The following lemma is a special property of Cauchy-Riemann equations in dimension two.

LEMMA 3.11. *Let u,u' and G be as defined above. Assume that $w(s_0,t_0) = 0$ for some $(s_0,t_0) \in G$, then (s_0,t_0) is an isolated zero and*

$$\deg(w,G,0) > 0.$$

PROOF. We start with deriving an equation for w. Since H is C^2 we can use Taylor expansions as follows:

$$\nabla_g H(t,u') = \nabla_g H(t,u) + R_1(t,u,u'-u)(u'-u),$$

where R_1 is a continuous function of its arguments. Upon substitution this gives

$$w_s + J(s)w_t + A(s,t)w = 0, \quad w(s_0,t_0) = 0,$$

where $A(s,t) = R_1(t,u,-w)$. The function $A(s,t)$ is continuous on G.

Define complex coordinates $z = s - s_0 + i(t - t_0)$. Then by [**39**, Appendix A.6], there exists a $\delta < 0$, sufficiently small, a disc $D_\delta = \{z \mid |z| \leq \delta\}$, a holomorphic map $h : D_\delta \to \mathbb{C}$, and a continuous mapping $\Phi : D_\delta \to \mathrm{GL}_{\mathbb{R}}(\mathbb{C})$ such that

$$\det\ \Phi(z) > 0, \qquad J(z)\Phi(z) = \Phi(z)i, \qquad w(z) = \Phi(z)h(z),$$

for all $z \in D_\delta$. Clearly, Φ can be represented by a real 2×2 matrix function of invertible matrices.

Since $w = \Phi h$, it holds that the condition $w(z_0) = 0$ implies that $h(z_0) = 0$. The analyticity of h then implies that either z_0 is an isolated zero in D_δ, or $h \equiv 0$ on D_δ. If the latter holds, then also $w \equiv 0$ on D_δ. If we repeat the above arguments we conclude that $w \equiv 0$ on G (compare analytic continuation), which is a contradiction with the boundary conditions. Therefore, all zeroes of w in G are isolated, and there are finitely many zeroes $z_i \in \mathrm{int}(G)$.

For the degree we have that, since $\det \Phi(z) > 0$,

$$\deg(w, G, 0) = \sum_{i=1}^{m} \deg(w, B_{\varepsilon_i}(z_i), 0) = \sum_{i=1}^{m} \deg(h, B_{\varepsilon_i}(z_i), 0),$$

and for an analytic function with an isolated zero z_i it holds that $\deg(h, B_{\varepsilon_i}(z_i), 0) = n_i \geq 1$, and thus $\deg(w, G, 0) > 0$. ☐

For a curve $\Gamma : I \to \mathbb{R}^2 \setminus \{(0,0)\}$, with I a bounded interval, we define the winding number about the origin by[3]

$$W(\Gamma, 0) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_I \Gamma^* \alpha = \frac{1}{2\pi} \int_\Gamma \alpha,$$

where $\alpha = \frac{-q dp + p dq}{p^2 + q^2}$ is a closed 1-form on $\mathbb{R}^2 \setminus \{(0,0)\}$. In particular, for curves $w(s, \cdot) : [\tau, \tau'] \to \mathbb{R}^2 \setminus \{(0,0)\}$, $s = \sigma, \sigma'$ we have the winding number

$$W(w(s, \cdot), 0) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{[\tau,\tau']} w^* \alpha = \frac{1}{2\pi} \int_w \alpha, \qquad \text{for } s = \sigma, \sigma'.$$

We denote these winding numbers by $W_\sigma^{[\tau,\tau']}(w)$ and $W_{\sigma'}^{[\tau,\tau']}(w)$ respectively. In the case that $[\tau, \tau'] = [0, 1]$ we simply write $W_\sigma(w) \stackrel{\text{def}}{=} W_\sigma^{[0,1]}(w)$. Similarly, we have winding numbers for the curves $w(\cdot, t) : [\sigma, \sigma'] \to \mathbb{R}^2 \setminus \{(0,0)\}$, $t = \tau, \tau'$, which we denote by $W_\tau^{[\sigma,\sigma']}(w)$ and $W_{\tau'}^{[\sigma,\sigma']}(w)$ respectively. The following lemma gives a relation between these (local) winding numbers and degree of the map $w : G \to \mathbb{R}^2$.

LEMMA 3.12. *Let $u, u' : G \to \mathbb{R}^2$ be local solutions of Equation* (3.5)*, with $w|_{\partial G} \neq 0$. Then*

$$\Big[W_{\sigma'}^{[\tau,\tau']}(w) - W_\sigma^{[\tau,\tau']}(w)\Big] - \Big[W_{\tau'}^{[\sigma,\sigma']}(w) - W_\tau^{[\sigma,\sigma']}(w)\Big] = \deg(w, G, 0). \tag{3.12}$$

In particular, for each zero $(s_0, t_0) \in \mathrm{int}(G)$, there exists an $\varepsilon_0 > 0$ such that

$$W_{s_0+\varepsilon}^{[\tau,\tau']}(w) - W_{s_0-\varepsilon}^{[\tau,\tau']}(w) > W_{\tau'}^{[s_0-\varepsilon, s_0+\varepsilon]}(w) - W_\tau^{[s_0-\varepsilon, s_0+\varepsilon]}(w),$$

[3]For closed curves $\Gamma \in \mathbb{C} \setminus \{0\}$ we have that $\frac{1}{2\pi i} \oint_\Gamma \frac{1}{z} dz = \frac{1}{2\pi} \int_\Gamma \alpha$. For a general path Γ, with starting point P and end point Q, we have

$$\frac{1}{2\pi i} \int_\Gamma \frac{1}{z} dz + \frac{1}{2\pi i} \log\Big(\frac{|P|}{|Q|}\Big) = \frac{1}{2\pi} \int_\Gamma \alpha.$$

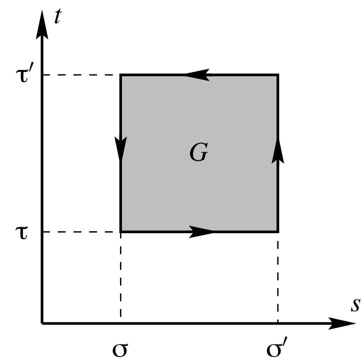

Figure 3.2: The contour around G.

for all $0 < \varepsilon \leq \varepsilon_0$.

PROOF. We abuse notation by regarding w as a map from the complex plane to itself. Let the contour $\gamma = \partial G$ be positively oriented, see Figure 3.2. The winding number of the contour $w(\gamma)$ about $0 \in \mathbb{C}$ in complex notation is given by

$$W(w(\gamma),0) = \frac{1}{2\pi i}\oint_{w(\gamma)} \frac{dz}{z} = \deg(w,G,0),$$

which is equal to the degree of $w : G \to \mathbb{R}^2$ with respect to the value 0. Using the special form of the contour γ we can write out the the Cauchy integral using the 1-form α:

$$\begin{aligned} \frac{1}{2\pi i}\oint_{w(\gamma)} \frac{dz}{z} &= \frac{1}{2\pi}\int_{w(\sigma',\cdot)} \alpha - \frac{1}{2\pi}\int_{w(\cdot,\tau')} \alpha - \frac{1}{2\pi}\int_{w(\sigma,\cdot)} \alpha + \frac{1}{2\pi}\int_{w(\cdot,\tau)} \alpha \\ &= \Big[W_{\sigma'}^{[\tau,\tau']}(w) - W_{\sigma}^{[\tau,\tau']}(w)\Big] - \Big[W_{\tau'}^{[\sigma,\sigma']}(w) - W_{\tau}^{[\sigma,\sigma']}(w)\Big], \end{aligned}$$

which proves the first statement.

As for the second statement we argue as follows. Lemma 3.11 states that all zeroes of w are isolated and have positive degree. Therefore, there exists an $\varepsilon_0 > 0$ such that $G_\varepsilon = [s_0 - \varepsilon, s_0 + \varepsilon] \times [\tau,\tau']$ contains no zeroes on the boundary, for all $0 < \varepsilon \leq \varepsilon_0$, from which the second statement follows. □

On the level of comparing two local solutions of Equation (3.5) the winding number behaves like a discrete Lyapunov function with respect to the time variable s. This can be further formalized for solutions of Cauchy-Riemann on $\overline{\Omega}^n$. For a closed braid $\mathbf{x} \in \Omega^n$, define the total crossing number

$$\mathrm{Cross}(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{k,k'} W\big(x^k - x^{k'},0\big) = 2\sum_{\substack{\{k,k'\}\\ k\neq k'}} W\big(x^k - x^{k'},0\big),$$

where the second sum is over all unordered pairs $\{k,k'\}$, using the fact that the winding number is invariant under the inversion $(p,q) \to (-p,-q)$. The number $\mathrm{Cross}(\mathbf{x})$ is equal to the total linking/self-linking number of all components in a closed braid $\mathbf{x}$. The local winding number as introduced above is not necessarily an integer. However, for closed curves the winding number is integer valued. We claim that the number $\mathrm{Cross}(\mathbf{x})$ as defined above is also an integer. One way to interpret Cross is via associated braid diagrams. One

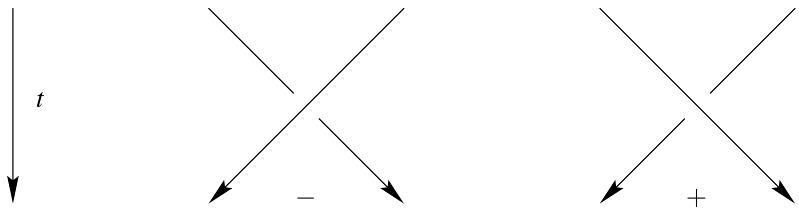

Figure 3.3: The time direction and the convention for negative and positive crossings.

can always project $\mathbf{x}$ onto a plane by projecting the coordinates (p,q) onto a line $L \subset \mathbb{R}^2$ and counting the number positive and negative crossings.

LEMMA 3.13. *The number* $\mathrm{Cross}(\mathbf{x})$ *is an integer, and*

$$\mathrm{Cross}(\mathbf{x}) \quad = \quad \#\ \textit{positive crossings} - \#\ \textit{negative crossings}$$

The (braid) crossing number is an invariant for a braid class, i.e. $\mathrm{Cross}(\mathbf{x}) = \mathrm{Cross}(\mathbf{x}')$ *for all* $\mathbf{x}, \mathbf{x}' \in [\mathbf{x}]$.

PROOF. The expression for $\mathrm{Cross}(\mathbf{x})$ is twice the sum of $\begin{pmatrix} n \\ 2 \end{pmatrix}$ local winding numbers. On the unordered pairs $\{k,k'\}$ there the exists the following equivalence relation. Two pairs $\{k,k'\}$ and $\{h,h'\}$ are equivalent if for some integer $d \geq 0$, $\{x^k(d), x^{k'}(d)\} = \{x^h(0), x^{h'}(0)\}$ as unordered pairs. The equivalence classes of unordered pairs $\{k,k'\}$ are denoted by π_j and the number of elements in π_j is denoted by $|\pi_j|$, see Figure 3.4. For each class π_j define $w_{\pi_j} = x^k - x^{k'}$, for some representative $\{k,k'\} \in \pi_j$. For $t \in [0, 2|\pi_j|]$, the functions $w_{\pi_j}(t)$ represent closed loops in $\mathbb{R}^2$ regardless of the choice of the representative $\{k,k'\} \in \pi_j$. Namely, note that $\{x^k(|\pi_j|), x^{k'}(|\pi_j|)\} = \{x^k(0), x^{k'}(0)\}$ as unordered pairs, which imlies that

$$x^k(|\pi_j|) = x^k(0) \qquad \text{or} \qquad x^k(|\pi_j|) = x^{k'}(0). \tag{3.13}$$

For the crossing number we have

$$\mathrm{Cross}(\mathbf{x}) = 2\sum_j \Big(\sum_{\{k,k'\}\in\pi_j} W\big(x^k - x^{k'}, 0\big) \Big) = 2\sum_j W^{[0,|\pi_j|]}(w_{\pi_j}, 0) = \sum_j W^{[0,|2\pi_j|]}(w_{\pi_j}, 0), \tag{3.14}$$

where the (outer) sum is over all equivalence classes π_j. For the final equality we have used (3.13) and the invariance of the winding number under the inversion $w \to -w$. Since the latter winding numbers are winding numbers for closed loops about 0 (linking numbers), they are all integers, and thus $\mathrm{Cross}(\mathbf{x})$ is an integer.

As for the expression in terms of positive and negative crossings we argue as follows. Upon inspection $W\big(x^k - x^{k'}, 0\big)$ equals all positive minus negative crossings between the two strands, see Figure 3.4. The invariance of $\mathrm{Cross}(\mathbf{x})$ with respect to $[\mathbf{x}]$ follows from the homotopy invariance of the winding number. □

Using the representation of the crossing number for a braid in terms of winding numbers, we can prove a Lyapunov property. Before stating the result, we introduce the *conjugate* crossing number as

$$\mathrm{Cross}^*(\mathbf{x}) = -\mathrm{Cross}(\mathbf{x}),$$

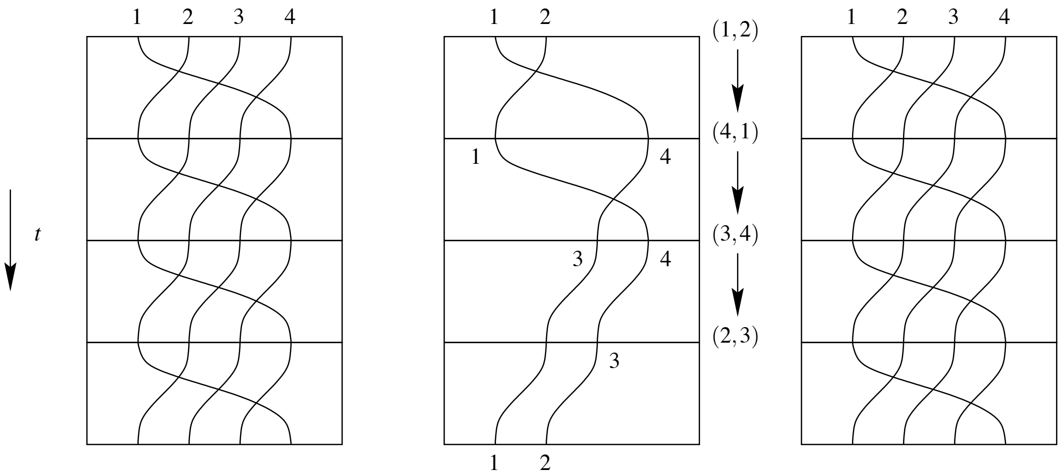

Figure 3.4: In this example the equivalence classes are $\pi_1 = \{(1,2),(4,1),(3,4),(2,3)\}$ and $\pi_2 = \{(1,3),(4,2)\}$. IS THE LAST PART OF THE FIGURE NEEDED?

which will be shown to be a *decreasing* Lyapunov function for the Cauchy-Riemann flow. Note that elements $\mathbf{u}$ of $\mathcal{M}$ are not necessarily in Ω^n for all s. Therefore $\mathrm{Cross}^*(\mathbf{u}(s,\cdot))$ is only well-defined whenever $\mathbf{u}(s,\cdot) \in \Omega^n$.

PROPOSITION 3.14. *Let* $\mathbf{u} \in \mathcal{M}$*, then* $\mathrm{Cross}^*(\mathbf{u}(s,\cdot))$*, where defined, is a non-increasing function as* $s \to \infty$*. To be more precise, if* $u^k(s_0,t_0) = u^{k'}(s_0,t_0)$ *for some* $(s_0,t_0) \in \mathbb{R}\times\mathbb{R}/\mathbb{Z}$*, and* $k \neq k'$*, then either there exists an* $\varepsilon_0 > 0$ *such that*

$$\mathrm{Cross}^*(\mathbf{u}(s_0-\varepsilon,\cdot)) > \mathrm{Cross}^*(\mathbf{u}(s_0+\varepsilon,\cdot)),$$

for all $0 < \varepsilon \le \varepsilon_0$*, or* $u^k \equiv u^{k'}$.

PROOF. Let $\mathbf{u} = \{u^k\} \in \mathcal{M}$, then $\mathrm{Cross}(\mathbf{u}(s,\cdot))$ is well-defined for all $s \in \mathbb{R}$ for which $\mathbf{u}(s,\cdot) \in \Omega^n$. As in the proof of Lemma 3.13 we define $w_{\pi_j}(s,t) = u^k(s,t) - u^{k'}(s,t)$ for some representative $\{k,k'\} \in \pi_j$. From the proof of Lemma 3.11 we know that (s_0,t_0) is either isolated, or $u^k \equiv u^{k'}$. In the case that (s_0,t_0) is an isolated zero there exists an $\varepsilon_0 > 0$, such that (s_0,t_0) is the only zero in $[s_0-\varepsilon, s_0+\varepsilon]\times[t_0-\varepsilon,t_0+\varepsilon]$, for all $0<\varepsilon\le\varepsilon_0$. By periodicity it holds that $w_{\pi_j}(s,t+|\pi_j|) = w_{\pi_j}(s,t)$, for all $(s,t)\in\mathbb{R}^2$, and therefore $W^{[\sigma,\sigma']}_{t_0-\varepsilon+|\pi_j|}(w_{\pi_j}) = W^{[\sigma,\sigma']}_{t_0-\varepsilon}(w_{\pi_j})$, for any $\sigma<\sigma'$. From Lemma 3.12 it then follows that

$$W^{[t_0-\varepsilon,t_0-\varepsilon+|\pi_j|]}_{s_0+\varepsilon}(w_{\pi_j}) > W^{[t_0-\varepsilon,t_0-\varepsilon+|\pi_j|]}_{s_0-\varepsilon}(w_{\pi_j}),$$

and since these terms make up the expression for $\mathrm{Cross}^*(\mathbf{u}(s,\cdot))$ in Equation (3.14), we obtain the desired inequality. □

From Lemma 3.12 we can also derive the following a priori estimate for solutions of the Cauchy-Riemann equations.

PROPOSITION 3.15. *Let* $u: G \to \mathbb{D}^2$ *be a local solution of Equation* (3.5)*, then either*

$$|u(s,t)| = 1, \qquad \text{or} \qquad |u(s,t)| < 1,$$

for all $(s,t) \in G$. *In particular, solutions* $\mathbf{u} \in \mathcal{M}$ *have the property that components* u^k *either lie entirely on* $\partial\mathbb{D}^2$*, or entirely in the interior of* $\mathbb{D}^2$.

PROOF. By the hypotheses (h3) the boundary of the disc is invariant for X_H, and thus consists of a solutions $x(t)$, with $|x(t)| = 1$. Assume that $u(s_0,t_0) = x(t_0)$ for some (s_0,t_0) and some boundary trajectory $x(t)$. For convenience, we write $u'(s,t) = x(t)$, and we consider the difference $w(s,t) = u'(s,t) - u(s,t) = x(t) - u(s,t)$. By the arguments presented in the proof of Lemma 3.11, we know that either all zeroes of w are isolated, or $w \equiv 0$. In the latter case $u \equiv x$, hence $|u(s,t)| \equiv 1$. We consider the remaining possibility, namely that (s_0,t_0) is an isolated zero of w, and we show that it leads to a contradiction.

We can choose a rectangle $G = [\sigma,\sigma'] \times [\tau,\tau']$ containing (s_0,t_0), such that $w|_{\partial G} \neq 0$. From Lemma 3.12 we have that, with $\gamma = \partial G$ positively oriented,

$$W(w(\gamma),0) = \deg(w,G,0) \geq 1.$$

The latter is due to the assumption that G contains a zero. Consider on the other hand the loops $u(\gamma)$ and $u'(\gamma)$. By assumption

$$|(u'-w)(\gamma)| = |u(\gamma)| < |u'(\gamma)| = 1.$$

If we now apply the 'Dog-on-a-Leash' Lemma[4] from the theory of winding numbers, we conclude that

$$1 \leq W(w(\gamma),0) = W(u'(\gamma),0) = 0,$$

which contradicts the assumption that u touches $\partial\mathbb{D}^2$. Hence $|u(s,t)| < 1$ for all (s,t). □

As a consequence of this proposition we have following result for connecting orbit spaces. For $\mathbf{x}_\pm \in \mathrm{Crit}(\overline{\Omega}^n)$, define

$$\mathcal{M}_{\mathbf{x}_-,\mathbf{x}_+} = \big\{\mathbf{u} \in \mathcal{M} \mid \lim_{s\to\pm\infty} \mathbf{u}(s,\cdot) = \mathbf{x}_\pm\big\}.$$

COROLLARY 3.16. *For* $\mathbf{u} \in \mathcal{M}_{\mathbf{x}_-,\mathbf{x}_+}$*, with* $|\mathbf{x}_\pm| < 1$*, it holds that*

$$|\mathbf{u}(s,t)| < 1,$$

for all $(s,t) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$.

This is an isolating property of the connecting orbit spaces that plays an important role later on in the definition of Floer homology.

REMARK 3.17. To get a sense for the evolution of the (conjugate) crossing number, consider the linear Cauchy-Riemann equations,

$$u_s + iu_t + 2\pi u = 0,$$

where we identify u with $p+iq \in \mathbb{C}$. Consider the solutions $u^1(s,t) = e^{-2\pi s}z_0 + e^{2\pi it}$, $z_0 \neq 0$, and $u^2 \equiv 0$, and the braid $\mathbf{u} = \{u^1,u^2\}$. Then the conjugate crossing number $\mathrm{Cross}^*(\mathbf{u})$ decreases from 0 as $s \to -\infty$ to -2 as $s \to \infty$. Similarly, for the solution $u^1 = e^{-4\pi s-2\pi it}$ and $u^2 = e^{2\pi s+4\pi it}$ we have that $\mathrm{Cross}^*(\mathbf{u})$ is decreasing from 2 to -4. Finally, when $u^1 = e^{-\pi s+\pi it}$ and $u^2 = -e^{-\pi s+\pi it}$, then $\mathrm{Cross}^*(\mathbf{u}) = -1$ for all s.

[4]The 'Dog-on-a-Leash' Lemma [**31**] can be viewed as an extension Rouché's theorem in the analytic case, and states that if two closed paths $\Gamma(t)$ — dog — and $\Gamma'(t)$ — walker — in $\mathbb{R}^2$, parameterized over $t \in I$, have the property $|\Gamma'(t)-\Gamma(t)| < |\Gamma'(t)-P|$ — leash is shorter then the walkers distance to the pole P — then $W(\Gamma,P) = W(\Gamma',P)$. Here we set $P = (0,0)$, $\Gamma = (u'-u)(\gamma) = w(\gamma)$, and $\Gamma' = u'(\gamma)$.

3.5. Relative braids

Given two braids $\mathbf{x} \in \Omega^n$, $\mathbf{y} \in \Omega^m$, one can define their union $\mathbf{x} \cup \mathbf{y} \in \overline{\Omega}^{n+m}$, as the union of all strands in $\mathbf{x}$ and $\mathbf{y}$. If the collection of all strands $\mathbf{x} \cup \mathbf{y} = \{x^k\} \cup \{y^\ell\}$ satisfies the conditions in Definition 3.2, then $\mathbf{x} \cup \mathbf{y} \in \Omega^{n+m}$ and $\mathbf{x} \cup \mathbf{y}$ is a braid. The crossing number $\mathrm{Cross}(\mathbf{x} \cup \mathbf{y})$ counts all crossings between strands in $\mathbf{x}$, strands in $\mathbf{y}$, and crossings between strands in $\mathbf{x}$ and $\mathbf{y}$.

The reason to consider braids with a split into two sub-braids is dictated by the application to the Hamilton equations Equation (3.1). One can think of $\mathbf{y}$ as a stationary braid of (3.1), i.e. these strands $\mathbf{y}$ are periodic solutions of the Hamilton equations. Now we try to find new solutions $\mathbf{x}$ that are weaved through $\mathbf{y}$ in a certain way. Question: Can $\mathbf{y}$, called a *skeleton*, force additional solutions? This leads us to the following definitions.

Relative braid classes and components

Consider the space

$$\Omega^{n,m} = \big\{(\mathbf{x},\mathbf{y}) \in \Omega^n \times \Omega^m \mid \mathbf{x} \cup \mathbf{y} \in \Omega^{n+m}\big\}.$$

In particular, for $(\mathbf{x},\mathbf{y}) \in \Omega^{n,m}$ it holds that $\mathbf{x} \in \Omega^n$ and $\mathbf{y} \in \Omega^m$. On $\Omega^{n,m}$ define the projection

$$\pi : \Omega^{n,m} \to \Omega^m, \quad (\mathbf{x},\mathbf{y}) \mapsto \mathbf{y}.$$

The projection π is surjective.

DEFINITION 3.18. The path connected components of $\Omega^{n,m}$ are called *relative braid classes* and are denoted by $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$. The elements $(\mathbf{x},\mathbf{y})$ in $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ are called relative braids and are usually denoted by $\mathbf{x}$ rel $\mathbf{y}$. For a given $\mathbf{y}' \in \pi\big([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]\big)$, the fiber

$$[\mathbf{x}'] \text{ rel } \mathbf{y}' = \pi^{-1}|_{[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]}(\mathbf{y}')$$

is called a *relative braid class with fixed skeleton* $\mathbf{y}'$. The fibers $[\mathbf{x}]$ rel $\mathbf{y}$ are not necessarily path connected. For any given $\mathbf{y} \in \Omega^m$, the fiber $\pi^{-1}(\mathbf{y}) = \Omega^n \text{ rel } \mathbf{y}$ denotes the space of relative braid classes with fixed skeleton $\mathbf{y}$.

Relative braid classes have the property that two relative braids $\mathbf{x} \text{ rel } \mathbf{y}, \mathbf{x}' \text{ rel } \mathbf{y} \in [\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ lie in the same $[\mathbf{x}]$ rel $\mathbf{y}$ if they lie in the same path component in $[\mathbf{x}]$ rel $\mathbf{y}$, but also, more generally, if there exists a continuous path $(\mathbf{x}(s),\mathbf{y}(s))$ in $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$, for all $s \in [0,1]$, and $\mathbf{x}(0) \text{ rel } \mathbf{y}(0) = \mathbf{x} \text{ rel } \mathbf{y}$ and $\mathbf{x}(1) \text{ rel } \mathbf{y}(1) = \mathbf{x}' \text{ rel } \mathbf{y}$.

This can also be characterized in terms of the invariants $\mathrm{Cross}(\mathbf{x})$, $\mathrm{Cross}(\mathbf{y})$, and $\mathrm{Cross}(\mathbf{x} \cup \mathbf{y})$. A fourth dependent invariant can added to the list:

$$\mathrm{Cross}(\mathbf{x},\mathbf{y}) = \mathrm{Cross}(\mathbf{x} \cup \mathbf{y}) - \mathrm{Cross}(\mathbf{x}) - \mathrm{Cross}(\mathbf{y}),$$

and is called the *relative crossing number*.

For the purpose of defining invariants for $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ we need to understand the closure of fiber Ω^n rel $\mathbf{y}$. As before this is achieved by allowing braids $\mathbf{x} \cup \mathbf{y}$ which do not necessarily satisfy Condition (ii) of Definition 3.2, and the closure is denoted by $\overline{\Omega}^n$ rel $\mathbf{y}$. The singular braids are

$$\Sigma^n \text{ rel } \mathbf{y} = \overline{\Omega}^n \text{ rel } \mathbf{y} \backslash \Omega^n \text{ rel } \mathbf{y},$$

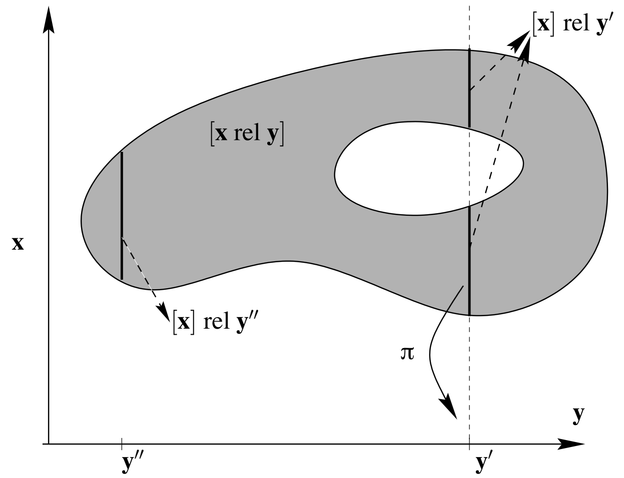

Figure 3.5: The relative braid classes $[\mathbf{x}]$ rel $\mathbf{y}$ and $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$.

and in particular contain the set Σ^n. The condition that $\mathbf{x} \cup \mathbf{y}$ is a braid, puts additional restriction on Ω^n, so that Ω^n rel $\mathbf{y}$ can be regarded as 'putting up' additional walls (the set Σ^n rel $\mathbf{y} \backslash \Sigma^n$). The collapsed relative braids can be defined as

$$\Sigma^n_- \text{ rel } \mathbf{y} \stackrel{\text{def}}{=} \Sigma^{n+m}_- \cap \big(\Sigma^n \text{ rel } \mathbf{y}\big),$$

which are singular braids for which $x^k(t) \equiv x^{k'}(t)$, $k \neq k'$, or $x^k(t) \equiv y^\ell(t)$, for all $t \in [0,1]$. This leads to the following essential definition.

DEFINITION 3.19. A relative braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ is called *proper* if

(i) for any $\mathbf{x}'$ rel $\mathbf{y} \in \mathrm{cl}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big)$ it holds that $|x^k(t)| \not\equiv 1$ for any strand;

(ii) $\mathrm{cl}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big) \cap \big(\Sigma^n_- \text{ rel } \mathbf{y}\big) = \varnothing$,

for any $\mathbf{y} \in \pi\big([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]\big)$ supported in $\mathrm{int}(\mathbb{D}^2)$. The 'closure' is with respect to the topology described in Definition 3.2. A relative braid $\mathbf{x}$ rel $\mathbf{y}$ in a proper braid class is called a proper braid.

An intuitive way of looking at this definition is that a braid class is proper when strands in $\mathbf{x}$ *cannot* collapse on each other, *nor* on strands of $\mathbf{y}$, *nor* on the boundary $\mathbb{D}^2 \times [0,1]$. Moreover, this has to be the case for *any* fiber in $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$. Note that braid classes $[\mathbf{x}]$ are never proper.

Isolation for proper relative braid classes

For proper braid classes $[\mathbf{x}]$ rel $\mathbf{y}$ introduced in the previous section, the Cauchy-Riemann equations have special properties. The most important property is that proper braid classes 'isolate' the set of bounded solutions of Cauchy-Riemann inside a relative braid class. Before stating the main result of this section, let us first introduce some notation. Following Floer **[29]** we define the set of bounded solutions inside a proper relative braid class $[\mathbf{x}]$ rel $\mathbf{y}$:

$$\mathcal{M}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big) \stackrel{\text{def}}{=} \Big\{\mathbf{u} \in \mathcal{M}(\overline{\Omega}^n) \mid \mathbf{u}(s,\cdot) \in [\mathbf{x}] \text{ rel } \mathbf{y},\ \forall\, s \in \mathbb{R}\Big\}.$$

We are also interested in the paths traversed (as a function of s) by these bounded solutions in phase space, hence we define, since the Cauchy-Riemann equations (3.5) are autonomous when seen as a flow in s,

$$\mathcal{S}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big) \stackrel{\text{def}}{=} \Big\{\mathbf{x} = \mathbf{u}(0,\cdot) \mid \mathbf{u} \in \mathcal{M}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big)\Big\}.$$

If there is no ambiguity about the relative braid class we simply write $\mathcal{S}$. We recall that $\mathcal{M}$ carries the $C^1_{\text{loc}}(\mathbb{R}\times[0,1];\mathbb{D}^2)^n$ topology, while $\mathcal{S}$ is endowed with the $C^1([0,1],\mathbb{D}^2)^n$ topology.

PROPOSITION 3.20. *For any proper relative braid class* $[\mathbf{x}]$ rel $\mathbf{y}$ *the set* $\mathcal{M}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big)$ *is compact, and* $\mathcal{S}$ *is a compact isolated set in* $[\mathbf{x}]$ rel $\mathbf{y}$*, i.e. (i)* $|\mathbf{u}(s,t)| < 1$*, for all* s,t *and (ii)* $\mathbf{u}(s,\cdot)\cap\Sigma^n$ rel $\mathbf{y} = \varnothing$*, for all* s*.*

PROOF. The set $\mathcal{M}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big)$ is contained in the compact set $\mathcal{M}(\overline{\Omega}^n)$ (Proposition 3.8). Let $\{\mathbf{u}_m\}\subset\mathcal{M}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big)$ be a sequence, then for any compact interval I, the limit $\mathbf{u} = \lim_{m'\to\infty}\mathbf{u}_{m'}$ lies in $\mathcal{M}(\overline{\Omega}^n)$ and has the property that $\mathbf{u}(s,\cdot)\in \text{cl}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big)$, for all $s\in I$. We will show now that $\mathbf{u}(s,\cdot)$ is in the relative braid class $[\mathbf{x}]$ rel $\mathbf{y}$, by eliminating the possible boundary behaviors.

If $|u^k(s_0,t_0)| = 1$, for some (s_0,t_0), and k, then Proposition 3.15 implies that $|u^k|\equiv 1$, hence $|u^k_{m'}|\to 1$ as $m'\to\infty$ uniformly on compact sets in (s,t). This contradicts the fact that $[\mathbf{x}]$ rel $\mathbf{y}$ is proper, and therefore the limit satisfies $|\mathbf{u}| < 1$.

If $u^k(s_0,t_0) = u^{k'}(s_0,t_0)$ for some (s_0,t_0) and some pair $\{k,k'\}$, then by Proposition 3.14 either $\text{Cross}^*(\mathbf{u}(s_0-\varepsilon,\cdot)) > \text{Cross}^*(\mathbf{u}(s_0+\varepsilon,\cdot))$, for some $0<\varepsilon\le\varepsilon_0$, or $u^k\equiv u^{k'}$. The former case will be dealt with a little later, while in the latter case $\mathbf{u}\in\Sigma^n_-$ rel $\mathbf{y}$, contradicting that $\mathbf{x}$ rel $\mathbf{y}$ is proper as before.

If $u^k(s_0,t_0) = y^\ell(t_0)$ for some (s_0,t_0) and k and $y^\ell\in\mathbf{y}$, then by Proposition 3.14 either $\text{Cross}^*(\mathbf{u}(s_0-\varepsilon,\cdot)\cup\mathbf{y}) > \text{Cross}^*(\mathbf{u}(s_0+\varepsilon,\cdot)\cup\mathbf{y})$, for some $0<\varepsilon\le\varepsilon_0$, or $u^k\equiv y^\ell$. Again, the former case will be dealt with below, while in the latter case $\mathbf{u}\in\Sigma^n_-$ rel $\mathbf{y}$, contradicting that $\mathbf{x}$ rel $\mathbf{y}$ is proper.

Finally, the two statements about the conjugate crossing numbers imply that both $\mathbf{u}(s_0-\varepsilon,\cdot),\mathbf{u}(s_0+\varepsilon,\cdot)\in\Omega^n$ rel $\mathbf{y}$, and thus $\mathbf{u}(s_0-\varepsilon,\cdot),\mathbf{u}(s_0+\varepsilon,\cdot)\in[\mathbf{x}]$ rel $\mathbf{y}$. On the other hand, since at least one crossing number at $s_0-\varepsilon$ has strictly decreased at $s_0+\varepsilon$, the braids $\mathbf{u}(s_0-\varepsilon,\cdot)$ and $\mathbf{u}(s_0+\varepsilon,\cdot)$ cannot belong to the same relative braid class, which is a contradiction. As a consequence $\mathbf{u}(s,\cdot)$ rel $\mathbf{y}\in[\mathbf{x}]$ rel $\mathbf{y}$ for all s, which proves that $\mathcal{M}\big([\mathbf{x}] \text{ rel } \mathbf{y}\big)$ is compact, and therefore also $\mathcal{S}\subset[\mathbf{x}]$ rel $\mathbf{y}$ is compact. □

3.6. The Maslov index for braids and Fredholm theory

The action $\mathbf{f}_H$ defined on $\overline{\Omega}^n$ has the property that stationary braids have a doubly unbounded spectrum, i.e., if we consider the $d^2\mathbf{f}_H(\mathbf{x})$ at a stationary braid $\mathbf{x}$, then $d^2\mathbf{f}_H(\mathbf{x})$ is a self-adjoint operator whose (real) spectrum consists of isolated eigenvalues and is not bounded from above nor below. The classical Morse index for stationary braids is therefore not well-defined. The theory of the Maslov index for Lagrangian subspaces can be used to replace the classical Morse index [**29, 60, 59**]; in combination with Fredholm theory the Maslov index will play the same role the Morse index.

The Maslov index

Let (E,ω) be a (real) symplectic vector space of dimension $\dim E = 2n$, with compatible almost complex structure $J \in \mathrm{Sp}^+(E,\omega)$. An n-dimensional subspace $V \subset E$ is called *Lagrangian* if $\omega(v,v') = 0$ for all $v,v' \in V$. Denote the space of Lagrangian subspaces of (E,ω) by $\mathcal{L}(E,\omega)$, or $\mathcal{L}$ for short.

LEMMA 3.21. *A subspace $V \subset E$ is Lagrangian if and only if $V = range(X)$ for some linear map $X : W \to E$ of rank n and some n-dimensional (real) vector space W, with X satisfying*

$$X^T JX = 0, \tag{3.15}$$

where the transpose is defined via the inner product $\langle\cdot,\cdot\rangle \stackrel{\text{def}}{=} \omega(\cdot,J\cdot)$.

PROOF. Let $V = [v_1,\cdots,v_n]$ which yields a map $X : \mathbb{R}^n \to E$ of rank n such that $V = X(\mathbb{R}^n)$. This establishes that any n-dimensional subspace is of the form $X(W)$. Let $V = X(W)$ and suppose V is Lagrangian. Then $\omega\big(Xw,Xw'\big) = 0$ for all $w,w' \in W$. It holds that

$$\omega\big(Xw,Xw'\big) = \big\langle Xw,-JXw'\big\rangle = \big\langle w,-X^T JX\, w'\big\rangle = 0, \ \ \forall w,w' \in W,$$

which implies that $X^T JX = 0$.

Conversely, if $X : W \to E$ is given and satisfies $X^T JX = 0$ then $\big\langle w,-X^T JX\, w'\big\rangle = 0$ for all $w,w' \in W$ and $V = X(W)$ is Lagrangian by retracing the steps above. □

The map X is called a *Lagrangian frame* for V. If we restrict to the special case $(E,\omega) = (\mathbb{R}^{2n},\overline{\omega}_0)$, with standard $J_0 \in \mathcal{J}^+$, then for a point x in $\mathbb{R}^{2n}$ one can choose symplectic coordinates $x = (p^1,\cdots,p^n,q^1,\cdots,q^n)$ and the standard symplectic form is given by $\overline{\omega}_0 = dp^1 \wedge dq^1 + \cdots + dp^n \wedge dq^n$, see Section 3.3. In this case a subspace $V \subset \mathbb{R}^{2n}$ is Lagrangian if $X = \begin{pmatrix} P \\ Q \end{pmatrix}$, with P,Q $n\times n$ matrices satisfying $P^T Q = Q^T P$, and X has rank n. The condition on P and Q follows immediately from Equation (3.15).

For any fixed $V \in \mathcal{L}$, the space $\mathcal{L}$ can be decomposed in strata $\Xi_k(V)$:

$$\mathcal{L} = \bigcup_{k=0}^{n} \Xi_k(V).$$

The strata $\Xi_k(V)$ of Lagrangian subspaces V' which intersect V in a subspace of dimension k are submanifolds of co-dimension $k(k+1)/2$. The *Maslov cycle* is defined as

$$\Xi(V) = \bigcup_{k=1}^{n} \Xi_k(V).$$

Let $\Lambda(t)$ be a smooth curve of Lagrangian subspaces and $X(t)$ a smooth Lagrangian frame for $\Lambda(t)$. A crossing is a number t_0 such that $\Lambda(t_0) \in \Xi(V)$, i.e., $X(t_0)w = v \in V$, for some $w \in W$, $v \in V$. For a curve $\Lambda : [a,b] \to \mathcal{L}$, the set of crossings in compact, and for each crossing $t_0 \in [a,b]$ we can define the crossing form on $\Lambda(t_0) \cap V$:

$$\Gamma(\Lambda,V,t_0)(v) \stackrel{\text{def}}{=} \omega\big(X(t_0)w,X'(t_0)w\big).$$

A crossing t_0 is called regular if Γ is a nondegenerate form. If $\Lambda : [a,b] \to \mathcal{L}$ is a Lagrangian curve that has only regular crossings then the *Maslov index* of the pair (Λ,V) is defined by

$$\mu(\Lambda,V) = \frac{1}{2}\mathrm{sign}\ \Gamma(\Lambda,V,a) + \sum_{a<t_0<b} \mathrm{sign}\ \Gamma(\Lambda,V,t_0) + \frac{1}{2}\mathrm{sign}\ \Gamma(\Lambda,V,b),$$

where $\Gamma(\Lambda,V,a)$ and $\Gamma(\Lambda,V,b)$ are zero when a or b are not crossings. The notation 'sign' is the signature of a quadratic form, i.e. the number of positive minus the number of negative eigenvalues and the sum is over the crossings $t_0\in(a,b)$. Since the Maslov index is homotopy invariant and every path is homotopic to a regular path the above definition extends to arbitrary continuous Lagrangian paths, using property (iii) below. In the special case of $(\mathbb{R}^{2n},\overline{\omega}_0)$ we have that

$$\begin{aligned}\Gamma(\Lambda,V,t_0)(v) &= \overline{\omega}_0\big(X(t_0)w,X'(t_0)w\big)\\ &= \langle P(t_0)w,Q'(t_0)w\rangle-\langle P'(t_0)w,Q(t_0)w\rangle.\end{aligned}$$

A list of properties of the Maslov index can be found (and is proved) in **[59]**, of which we mention the most important ones:

(i) for any $\Psi\in\mathrm{Sp}(E)$, $\mu(\Psi\Lambda,\Psi V)=\mu(\Lambda,V)$;[5]
(ii) for $\Psi:[a,b]\to\mathcal{L}$ it holds that $\mu(\Lambda,V)=\mu(\Lambda|_{[a,c]},V)+\mu(\Lambda|_{[c,b]},V)$, for any $a<c<b$;
(iii) two paths $\Lambda_0,\Lambda_1:[a,b]\to\mathcal{L}$ with the same end points are homotopic if and only if $\mu(\Lambda_0,V)=\mu(\Lambda_1,V)$;
(iv) for any path $\Lambda:[a,b]\to\Xi_k(V)$ it holds that $\mu(\Lambda,V)=0$.

The same can be carried out for pairs of Lagrangian curves $\Lambda,\Lambda^\dagger:[a,b]\to\mathcal{L}$. The crossing form on $\Lambda(t_0)\cap\Lambda^\dagger(t_0)$ is then given by

$$\Gamma(\Lambda,\Lambda^\dagger,t_0)\stackrel{\text{def}}{=}\Gamma(\Lambda,\Lambda^\dagger(t_0),t_0)-\Gamma(\Lambda^\dagger,\Lambda(t_0),t_0).$$

For pairs $(\Lambda,\Lambda^\dagger)$ with only regular crossings the Maslov index $\mu(\Lambda,\Lambda^\dagger)$ is defined in the same way as above using the crossing form for Lagrangian pairs. By setting $\Lambda^\dagger(t)\equiv V$ we retrieve the previous case, and $\Lambda(t)\equiv V$ yields $\Gamma(V,\Lambda^\dagger,t_0)=-\Gamma(\Lambda^\dagger,V,t_0)$. Consider the symplectic space $(\overline{E},\overline{\omega})=(E\times E,(-\omega)\times\omega)$, with almost complex structure $(-J)\times J$. A crossing $\Lambda(t_0)\cap\Lambda^\dagger(t_0)\neq\varnothing$ is equivalent to a crossing $(\Lambda\times\Lambda^\dagger)(t_0)\in\Xi(\Delta)$, where $\Delta\subset\overline{E}$ is the diagonal Lagrangian plane, and $\Lambda\times\Lambda^\dagger$ a Lagragian curve in $\overline{E}$, which follows from Equation (3.15) using the Lagrangian frame $\overline{X}(t)=\begin{pmatrix}X(t)\\X^\dagger(t)\end{pmatrix}$. Let $\overline{v}=(v,v)=\overline{X}(t_0)w$, then

$$\begin{aligned}\Gamma(\Lambda\times\Lambda^\dagger,\Delta,t_0)(\overline{v}) &= \overline{\omega}\big(\overline{X}(t_0)w,\overline{X'}(t_0)w\big)\\ &= -\omega\big(X(t_0)w,X'(t_0)w\big)+\omega\big(X^\dagger(t_0)w,X^{\dagger\prime}(t_0)w\big)\\ &= -\Gamma(\Lambda,\Lambda^\dagger(t_0),t_0)(v)+\Gamma(\Lambda^\dagger,\Lambda(t_0),t_0)(v).\end{aligned}$$

This justifies the identity

$$\mu(\Lambda,\Lambda^\dagger)=\mu(\Delta,\Lambda\times\Lambda^\dagger). \tag{3.16}$$

Equation (3.16) is used to define the Maslov index for continuous pairs of Lagrangian curves, and is a special case of the more general formula below. Let $\Psi:[a,b]\to\mathrm{Sp}(E)$ be a symplectic curve, then

$$\mu(\Psi\Lambda,\Lambda^\dagger)=\mu(\mathrm{gr}(\Psi),\Lambda\times\Lambda^\dagger), \tag{3.17}$$

where $\mathrm{gr}(\Psi)=\{(x,\Psi x)\mid x\in E\}$ is the graph of Ψ. The curve $\mathrm{gr}(\Psi)(t)$ is a Lagrangian curve in $(\overline{E},\overline{\omega})$ and $X_\Psi(t)=\begin{pmatrix}\mathrm{Id}\\\Psi(t)\end{pmatrix}$ is a Lagrangian frame for $\mathrm{gr}(\Psi)$. Indeed, via (3.15)

[5] This property shows that we can assume E to be the standard symplectic space without loss of generality.

we have

$$\begin{pmatrix} \mathrm{Id} & \Psi^T(t) \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ \Psi(t) \end{pmatrix} = \Psi^T(t) J \Psi(t) - J = 0,$$

which proves that $\mathrm{gr}(\Psi)(t)$ is a Lagrangian curve in $\overline{E}$. Via $\overline{E} \times \overline{E}$ the crossing form is given by

$$\Gamma\big(\mathrm{gr}(\Psi), \Lambda \times \Lambda^\dagger, t_0\big) = \Gamma\big(\mathrm{gr}(\Psi), (\Lambda \times \Lambda^\dagger)(t_0), t_0\big) - \Gamma\big(\Lambda \times \Lambda^\dagger, \mathrm{gr}(\Psi)(t_0), t_0\big).$$

and upon inspection consists of the three terms making up the crossing form of $(\Psi\Lambda, \Lambda^\dagger)$ in $\overline{E}$. More specifically, let $\xi = X_\Psi(t_0)\xi_0 = \overline{X}(t_0)\eta_0 = \eta$, so that $\Psi X \eta_0 = \Psi \xi_0 = X^\dagger \eta_0$, which yields

$$\begin{aligned} \Gamma\big(\mathrm{gr}(\Psi), (\Lambda \times \Lambda^\dagger)(t_0), t_0\big)(\xi) &= \omega(\Psi(t_0)\xi_0, \Psi'(t_0)\xi_0) \\ &= \omega(\Psi(t_0)X(t_0)\eta_0, \Psi'(t_0)X(t_0)\eta_0), \end{aligned}$$

and

$$\begin{aligned} &\Gamma\big(\Lambda \times \Lambda^\dagger, \mathrm{gr}(\Psi)(t_0), t_0\big)(\eta) \\ &= -\omega\big(X(t_0)\eta_0, X'(t_0)\eta_0\big) + \omega\big(X^\dagger(t_0)\eta_0, X^{\dagger\prime}(t_0)\eta_0\big) \\ &= -\omega\big(\Psi(t_0)X(t_0)\eta_0, \Psi(t_0)X'(t_0)\eta_0\big) + \omega\big(X^\dagger(t_0)\eta_0, X^{\dagger\prime}(t_0)\eta_0\big) \end{aligned}$$

which proves Equation (3.17). The crossing form for a more general Lagrangian pair of the form $(\mathrm{gr}(\Psi), \Lambda)$, where $\overline{\Lambda}(t)$ is a Lagrangian curve in $\overline{E}$, is given by $\Gamma\big(\mathrm{gr}(\Psi), \overline{\Lambda}, t_0\big)$ as described above. In the special case that $\overline{\Lambda}(t) \equiv V \times V$, then

$$\Gamma\big(\mathrm{gr}(\Psi), \overline{\Lambda}, t_0\big)(\overline{v}) = \omega(\Psi(t_0)w, \Psi'(t_0)w),$$

where $\overline{v} = X_\Psi(t_0)w$.

A particular example of the Maslov index for symplectic paths is the Conley-Zehnder index on $(E, \omega) = (\mathbb{R}^{2n}, \overline{\omega}_0)$, which is defined as $\mu_{CZ}(\Psi) \stackrel{\mathrm{def}}{=} \mu(\mathrm{gr}(\Psi), \Delta)$ for paths $\Psi : [a,b] \to \mathrm{Sp}(2n, \mathbb{R})$, with $\Psi(a) = \mathrm{Id}$ and $\mathrm{Id} - \Psi(b)$ invertible. It holds that $\Psi' = \overline{J}_0 K(t)\Psi$, for some smooth path $t \mapsto K(t)$ of symmetric matrices. An intersection of $\mathrm{gr}(\Psi)$ and Δ is equivalent to the condition $\det(\Psi(t_0) - \mathrm{Id}) = 0$, i.e. for $\xi_0 \in \ker\,(\Psi(t_0) - \mathrm{Id})$ it holds that $\Psi(t_0)\xi_0 = \xi_0$. The crossing form is given by

$$\begin{aligned} \Gamma(\mathrm{gr}(\Psi), \Delta, t_0)(\overline{\xi}_0) &= \overline{\omega}_0(\Psi(t_0)\xi_0, \Psi'(t_0)\xi_0) \\ &= \langle \Psi(t_0)\xi_0, K(t_0)\Psi(t_0)\xi_0 \rangle \\ &= \langle \xi_0, K(t_0)\xi_0 \rangle. \end{aligned}$$

In the case of a symplectic path $\Psi : [0, \tau] \to \mathrm{Sp}(2n, \mathbb{R})$, with $\Psi(0) = \mathrm{Id}$, the extended Conley-Zehnder index is defined as $\mu_{CZ}(\Psi, \tau) = \mu(\mathrm{gr}(\Psi), \Delta)$.

The permuted Conley-Zehnder index

We now define a variation on the Conley-Zehnder index suitable for the application to braids. Consider the symplectic space

$$E = \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \qquad \omega = (-\overline{\omega}_0) \times \overline{\omega}_0.$$

In E we choose coordinates $(x,\tilde{x})$, with $x=(p^1,\cdots,p^n,q^1,\cdots,q^n)$ and $\tilde{x}=(\tilde{p}^1,\cdots,\tilde{p}^n,\tilde{q}^1,\cdots,\tilde{q}^n)$ both in $\mathbb{R}^{2n}$. Let $\sigma\in S_n$ be a permutation, then the permuted diagonal Δ_σ is defined by:

$$\Delta_\sigma \stackrel{\text{def}}{=} \big\{(x,\tilde{x}) \mid (\tilde{p}^k,\tilde{q}^k)=(p^{\sigma(k)},q^{\sigma(k)}),\ \ 1\le k\le n\big\}. \tag{3.18}$$

It holds that $\Delta_\sigma=\mathrm{gr}(\boldsymbol{\sigma})$, where $\boldsymbol{\sigma}=\begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}$ and the permuted diagonal Δ_σ is a Lagrangian subspace of E. Let $\Psi:[0,\tau]\to\mathrm{Sp}(2n,\mathbb{R})$ be a symplectic path with $\Psi(0)=\mathrm{Id}$. A crossing $t=t_0$ is defined by the condition ker $(\Psi(t_0)-\boldsymbol{\sigma})\neq\{0\}$ and the crossing form is given by

$$\begin{aligned}\Gamma(\mathrm{gr}(\Psi),\Delta_\sigma,t_0)(\xi_0^\sigma) &= \overline{\omega}_0(\Psi(t_0)\xi_0,\Psi'(t_0)\xi_0)\\ &= \langle\Psi(t_0)\xi_0,K(t_0)\Psi(t_0)\xi_0\rangle\\ &= \langle\boldsymbol{\sigma}\xi_0,K(t_0)\boldsymbol{\sigma}\xi_0\rangle=\langle\xi_0,\boldsymbol{\sigma}^TK(t_0)\boldsymbol{\sigma}\xi_0\rangle,\end{aligned} \tag{3.19}$$

where $\xi_0^\sigma=X_\sigma\xi_0$, and X_σ the frame for Δ_σ. The *permuted Conley-Zehnder index* is defined as

$$\mu_\sigma(\Psi,\tau)\stackrel{\text{def}}{=}\mu(\mathrm{gr}(\Psi),\Delta_\sigma). \tag{3.20}$$

Based on the properties of the Maslov index the following list of basic properties of the index μ_σ can be derived.

LEMMA 3.22. *Let* $\Psi:[0,\tau]\to\mathrm{Sp}(2n,\mathbb{R})$ *be a symplectic path with* $\Psi(0)=\mathrm{Id}$*, then*

(i) $\mu_\sigma(\Psi\times\Psi^\dagger,\tau)=\mu_\sigma(\Psi,\tau)+\mu_\sigma(\Psi^\dagger,\tau)$*;*

(ii) *let* $\Phi_k(t):[0,\tau]\to\mathrm{Sp}(2n,\mathbb{R})$ *be a symplectic loop (rotation) given by* $\Phi_k(t)=e^{\frac{2\pi k}{\tau}J_0t}$*, then* $\mu_\sigma(\Phi_k\Psi,\tau)=\mu_\sigma(\Psi,\tau)+2kn$*,*

PROOF. Property (i) follows from the fact that the equations for the crossings uncouple. As for (ii) we argue as follows. Consider the symplectic curves (using $\Psi(0)=\mathrm{Id}$)

$$\Psi_0(t)=\begin{cases}\Phi_k(t)\Psi(t) & t\in[0,\tau]\\ \Psi(\tau) & t\in[\tau,2\tau],\end{cases}\qquad \Psi_1(t)=\begin{cases}\Phi_k(t) & t\in[0,\tau]\\ \Psi(t-\tau) & t\in[\tau,2\tau].\end{cases}$$

The curves Ψ_0 and Ψ_1 are homotopic via the homotopy

$$\Psi_\lambda(t)=\begin{cases}\Phi_k(t)\Psi((1-\lambda)t) & t\in[0,\tau]\\ \Psi\big(\tau+\lambda(t-2\tau)\big) & t\in[\tau,2\tau],\end{cases}$$

with $\lambda\in[0,1]$, and $\mu_\sigma(\Psi_0,2\tau)=\mu_\sigma(\Psi_1,2\tau)$. By the definition of Ψ_0 it follows that $\mu_\sigma(\Phi_k\Psi,\tau)=\mu_\sigma(\Psi_0,2\tau)$. Using property (iii) of the Maslov index mentioned before, we obtain

$$\begin{aligned}\mu_\sigma(\Phi_k\Psi,\tau) &= \mu_\sigma(\Psi_0,2\tau)=\mu_\sigma(\Psi_1,2\tau)=\mu(\mathrm{gr}(\Psi_1),\Delta_\sigma)\\ &= \mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]},\Delta_\sigma\right)+\mu\left(\mathrm{gr}(\Psi(t-\tau))|_{[\tau,2\tau]},\Delta_\sigma\right)\\ &= \mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]},\Delta_\sigma\right)+\mu_\sigma(\Psi,\tau).\end{aligned}$$

It remains to evaluate $\mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]},\Delta_\sigma\right)$. Recall from [**59**], Remark 2.6, that for a Lagrangian loop $\Lambda(t+1)=\Lambda(t)$ and any Lagrangian subspace V the Maslov index is given by

$$\mu(\Lambda,V)=\frac{\alpha(1)-\alpha(0)}{\pi},\quad \det(P(t)+iQ(t))=e^{i\alpha(t)},$$

where $X = (P,Q)^t$ is a unitary Lagrangian frame for Λ. In particular, the index of the loop is independent of the Lagrangian subspace V. From this we derive that

$$\mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta_\sigma\right) = \mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta\right),$$

and the latter is computed as follows. Consider the crossings of Φ_k: $\det\left(e^{\frac{2\pi k}{\tau}\overline{J}_0 t_0} - \mathrm{Id}\right) = 0$, which holds for $t_0 = \frac{\tau n}{k}$, $n = 0,\cdots,k$. Since Φ_k satisfies $\Phi_k' = \frac{2\pi k}{\tau}\overline{J}_0\Phi_k$, the crossing form is given by $\Gamma\left(\mathrm{gr}(\Phi_k),\Delta,t_0\right)\xi_0 = \left\langle \xi_0, \frac{2\pi k}{\tau}\xi_0\right\rangle = \frac{2\pi k}{\tau}|\xi_0|^2$, with $\xi_0 \in \ker\left(\Psi(t_0) - \mathrm{Id}\right) \neq \{0\}$, and $\mathrm{sign}\,\Gamma\left(\mathrm{gr}(\Phi_k),\Delta,t_0\right) = 2n$ (the dimension of the kernel is $2n$). From this we derive that $\mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]},\Delta\right) = 2kn$ and consequently $\mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]},\Delta_\sigma\right) = 2kn$. □

Fredholm theory and the Maslov index for closed braids

The main result of this section concerns the relation between the permuted Conley-Zehnder index μ_σ and the Fredholm index of the linearized Cauchy-Riemann operator

$$\bar{\partial}_{K,\Delta_\sigma} = \frac{\partial}{\partial s} + \overline{J}_0\frac{\partial}{\partial t} + K(s,t),$$

where $K(s,t)$ is a family of symmetric[6] $2n\times 2n$ matrices parameterized by $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. The matrix $\overline{J}_0$ is the standard symplectic on $\mathbb{R}^{2n}$, with $\langle\cdot,\cdot\rangle = \overline{\omega}_0(\cdot,\overline{J}_0\cdot)$. The operator $\bar{\partial}_{K,\Delta_\sigma}$ acts on functions satisfying the non-local boundary conditions $\left(\xi(s,0),\xi(s,1)\right)\in\Delta_\sigma$, or in other words $\xi(s,1) = \boldsymbol{\sigma}\xi(s,0)$. On K we impose the following hypotheses:

(k1) there exist continuous functions $K_\pm : \mathbb{R}/\mathbb{Z} \to M(2n,\mathbb{R})$,[7] such that $\lim_{s\to\pm\infty} K(s,t) = K_\pm(t)$, uniform in $t\in[0,1]$;

(k2) the solutions $\Psi_\pm$ of the initial value problem

$$\frac{d}{dt}\Psi_\pm - \overline{J}_0 K_\pm(t)\Psi_\pm = 0, \quad \Psi_\pm(0) = \mathrm{Id},$$

have the property that $\mathrm{gr}\left(\Psi_\pm(1)\right)$ is transverse to Δ_σ.

Hypothesis (k2) can be rephrased as $\det\left(\Psi_\pm(1) - \boldsymbol{\sigma}\right) \neq 0$. It follows from the proof below that this is equivalent to saying that the mappings $L_\pm = -\overline{J}_0\frac{d}{dt} - K_\pm(t)$ are invertible.

In [**60**] the following result was proved. Define the function spaces

$$\begin{aligned} W_\sigma^{1,2}([0,1];\mathbb{R}^{2n}) &\stackrel{\text{def}}{=} \left\{\eta\in W^{1,2}([0,1]) \mid \left(\eta(0),\eta(1)\right)\in\Delta_\sigma\right\} \\ W_\sigma^{1,2}(\mathbb{R}\times[0,1];\mathbb{R}^{2n}) &\stackrel{\text{def}}{=} \left\{\xi\in W^{1,2}(\mathbb{R}\times[0,1]) \mid \left(\xi(s,0),\xi(s,1)\right)\in\Delta_\sigma\right\}. \end{aligned}$$

PROPOSITION 3.23. *Suppose that Hypotheses (k1) and (k2) are satisfied. Then the operator $\bar{\partial}_{K,\Delta_\sigma} : W^{1,2}_{\Delta_\sigma} \to L^2$ is Fredholm and the Fredholm index is given by*

$$\mathrm{ind}\,\bar{\partial}_{K,\Delta_\sigma} = \mu_\sigma(\Psi_+,1) - \mu_\sigma(\Psi_-,1).$$

As a matter of fact $\bar{\partial}_{K,\Delta_\sigma}$ is a Fredholm operator from $W_\sigma^{1,p}$ to L^p, $1<p<\infty$, with the same Fredholm index.

---

[6]The theory also holds for families $K(s,t)$ for which only the limits are symmetric.

[7]The $2n\times 2n$ real matrices are denoted by $M(2n,\mathbb{R})$.

PROOF. In [**60**] this result is proved that under Hypotheses (k1) and (k2) on the operator $\bar{\partial}_{K,\Delta_\sigma}$. We will sketch the proof adjusted to the special situation here. Regard the linearized Cauchy-Riemann operator as an unbounded operator

$$D_L = \frac{d}{ds} - L(s),$$

on $L^2\big(\mathbb{R};L^2([0,1];\mathbb{R}^{2n})\big)$, where $L(s) = -\bar{J}_0\frac{d}{dt} - K(s,t)$ is a family of unbounded, self-adjoint operators on $L^2([0,1];\mathbb{R}^{2n})$, with (dense) domain $W^{1,2}_\sigma([0,1];\mathbb{R}^{2n})$. In this special case the result follows from the spectral flow of $L(s)$, which can be described as follows. For the path $s \mapsto L(s)$ a number $s_0 \in \mathbb{R}$ is a crossing if $\ker L(s) \neq \{0\}$. On $\ker L(s)$ we have the crossing form

$$\Gamma(L,s_0)\xi \stackrel{\text{def}}{=} (\xi, L'(s)\xi)_{L^2} = -\int_0^1 \Big\langle \xi(t), \frac{\partial K(s,t)}{\partial s}\xi(t)\Big\rangle dt,$$

with $\xi \in \ker L(s)$. If the path $s \mapsto L(s)$ has only regular crossings — crossings for which Γ is non-degenerate — then the main result in [**60**] states that D_L is Fredholm and the Fredholm index is given by

$$\text{ind}\, D_L = -\sum_{s_0} \text{sign}\, \Gamma(L,s_0) \stackrel{\text{def}}{=} -\mu_{\text{spec}}(L).$$

Let $\Psi(s,t)$ be the solution of the parametrized (by s) family of ODEs

$$\begin{cases} L(s)\Psi(s,t) = 0, \\ \Psi(s,0) = \text{Id}. \end{cases}$$

Note that $\xi \in \ker\, L(s)$ if and only if $\xi(t) = \Psi(s,t)\xi_0$ and $\Psi(s,1)\xi_0 = \boldsymbol{\sigma}\xi_0$, i.e. $\xi_0 \in \ker(\Psi(s,1) - \boldsymbol{\sigma})$. The crossing form for L can be related to the crossing form for $(\text{gr}(\Psi),\Delta_\sigma)$. We have that $L(s)\Psi(s,\cdot) = 0$, and thus by differentiating

$$\frac{\partial K(s,t)}{\partial s}\Psi(s,t) + K(s,t)\frac{\partial \Psi(s,t)}{\partial s} = -\bar{J}_0\frac{\partial^2\Psi(s,t)}{\partial s\partial t}.$$

From this we derive

$$\begin{aligned}
&-\Big\langle \Psi(s,t)\xi_0, \frac{\partial K(s,t)}{\partial s}\Psi(s,t)\xi_0\Big\rangle \\
&\quad = \Big\langle \Psi(s,t)\xi_0, K(s,t)\frac{\partial\Psi(s,t)}{\partial s}\xi_0\Big\rangle + \Big\langle \Psi(s,t)\xi_0, \bar{J}_0\frac{\partial^2\Psi(s,t)}{\partial s\partial t}\xi_0\Big\rangle \\
&\quad = \Big\langle K(s,t)\Psi(s,t)\xi_0, \frac{\partial\Psi(s,t)}{\partial s}\xi_0\Big\rangle + \Big\langle \Psi(s,t)\xi_0, \bar{J}_0\frac{\partial^2\Psi(s,t)}{\partial s\partial t}\xi_0\Big\rangle \\
&\quad = -\Big\langle \bar{J}_0\frac{\partial\Psi(s,t)}{\partial t}\xi_0, \frac{\partial\Psi(s,t)}{\partial s}\xi_0\Big\rangle + \Big\langle \Psi(s,t)\xi_0, \bar{J}_0\frac{\partial^2\Psi(s,t)}{\partial s\partial t}\xi_0\Big\rangle,
\end{aligned}$$

which yields that

$$-\Big\langle \Psi(s,t)\xi_0, \frac{\partial K(s,t)}{\partial s}\Psi(s,t)\xi_0\Big\rangle = \frac{\partial}{\partial t}\Big\langle \Psi(s,t)\xi_0, \bar{J}_0\frac{\partial\Psi(s,t)}{\partial s}\xi_0\Big\rangle.$$

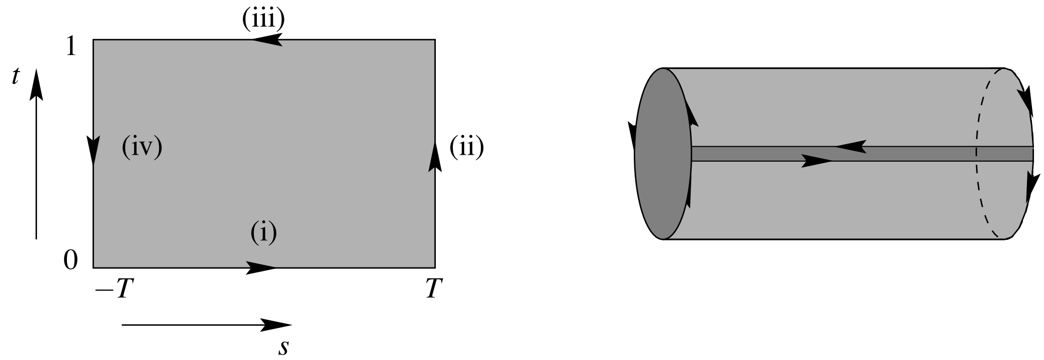

Figure 3.6: The symplectic contour in $\mathbb{R}^2$ and as cylinder $[-T,T]\times\mathbb{R}/\mathbb{Z}$.

We substitute this identity in the integral crossing form for $L(s)$ at a crossing $s=s_0$:

$$\begin{aligned}\Gamma(L,s_0)(\xi) &= -\int_0^1\Big\langle\xi(t),\frac{\partial K(s,t)}{\partial s}\xi(t)\Big\rangle dt\\ &= -\int_0^1\Big\langle\Psi(s,t)\xi_0,\frac{\partial K(s,t)}{\partial s}\Psi(s,t)\xi_0\Big\rangle dt\\ &= \Big\langle\Psi(s,t)\xi_0,\overline{J}_0\frac{\partial\Psi(s,t)}{\partial s}\xi_0\Big\rangle\Big|_0^1=\Big\langle\Psi(s,1)\xi_0,\overline{J}_0\frac{\partial\Psi(s,1)}{\partial s}\xi_0\Big\rangle\\ &= -\overline{\omega}_0\Big(\Psi(s,1)\xi_0,\frac{\partial\Psi(s,1)}{\partial s}\xi_0\Big)=-\Gamma\big(\mathrm{gr}(\Psi(s,1),\Delta_\sigma,s_0\big)(\xi_0^\sigma).\end{aligned}$$

The boundary term at $t=0$ is zero since $\Psi(s,0)=\mathrm{Id}$ for all s. The relation between the crossing forms proves that the curves $s\mapsto L(s)$ and $s\mapsto\Psi(s,1)$ have the same crossings, and $\mu(\mathrm{gr}(\Psi(s,1)),\Delta_\sigma)=-\mu_{\mathrm{spec}}(L)$. We assume that $\Psi(\pm T,t)=\Psi_\pm(t)$, and that the crossings $s=s_0$ are regular, as the general case follows from homotopy invariance. The symplectic path along the boundary of the cylinder $[-T,T]\times\mathbb{R}/\mathbb{Z}\subset\mathbb{R}\times\mathbb{R}/\mathbb{Z}$ yields

$$\mu(\Delta,\Delta_\sigma)+\mu_\sigma(\Psi_+,1)+\mu_{\mathrm{spec}}(L)-\mu_\sigma(\Psi_-,1)=0.$$

Indeed, since the loop is contractible the sum of the terms is zero. The individual terms along the boundary components are found as follows, see Figure 3.6: (i) for $-T\le s\le T$, it holds that $\Psi(s,0)=\mathrm{Id}$, and thus $\mathrm{gr}(\Psi(s,0))=\Delta$ and $\mu(\mathrm{gr}(\Psi(s,0)),\Delta_\sigma)=\mu(\Delta,\Delta_\sigma)$; (ii) for $0\le t\le 1$, we have $\Psi(T,t)=\Psi_+(t)$, and therefore $\mu(\mathrm{gr}(\Psi_+),\Delta_\sigma)=\mu_\sigma(\Psi_+,1)$; (iii) for $-T\le s\le T$ (opposite direction) the previous calculations with the crossing form for $L(s)$ show that $\mu(\mathrm{gr}(\Psi(s,1)),\Delta_\sigma)=-\mu_{\mathrm{spec}}(L)$; (iv) for $0\le t\le 1$ (opposite direction), it holds that $\Psi(-T,t)=\Psi_-(t)$, and therefore $\mu(\mathrm{gr}(\Psi_-),\Delta_\sigma)=\mu_\sigma(\Psi_-,1)$. Since $\mathrm{ind}\,D_L=-\mu_{\mathrm{spec}}(L)$ we obtain

$$\mathrm{ind}\,D_L=\mathrm{ind}\,\bar\partial_{K,\Delta_\sigma}=\mu_\sigma(\Psi_+,1)-\mu_\sigma(\Psi_-,1)+\mu(\Delta,\Delta_\sigma).$$

Since Δ_σ and Δ are both constant Lagrangian curves, it follows that $\mu(\Delta,\Delta_\sigma)=0$, which concludes the proof of the Theorem. □

We recall from Section 3.3 that the Hamiltonian for multi-strand braids is defined as $\overline{H}(t,\mathbf{x}(t)) = \sum_{k=1}^{n} H(t,x^k(t))$. The linearization around a braid $\mathbf{x}$ is given by

$$L_{\mathbf{x}} \stackrel{\text{def}}{=} d^2\mathbf{f}_H(\mathbf{x}) = -\overline{J}_0\frac{d}{dt} - d^2\overline{H}(t,\mathbf{x}). \tag{3.21}$$

Define the symplectic path $\Psi : [0,1] \to \mathrm{Sp}(2n,\mathbb{R})$ by

$$\frac{d\Psi}{dt} - \overline{J}_0 d^2\overline{H}(t,\mathbf{x}(t))\Psi = 0, \qquad \Psi(0) = \mathrm{Id}. \tag{3.22}$$

For convenience we write $K(t) = d^2\overline{H}(t,\mathbf{x}(t))$, so that the linearized equation becomes $\frac{d}{dt}\Psi - \overline{J}_0 K(t)\Psi = 0$.

LEMMA 3.24. *If* $\det\big(\Psi(1) - \boldsymbol{\sigma}\big) \neq 0$, *then* $\mu_\sigma(\Psi,1)$ *is an integer.*

PROOF. Since crossings between $\mathrm{gr}(\Psi)$ and Δ_σ exactly occur when $\det\big(\Psi(t) - \boldsymbol{\sigma}\big) = 0$, the only endpoint that may lead to a non-integer contribution is the starting point. There the crossing form is given by, see (3.19),

$$\Gamma(\mathrm{gr}(\Psi),\Delta_\sigma,0)(\xi^\sigma) = \omega(\xi,\boldsymbol{\sigma}^T K(0)\boldsymbol{\sigma}\xi),$$

for all $\xi \in \ker(\Psi(0) - \boldsymbol{\sigma})$. The kernel of $\Psi(0) - \boldsymbol{\sigma} = \mathrm{Id} - \boldsymbol{\sigma}$ is even dimensional, since in coordinates (3.18) it is of the form

$$\ker(\mathrm{Id}_{2n} - \boldsymbol{\sigma}) = \ker(\mathrm{Id}_n - \sigma) \times \ker(\mathrm{Id}_n - \sigma).$$

Therefore, $\mathrm{sign}\,\Gamma(\mathrm{gr}(\Psi),\Delta_\sigma,0)$ is always even, and $\mu_\sigma(\Psi,1)$ is an integer. □

The non-degeneracy condition leads to an integer valued Conley-Zehnder index for braids.

DEFINITION 3.25. A stationary braid $\mathbf{x}$ is said to be non-degenerate if $\det\big(\Psi(1) - \boldsymbol{\sigma}\big) \neq 0$. The Conley-Zehnder index of a non-degenerate stationary braid $\mathbf{x}$ is defined by

$$\mu(\mathbf{x}) \stackrel{\text{def}}{=} -\mu_\sigma(\Psi,1),$$

where $\sigma \in S_n$ is the associated permutation of $\mathbf{x}$.

3.7. Transversality and connecting orbit spaces

Central to the analysis of the Cauchy-Riemann equations are various generic non-degeneracy and transversality properties. The first important statement in this direction involves the generic non-degeneracy of critical points.

Generic properties of critical points

Define

$$\mathrm{Crit}_H\left([\mathbf{x}] \text{ rel } \mathbf{y}\right) \stackrel{\text{def}}{=} \mathrm{Crit}_H(\overline{\Omega}^n) \cap [\mathbf{x}] \text{ rel } \mathbf{y}.$$

PROPOSITION 3.26. *Let* $[\mathbf{x}]$ rel $\mathbf{y}$ *be a proper relative braid class. Then, for any Hamiltonian* $H \in \mathcal{H}$, *with* $\mathbf{y} \in \mathrm{Crit}_H(\overline{\Omega}^m)$, *there exists a* $\delta_* > 0$ *such that for any* $\delta < \delta_*$ *there exists a nearby Hamiltonian* $H' \in \mathcal{H}$ *satisfying* $\|H - H'\|_{C^2} < \delta$, *with* $\mathbf{y} \in \mathrm{Crit}_{H'}(\overline{\Omega}^m)$ *and such that* $\mathrm{Crit}_{H'}\left([\mathbf{x}] \text{ rel } \mathbf{y}\right)$ *consists of only finitely many non-degenerate critical points for the action* $\mathbf{f}_{H'}$.

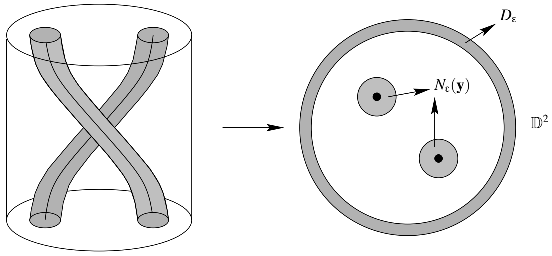

Figure 3.7: Tubular neighborhoods of a skeleton $\mathbf{y}$ [left] and a cross section indicating the set A_ε [right].

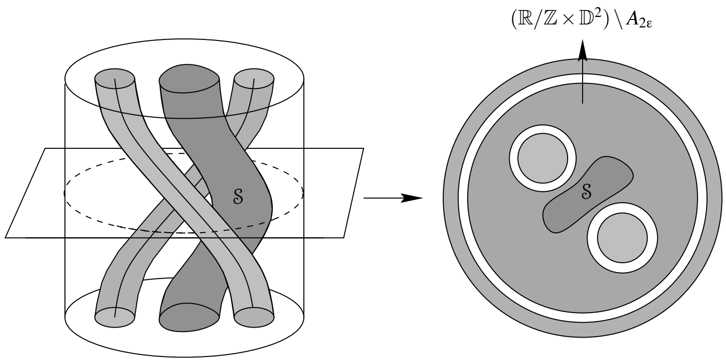

Figure 3.8: The invariant set $\mathcal{T}$ avoiding both $N_\varepsilon(\mathbf{y})$ and D_ε [left] and a cross section which shows how $\mathcal{T}$ is contained in $\mathbb{R}/\mathbb{Z}\times\mathbb{D}^2\setminus A_{2\varepsilon}$ [right].

We say that the property that $\mathrm{Crit}_H([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$ consists of only non-degenerate critical points is a *generic* property, and is satisfied by *generic* Hamiltonians in the above sense.

PROOF. Given $H\in\mathcal{H}$ we start off with defining a class of perturbations. For a braid $\mathbf{y}\in\Omega^m$, define the tubular neighborhood $N_\varepsilon(\mathbf{y})$ of $\mathbf{y}$ in $\mathbb{R}/\mathbb{Z}\times\mathbb{D}^2$ by :

$$N_\varepsilon(\mathbf{y}) = \bigcup_{\substack{k=1,\cdots,m\\ t\in[0,1]}} B_\varepsilon(y^k(t)).$$

If $\varepsilon>0$ is sufficiently small, then a neighborhood $N_\varepsilon(\mathbf{y})$ consists of m disjoint cylinders. Let $D_\varepsilon=\{x\in\mathbb{D}^2 \mid 1-\varepsilon<|x|\le 1\}$ be a small neighborhood of the boundary, and define

$$A_\varepsilon = N_\varepsilon(\mathbf{y})\cup\big(\mathbb{R}/\mathbb{Z}\times D_\varepsilon\big),\quad A_\varepsilon^c=\big(\mathbb{R}/\mathbb{Z}\times\mathbb{D}^2\big)\setminus A_\varepsilon,$$

see Figure 3.7. Let $\mathcal{T}^{J,H}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$ represent the paths in the cylinder traced out by the elements of $\mathcal{S}^{J,H}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$:

$$\mathcal{T}^{J,H}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y}) \stackrel{\mathrm{def}}{=} \big\{x^k(t)\ \big|\ 1\le k\le n, t\in[0,1], \mathbf{x}\in\mathcal{S}^{J,H}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})\big\}.$$

Since $[\mathbf{x}]$ rel $\mathbf{y}$ is proper, there exists an $\varepsilon_*>0$, such that for all $\varepsilon\le\varepsilon_*$ it holds that $\mathcal{T}^{J,H}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})\subset\mathrm{int}(A_{2\varepsilon}^c)$, see Figure 3.8. Now fix $\varepsilon\in(0,\varepsilon_*]$. Let

$$\begin{aligned}\mathcal{V}_\varepsilon &\stackrel{\mathrm{def}}{=} \{h\in C^2(\mathbb{R}/\mathbb{Z}\times\mathbb{D}^2;\mathbb{R}) \mid \mathrm{supp}\,h\subset A_\varepsilon^c\},\\ \mathcal{V}_{\delta,\varepsilon} &\stackrel{\mathrm{def}}{=} \{h\in\mathcal{V}_\varepsilon \mid \|h\|_{C^2}<\delta\},\end{aligned}$$

and consider Hamiltonians of the form $H' = H + h_\delta \in \mathcal{H}$, with $h_\delta \in \mathcal{V}_{\delta,\varepsilon}$. Then, by construction $\mathbf{y} \in \mathrm{Crit}_{H'}(\overline{\Omega}^m)$, and by Proposition 3.20 the set $\mathcal{S}^{J,H'}([\mathbf{x}] \text{ rel } \mathbf{y})$ is compact and isolated in the *proper* braid class $[\mathbf{x}]$ rel $\mathbf{y}$ for all perturbation $h_\delta \in \mathcal{V}_{\delta,\varepsilon}$. A straightforward compactness argument, using the compactness result of Proposition 3.20, shows that $\mathcal{T}^{J,H+h_\delta}([\mathbf{x}] \text{ rel } \mathbf{y})$ converges to $\mathcal{T}^{J,H}([\mathbf{x}] \text{ rel } \mathbf{y})$ in the Hausdorff metric as $\delta \to 0$. Therefore, there exists a $\delta_* > 0$, such that $\mathcal{T}^{J,H+h_\delta}([\mathbf{x}] \text{ rel } \mathbf{y}) \subset \mathrm{int}(A^c_{2\varepsilon})$, for all $0 \le \delta \le \delta_*$. In particular $\mathrm{Crit}_{H+h_{\delta,\varepsilon}} \subset \mathrm{int}(A^c_{2\varepsilon})$, for all $0 \le \delta \le \delta_*$. Now fix $\delta \in (0, \delta_*]$.

The Hamilton equations for H' are $x^k_t - J_0 \nabla H(t,x^k) - J_0 \nabla h(t,x) = 0$, or

$$-J_0 x^k_t - \nabla H(t,x^k) - \nabla h(t,x) = 0,$$

with the boundary conditions given in Definition 3.2. Define $\mathcal{U}_\varepsilon \subset W^{1,2}_\sigma([0,1];\mathbb{R}^{2n})$ to be the open subset of functions $\mathbf{x} = \{x^k\}$ such that $x^k(t) \in \mathrm{int}(A^c_{2\varepsilon})$ and define the nonlinear mapping

$$\mathcal{G} : \mathcal{U}_\varepsilon \times \mathcal{V}_{\delta,\varepsilon} \to L^2([0,1];\mathbb{R}^{2n}),$$

which represents the above system of equations and boundary conditions. Explicitly,

$$\mathcal{G}(\mathbf{x},h) = -\overline{J}_0 \mathbf{x}_t - \nabla \overline{H}(t,\mathbf{x}) - \nabla \overline{h}(t,\mathbf{x}),$$

where $\overline{H}(t,\mathbf{x}) = \sum_k H(t,x^k)$, and the same for $\overline{h}$. The mapping $\mathcal{G}$ is linear in h. Since $\mathcal{G}$ is defined on $\mathcal{U}_\varepsilon$ and both H and h are of class C^2, the mapping $\mathcal{G}$ is of class C^1. The derivative with respect to variations $(\xi,\eta) \in W^{1,2}_\sigma([0,1];\mathbb{R}^{2n}) \times \mathcal{V}_\varepsilon$ is given by

$$\begin{aligned} d\mathcal{G}(\mathbf{x},h)(\xi,\eta) &= -\overline{J}_0 \xi_t - d^2\overline{H}(t,\mathbf{x})\xi - d^2\overline{h}(t,\mathbf{x})\xi - \nabla\overline{\eta}(t,\mathbf{x}) \\ &= L_\mathbf{x}\xi - \nabla\overline{\eta}(t,\mathbf{x}), \end{aligned}$$

where $L_\mathbf{x} = -\overline{J}_0 \frac{d}{dt} - d^2\overline{H}(t,\mathbf{x}) - d^2\overline{h}(t,\mathbf{x})$, analogous to (3.21) in the previous section. We see that there is a one-to-one correspondence between elements Ψ in the kernel of $L_\mathbf{x}$ and symplectic paths described by (3.22) with $\det\big(\Psi(1) - \boldsymbol{\sigma}\big) = 0$. In other words, the stationary braid $\mathbf{x}$ is non-degenerate if and only if $L_\mathbf{x}$ has trivial kernel.

The operator $L_\mathbf{x}$ is a self-adjoint operator on $L^2([0,1];\mathbb{R}^{2n})$ with domain $W^{1,2}_\sigma([0,1];\mathbb{R}^{2n})$ and is Fredholm with $\mathrm{ind}(L_\mathbf{x}) = 0$. Therefore $\mathcal{G}_h \stackrel{\text{def}}{=} \mathcal{G}(\cdot,h)$ is a (proper) nonlinear Fredholm operator with

$$\mathrm{ind}(\mathcal{G}_h) = \mathrm{ind}(L_\mathbf{x}) = 0.$$

Define the set

$$Z = \big\{(\mathbf{x},h) \in \mathcal{U}_\varepsilon \times \mathcal{V}_{\delta,\varepsilon} \mid \mathcal{G}(\mathbf{x},h) = 0\big\} = \mathcal{G}^{-1}(0),$$

and we show that Z is a Banach manifold. In order to prove this we show that $d\mathcal{G}(\mathbf{x},h)$ is surjective for all $(\mathbf{x},h) \in Z$. Since $d\mathcal{G}(\mathbf{x},h)(\xi,\eta) = L_\mathbf{x}\xi - \nabla\overline{\eta}(t,\mathbf{x})$, and the (closed) range of $L_\mathbf{x}$ has finite codimension, we need to show there is a (finite dimensional) complement of $R(L_\mathbf{x})$ in the image of $\nabla\overline{\eta}(t,\mathbf{x})$. It suffices to show that $\nabla\overline{\eta}(t,\mathbf{x})$ is dense in $L^2([0,1];\mathbb{R}^{2n})$.

Recall that for any pair $(\mathbf{x},h) \in Z$, it holds that $\mathbf{x} \in \mathrm{Crit}_{H'} \subset \mathrm{int}(A^c_{2\varepsilon})$. As before consider a neighborhood $N_\varepsilon(\mathbf{x})$, so that $N_\varepsilon(\mathbf{x}) \subset \mathrm{int}(A^c_\varepsilon)$ and consists of n disjoint cylinders $N_\varepsilon(x^k)$. Let $\phi^k_\varepsilon(t,x) \in C^\infty_0(N_\varepsilon(x^k))$, such that $\phi^k_\varepsilon \equiv 1$ on $N_{\varepsilon/2}(x^k)$. Define, for arbitrary $f^k \in C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{R}^2)$,

$$\eta(t,x) = \sum_{k=1}^n \phi^k_\varepsilon(t,x)\langle f^k(t), x\rangle_{L^2}.$$

Since $\phi^k_\varepsilon(t, x^k(t)) \equiv 1$ it holds that $\overline{\eta}(t,\mathbf{x}) = \sum_{k=1}^n \langle f^k(t), x^k\rangle_{L^2}$ for $\mathbf{x} \in \mathrm{Crit}_{H'}$, and therefore the gradient satisfies $\nabla\overline{\eta}(t,\mathbf{x}) = \mathbf{f} = (f^k) \in C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{R}^{2n})$. Moreover, $\eta \in \mathcal{V}_\varepsilon$ by construction, and because $C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{R}^{2n})$ is dense in $L^2([0,1];\mathbb{R}^{2n})$ it follows that $d\mathcal{G}(\mathbf{x},h)$ is surjective.

Consider the projection $\pi : Z \to \mathcal{V}_{\delta,\varepsilon}$, defined by $\pi(\mathbf{x},h) = h$. The projection π is a Fredholm operator. Indeed, $d\pi : T_{(\mathbf{x},h)}Z \to \mathcal{V}_\varepsilon$, with $d\pi(\mathbf{x},h)(\xi,\eta) = \eta$, and

$$T_{(\mathbf{x},h)}Z = \left\{(\xi,\eta) \in W^{1,2}_\sigma \times \mathcal{V}_\varepsilon \mid L_\mathbf{x}\xi - \nabla\overline{\eta} = 0\right\}.$$

From this it follows that $\mathrm{ind}(d\pi) = \mathrm{ind}(L_\mathbf{x}) = 0$. The Sard-Smale Theorem [**66**] implies that the set of perturbations $h \in \mathcal{V}^{\mathrm{reg}}_{\delta,\varepsilon} \subset \mathcal{V}_{\delta,\varepsilon}$ for which h is a regular value of π is an open and dense subset. It remains to show that $h \in \mathcal{V}^{\mathrm{reg}}_{\delta,\varepsilon}$ yields that $L_\mathbf{x}$ is surjective. Let $h \in \mathcal{V}^{\mathrm{reg}}_{\delta,\varepsilon}$, and $(\mathbf{x},h) \in Z$, then $d\mathcal{G}(\mathbf{x},h)$ is surjective, i.e., for any $\zeta \in L^2([0,1];\mathbb{R}^{2n})$ there are (ξ,η) such that $d\mathcal{G}(\mathbf{x},h)(\xi,\eta) = \zeta$. On the other hand, since since h is a regular value for π, there exists a $\widehat{\xi}$ such that $d\pi(\mathbf{x},h)(\widehat{\xi},\eta) = \eta$, $(\widehat{\xi},\eta) \in T_{(\mathbf{x},h)}Z$, i.e. $L_\mathbf{x}\widehat{\xi} - \nabla\overline{\eta} = 0$. Now

$$\begin{aligned} L_\mathbf{x}(\xi - \widehat{\xi}) &= d\mathcal{G}(\mathbf{x},h)(\xi - \widehat{\xi}, 0) \\ &= d\mathcal{G}(\mathbf{x},h)\big((\xi,\eta) - (\widehat{\xi},\eta)\big) = \zeta - 0 = \zeta, \end{aligned}$$

which proves that for all $h \in \mathcal{V}^{\mathrm{reg}}_{\delta,\varepsilon}$ the operator $L_\mathbf{x}$ is surjective, and hence also injective, implying that $\mathbf{x}$ is non-degenerate. □

For $\mathbf{x}_\pm \in \mathrm{Crit}_H$, let $\mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ be the space of all bounded solutions in $\mathbf{u} \in \mathcal{M}^{J,H}([\mathbf{x}] \text{ rel } \mathbf{y})$ such that $\lim_{s\to\pm\infty} \mathbf{u}(s,\cdot) = \mathbf{x}_\pm(\cdot)$, i.e., connecting orbits in the relative braid class. If $\mathbf{x}_- = \mathbf{x}_+$ then the set consists of just this one critical point. The space $\mathcal{S}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$, as usual, consists of the corresponding trajectories.

LEMMA 3.27. *Let* $[\mathbf{x}]$ rel $\mathbf{y}$ *be a proper braid class and let* $H \in \mathcal{H}$ *be a generic Hamiltonian. Then*

$$\mathcal{M}^{J,H}([\mathbf{x}] \text{ rel } \mathbf{y}) \subset \bigcup_{\mathbf{x}_\pm \in \mathrm{Crit}_H} \mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y}),$$

where $\mathrm{Crit}_H = \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$.

PROOF. Using the a priori estimate (3.10) we establish that bounded solutions have limits:

$$\lim_{s\to\pm\infty} \mathbf{u}(s,t) = \mathbf{x}_\pm,$$

for some $\mathbf{x}_\pm$ in $\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$, where the convergence is uniform in t, and $\lim_{s\to\pm\infty} \mathbf{u}_s(s,t)$ goes to 0, uniformly in t. Indeed, by assuming the contrary we have a sequence (s_n,t_n), with $|s_n| \to \infty$, such that $\mathbf{u}(s_n,t_n)$ stays strictly away from $\mathbf{x}(t_n)$ for any of the finitely many $\mathbf{x} \in \mathrm{Crit}_H(\mathbf{x} \text{ rel } \mathbf{y})$ as $n \to \infty$. We may assume that $t_n \to t_*$, and thus

$$|\widehat{\mathbf{u}}(0,t_*) - \mathbf{x}(t_*)| \geq \delta > 0 \qquad \text{for all } \mathbf{x} \in \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y}). \tag{3.23}$$

Define $\mathbf{u}_n(s,t) = \mathbf{u}(s + s_n, t)$. For the sequence $\{\mathbf{u}_n\}$ we have the a priori estimate

$$\|\mathbf{u}_n\|_{C^{1,\lambda}(\mathbb{R}^2)} \leq C(J,H), \quad n \to \infty,$$

where $0 < \lambda \leq 1 - 2/p$. In view of the compactness of the embedding

$$C^{1,\lambda}(\mathbb{R}^2) \hookrightarrow C^{1,\lambda'}(K),$$

for any compact domain $K \subset \mathbb{R}^2$ and $0 \le \lambda' < \lambda$, there exists a subsequence, again denoted by $\mathbf{u}_n$, and a function function $\widehat{\mathbf{u}} \in C^{1,\lambda'}(K)$ such that

$$\mathbf{u}_n \longrightarrow \widehat{\mathbf{u}}, \qquad \text{in} \quad C^{1,\lambda'}(K), \quad \text{as} \quad n \to \infty.$$

The limit function $\widehat{\mathbf{u}}$ satisfies the equation $\bar{\partial}_{J,H}(\widehat{\mathbf{u}}) = 0$ and the boundary conditions, and therefore $\widehat{\mathbf{u}} \in \mathcal{M}^{J,H}$. For any $T > 0$ we have that $\int_{-T}^{T}\int_0^1 |\widehat{\mathbf{u}}_s|^2 dtds = \lim_{n\to\infty}\int_{-T}^{T}\int_0^1 |(\mathbf{u}_n)_s|^2 dtds$. By definition, since $|s_n| \to \infty$, it holds that

$$\lim_{n\to\infty}\int_{-T}^{T}\int_0^1 |(\mathbf{u}_n)_s|^2 dtds = \lim_{n\to\infty}\int_{-T-s_n}^{T-s_n}\int_0^1 |\mathbf{u}_s|^2 dtds = 0.$$

Therefore, the limit function $\widehat{\mathbf{u}}$ is independent of s and $\widehat{\mathbf{u}} \in \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$, contradiction (3.23). □

COROLLARY 3.28. *Let* $[\mathbf{x}]$ rel $\mathbf{y}$ *be a proper relative braid class and let H be a generic Hamiltonian with* $\mathbf{y} \in \mathrm{Crit}_H(\overline{\Omega}^m)$. *Then the space of bounded solutions is given by*

$$\mathcal{M}^{J,H}([\mathbf{x}] \text{ rel } \mathbf{y}) = \bigcup_{\mathbf{x}_\pm \in \mathrm{Crit}_H} \mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y}).$$

PROOF. The key observation is that since $\mathbf{x}_\pm$ rel $\mathbf{y} \in [\mathbf{x}]$ rel $\mathbf{y}$, also $\mathbf{u}(s,\cdot)$ rel $\mathbf{y} \in [\mathbf{x}]$ rel $\mathbf{y}$, for all $s \in \mathbb{R}$ (the crossing number cannot change). Therefore, any $\mathbf{u} \in \mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ is contained in $[\mathbf{x}]$ rel $\mathbf{y}$, and thus $\mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y}) \subset \mathcal{M}^{J,H}([\mathbf{x}] \text{ rel } \mathbf{y})$. The remainder of the proof follows from Lemma 3.27. □

Note that the sets $\mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ are not necessarily compact in $\overline{\Omega}^n$. The following corollary gives a more precise statement about the compactness of the spaces $\mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$, which will be referred to as *geometric convergence* [**62**].

COROLLARY 3.29. *Let* $[\mathbf{x}]$ rel $\mathbf{y}$ *be a proper relative braid class and H be a generic Hamiltonian with* $\mathbf{y} \in \mathrm{Crit}_H(\overline{\Omega}^m)$. *Then for any sequence* $\{\mathbf{u}_n\} \subset \mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ *(along a subsequence) there exist stationary braids* $\mathbf{x}^i \in \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$, $i = 0,\cdots,m$, *orbits* $\mathbf{u}^i \in \mathcal{M}^{J,H}_{\mathbf{x}^i,\mathbf{x}^{i-1}}([\mathbf{x}] \text{ rel } \mathbf{y})$ *and times* s_n^i, $i = 1,\cdots,m$, *such that*

$$\mathbf{u}_n(\cdot + s_n^i,\cdot) \longrightarrow \mathbf{u}^i, \quad n \to \infty,$$

in $C^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}/\mathbb{Z})$. *Moreover,* $\mathbf{x}^0 = \mathbf{x}_+$ *and* $\mathbf{x}^m = \mathbf{x}_-$ *and* $\mathbf{f}_H(\mathbf{x}^i) > \mathbf{f}_H(\mathbf{x}^{i-1})$ *for* $i = 1,\cdots,m$. *The sequence* $\mathbf{u}_n$ *is said to geometrically converge to the broken trajectory* $(\mathbf{u}^1,\cdots,\mathbf{u}^m)$.

REMARK 3.30. In the case of smooth Hamiltonian $H \in \mathcal{H}\cap C^\infty$ we can find generic Hamiltonians in the same smooth class $\mathcal{H}\cap C^\infty$. Then Proposition 3.26 holds with respect to smooth Hamiltonians.

Generic properties for connecting orbits

As for critical points, non-degeneracy can also be defined for connecting orbits. This closely follows the ideas in the previous subsection. Set $W^{1,p}_\sigma = W^{1,p}_\sigma(\mathbb{R}\times[0,1];\mathbb{R}^{2n})$ and $L^p = L^p(\mathbb{R}\times[0,1];\mathbb{R}^{2n})$. In order to define transversality of connecting orbits we embed the Cauchy-Riemann equations for braids in $\mathbb{R}^{2n}$. Recall that $\overline{\omega}_0 = \omega_0 \times\cdots\times\omega_0$ is the standard symplectic form on $(\mathbb{D}^2)^n$ and $\overline{J} \in \mathrm{Sp}^+(2n,\mathbb{R})$ is defined from $J \in \mathcal{J}^+ \subset \mathrm{Sp}(2,\mathbb{R})$ via the relation $\overline{\omega}_0(\cdot,\overline{J}\cdot) = \overline{g}(\cdot,\cdot)$, where $\overline{g} = \big(g\times\cdots\times g\big)$. Define $\overline{X}_H$ via the relation $i_{\overline{X}_H}\overline{\omega}_0 = -d\overline{H}$.

DEFINITION 3.31. Let $\mathbf{x}_-, \mathbf{x}_+ \in \mathrm{Crit}_H(\overline{\Omega}^n)$ be non-degenerate stationary braids. A connecting orbit $\mathbf{u} \in \mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}$ is said to be *non-degenerate*, or *transverse*, if the linearized Cauchy-Riemann operator

$$\frac{\partial}{\partial s} + \overline{J}\frac{\partial}{\partial t} - \overline{J}d\overline{X}_H(t,\mathbf{u}(s,t)) : W^{1,p}_\sigma \to L^p,$$

is a surjective operator (for all $1 < p < \infty$).

For smooth perturbations $\mathfrak{h} \in C^\infty\left(\mathbb{R}/\mathbb{Z} \times (\mathbb{D}^2)^n; \mathbb{R}\right)$, consider the following extension of the nonlinear Cauchy-Riemann equations

$$\mathbf{u}_s + \overline{J}\mathbf{u}_t + \nabla_{\overline{g}}\overline{H}(t,\mathbf{u}) + \nabla_{\overline{g}}\mathfrak{h}(t,\mathbf{u}) = 0.$$

Note that we allow perturbations $\mathfrak{h}$ that change the dynamics for each of the n strands in a (slightly) different way. The space of bounded solutions and trajectories are denoted by $\mathcal{M}^{J,H,\mathfrak{h}}$ and $\mathcal{S}^{J,H,\mathfrak{h}}$, respectively. The compactness properties of these spaces are completely analogous to the ones of $\mathcal{M}^{J,H}$ and $\mathcal{S}^{J,H}$. In order to get genericity for connecting orbits we start off with smooth Hamiltonians and smooth braids, i.e. we assume that $\mathbf{y}$ is a smooth skeleton. On C^∞ we define a Banach space structure by defining the norm

$$\|\mathfrak{h}\|_{C^\infty} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \varepsilon_k \|\mathfrak{h}\|_{C^k},$$

for a sufficiently fast decaying sequence $\varepsilon_k > 0$, such that equipped with this norm C^∞ is a separable Banach space, that is dense in L^2.

PROPOSITION 3.32. *Let* $[\mathbf{x}]$ rel $\mathbf{y}$ *be a proper relative braid class and let* $\mathbf{x}_-, \mathbf{x}_+ \in \mathrm{Crit}_H([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$ *be non-degenerate stationary braids for some Hamiltonian* $H \in \mathcal{H} \cap C^\infty$. *Then there exists a* $\delta_* > 0$ *such that for any* $\delta \le \delta_*$ *there exists a Hamiltonian perturbation* $\mathfrak{h} \in C^\infty(\mathbb{R}/\mathbb{Z} \times (\mathbb{D}^2)^n; \mathbb{R})$*, with* $\|\mathfrak{h}\|_{C^\infty} < \delta$*, so that that*

(i) $\mathcal{S}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$ *is isolated in* $[\mathbf{x}]$ rel $\mathbf{y}$*;*
(ii) $\mathbf{x}_\pm \subset \mathrm{Crit}_{\overline{H}+\mathfrak{h}}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$*;*
(iii) $\mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$ *consists of non-degenerate connecting orbits;*
(iv) $\mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$ *are smooth manifolds without boundary and*

$$\dim \mathcal{M}^{J,H''}_{\mathbf{x}_-,\mathbf{x}_+} = \mu(\mathbf{x}_-) - \mu(\mathbf{x}_+),$$

where μ *is the Conley-Zehnder index defined in Definition 3.25.*

PROOF. By assumption $\mathcal{S}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}$ is isolated in $[\mathbf{x}]$ rel $\mathbf{y}$. In order to prove that the same holds for $\mathcal{S}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}$ when δ_* is sufficiently small we argue by contradiction. Suppose no such δ_* exists, then there is sequence $\delta_k \to 0$ as $k \to \infty$, functions $\mathfrak{h}_k$ with $\|\mathfrak{h}_k\|_{C^\infty} < \delta_k$ and solutions $\mathbf{u}_k \in \mathcal{M}^{J,H,\mathfrak{h}_k}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$ such that either

(a) $|\mathbf{u}_k(s_k,t_k)| = 1$ for some $(s_k,t_k) \in \mathbb{R} \times [0,1]$, or
(b) $\mathbf{u}_k(s_k,\cdot) \cap \Sigma^n$ rel $\mathbf{y} \ne \varnothing$ for some s_k.

By shifting with s_k in s we may assume without loss of generality that $s_k = 0$. By compactness $\lim_{k\to\infty} \mathbf{u}_k = \mathbf{u} \in \mathcal{M}^{J,H}([\mathbf{x}]\ \mathrm{rel}\ \mathbf{y})$ with either (a) or (b) satisfied. This contradicts the isolation of $\mathcal{S}^{J,H}_{\mathbf{x}_-,\mathbf{x}_+}$ and proves (i).

As for the transversality properties we refer to Salamon and Zehnder [**63**], where perturbations in in $\mathbb{R}^{2n}$ are considered. The proof is similar in spirit to the genericity of critical points. We sketch the main steps based on the proof in [**63**].

Denote by $C_0^\infty(\mathbb{R}/\mathbb{Z}\times(\mathbb{D}^2)^n, H)$ the subset of C^∞ of perturbations $\mathfrak{h}$ whose support is bounded away from $\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$. Therefore $\mathrm{Crit}_{\overline{H}+\mathfrak{h}}([\mathbf{x}] \text{ rel } \mathbf{y}) = \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$ for $\mathfrak{h}$ small enough (by the compactness properties). As in the proof of Proposition 3.26 define the Cauchy-Riemann equations

$$\mathcal{G}(\mathbf{u},\mathfrak{h}) = \mathbf{u}_s + \overline{J}\mathbf{u}_t + \nabla_{\overline{g}}\overline{H}(t,\mathbf{u}) + \nabla_{\overline{g}}\mathfrak{h}(t,\mathbf{u}).$$

Based on the a priori regularity of bounded solutions of the Cauchy-Riemann equations we define for $2 < p < \infty$ the affine spaces

$$\mathcal{U}^{1,p}(\mathbf{x}_-,\mathbf{x}_+) \stackrel{\text{def}}{=} \big\{\gamma + \xi \mid \xi \in W_\sigma^{1,p}\big\}, \tag{3.24}$$

and balls $\mathcal{U}_\varepsilon^{1,p} = \{\mathbf{u} \in \mathcal{U}^{1,p} \mid \|\xi\|_{W_\sigma^{1,p}} < \varepsilon\}$, where $\gamma(s,t) \in C^2\big(\mathbb{R}\times[0,1];(\mathbb{D}^2)^n\big)$ is a fixed connecting path such that $\lim_{s\to\pm\infty}\gamma(s,\cdot) = \mathbf{x}_\pm$ and $\gamma(s,t) \in \mathrm{int}(\mathbb{D}^2)^n$ for all $(s,t) \in \mathbb{R}\times[0,1]$. Therefore, for $p > 2$, functions $\mathbf{u} \in \mathcal{U}_\varepsilon^{1,p}(\mathbf{x}_-,\mathbf{x}_+)$ satisfy the limits $\lim_{s\to\pm\infty}\mathbf{u}(s,\cdot) = \mathbf{x}_\pm$ and if $\varepsilon > 0$ is chosen sufficiently small then also $\mathbf{u}(s,t) \in \mathrm{int}(\mathbb{D}^2)^n$ for all $(s,t) \in \mathbb{R}\times[0,1]$. The mapping

$$\mathcal{G} : \mathcal{U}_\varepsilon^{1,p}(\mathbf{x}_-,\mathbf{x}_+)\times C_0^\infty \to L^p(\mathbb{R}\times[0,1];\mathbb{R}^{2n}),$$

is a smooth mapping. Define

$$Z_{\mathbf{x}_-,\mathbf{x}_+} \stackrel{\text{def}}{=} \big\{(\mathbf{u},\mathfrak{h}) \in \mathcal{U}_\varepsilon^{1,p}(\mathbf{x}_-,\mathbf{x}_+)\times C_0^\infty \mid \mathcal{G}(\mathbf{u},\mathfrak{h}) = 0\big\} = \mathcal{G}^{-1}(0),$$

which is Banach manifold provided that $d\mathcal{G}(\mathbf{u},\mathfrak{h})$ is onto on for all $(\mathbf{u},\mathfrak{h}) \in Z_{\mathbf{x}_-,\mathbf{x}_+}$, where

$$d\mathcal{G}(\mathbf{u},\mathfrak{h})(\xi,\delta\mathfrak{h}) = d_1\mathcal{G}(\mathbf{u},\delta\mathfrak{h})\xi + \nabla_{\overline{g}}\delta\mathfrak{h}.$$

We summarize the most important ingredients of the proof and for details we refer to [**63**]. Assume that $d\mathcal{G}(\mathbf{u},\mathfrak{h})$ is not onto. Then there exists a non-zero function $\eta \in L^q$ which annihilates the range of $d\mathcal{G}(\mathbf{u},\mathfrak{h})$ and thus also the range of $d_1\mathcal{G}(\mathbf{u},\mathfrak{h})$, which is a Fredholm operator of index $\mu(\mathbf{x}_-) - \mu(\mathbf{x}_+)$, see Proposition 3.23. The relation $\langle d_1\mathcal{G}(\mathbf{u},\mathfrak{h})(\xi),\eta\rangle = 0$ for all ξ implies that

$$d_1\mathcal{G}(\mathbf{u},\mathfrak{h})^*\eta = -\eta_s + \overline{J}\eta_t + d\nabla_{\overline{g}}\overline{H}(t,\mathbf{u})\eta = 0,$$

and since also $\langle d\mathcal{G}(\mathbf{u},\mathfrak{h})(\xi,\delta\mathfrak{h}),\eta\rangle = 0$ it follows that

$$\int_{-\infty}^{\infty}\int_0^1 \langle \eta(s,t), \nabla_{\overline{g}}\delta\mathfrak{h}\rangle_{\mathbb{R}^{2n}}\,dtds = 0, \qquad \text{for all } \delta\mathfrak{h}. \tag{3.25}$$

Due to the assumptions on $\mathfrak{h}$ and H the regularity theory for the linear Cauchy-Riemann operator implies that η is smooth. It remains to show that no such non-zero function η exists.

Step 1: Since η satisfies a perturbed Laplace's equation it follows from Aronszajn's unique continuation [**8**] theorem that $\eta(s,t) \neq 0$ for almost all $(s,t) \in \mathbb{R}\times[0,1]$.

Step 2: The vectors $\eta(s,t)$ and $\mathbf{u}_s(s,t)$ are linearly dependent for all s and t. Suppose not, then these vector are linearly independent in some small neighborhood of (s_0,t_0). This allows the construction of $\delta\mathfrak{h}(\mathbf{u},t) \in C_0^\infty$ which violates the integral condition in (3.25). See [**63**] for the details.

Step 3: The previous step implies the existence of a function $\lambda : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that

$$\eta(s,t) = \lambda(s,t)\frac{\partial \mathbf{u}}{\partial s}(s,t),$$

for all s,t for which $\eta(s,t) \neq 0$. Using a contradiction argument with respect to equation (3.25) yields that

$$\frac{\partial \lambda}{\partial s}(s,t) = 0,$$

for almost all (s,t). In particular we obtain that λ is s-independent and we can assume that $\lambda(t) \geq \delta > 0$ for all $t \in [0,1]$ (follows again from Aronszajn's unique continuation theorem).

Step 4: This final step provides a contradiction to the assumption that $d\mathcal{G}$ is not onto. It holds that

$$\int_0^1 \left\langle \frac{\partial \mathbf{u}}{\partial s}(s,t), \eta(s,t) \right\rangle dt = \int_0^1 \lambda(t) \left|\frac{\partial \mathbf{u}}{\partial s}(s,t)\right|^2 dt \geq \delta \int_0^1 \left|\frac{\partial \mathbf{u}}{\partial s}(s,t)\right|^2 dt > 0.$$

The functions $\mathbf{u}_s$ and η satisfy the equations

$$d_1\mathcal{G}(\mathbf{u},\mathfrak{h})\mathbf{u}_s = 0, \quad d_1\mathcal{G}(\mathbf{u},\mathfrak{h})^*\eta = 0,$$

respectively. From these equations we can derive expressions for $\mathbf{u}_{ss}$ and η_s from which we get that

$$\frac{d}{ds}\int_0^1 \left\langle \frac{\partial \mathbf{u}}{\partial s}(s,t), \eta(s,t) \right\rangle dt = 0.$$

Combining this with the previous estimate yields that $\int_{-\infty}^{\infty}\int_0^1 |\mathbf{u}_s(s,t)|^2 dt = \infty$, which, combined with the compactness properties, contradicts the fact that $u \in \mathcal{M}_{\mathbf{x}_-,\mathbf{x}_+}$, and thus $d\mathcal{G}(\mathbf{u},\mathfrak{h})$ is onto for all $(\mathbf{u},\mathfrak{h}) \in Z_{\mathbf{x}_-,\mathbf{x}_+}$.

We can now apply the Sard-Smale theorem as in the proof of Proposition 3.26. The only difference here is that application of the Sard-Smale requires $(\mu(\mathbf{x}_-) - \mu(\mathbf{x}_+) + 1)$-smoothness of $\mathcal{G}$ which is guaranteed by the smoothness of $\mathbf{y}, H$ and $\mathfrak{h}$. □

For generic pairs $(H,\mathfrak{h})$ the convergence of Corollary 3.29 can be extended with estimates on the Conley-Zehnder indices of the stationary braids.

COROLLARY 3.33. *Let* $[\mathbf{x}]$ rel $\mathbf{y}$ *be a proper relative braid class and* H *be a generic Hamiltonian with* $\mathbf{y} \in \mathrm{Crit}_H(\overline{\Omega}^m)$. *If* $\mathbf{u}_n$ *geometrically converges to the broken trajectory* $(\mathbf{u}^1, \cdots, \mathbf{u}^m)$, *with* $\mathbf{u}^i \in \mathcal{M}^{J,H}_{\mathbf{x}^i,\mathbf{x}^{i-1}}([\mathbf{x}] \text{ rel } \mathbf{y})$, $i = 1, \cdots, m$ *and* $\mathbf{x}^i \in \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$, $i = 0, \cdots, m$, *then*

$$\mu(\mathbf{x}^i) > \mu(\mathbf{x}^{i-1}),$$

for $i = 1, \cdots, m$.

Since Proposition 3.32 provides a dense set of perturbations $\mathfrak{h}$ the intersection of dense sets over all pairs $(\mathbf{x}_-,\mathbf{x}_+)$ yields a dense set of perturbations $\mathfrak{h}$ for which (i)-(iv) in Proposition 3.32 holds for all pairs pairs $(\mathbf{x}_-,\mathbf{x}_+)$ and thus for all of $\mathcal{M}^{J,H,\mathfrak{h}}([\mathbf{x}] \text{ rel } \mathbf{y})$.

The above proof also carries over to the Cauchy-Riemann equations with s-dependent Hamiltonian $H(s,\cdot,\cdot)$. The only difference will be that one needs to consider perturbations $\mathfrak{h} : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times (\mathbb{D}^2)^n \to \mathbb{R}$. Exploiting the Fredholm index property for the s-dependent case we obtain the following corollary. Let $s \mapsto H(s,\cdot,\cdot)$ be a smooth path in $\mathcal{H}$ with the property

$H_s = 0$ for $|s| \geq R$. We have the following non-autonomous version of Propostion 3.32, see [**63**].

COROLLARY 3.34. *Let* $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ *be a proper relative braid class with fibers* $[\mathbf{x}_-] \text{ rel } \mathbf{y}_-$, $[\mathbf{x}_+] \text{ rel } \mathbf{y}_+$ *in* $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$. *Let* $s \mapsto H(s,\cdot,\cdot)$ *be a smooth path in* $\mathcal{H}$ *such that* $\mathrm{Crit}_{H_\pm}([\mathbf{x}_\pm] \text{ rel } \mathbf{y}_\pm)$, *with* $H_\pm = H(\pm\infty,\cdot,\cdot)$, *consist of non-degenerate stationary braids and* $\mathcal{M}^{J,H_\pm}([\mathbf{x}_\pm] \text{ rel } \mathbf{y}_\pm)$ *of only non-degenerate connections. Then there exists a* $\delta_* > 0$ *such that for any* $\delta \leq \delta_*$ *there exist s-dependent perturbations* $\mathfrak{h} \in C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times (\mathbb{D}^2)^n;\mathbb{R})$, *with* $\|\mathfrak{h}\|_{C^2} \leq \delta$ *and* $\mathfrak{h}_s = 0$ *for* $|s| \geq R$ *(and* $\mathfrak{h}_\pm = \mathfrak{h}(\pm\infty,\cdot,\cdot)$*), so that for* $x'_\pm \mathrm{Crit}_{H_\pm}([\mathbf{x}_\pm] \text{ rel } \mathbf{y}_\pm)$

(i) $\mathcal{S}^{J,H_\pm,\mathfrak{h}_\pm}_{\mathbf{x}'_-,\mathbf{x}'_+}([\mathbf{x}_\pm] \text{ rel } \mathbf{y}_\pm)$ *are isolated in* $[\mathbf{x}_\pm] \text{ rel } \mathbf{y}_\pm$ *respectively;*
(ii) $\mathrm{Crit}_{\overline{H}_\pm+\mathfrak{h}_\pm}([\mathbf{x}_\pm] \text{ rel } \mathbf{y}_\pm) = \mathrm{Crit}_{H_\pm}([\mathbf{x}_\pm] \text{ rel } \mathbf{y}_\pm)$*;*
(iii) $\mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}'_-,\mathbf{x}'_+}$ *consist of non-degenerate connecting orbits with respect to the s-dependent Cauchy-Riemann equations;*
(iv) $\mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}'_-,\mathbf{x}'_+}$ *are smooth manifolds without boundary with*

$$\dim \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}'_-,\mathbf{x}'_+} = \mu(\mathbf{x}'_-) - \mu(\mathbf{x}'_+) + 1,$$

where $\mu(\mathbf{x}'_\pm)$ *are the Conley-Zehnder indices with respect to the Hamiltonians* $H_\pm$.

3.8. Floer homology for proper braid classes

Since proper relative braid classes have the property that the sets $\mathcal{S}([\mathbf{x}] \text{ rel } \mathbf{y})$ are isolated in $[\mathbf{x}] \text{ rel } \mathbf{y}$ we can assign Floer homology groups following the celebrated construction of building a chain complex due to Floer [**29**]. As pointed out before the isolating neighborhoods are found via proper relative braid classes $[\mathbf{x}] \text{ rel } \mathbf{y}$ and we embed the defining system of Cauchy-Riemann equations $\bar{\partial}_{\overline{J},\overline{H}}(\mathbf{u}) = 0$ as Cauchy-Riemann equations in the symplectic product

$$(M,\omega) = \left(\mathbb{D}^2 \times \cdots \times \mathbb{D}^2, \omega_0 \times \cdots \times \omega_0\right),$$

which are given in (3.7). In [**29**] the idea of defining Floer homology for isolating neighborhoods was introduced and it provides the natural framework for relative braid classes. In this section we outline the definition of Floer homology for isolating neighborhoods applied to proper braid classes.

Definition

Let $\mathbf{y} \in \Omega^m$ be a smooth braid and $[\mathbf{x}] \text{ rel } \mathbf{y}$ a proper relative braid class. Let $H \in \mathcal{H} \cap C^\infty$ be a generic Hamiltonian with respect to the proper braid class $[\mathbf{x}] \text{ rel } \mathbf{y}$ (Proposition 3.26). Then the set of bounded solutions $\mathcal{M}^{J,H}([\mathbf{x}] \text{ rel } \mathbf{y})$ is compact, $\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$ is non-degenerate and $\mathcal{S}^{J,H}([\mathbf{x}] \text{ rel } \mathbf{y})$ is isolated in $[\mathbf{x}] \text{ rel } \mathbf{y}$. Since $\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$ is a finite set we can define the chain groups

$$C_k\big([\mathbf{x}] \text{ rel } \mathbf{y}, H; \mathbb{Z}_2\big) \stackrel{\text{def}}{=} \bigoplus_{\substack{\mathbf{x}' \in \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y}) \\ \mu(\mathbf{x}')=k}} \mathbb{Z}_2 \cdot \mathbf{x}', \tag{3.26}$$

which are free abelian groups isomorphic to $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. In order to have a chain complex also a boundary operator $\partial_k : C_k \to C_{k-1}$ is needed. By Proposition 3.32 we can choose a perturbation $\mathfrak{h}$ such that the set $\mathcal{S}^{J,H,\mathfrak{h}}([\mathbf{x}] \text{ rel } \mathbf{y})$ is isolated in $[\mathbf{x}] \text{ rel } \mathbf{y}$ and the orbits $\mathbf{u} \in$

$\mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ are non-degenerate for all pairs $\mathbf{x}_-, \mathbf{x}_+ \in \text{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$. Let $\widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+} = \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}/\mathbb{R}$ be the equivalence classes of orbits identified by translation in the s-variable. Consequently, $\widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}$ are smooth manifolds of dimension $\dim \widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+} = \mu(\mathbf{x}_-) - \mu(\mathbf{x}_+) - 1$.

LEMMA 3.35. *If $\mu(\mathbf{x}_-) - \mu(\mathbf{x}_+) = 1$, then $\widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ consists of finitely many equivalence classes.*

PROOF. From the compactness Theorem 3.8 and the geometric convergence in Corollaries 3.29 and 3.33 we derive that for any sequence $\{\mathbf{u}_n\} \subset \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ geometrically converges to a broken trajectory $(\mathbf{u}^1, \cdots, \mathbf{u}^m)$, with $\mathbf{u}^i \in \mathcal{M}^{J,H}_{\mathbf{x}^i,\mathbf{x}^{i-1}}([\mathbf{x}] \text{ rel } \mathbf{y})$, $i = 1, \cdots, m$ and $\mathbf{x}^i \in \text{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$, $i = 0, \cdots, m$, such that $\mu(\mathbf{x}^i) > \mu(\mathbf{x}^{i-1})$, for $i = 1, \cdots, m$. Since by assumption $\mu(\mathbf{x}_-) = \mu(\mathbf{x}_+) + 1$ it follows that $m = 1$ and $\mathbf{u}_n$ converges to a single orbit $\mathbf{u}^1 \in \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$. Therefore, the set $\widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ is compact. From Proposition 3.32 it follows that the orbits in $\widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ occur as isolated points and therefore $\widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ is a finite set. □

Define the boundary operator by

$$\partial_k(J,H,\mathfrak{h})\mathbf{x} \stackrel{\text{def}}{=} \sum_{\substack{\mathbf{x}' \in \text{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y}) \\ \mu(x')=k-1}} n(\mathbf{x},\mathbf{x}';J,H,\mathfrak{h})\mathbf{x}', \tag{3.27}$$

where

$$n(\mathbf{x},\mathbf{x}';J,H,\mathfrak{h}) = \left[\# \widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x},\mathbf{x}'}\right] \bmod 2 \in \mathbb{Z}_2.$$

The boundary operator ∂_k can be represented by a matrix consisting of 0's and 1's. The final property that the boundary operator has to satisfy is $\partial_{k-1} \circ \partial_k = 0$. Let us compute the expression

$$\begin{aligned}
\partial_{k-1}\big(\partial_k \mathbf{x}\big) &= \partial_{k-1}\left(\sum_{\substack{\mathbf{x}' \in \text{Crit}_H(\mathbf{x} \text{ rel } \mathbf{y}) \\ \mu(\mathbf{x}')=k-1}} n(\mathbf{x},\mathbf{x}';J,H,\mathfrak{h})\mathbf{x}' \right) \\
&= \sum_{\substack{\mathbf{x}' \in \text{Crit}_H(\mathbf{x} \text{ rel } \mathbf{y}) \\ \mu(\mathbf{x}')=k-1}} n(\mathbf{x},\mathbf{x}';J,H,\mathfrak{h})\partial_{k-1}\mathbf{x}' \\
&= \sum_{\substack{\mathbf{x}' \in \text{Crit}_H \\ \mu(\mathbf{x}')=k-1}} n(\mathbf{x},\mathbf{x}';J,H,\mathfrak{h}) \left(\sum_{\substack{\mathbf{x}'' \in \text{Crit}_H \\ \mu(\mathbf{x}'')=k-2}} n(\mathbf{x}',\mathbf{x}'';J,H,\mathfrak{h})\mathbf{x}'' \right) \\
&= \sum_{\substack{\mathbf{x}'' \in \text{Crit}_H \\ \mu(\mathbf{x}'')=k-2}} \sum_{\substack{\mathbf{x}' \in \text{Crit}_H \\ \mu(\mathbf{x}')=k-1}} n(\mathbf{x},\mathbf{x}';J,H,\mathfrak{h}) n(\mathbf{x}',\mathbf{x}'';J,H,\mathfrak{h})\mathbf{x}''.
\end{aligned}$$

The sum $\sum_{\substack{\mathbf{x}' \in \text{Crit}_H \\ \mu(\mathbf{x}')=k-1}} n(\mathbf{x},\mathbf{x}';J,H,\mathfrak{h}) n(\mathbf{x}',\mathbf{x}'';J,H,\mathfrak{h}) = m(\mathbf{x},\mathbf{x}'')$ is the number of 'broken connections' from $\mathbf{x}$ to $\mathbf{x}''$ modulo 2.

LEMMA 3.36. *If $\mu(\mathbf{x}_-) - \mu(\mathbf{x}_+) = 2$, then $\widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ is a smooth 1-dimensional manifold with finitely many connected components. The* non-compact *components can be identified with $(0,1)$ and the "closure" with $[0,1]$*[8]. *The limits ("boundary") $\{0,1\}$ correspond to unique pairs of distinct broken trajectories*

$$(\mathbf{u}^1, \mathbf{u}^2) \in \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}'}([\mathbf{x}] \text{ rel } \mathbf{y}) \times \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}',\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y}),$$

and

$$(\tilde{\mathbf{u}}^1, \tilde{\mathbf{u}}^2) \in \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}''}([\mathbf{x}] \text{ rel } \mathbf{y}) \times \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}'',\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y}),$$

with $\mu(\mathbf{x}'') = \mu(\mathbf{x}') = \mu(\mathbf{x}_-) - 1$.

PROOF. As in the proof of Lemma 3.35, from the compactness Theorem 3.8 and the geometric convergence in Corollaries 3.29 and 3.33 we derive that any sequence $\{\mathbf{u}_n\} \subset \mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}([\mathbf{x}] \text{ rel } \mathbf{y})$ geometrically converges to a broken trajectory $(\mathbf{u}^1, \mathbf{u}^2)$.

The results by Floer [**29**] show that for each pair $(\mathbf{u}^1, \mathbf{u}^2) \in \widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}'} \times \widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}',\mathbf{x}_+}$ there exists a unique local family of connecting orbits $\mathbf{u}'' \in \widehat{\mathcal{M}}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}$ — the gluing construction. Due to the local uniqueness of the gluing construction, the 'ends' 0 and 1 cannot coincide, and the "closure" of the non-compact components can be identified with $[0,1]$. □

This implies that the total number of broken connections from $\mathbf{x}$ to $\mathbf{x}''$ is even and thus $m(\mathbf{x}, \mathbf{x}'') = 0$, which proves that $\partial_{k-1} \circ \partial_k = 0$. Consequently,

$$\Big(C_*\big([\mathbf{x}] \text{ rel } \mathbf{y}, H; \mathbb{Z}_2\big), \partial_*(J,H,\mathfrak{h})\Big)$$

is a (finite) chain complex.

The homology of the chain complex (C_*, ∂_*) is defined as

$$FH_k\big([\mathbf{x}] \text{ rel } \mathbf{y}, J, H, \mathfrak{h}; \mathbb{Z}_2\big) \stackrel{\text{def}}{=} \frac{\ker \partial_k}{\text{im} \partial_{k+1}}, \tag{3.28}$$

and is called the Floer homology of $\big([\mathbf{x}] \text{ rel } \mathbf{y}, J, H, \mathfrak{h}\big)$. The Floer homology takes values in $\mathbb{Z}_2$ and is finite. It is not clear at this point that FH_* is independent of $J, H, \mathfrak{h}$ and whether FH_* is an invariant for proper relative braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$.

Continuation

Floer homology has a powerful invariance property with respect to 'large' variations in its parameters [**29**]. Let $[\mathbf{x}]$ rel $\mathbf{y}$ be a proper relative braid class and consider almost complex structures $J, \widetilde{J} \in \mathcal{J}^+$, generic Hamiltonians $H, \widetilde{H}$ such that $\mathbf{y} \in \text{Crit}_H \cap \text{Crit}_{\widetilde{H}}$ and functions $\mathfrak{h}, \widetilde{\mathfrak{h}}$ such that the connecting orbits are non-degenerate. Then the Floer homologies

$$FH_*\big([\mathbf{x}] \text{ rel } \mathbf{y}, J, H, \mathfrak{h}; \mathbb{Z}_2\big) \quad \text{and} \quad FH_*\big([\mathbf{x}] \text{ rel } \mathbf{y}, \widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}}; \mathbb{Z}_2\big),$$

are well-defined.

PROPOSITION 3.37. *Given a proper relative braid class* $[\mathbf{x}]$ rel $\mathbf{y}$ *it holds that*

$$FH_*\big([\mathbf{x}] \text{ rel } \mathbf{y}, J, H, \mathfrak{h}; \mathbb{Z}_2\big) \cong FH_*\big([\mathbf{x}] \text{ rel } \mathbf{y}, \widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}}; \mathbb{Z}_2\big),$$

under the hypotheses on $(J, H, \mathfrak{h})$ and $(\widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}})$ as stated above.

[8] This can more easily be formalized by considering the corresponding set $\mathcal{S}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}_+}$, but we do not go into details here.

In order to prove the isomorphism we follow the standard procedure in Floer homology. Consider the chain complexes

$$\Big(C_*\big([\mathbf{x}] \text{ rel } \mathbf{y}, H; \mathbb{Z}_2\big), \partial_*(J,H,\mathfrak{h})\Big) \quad \text{and} \quad \Big(C_*\big([\mathbf{x}] \text{ rel } \mathbf{y}, \widetilde{H}; \mathbb{Z}_2\big), \partial_*(\widetilde{J},\widetilde{H}.\widetilde{\mathfrak{h}})\Big)$$

and construct homomorphisms h_k satisfying the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_k(H) & \xrightarrow{\partial_k(J,H,\mathfrak{h})} & C_{k-1}(H) & \xrightarrow{\partial_{k-1}(J,H,\mathfrak{h})} & C_{k-2}(H) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h_{k-1}} & & \downarrow{\scriptstyle h_{k-2}} & & \\ \cdots & \longrightarrow & C_k(\widetilde{H}) & \xrightarrow{\partial_k(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})} & C_{k-1}(\widetilde{H}) & \xrightarrow{\partial_{k-1}(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})} & C_{k-2}(\widetilde{H}) & \longrightarrow & \cdots \end{array}$$

To define h_k consider the homotopies $\lambda \mapsto (J_\lambda, H_\lambda, \mathfrak{h}_\lambda)$ in $\mathcal{J}^+ \times \mathcal{H} \times C^\infty$ with $\lambda \in [0,1]$. In particular choose $H_\lambda = (1-\lambda)H + \lambda\widetilde{H}$ so that $\mathbf{y} \in \mathrm{Crit}_{H_\lambda}$ for all $\lambda \in [0,1]$ and $\mathfrak{h}_\lambda = (1-\lambda)\mathfrak{h} + \lambda\widetilde{\mathfrak{h}}$. It is important to notice that at the end points $\lambda = 0,1$ the systems are generic, but this is not necessarily true for all $\lambda \in (0,1)$. Define the smooth function $\lambda(s)$ as a function that satisfies $0 \le \lambda(s) \le 1$ and

$$\lambda(s) = \begin{cases} 0 & \text{for } s \le -R \\ 1 & \text{for } s \ge R, \end{cases}$$

for some $R > 0$. The non-autonomous Cauchy-Riemann equations in $(\mathbb{D}^2)^n$ then are

$$\mathbf{u}_s + \overline{J}_{\lambda(s)}\mathbf{u}_t + \nabla_{\overline{g}}\overline{H}_{\lambda(s)}(t,\mathbf{u}) + \nabla_{\overline{g}}\mathfrak{h}_{\lambda(s)}(t,\mathbf{u}) + \nabla_{\overline{g}}\mathfrak{h}'(s,t,\mathbf{u}) = 0, \tag{3.29}$$

where $\mathfrak{h}'$ is a perturbation as described in Corollary 3.34. By setting $J(s) = J_{\lambda(s)}$, $H(s,\cdot,\cdot) = H_{\lambda(s)}$ and $\mathfrak{h}(s,\cdot,\cdot) = \mathfrak{h}_{\lambda(s)}$ then Equation (3.29) fits in the framework of Equation (3.7) and for generic perturbations $\mathfrak{h}'$ the bounded orbits are non-degenerate. To simplify notation we will write $\mathfrak{h}'_\lambda = \mathfrak{h}_\lambda + \mathfrak{h}'$.

As before, denote the space of bounded solutions of the augmented Cauchy-Riemann equations by $\mathcal{M}^{J_\lambda,H_\lambda,\mathfrak{h}'_\lambda} = \mathcal{M}^{J_\lambda,H_\lambda,\mathfrak{h}'_\lambda}(\overline{\Omega}^m)$, and we derive the following basic compactness result.

PROPOSITION 3.38. *The space $\mathcal{M}^{J_\lambda,H_\lambda,\mathfrak{h}'_\lambda}$ is compact in the topology of uniform convergence on compact sets in $(s,t) \in \mathbb{R}^2$, with derivatives up to order 1. Moreover, $\mathbf{f}_{H_\lambda}$ is uniformly bounded along trajectories $\mathbf{u} \in \mathcal{M}^{J_\lambda,H_\lambda,\mathfrak{h}'_\lambda}$, and*

$$\lim_{s\to\pm\infty} |\mathbf{f}_{H_\lambda}(\mathbf{u}(s,\cdot))| = |c_\pm(\mathbf{u})| \le C(J,H),$$
$$\int_{\mathbb{R}}\int_0^1 |\mathbf{u}_s|^2 dt ds = \sum_{k=1}^n \int_{\mathbb{R}}\int_0^1 |u_s^k|^2 dt ds \le C'(J,H),$$

for all $\mathbf{u} \in \mathcal{M}$. Moreover,

$$\lim_{s\to-\infty} \mathbf{u}(s,\cdot) \in \mathrm{Crit}_H, \qquad \textit{and} \qquad \lim_{s\to+\infty} \mathbf{u}(s,\cdot)\mathrm{Crit}_{\widetilde{H}},$$

for any $\mathbf{u} \in \mathcal{M}^{J_\lambda,H_\lambda,\mathfrak{h}'_\lambda}$.

PROOF. The compactness follows from the estimates in Section 3.3 and the compactness in Proposition 3.8. Due to genericity, bounded solutions have limits in $\mathrm{Crit}_H \cup \mathrm{Crit}_{\widetilde{H}}$, see Corollary 3.28. □

We can define a homomorphism $h_k = h_k(J_\lambda, H_\lambda, \mathfrak{h}'_\lambda)$ Via the non-autonomous Cauchy-Riemann equations between the chain groups as follows. We recall that when we write $\mu(\mathbf{x})$ for $\mathbf{x} \in \mathrm{Crit}_H$ or $\mathbf{x} \in \mathrm{Crit}_{\widetilde{H}}$, then this is the Conley-Zehnder index $\mu(\mathbf{x})$ with respect to the Hamiltonian H or $\widetilde{H}$, respectively. For any $\mathbf{x} \in \mathrm{Crit}_H$ with $\mu(\mathbf{x}) = k$ we define

$$h_k \mathbf{x} = \sum_{\substack{\mathbf{x}' \in \mathrm{Crit}_{\widetilde{H}} \\ \mu(\mathbf{x}')=k}} n(\mathbf{x}, \mathbf{x}'; J_\lambda, H_\lambda, \mathfrak{h}'_\lambda)\mathbf{x}',$$

where

$$n(\mathbf{x}, \mathbf{x}'; J_\lambda, H_\lambda, \mathfrak{h}'_\lambda) = \left[\# \widehat{\mathcal{M}}^{J_\lambda, H_\lambda, \mathfrak{h}'_\lambda}_{\mathbf{x},\mathbf{x}'}\right] \bmod 2 \in \mathbb{Z}_2.$$

LEMMA 3.39. *The homomorphisms h_k defined above are chain homomorphisms, i.e.,*

$$\partial_k(\widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}}) \circ h_k = h_{k-1} \circ \partial_k(J, H, \mathfrak{h}),$$

for all $k \in \mathbb{Z}$.

PROOF. As for the boundary operator it holds that

$$\partial_k(\widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}})(h_k \mathbf{x}) = \sum_{\substack{\mathbf{x}'' \in \mathrm{Crit}_{\widetilde{H}} \\ \mu(\mathbf{x}'')=k-1}} \sum_{\substack{\mathbf{x}' \in \mathrm{Crit}_{\widetilde{H}} \\ \mu(\mathbf{x}')=k}} n(\mathbf{x}, \mathbf{x}'; J_\lambda, H_\lambda, \mathfrak{h}'_\lambda) n(\mathbf{x}', \mathbf{x}''; \widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}})\mathbf{x}'',$$

and

$$h_{k-1}\big(\partial_k(J, H, \mathfrak{h})\mathbf{x}\big) = \sum_{\substack{\mathbf{x}'' \in \mathrm{Crit}_{\widetilde{H}} \\ \mu(\mathbf{x}'')=k-1}} \sum_{\substack{\mathbf{x}' \in \mathrm{Crit}_{H} \\ \mu(\mathbf{x}')=k-1}} n(\mathbf{x}, \mathbf{x}'; J, H, \mathfrak{h}) n(\mathbf{x}', \mathbf{x}''; J_\lambda, H_\lambda, \mathfrak{h}'_\lambda)\mathbf{x}'',$$

We need to show that

$$\begin{aligned} &\sum_{\substack{\mathbf{x}' \in \mathrm{Crit}_{\widetilde{H}} \\ \mu(\mathbf{x}')=k}} n(\mathbf{x}, \mathbf{x}'; J_\lambda, H_\lambda, \mathfrak{h}'_\lambda) n(\mathbf{x}', \mathbf{x}''; \widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}}) \\ &\qquad = \sum_{\substack{\mathbf{x}' \in \mathrm{Crit}_{H} \\ \mu(\mathbf{x}')=k-1}} n(\mathbf{x}, \mathbf{x}'; J, H, \mathfrak{h}) n(\mathbf{x}', \mathbf{x}''; J_\lambda, H_\lambda, \mathfrak{h}'_\lambda) \mod 2 \end{aligned} \tag{3.30}$$

In order to establish this, one has to investigate space $\mathcal{M}^{J_\lambda, H_\lambda, \mathfrak{h}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}$, with $\mu(\mathbf{x}_-) = \mu(\mathbf{x}_+) + 1$; see [**?**] for all details. As before we use the characterization of $\mathcal{M}^{J_\lambda, H_\lambda, \mathfrak{h}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}$ and a non-autonomous version of the gluing lemma. From genericity and compactness we conclude that $\dim \mathcal{M}^{J_\lambda, H_\lambda, \mathfrak{h}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+} = 1$ and the closure of non-compact components can be identified with $[0,1]$. A sequence $\{\mathbf{u}_n\} \subset \mathcal{M}^{J_\lambda, H_\lambda, \mathfrak{h}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}$ geometrically to broken trajectories $(\mathbf{u}^1, \mathbf{u}^2)$ in either $\mathcal{M}^{J,H}_{\mathbf{x}_-,\mathbf{x}} \times \mathcal{M}^{J_\lambda, H_\lambda, \mathfrak{h}'_\lambda}_{\mathbf{x},\mathbf{x}_+}$ or $\mathcal{M}^{J_\lambda, H_\lambda, \mathfrak{h}'_\lambda}_{\mathbf{x}_-,\widetilde{\mathbf{x}}} \times \mathcal{M}^{\widetilde{J},\widetilde{H}}_{\widetilde{\mathbf{x}},\mathbf{x}_+}$, with $\mathbf{x} \in \mathrm{Crit}_H$ or $\widetilde{\mathbf{x}} \in \mathrm{Crit}_{\widetilde{H}}$, respectively. A non-autonomous version of the gluing principle shows that near broken trajectories of the above type there exist unique connecting orbits in $\mathcal{M}^{J_\lambda, H_\lambda, \mathfrak{h}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}$. This reveals three possible possible types of boundaries for the non-compact components, see Figure 3.9. The left and right diagrams contribute a broken connection to each of the sums in (3.30), whereas the middle one contributes two broken connections to the sum on the lefthand side, and none to the righthand side. Consequently, the sums in (3.30) can only differ by an even integer, which proves the lemma. □

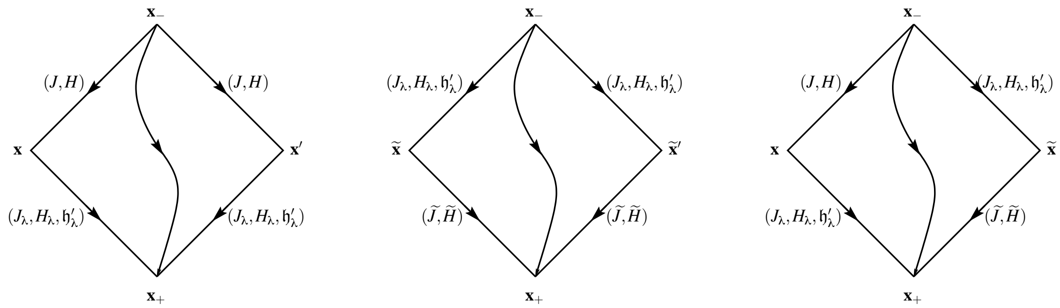

Figure 3.9: The three types of boundary behavior of non-compact components.

The mappings h_k are chain homomorphisms and induce a homomorphisms h_k^* on Floer homology:

$$h_k^*(J_\lambda,H_\lambda,\mathfrak{h}'_\lambda) : FH_*(\mathbf{x} \text{ rel } \mathbf{y};J,H,\mathfrak{h}) \to FH_*(\mathbf{x} \text{ rel } \mathbf{y};\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}}).$$

From a further analysis of the non-autonomous Cauchy-Riemann equations the standard procedures in Floer homology theory show that any two homotopies $(J_\lambda,H_\lambda,\mathfrak{h}'_\lambda)$ and $(\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda)$ between $(J,H,\mathfrak{h})$ and $(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})$ yield the same homomorphism in Floer homology:

LEMMA 3.40. *It holds, for any two homotopies* $(J_\lambda,H_\lambda,\mathfrak{h}'_\lambda)$ *and* $(\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda)$ *between* $(J,H,\mathfrak{h})$ *and* $(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})$*, that*

$$h_k^*(J_\lambda,H_\lambda,\mathfrak{h}'_\lambda) = h_k^*(\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda).$$

Moreover, for a homotopy $(J_\lambda,H_\lambda,\mathfrak{h}'_\lambda)$ *between* $(J,H,\mathfrak{h})$ *and* $(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})$ *and a homotopy* $(\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda)$ *between* $(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})$ *and* $(\check{J},\check{H},\check{\mathfrak{h}})$ *the induced homomorphism between the Floer homologies is given by*

$$h_k^* : FH_*([\mathbf{x}] \text{ rel } \mathbf{y},J,H,\mathfrak{h}) \to FH_*([\mathbf{x}] \text{ rel } \mathbf{y},\check{J},\check{H},\check{\mathfrak{h}}'),$$

where $h_k^* = h_k^*(\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda) \circ h_k^*(J_\lambda,H_\lambda,\mathfrak{h}'_\lambda)$.

PROOF. Let $(J_\lambda,H_\lambda,\mathfrak{h}'_\lambda)$ and $(\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda)$ be two generic homotopies from $(J,H,\mathfrak{h})$ to $(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})$ and h_* and $\hat{h}_*$ the corresponding chain homomorphisms. We will define a sequence of homomorphisms $\phi_k : C_k([\mathbf{x}] \text{ rel } \mathbf{y},H) \to C_{k+1}([\mathbf{x}] \text{ rel } \mathbf{y},\widetilde{H})$ such that

$$\hat{h}_k - h_k = \partial_{k+1}(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}}) \circ \phi_k - \phi_{k-1} \circ \partial_k(J,H,\mathfrak{h}). \tag{3.31}$$

Since ϕ is thus a chain homotopy between h and $\hat{h}$, it follows from standard arguments that h and $\hat{h}$ induce the same homomorphisms on homology. We start with the definition of ϕ_k. Let $J_\lambda^\nu,H_\lambda^\nu,\mathfrak{h}_\lambda^\nu$, $\nu \in [0,1]$, be a smooth homotopy between $(J_\lambda,H_\lambda,\mathfrak{h}'_\lambda)$ and $(\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda)$. Consider the spaces

$$\mathcal{M}^\nu_{\mathbf{x}_-,\mathbf{x}_+} = \big\{(\nu,\mathbf{u}) \mid \nu \in [0,1],\ \ \mathbf{u} \in \mathcal{M}^{J_\lambda^\nu,H_\lambda^\nu,\mathfrak{h}_\lambda^\nu}_{\mathbf{x}_-,\mathbf{x}_+}\big\},$$

for any $\mathbf{x}_- \in \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$ and $\mathbf{x}_+ \in \mathrm{Crit}_{\widetilde{H}}([\mathbf{x}] \text{ rel } \mathbf{y})$. For generic homotopies $J_\lambda^\nu,H_\lambda^\nu,\mathfrak{h}_\lambda^\nu$ the space $\mathcal{M}^\nu_{\mathbf{x}_-,\mathbf{x}_+}$ is a smooth manifold with boundary of dimension $\dim \mathcal{M}^\nu_{\mathbf{x}_-,\mathbf{x}_+} = \mu(x_-) - \mu(x_+) + 1$. If $\mu(x_-) - \mu(x_+) + 1 = 0$ then $\mathcal{M}^\nu_{\mathbf{x}_-,\mathbf{x}_+}$ consists of finitely many pairs $(\nu_i,\mathbf{u}_i)$ and

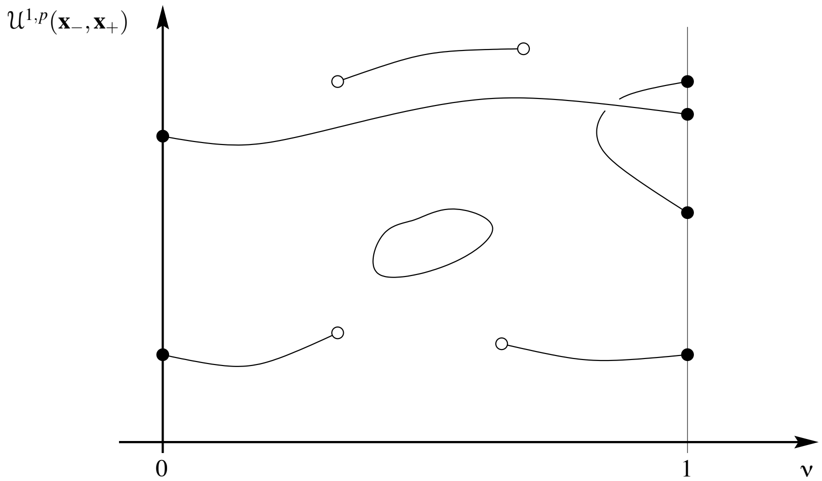

Figure 3.10: A schematic picture of $\mathcal{M}^{\nu}_{\mathbf{x}_-,\mathbf{x}_+}$ and the different possible connected components.

since the 'ends' are regular, $\mathcal{M}^{J^\nu_\lambda,H^\nu_\lambda,\mathfrak{h}^\nu_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}=\varnothing$ for $\nu=0,1$. Now define, for $\mathbf{x}\in\mathrm{Crit}_H$ with $\mu(\mathbf{x})=k$,

$$\phi_k(J^\nu_\lambda,H^\nu_\lambda,\mathfrak{h}^\nu_\lambda)\mathbf{x}=\sum_{\substack{\mathbf{x}'\in\mathrm{Crit}_{\widetilde{H}}\\ \mu(\mathbf{x}')=k+1}} n(\mathbf{x},\mathbf{x}';J^\nu_\lambda,H^\nu_\lambda,\mathfrak{h}^\nu_\lambda)\mathbf{x}',$$

where

$$n(\mathbf{x},\mathbf{x}';J^\nu_\lambda,H^\nu_\lambda,\mathfrak{h}^\nu_\lambda)=\left[\#\,\mathcal{M}^{\nu}_{\mathbf{x},\mathbf{x}'}\right]\bmod 2\in\mathbb{Z}_2.$$

Let us start with $\hat{h}_k-h_k$:

$$\left[\hat{h}_k-h_k\right]\mathbf{x}=\sum_{\substack{\mathbf{x}''\in\mathrm{Crit}_{\widetilde{H}}\\ \mu(\mathbf{x}'')=k}}\left(n\big(\mathbf{x},\mathbf{x}'';\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda\big)-n\left(\mathbf{x},\mathbf{x}'';J_\lambda,H_\lambda,\mathfrak{h}'_\lambda\right)\right)\mathbf{x}'',$$

and for $\tilde{\partial}_{k+1}\circ\phi_k-\phi_{k-1}\circ\partial_k$:

$$\begin{aligned}\left[\tilde{\partial}_{k+1}\circ\phi_k-\phi_{k-1}\circ\partial_k\right]\mathbf{x}&\\ =&\sum_{\substack{\mathbf{x}''\in\mathrm{Crit}_{\widetilde{H}}\\ \mu(\mathbf{x}'')=k}}\sum_{\substack{\mathbf{x}'\in\mathrm{Crit}_{\widetilde{H}}\\ \mu(\mathbf{x}')=k+1}} n(\mathbf{x},\mathbf{x}';J^\nu_\lambda,H^\nu_\lambda,\mathfrak{h}^\nu_\lambda)n(\mathbf{x}',\mathbf{x}'';\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})\mathbf{x}''\\ &-\sum_{\substack{\mathbf{x}''\in\mathrm{Crit}_{\widetilde{H}}\\ \mu(\mathbf{x}'')=k}}\sum_{\substack{\mathbf{x}'\in\mathrm{Crit}_{H}\\ \mu(\mathbf{x}')=k-1}} n(\mathbf{x},\mathbf{x}';J,H,\mathfrak{h})n(\mathbf{x}',\mathbf{x}'';J^\nu_\lambda,H^\nu_\lambda,\mathfrak{h}^\nu_\lambda)\mathbf{x}''.\end{aligned}$$

In order to prove that the two expressions are equal modulo 2 we consider the spaces $\mathcal{M}^\nu_{\mathbf{x}_-,\mathbf{x}_+}$ with $\mu(\mathbf{x}_-)=\mu(\mathbf{x}_+)$. Then $\mathcal{M}^\nu_{\mathbf{x}_-,\mathbf{x}_+}$ is a smooth manifold with boundary with $\dim\mathcal{M}^\nu_{\mathbf{x}_-,\mathbf{x}_+}=1$. As a matter of fact $\mathcal{M}^\nu_{\mathbf{x}_-,\mathbf{x}_+}\subset\mathcal{U}^{1,p}(\mathbf{x}_-,\mathbf{x}_+)$, see (3.24), and it is schematically depicted in Figure 3.10. The connected components are identified with either $[0,1]$, S^1, $[0,\infty)$, $(-\infty,1]$, or $(-\infty,\infty)$. Via yet another version of the gluing principle the 'open' ends are identified with broken trajectories [**?**]. Let us go through the different cases in order to prove the desired identity. Components diffeomorphic to S^1 do not contribute to

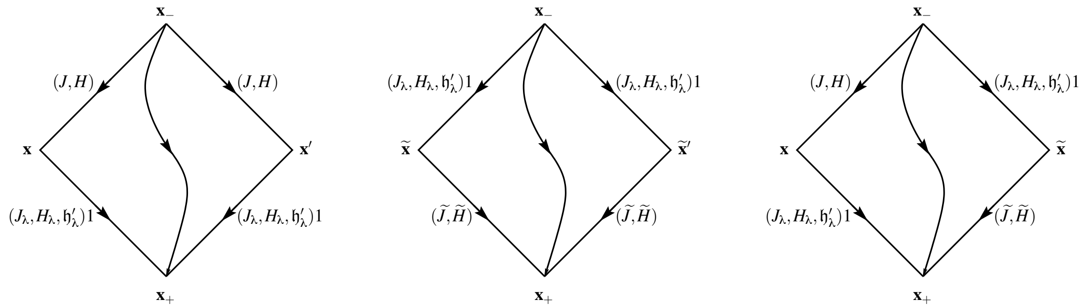

Figure 3.11: The three types of boundary behavior of non-compact components.

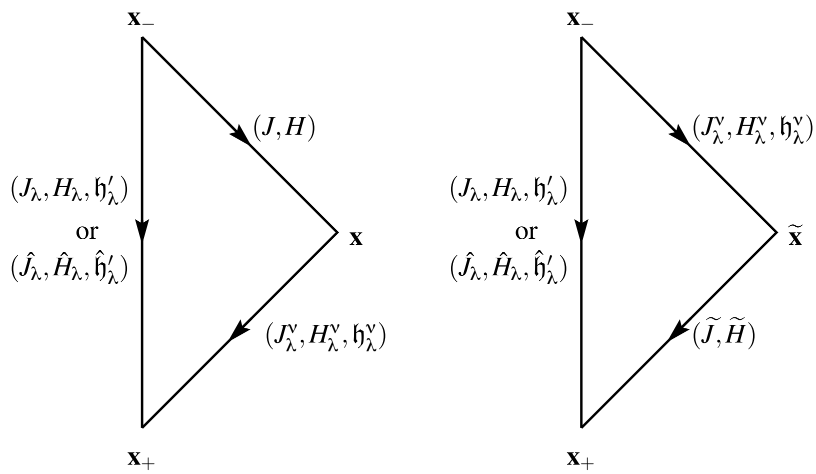

Figure 3.12: The two types of boundary behavior of non-compact components.

the above expressions. If a component corresponds to a closed interval $[0,1]$ then the limits $(0,\mathbf{u})$ and $(1,\mathbf{u})$ lie in $\mathcal{M}^{J_\lambda,H_\lambda,\mathfrak{h}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}$ or $\mathcal{M}^{\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}$ respectively. Therefore,

$$\left[n\big(\mathbf{x},\mathbf{x}'';\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda\big) - n\left(\mathbf{x},\mathbf{x}'';J_\lambda,H_\lambda,\mathfrak{h}'_\lambda\right)\right] \mod 2$$

counts the 'closed' ends modulo 2. For components corresponding to $(-\infty,\infty)$ we have to following cases of geometric convergence to the 'open' ends: the broken trajectories $(\mathbf{u}^1,\mathbf{u}^2)$ lie in either $\mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}} \times \mathcal{M}^{J^\vee_\lambda,H^\vee_\lambda,\mathfrak{h}^\vee_\lambda}_{\mathbf{x},\mathbf{x}_+}$, or $\mathcal{M}^{J^\vee_\lambda,H^\vee_\lambda,\mathfrak{h}^\vee_\lambda}_{\mathbf{x}_-,\widetilde{\mathbf{x}}} \times \mathcal{M}^{\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}}}_{\widetilde{\mathbf{x}},\mathbf{x}_+}$. Figure 3.11 shows the different possibilities. The terms in $\tilde{\partial}_{k+1}\circ\phi_k - \phi_{k-1}\circ\partial_k$ obtain an even contribution from the ends described in Figure 3.11. Modulo 2 these terms contribute 0 in $\tilde{\partial}_{k+1}\circ\phi_k - \phi_{k-1}\circ\partial_k$. Finally for components corresponding to $[0,\infty)$, or $(-\infty,1]$ the broken trajectories also lie in either $\mathcal{M}^{J,H,\mathfrak{h}}_{\mathbf{x}_-,\mathbf{x}} \times \mathcal{M}^{J^\vee_\lambda,H^\vee_\lambda,\mathfrak{h}^\vee_\lambda}_{\mathbf{x},\mathbf{x}_+}$, or $\mathcal{M}^{J^\vee_\lambda,H^\vee_\lambda,\mathfrak{h}^\vee_\lambda}_{\mathbf{x}_-,\widetilde{\mathbf{x}}} \times \mathcal{M}^{\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}}}_{\widetilde{\mathbf{x}},\mathbf{x}_+}$. Figure 3.12 shows the different possibilities. Clearly, the broken trajectories are balanced by trajectories in $\mathcal{M}^{J_\lambda,H_\lambda,\mathfrak{h}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}$ or $\mathcal{M}^{\hat{J}_\lambda,\hat{H}_\lambda,\hat{\mathfrak{h}}'_\lambda}_{\mathbf{x}_-,\mathbf{x}_+}$, and therefore modulo 2 the terms in $\tilde{\partial}_{k+1}\circ\phi_k - \phi_{k-1}\circ\partial_k$ add up to $\big(\hat{h}_k - h_k\big)$ modulo 2, which proves the identity. The identity implies now that $\hat{h}_k^* = h_k^*$ as induced homomorphisms in Floer homology.

To prove the composition property we construct yet another homotopy. Let $(J_\lambda, H_\lambda, \mathfrak{h}'_\lambda)$ and $(\hat{J}_\lambda, \hat{H}_\lambda, \hat{\mathfrak{h}}'_\lambda)$ be two homotopies from $(J,H,\mathfrak{h})$ to $(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})$ and form $(\widetilde{J},\widetilde{H},\widetilde{\mathfrak{h}})$ to $(\check{J},\check{H},\check{\mathfrak{h}})$ respectively. Define

$$(J_\lambda^R, H_\lambda^R, \mathfrak{h}_\lambda^R) = \begin{cases} \big(J_\lambda(s+R), H_\lambda(s+R,\cdot,\cdot), \mathfrak{h}'_\lambda(s+R,\cdot,\cdot)\big) & \text{for } s \le 0, \\ \big(\hat{J}_\lambda(s-R), \hat{H}_\lambda(s-R,\cdot,\cdot), \hat{\mathfrak{h}}'_\lambda(s-R,\cdot,\cdot)\big) & \text{for } s \ge 0, \end{cases}$$

for R sufficiently large. Using convergence and the gluing principle we conclude that for R sufficiently large it holds that

$$h_k^R = \hat{h}_k \circ h_k$$

where $h_k, \hat{h}_k$ are chain homomorphisms corresponding to $(J_\lambda, H_\lambda, \mathfrak{h}'_\lambda)$ and $(\hat{J}_\lambda, \hat{H}_\lambda, \hat{\mathfrak{h}}'_\lambda)$ respectively. By the previous we know that the choice of homotopy is arbitrary, and therefore $(h_k^R)^* = \hat{h}_k^* \circ h_k^*$ is the homomorphism between $FH_*([\mathbf{x}] \text{ rel } \mathbf{y}, J, H, \mathfrak{h})$ and $FH_*([\mathbf{x}] \text{ rel } \mathbf{y}, \check{J}, \check{H}, \check{\mathfrak{h}}')$. □

PROOF. (of Proposition 3.37). The properties given by this lemma guarantee that h_k^* is an isomorphism. To be more precise, consider the homotopies

$$h_k^* : FH_*(\mathbf{x} \text{ rel } \mathbf{y}; J, H, \mathfrak{h}) \to FH_*(\mathbf{x} \text{ rel } \mathbf{y}; \widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}}),$$

and

$$\bar{h}_k^* : FH_*(\mathbf{x} \text{ rel } \mathbf{y}; \widetilde{J}, \widetilde{H}, \widetilde{\mathfrak{h}}) \to FH_*(\mathbf{x} \text{ rel } \mathbf{y}; J, H, \mathfrak{h}),$$

then

$$\bar{h}_k^* \circ h_k^* : FH_*(\mathbf{x} \text{ rel } \mathbf{y}; J, H, \mathfrak{h}) \to FH_*(\mathbf{x} \text{ rel } \mathbf{y}; J, H, \mathfrak{h}).$$

Since a homotopy from $(J,H,\mathfrak{h})$ to itself induces the identity homomorphism on homology, it holds that $\bar{h}_k^* \circ h_k^* = \mathrm{Id}$. By the same token it follows that $h_k \circ \bar{h}_k^* = \mathrm{Id}$, which proves that $\bar{h}_k^* = (h_k^*)^{-1}$ and therefore the proposition. □

3.9. Admissible triples and independence of the skeleton

By proposition 3.37 the Floer homology of $[\mathbf{x}]$ rel $\mathbf{y}$, under the assumption that $\mathbf{y}$ is smooth and $\mathbf{y} \in \mathrm{Crit}_H$ for some Hamiltonian $H \in \mathcal{H}$, is independent of a generic triple $(J,H,\mathfrak{h})$, which justifies the notation $FH_*([\mathbf{x}] \text{ rel } \mathbf{y}; \mathbb{Z}_2)$. It remains to show that, firstly, for any braid class $[\mathbf{x}]$ rel $\mathbf{y}$ a triple exists, and thus the Floer homology is defined, and secondly that the Floer homology only depends on the braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$.

LEMMA 3.41. *Let $\mathbf{y} \in \Omega^m \cap C^3$, then there exists a Hamiltonian $H \in \mathcal{H}$ such that $\mathbf{y} \in \mathrm{Crit}_H$. In particular, when $\mathbf{y}$ is smooth, then H can be chosen to be in $\mathcal{H} \cap C^\infty$.*

PROOF. Let $\mathbf{y} = \{y^k\}_{k=1}^m$ and define $H^k(t,x) = \langle y_t^k, J_0 x\rangle$, which is a C^2-function on $\mathbb{R} \times \mathbb{R}^2$. Note that $H^k(t+1,x) = H^{\sigma(k)}(t,x)$, and H^k is smooth if $\mathbf{y}$ is smooth. The strand y^k is a solution of the Hamilton equations for H^k. This construction works for any $k = 1, \ldots, m$. Define tubular neighborhoods $A_\varepsilon^k = \bigcup_{t\in\mathbb{R}} B_\varepsilon(y^k(t)) \subset \mathbb{R} \times \mathbb{D}^2$, and $D_\varepsilon = \{x \in \mathbb{D}^2 \mid 1-\varepsilon < |x| \le 1\}$. Choose $\varepsilon > 0$ so small that the sets $\{A_\varepsilon^k\}_{k=1}^m$ and D_ε are all disjoint.

Define a cut-off function $\lambda_\varepsilon \in C^\infty([0,\infty),\mathbb{R})$ such that $\lambda(r) = 1$ for $0 \le r \le \varepsilon/4$ and $\lambda(r) = 0$ for $r \ge \varepsilon/2$. Let $\lambda_\varepsilon^k(t,x) = \lambda_\varepsilon(|x - y^k(t)|)$. Then λ_ε^k is a C^3 function with support in A_ε^k, and $\lambda_\varepsilon^k(t+1,x) = \lambda_\varepsilon^{\sigma(k)}(t,x)$.

Now define

$$H(t,x) \stackrel{\text{def}}{=} \sum_{k=1}^{m} \lambda_\varepsilon^k(t,x) H^k(t,x).$$

We claim that $H \in \mathcal{H}$. Indeed,

$$\begin{aligned} H(t+1,x) &= \sum_{k=1}^{m} \lambda_\varepsilon^k(t+1,x) H^k(t+1,x) \\ &= \sum_{k=1}^{m} \lambda_\varepsilon^{\sigma(k)}(t,x) H^{\sigma(k)}(t,x) = H(t,x). \end{aligned}$$

By the construction of H, it holds that $y_t^k = X_{H^k}(t,y^k) = X_H(t,y^k)$, since H restricts to H^k in a neighborhood of y^k. □

Lemma 3.41 establishes that the Floer homology $FH_*([\mathbf{x}] \text{ rel } \mathbf{y})$ is defined for any proper relative braid class $[\mathbf{x}] \text{ rel } \mathbf{y} \in \Omega^n \text{ rel } \mathbf{y}$ with $\mathbf{y} \in \Omega^m \cap C^\infty$. In order to refer to the Floer homology as an invariant we need to establish independence of the braid class in $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$, i.e. the Floer homology is the same for any two relative braid classes $[\mathbf{x}] \text{ rel } \mathbf{y}$, $[\mathbf{x}'] \text{ rel } \mathbf{y}'$such that $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!] = [\![\mathbf{x}' \text{ rel } \mathbf{y}']\!]$. This leads to the first main result of this paper.

THEOREM 3.42. *Let $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ be a proper relative braid class. It holds that*

$$FH_*([\mathbf{x}] \text{ rel } \mathbf{y}) \cong FH_*([\mathbf{x}]' \text{ rel } \mathbf{y}'),$$

for any two fibers $[\mathbf{x}] \text{ rel } \mathbf{y}$ and $[\mathbf{x}'] \text{ rel } \mathbf{y}'$ that are both in $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$. In particular,

$$FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]; \mathbb{Z}_2) \stackrel{\text{def}}{=} FH_*([\mathbf{x}] \text{ rel } \mathbf{y})$$

is an invariant of $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$.

PROOF. Let $\mathbf{y}, \mathbf{y}' \in \Omega^m \cap C^\infty$ and let $(\mathbf{x}(\lambda), \mathbf{y}(\lambda))$, $\lambda \in [0,1]$ be a smooth path[9] $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ which connects the pairs $\mathbf{x} \text{ rel } \mathbf{y}$ and $\mathbf{x}' \text{ rel } \mathbf{y}'$. Since $\mathbf{x}(\lambda) \text{ rel } \mathbf{y}(\lambda) \in [\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$, for all $\lambda \in [0,1]$, the sets $\mathcal{N}_\lambda = [\mathbf{x}(\lambda)] \text{ rel } \mathbf{y}(\lambda)$ are isolating neighborhoods for all λ. Choose smooth Hamiltonians H_λ such that $\mathbf{y}(\lambda) \in \mathrm{Crit}_{H_\lambda}$. The construction in the proof of Lemma 3.41 allows us to construct H_λ such that H_λ depends smoothly on the parameter λ. There are two philosophies one can follow now to prove this theorem. On the one hand, using the genericity theory in Section 3.7 (Corollary 3.34) we can choose a generic family $(J_\lambda, H_\lambda, \mathfrak{h}'_\lambda)$, for any smooth homotopy of almost complex structures J_λ. Then by repeating the proof (of Proposition 3.37) for this homotopy, we conclude that

$$FH_*([\mathbf{x}] \text{ rel } \mathbf{y}) \cong FH_*([\mathbf{x}'] \text{ rel } \mathbf{y}').$$

On the other hand, without having to redo to homotopy theory we note that $\mathcal{S}^{J_\lambda, H_\lambda}([\mathbf{x}(\lambda)] \text{ rel } \mathbf{y}(\lambda))$ is compact and isolated in $\mathcal{N}_\lambda$. Due to compactness and isolation there exists an ε_λ for each $\lambda \in [0,1]$ such that $\mathcal{N}_\lambda$ isolates $\mathcal{S}(\mathbf{x}(\lambda') \text{ rel } \mathbf{y}(\lambda'))$ for all λ' in $[\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda]$. Fix $\lambda_0 \in (0,1)$, then, by arguments similar to those used in the proof of Proposition 3.37, we have

$$FH_*(\mathcal{N}_{\lambda_0}, J_{\lambda_0}, H_{\lambda_0}, \mathfrak{h}_{\lambda_0}) \cong FH_*(\mathcal{N}_{\lambda_0}, J_{\lambda'}, H_{\lambda'}, \mathfrak{h}_{\lambda'}),$$

[9] A property of the braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ is that continuous paths can be approximated arbitrarily close by smooth paths.

for all $\lambda' \in [\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}]$. A compactness argument shows that, for ε'_{λ_0} sufficiently small, the sets of bounded solutions $\mathcal{M}^{J_{\lambda'},H_{\lambda'},\mathfrak{h}_{\lambda'}}(\mathcal{N}_{\lambda'})$ and $\mathcal{M}^{J_{\lambda'},H_{\lambda'},\mathfrak{h}_{\lambda'}}(\mathcal{N}_{\lambda_0})$ are identical, for all $\lambda' \in [\lambda_0 - \varepsilon'_{\lambda_0}, \lambda_0 + \varepsilon'_{\lambda_0}]$. Together these imply that

$$FH_*\big([\mathbf{x}(\lambda') \text{ rel } \mathbf{y}(\lambda')]\big) \cong FH_*\big([\mathbf{x}(\lambda_0) \text{ rel } \mathbf{y}(\lambda_0)]\big)$$

for $|\lambda' - \lambda_0| \leq \min\{\varepsilon_{\lambda_0}, \varepsilon'_{\lambda_0}\}$. Since $[0,1]$ is compact, any covering has a finite subcovering, which proves that $FH_*([\mathbf{x}] \text{ rel } \mathbf{y}) \cong FH_*([\mathbf{x}'] \text{ rel } \mathbf{y}')$.

Finally, since any skeleton $\mathbf{y}$ in $\pi([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$ can be approximated by a smooth skeleton $\mathbf{y}'$, the isolating neighborhood $\mathcal{N} = \pi^{-1}(\mathbf{y}) \cap [\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ is also isolating for $\mathbf{y}'$, i.e., we can define $FH_*(\mathcal{N}) \stackrel{\text{def}}{=} FH_*(\mathcal{N}')$. This defines $FH_*([\mathbf{x}] \text{ rel } \mathbf{y}) = FH_*(\mathcal{N})$ for any $\mathbf{y} \in \pi([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$. $\square$

CHAPTER 4

Properties and applications

4.1. Properties and interpretation of the braid class invariant

The braid class invariant $FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$ for proper relative braid classes has specific properties with respect to braided solutions of the Hamilton equations (3.1) on the 2-disc $\mathbb{D}^2$; non-triviality of the invariant yields braided solutions.

THEOREM 4.1. *Let $H \in \mathcal{H}$ and let $\mathbf{y} \in \mathrm{Crit}_H(\overline{\Omega}^m)$. Let $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ be a proper relative braid class. If*

$$FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]) \neq 0,$$

then $\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y}) \neq \varnothing$.

PROOF. Let $H_n \in \mathcal{H}$ be a sequence of Hamiltonians such that $H_n \to H$ in $\mathcal{H}$, i.e. convergence in $C^2(\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2; \mathbb{R})$. If $FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]) \neq 0$, then $C_*([\mathbf{x}] \text{ rel } \mathbf{y}, H_n; \mathbb{Z}_2) \neq 0$, for any n since

$$H_*\big(C_*([\mathbf{x}] \text{ rel } \mathbf{y}, H_n; \mathbb{Z}_2), \partial_*\big) \cong FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]) \neq 0,$$

where $\partial_* = \partial_*(J, H_n, \mathfrak{h}_n)$ (see Section 3.8). Consequently, $\mathrm{Crit}_{H_n}([\mathbf{x}] \text{ rel } \mathbf{y}) \neq \varnothing$. The strands x_n^k satisfy the equation $x_n^{k\prime} = X_{H_n}(t, x_n^k)$ and therefore $\|x_n^k\|_{C^1([0,1])} \leq C$. By the compactness of $C^1([0,1]) \hookrightarrow C^0([0,1])$ it follows that (along a subsequence) $x_n^k \to x^k \in C^0([0,1])$. The right hand side of the Hamilton equations now converges, i.e., $X_{H_n}(t, x_n^k(t)) \to X_H(t, x(t))$ pointwise in $t \in [0,1]$, and thus also $x_n^k \to x^k$ in $C^1([0,1])$. This holds for any strand x^k and therefore produces a limit $\mathbf{x} \in \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$, which proves the theorem. □

Let $\beta_k = \dim FH_k([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]; \mathbb{Z}_2)$ be the $\mathbb{Z}_2$-Betti numbers of the braid class invariant. Its Poincaré series is defined as

$$P_t([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]) = \sum_{k \in \mathbb{Z}} \beta_k([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]) t^k.$$

A fundamental property of the braid class invariant can be expressed as follows.

THEOREM 4.2. *Let $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ be a proper relative braid class. Then, there exists an integer $n_0 \geq 0$ such that*

$$t^{n_0} P_t([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]) \in \mathbb{Z}^+[t],$$

i.e. $t^{n_0} P_t$ is polynomial with coefficients in $\mathbb{Z}^+$.

PROOF. Assume without loss of generality that $\mathbf{y}$ is a smooth skeleton and choose a smooth generic Hamiltonian H such that $\mathbf{y} \in \mathrm{Crit}_H$. Since the Floer homology is the same for all braid classes $[\mathbf{x}]$ rel $\mathbf{y} \in [\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ and all Hamiltonians H satisfying the above, $FH_*([\mathbf{x}] \text{ rel } \mathbf{y}, J, H, \mathfrak{h}) \cong FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$. Let $c_k = \dim C_k$, then by definition of the Betti numbers

$$c_k([\mathbf{x}] \text{ rel } \mathbf{y}, H) \geq \dim\ker C_k \geq \beta_k([\mathbf{x}] \text{ rel } \mathbf{y}, J, H, \mathfrak{h}) = \beta_k([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]).$$

Since H is generic, all critical points are non-degenerate, and it follows from compactness that $\sum_k c_k < \infty$. Therefore $c_k < \infty$, and $c_k \neq 0$ for finitely many k. By the above bound $\beta_k \leq c_k < \infty$, which proves the finiteness of the Floer homology. Now choose $n_0 = \min\{n \geq 0 \mid c_k = 0,\ \forall k < -n\} > 0$, which completes the proof. □

In the case that H is a generic Hamiltonian a more detailed result follows. Both $\bigoplus_k FH_k([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]; \mathbb{Z}_2)$ and $\bigoplus_k C_k([\mathbf{x}] \text{ rel } \mathbf{y}, H; \mathbb{Z}_2)$ are graded $\mathbb{Z}_2$-modules and their Poincaré series are well-defined and

$$P_t\big(\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})\big) = \sum_{k\in\mathbb{Z}} c_k([\mathbf{x}] \text{ rel } \mathbf{y}, H) t^k,$$

where $c_k = \dim C_k([\mathbf{x}] \text{ rel } \mathbf{y}, H; \mathbb{Z}_2)$.

THEOREM 4.3. *Let $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ be a proper relative braid class and H a generic Hamiltonian such that $\mathbf{y} \in \mathrm{Crit}_H$ [1] for a given skeleton $\mathbf{y}$. Then*

$$P_t\big(\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})\big) = P_t([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]) + (1+t)Q_t, \tag{4.1}$$

where $Q_t \geq 0$. In addition, $\#\, \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y}) \geq P_1([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$.

PROOF. Let $\mathbf{y}'$ be a smooth skeleton that approximates $\mathbf{y}$ arbitrarily close in C^2 and let H' be an associated smooth generic Hamiltonian. We start by proving (4.1) in the smooth case. Define $Z_k = \ker \partial_k$, $B_k = \mathrm{im}\ \partial_{k+1}$ and $B_k \subset Z_k \subset C_k([\mathbf{x}] \text{ rel } \mathbf{y}', H')$ by the fact that ∂_* is a boundary map. This yields the following short exact sequence

$$0 \xrightarrow{\ \mathrm{Id}\ } B_k \xrightarrow{\ i_k\ } Z_k \xrightarrow{\ j_k\ } FH_k = \frac{Z_k}{B_k} \xrightarrow{\ 0\ } 0.$$

The maps i_k and j_k are defined as follows: $i_k(\mathbf{x}) = \mathbf{x}$ and $j_k(\mathbf{x}) = \{\mathbf{x}\}$, the equivalence class in FH_k. Exactness is satisfied since $\ker i_k = 0 = \mathrm{im}\ \mathrm{Id}$, $\ker j_k = B_k = \mathrm{im}\ i_k$ and $\ker 0 = FH_k = \mathrm{im}\ j_k$. Upon inspection of the short exact sequence we obtain that

$$\dim Z_k = \dim B_k + \dim FH_k.$$

Indeed, by exactness, $Z_k \supset \ker j_k = B_k$ and $\mathrm{im}\ j_k = FH_k$ (onto) and therefore $\dim Z_k = \dim\ker j_k + \dim\mathrm{im}\ j_k = \dim B_k + \dim FH_k$. Similarly, we have the short exact sequence

$$0 \xrightarrow{\ \mathrm{Id}\ } Z_k \xrightarrow{\ i_k\ } C_k \xrightarrow{\ \partial_k\ } B_{k-1} \xrightarrow{\ 0\ } 0.$$

Hence $C_k \cong Z_k \oplus B_{k-1}$, and it holds that

$$\dim C_k = \dim Z_k + \dim B_{k-1}.$$

[1] We do not assume that $\mathbf{y}$ is a smooth skeleton!

Combining these equalities gives $\dim C_k = \dim FH_k + \dim B_{k-1} + B_k$. On the level of Poincaré series this gives

$$P_t(\oplus_k C_k) = P_t(\oplus_k FH_k) + (1+t)P_t(\oplus_k B_k),$$

which proves (4.1) in the case of smooth skeletons.

Now choose sequences $\mathbf{y}_n \to \mathbf{y}$ and $H_n \to H$ in C^2 (H_n generic). For each n the above identity is satisfied and since also H is generic (by assumption), it follows from hyperbolicity that $P_t\big(\mathrm{Crit}_{H_n}([\mathbf{x}] \text{ rel } \mathbf{y}_n)\big) = P_t\big(\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})\big)$ for n large enough. This then proves (4.1). Using the fact that all series are positive, the substitution $t = 1$ gives the lower bound on the number of stationary braids. □

An important question is whether $FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$ also contains information about $\mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y})$ in the non-generic case besides the result in Theorem 4.1. In [**33**] such a result was indeed obtained for the Conley index of discretized relative braid classes, and a detailed study of the spectral properties of stationary braids will most likely reveal a similar property here. We conjecture that $\#\ \mathrm{Crit}_H([\mathbf{x}] \text{ rel } \mathbf{y}) \geq \mathrm{length}\big(FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])\big)$, where $\mathrm{length}(FH_*)$ equals the number of monomial terms in $P_t\big([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]\big)$.

4.2. Homology shifts and Garside's normal form

In this section we show that composing a braid class with full twists yields a shift in Floer homology.

Full twists and homology shifts

In the Definition 3.25 the Conley-Zehnder index of a stationary braid $\mathbf{x} \in \mathrm{Crit}_H$ was defined as the permuted Conley-Zehnder index of the symplectic path $\Psi : [0,1] \to \mathrm{Sp}(2n, \mathbb{R})$ defined by

$$\frac{d\Psi}{dt} - \overline{J}_0 d^2\overline{H}(t, \mathbf{x}(t))\Psi = 0, \qquad \Psi(0) = \mathrm{Id}. \tag{4.2}$$

Consider the symplectic path $S : [0,1] \to \mathrm{Sp}(2,\mathbb{R})$ defined by $S(t) = e^{2\pi J_0 t}$, which rotates the variables over 2π as t goes from 0 to 1. On the product $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \cong \mathbb{R}^{2n}$ this yields $\overline{S}(t) = e^{2\pi \overline{J}_0 t}$, which is a path in $\mathrm{Sp}(2n,\mathbb{R})$. Let $\widehat{\mathbf{x}} = \overline{S}(t)\mathbf{x}$, or equivalently $\mathbf{x} = \overline{S}(-t)\widehat{\mathbf{x}}$, then $\widehat{\mathbf{x}} \in \mathrm{Crit}_{\widehat{H}}$, where $\widehat{H}(t,\widehat{x}) = H(t, e^{-2\pi J_0 t}\widehat{x}) + \pi|\widehat{x}|^2 - \pi$ and $\widehat{H} \in \mathcal{H}$. Indeed, upon substitution in (3.1) we obtain the transformed Hamilton equations for $\widehat{\mathbf{x}}$:

$$\widehat{x}^k_t - e^{2\pi J_0 t} J_0 \nabla H(t, e^{-2\pi J_0 t}\widehat{x}^k) - 2\pi J_0 \widehat{x}^k = 0, \tag{4.3}$$

which are the Hamilton equations for $\widehat{H}$. There exists a relation between the Conley-Zehnder indices $\mu(\mathbf{x})$ and $\mu(\widehat{\mathbf{x}})$:

LEMMA 4.4. *Let $\mathbf{x} \in \mathrm{Crit}_H$ and $\widehat{\mathbf{x}} = \overline{S}(t)\mathbf{x}$, with $\widehat{\mathbf{x}} \in \mathrm{Crit}_{\widehat{H}}$, then*

$$\mu(\widehat{\mathbf{x}}) = \mu(\mathbf{x}) + 2n, \tag{4.4}$$

where n equals the number of strands in $\mathbf{x}$. More generally, for any $g \in \mathbb{Z}$ and $\widehat{\mathbf{x}} = \overline{S}^g(t)\mathbf{x}$ it holds that $\mu(\widehat{\mathbf{x}}) = \mu(\mathbf{x}) + 2ng$.

PROOF. In order to compute the Conley-Zehnder index of $\widehat{\mathbf{x}}$ we linearize Equation (4.3) in $\widehat{\mathbf{x}}$, which yields

$$\frac{d\widehat{\Psi}}{dt} - \overline{S}(t)J_0 d^2\overline{H}(t,\overline{S}(-t)\widehat{\mathbf{x}}(t))\overline{S}(-t)\widehat{\Psi} - 2\pi\overline{J}_0\widehat{\Psi} = 0, \qquad \widehat{\Psi}(0) = \mathrm{Id}.$$

One verifies that Ψ in Equation (4.2) and $\widehat{\Psi}$ are related as follows: $\Psi = \overline{S}(-t)\widehat{\Psi}$. From Lemma 3.22 and the fact that $\mu(e^{2\pi J_0 t}) = 2$, it follows that

$$\begin{aligned}
\mu(\widehat{\mathbf{x}}) &= \mu_\sigma(\widehat{\Psi},1) = \mu(\overline{S}\Psi,1)\\
&= \mu_\sigma(\Psi,1) + \mu(\overline{S}) = \mu_\sigma(\Psi,1) + n\mu(e^{2\pi J_0 t})\\
&= \mu(\mathbf{x}) + 2n.
\end{aligned}$$

The proof of the second statement is a straightforward generalization of this argument. □

Consider the braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ and define the rotation

$$(\widehat{\mathbf{x}},\widehat{\mathbf{y}}) := \overline{S}^g(t)(\mathbf{x},\mathbf{y}) = \big(\overline{S}^g_n(t)\mathbf{x}, \overline{S}^g_m(t)\mathbf{y}\big),$$

Via the rotation $\overline{S}^g$ we can relate the Floer homologies of $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ with Hamiltonian H and $[\![\widehat{\mathbf{x}} \text{ rel } \widehat{\mathbf{y}}]\!]$ with Hamiltonian $\widehat{H}$ via the index shift in Lemma 4.4. Since the Floer homologies do not depend on the choice of Hamiltonian we obtain the following relation.

THEOREM 4.5. *Let* $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ *be a proper relative braid class and let* $[\![\widehat{\mathbf{x}} \text{ rel } \widehat{\mathbf{y}}]\!]$ *be as defined above. Then*

$$FH_k([\![\widehat{\mathbf{x}} \text{ rel } \widehat{\mathbf{y}}]\!]) \cong FH_{k-2ng}([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]), \quad \forall k \in \mathbb{Z},$$

for any $g \in \mathbb{Z}$.

PROOF. The Floer homology for $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ is defined by choosing a generic Hamiltonian H. From Lemma 4.4 we have that $\mu(\widehat{\mathbf{x}}) = \mu(\mathbf{x}) + 2ng$ and therefore

$$C_k([\widehat{\mathbf{x}}] \text{ rel } \widehat{\mathbf{y}}, \widehat{H}; \mathbb{Z}_2) = C_{k-2ng}([\mathbf{x}] \text{ rel } \mathbf{y}, H; \mathbb{Z}_2).$$

Since the solutions $\widehat{\mathbf{u}} \in \mathcal{M}^{J,\widehat{H},\widehat{\mathfrak{h}}}$ are obtained from the solutions $\mathbf{u} \in \mathcal{M}^{J,H,\mathfrak{h}}$ via $\widehat{\mathbf{u}} = \overline{S}^g_n(t)\mathbf{u}$, it also holds that

$$\partial_k(J,\widehat{H},\widehat{\mathfrak{h}}) = \partial_{k-2ng}(J,H,\mathfrak{h}),$$

and thus $FH_k([\![\widehat{\mathbf{x}} \text{ rel } \widehat{\mathbf{y}}]\!]) \cong FH_{k-2ng}([\![\mathbf{x} \text{ rel } \mathbf{y}]\!])$. □

Garside's normal form

For a given braid $\mathbf{x} \in \Omega^n$ denote by $\beta(\mathbf{x}) \in \mathcal{B}_n$ a representation of $\mathbf{x}$ in the braid group, and let β_i be the positive generators of $\mathcal{B}_n$. For any given closed braid $\mathbf{x} \in \Omega^n$ there exists a unique representation of the form

$$\beta(\mathbf{x}) = \Delta^g \cdot \beta^+ = \hat{\beta}^+ \cdot \Delta^g, \quad g \in \mathbb{Z}, \quad \beta^+ \in \mathcal{S}_n, \tag{4.5}$$

where $\Delta = (\beta_1 \cdots \beta_{n-1})(\beta_1 \cdots \beta_{n-2}) \cdots (\beta_1\beta_2)\beta_1$ represents a (positive) half-twist, or Garside element, in braid group $\mathcal{B}_n$ and β^+ is a unique word in $\mathcal{S}_n$, the semi-group of positive braids in $\mathcal{B}_n$. The word $\hat{\beta}^+$ is found from β^+ by replacing all β_i by β_{n-i}. The integer g is called the *power* of β, and β^+ the *tail* of β. Equation (4.5) is referred to as the *Garside normal form* of β, see e.g. [**?**], [**32**]. Roughly speaking g is the *largest* integer such that $\Delta^{-g} \cdot \beta(\mathbf{x})$ is equivalent in $\mathcal{B}_n$ to a positive braid word. The unique choice of the positive word β^+ follows from the algorithm, see [**?**], [**32**]. The Garside normal form provides a solution to the word problem in $\mathcal{B}_n$: two words $\beta, \beta' \in \mathcal{B}_n$ are equivalent in $\mathcal{B}_n$ if and only if their Garside normal forms coincide.

For example consider $\mathcal{B}_3$ with generators β_1 and β_2 and consider the word $\beta = \beta_1\beta_2^{-1}$; its Garside normal form is given by

$$\beta = \Delta^{-1} \cdot \beta_2^2\beta_1 = \beta_1^2\beta_2 \cdot \Delta^{-1}.$$

Since Ω^n consists of equivalence classes of closed braids, the braid class $[\mathbf{x}]$ corresponds to the *conjugacy* class of $[\beta(\mathbf{x})]$ in the braid group $\mathcal{B}_n$. For instance the words $\beta = \beta_1\beta_2^{-1}$ and $\beta' = \beta_2^{-1}\beta_1$ are conjugate in $\mathcal{B}_3$ but the left normal forms are different:

$$\beta = \Delta^{-1} \cdot \beta_2^2\beta_1, \quad \beta' = \Delta^{-1} \cdot \beta_2\beta_1^2.$$

Clearly, the Garside normal form is not a normal form for conjugacy classes and therefore not suited to solving the conjugacy problem in $\mathcal{B}_n$. However, in a similar spirit one can derive normal forms for conjugacy classes from Garside's normal form. These are used to solve the more complicated conjugacy problem, see e.g. [**?**], [**32**].

A second unique normal form of a braid $\mathbf{x} \in \Omega^n$ is obtained by considering *full* twists:

$$\beta(\mathbf{x}) = \square^{\lfloor g/2 \rfloor} \cdot (\Delta^r \beta^+), \quad g \in \mathbb{Z}, \quad \beta^+ \in \mathcal{S}_n, \tag{4.6}$$

where $\square = \Delta^2 = (\beta_1 \cdots \beta_{n-1})^n$ represents a full twist, and generates the center of the braid group $\mathcal{B}_n$, $\lfloor g/2 \rfloor$ is the largest integer less or equal to $g/2$, and $r = g - 2\lfloor g/2 \rfloor \in \{0,1\}$. This form is derived from the Garside normal form for β. Slightly abusing terminilogy, we will again call this the Garside normal form. Since full twists generate the center of $\mathcal{B}_n$ they commute with all elements in $\mathcal{B}_n$.

LEMMA 4.6. *Let* $\mathbf{x}, \widehat{\mathbf{x}} \in \Omega^n$ *be such that* $\beta(\widehat{\mathbf{x}}) = \square^\ell \cdot \beta(\mathbf{x})$, *for some* $\ell \in \mathbb{Z}$. *Then* $\overline{S}^\ell(t)\mathbf{x} \in [\widehat{\mathbf{x}}]$. *If* ℓ *is sufficiently large then* $\beta(\widehat{\mathbf{x}})$ *is conjugate to a positive braid word and* $[\widehat{\mathbf{x}}]$ *is a positive braid class.*

PROOF. Define the homotopy $\lambda \mapsto \mathbf{x}_{1+(\varepsilon-1)\lambda}$ in $[\mathbf{x}]$, $\lambda \in [0,1]$, where

$$\mathbf{x}_{1+(\varepsilon-1)\lambda}(t) = \begin{cases} \mathbf{x}\big(t/(1+(\varepsilon-1)\lambda)\big) & \text{for } t \in [0, 1+(\varepsilon-1)\lambda] \\ \mathbf{x}(1) & \text{for } t \in [1+(\varepsilon-1)\lambda, 1]. \end{cases}$$

If $\varepsilon > 0$ is sufficiently small then $\beta(\overline{S}^\ell(t)\mathbf{x}_\varepsilon)$ is a composition of $\beta(\mathbf{x}_\varepsilon) = \beta(\mathbf{x})$ with ℓ full twists:

$$\beta\big(\overline{S}^\ell(t)\mathbf{x}_\varepsilon\big) = \square^\ell \cdot \beta(\mathbf{x}_\varepsilon) = \square^\ell \cdot \beta(\mathbf{x}) = \beta(\widehat{\mathbf{x}}).$$

By construction $\overline{S}^\ell(t)\mathbf{x}_{1+(\varepsilon-1)\lambda}$ is a path in Ω^n, and therefore $\overline{S}^\ell(t)\mathbf{x}$ is homotopic to $\overline{S}^\ell(t)\mathbf{x}_\varepsilon$, i.e., $\big[\overline{S}^\ell(t)\mathbf{x}_\varepsilon\big] = \big[\overline{S}^\ell(t)\mathbf{x}\big]$, and consequently

$$\beta\big(\overline{S}^\ell(t)\mathbf{x}\big) = \beta\big(\overline{S}^\ell(t)\mathbf{x}_\varepsilon\big) = \beta(\widehat{\mathbf{x}}),$$

which proves that $\overline{S}^\ell(t)\mathbf{x} \in [\widehat{\mathbf{x}}]$. From (4.6) we choose $\ell = -\lfloor g/2 \rfloor$, which then guarantees that $\beta(\widehat{\mathbf{x}})$ represents a positive conjugacy class. $\square$

If we apply the above to $\mathbf{x}\cup\mathbf{y}$, we obtain a minimal integer ℓ such that

$$\beta(\mathbf{x}^+\cup\mathbf{y}^+) = \square^\ell\cdot\beta(\mathbf{x}\cup\mathbf{y})$$

is a positive braid, and for the pair $(\widehat{\mathbf{x}},\widehat{\mathbf{y}}) = \overline{S}^\ell(t)(\mathbf{x},\mathbf{y})$ it holds that $\widehat{\mathbf{x}}\,\mathrm{rel}\,\widehat{\mathbf{y}} \in [\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]$. This yields the following theorem that relates the Floer homology of arbitrary proper braid classes to the Floer homology of proper positive relative braid classes.

THEOREM 4.7. *Let* $[\![\mathbf{x}\ \mathrm{rel}\ \mathbf{y}]\!]$ *be a proper relative braid type, and its Garside normal form be given by* $\beta(\mathbf{x}\cup\mathbf{y}) = \square^{-\ell}\cdot\beta(\mathbf{x}^+\cup\mathbf{y}^+)$. *Then* $[\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]$ *is a proper positive relative braid class, and*

$$FH_k([\![\mathbf{x}\ \mathrm{rel}\ \mathbf{y}]\!]) \cong FH_{k+2n\ell}([\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]),\quad \textit{for all } k\in\mathbb{Z}.$$

PROOF. Let $(\widehat{\mathbf{x}},\widehat{\mathbf{y}}) = \overline{S}^\ell(t)(\mathbf{x},\mathbf{y})$, then by Lemma 4.6 $\widehat{\mathbf{x}}\,\mathrm{rel}\,\widehat{\mathbf{y}} \in [\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]$, and since FH_* is an invariant of $[\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]$ it follows that $FH_k([\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]) \cong FH_k([\![\widehat{\mathbf{x}}\ \mathrm{rel}\ \widehat{\mathbf{y}}]\!])$. By Theorem 4.5, $FH_k([\![\widehat{\mathbf{x}}\ \mathrm{rel}\ \widehat{\mathbf{y}}]\!]) \cong FH_{k-2n\ell}([\![\mathbf{x}\ \mathrm{rel}\ \mathbf{y}]\!])$, which proves the second part of the theorem.

We still need to prove that $[\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]$ is proper. Suppose, by contradiction, that $[\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]$ is not proper. Then there is a fiber $[\mathbf{x}^+]\ \mathrm{rel}\ \mathbf{y}^+ \subset [\![\mathbf{x}^+\ \mathrm{rel}\ \mathbf{y}^+]\!]$ such that there is a "collapsing" continuous path $\mathbf{x}_s$, $s\in[0,1]$, i.e., $\mathbf{x}_s\in[\mathbf{x}^+]\ \mathrm{rel}\ \mathbf{y}^+$ for $s\in[0,1)$, whereas $\mathbf{x}_1 \in (\Sigma_-\ \mathrm{rel}\ \mathbf{y}^+)\cup\partial\mathbb{D}^2$.

Let us define the ℓ-rotated extension of $\mathbf{x}$ as

$$E^\ell(\mathbf{x})(t) = \begin{cases} \mathbf{x}(2t) & 0\le t\le \frac12, \\ \overline{S}^\ell(2(t-\frac12))\mathbf{x}(1) & \frac12\le t\le 1. \end{cases}$$

For any integer ℓ and any braid $\mathbf{x}$, the rotated extension $E^\ell(\mathbf{x})$ is again a braid, since $E^\ell(\mathbf{x})^k(1) = x^k(1) = x^{\sigma(k)}(0) = E^\ell(\mathbf{x})^{\sigma(k)}(0)$. Similarly, if $\mathbf{x}\,\mathrm{rel}\,\mathbf{y}$ is a relative braid, then $E^\ell(\mathbf{x}\cup\mathbf{y})$ is a braid, hence $E^\ell(\mathbf{x})\ \mathrm{rel}\ E^\ell(\mathbf{y})$ is again a relative braid.

We conclude that $E^{-\ell}(\mathbf{x}_s)$ is a collapsing path in the fiber $[E^{-\ell}(\mathbf{x}^+)]\ \mathrm{rel}\ E^{-\ell}(\mathbf{y}^+) \subset [\![\mathbf{x}\ \mathrm{rel}\ \mathbf{y}]\!]$, contradicting the properness of the relative braid class $[\![\mathbf{x}\ \mathrm{rel}\ \mathbf{y}]\!]$. $\square$

4.3. Cyclic braid classes and their Floer homology

In this section we compute the Floer homology groups of various braid classes of cyclic type. The cases we consider here can be computed by continuing the skeletion and the Hamiltonians to a Hamiltonian system for which the space of bounded solutions can be determined.

Single-strand rotations and symplectic polar coordinates

Consider Hamiltonians of the form

$$H(x) = F(|x|) + \omega_\delta(|x|)G\big(\arg(x)\big),$$

where $\arg(x)=\theta$ is the argument and $G(\theta+2\pi) = G(\theta)$. The cut-off function ω_δ is chosen such that $\omega_\delta(|x|) = 0$ for $|x|\le\delta$ and $|x|\ge 1-\delta$, and $\omega_\delta(|x|) = 1$ for $2\delta\le|x|\le 1-2\delta$. In the special case that $G(\theta)\equiv 0$, then the Hamilton equations are given by

$$x_t = J_0\nabla H(x) = J_0\frac{f(|x|)}{|x|}x,$$

where $f(r) = F'(r)$. Solutions of the Hamilton equations are given by $x(t) = l(r\cos l(\frac{f(r)}{r}t + \theta^0\boldsymbol{r}), r\sin\left(\frac{f(r)}{r}t + \theta^0\boldsymbol{r}\right)$, where $r = |x|$. This yields a relation between the period T and the radius r: $T = \frac{2\pi r}{f(r)}$. Since H is autonomous all solutions of the Hamilton equations occurs as circles of solutions. On order to compute Floer homology from an explicit system we need the autonomous Hamiltonians given above, i.e. choose G appropriately. To construct such a Hamiltonian we perform a change of coordinates. The Cauchy-Riemann equations are given by

$$u_s + J_0 u_t + \nabla H(u) = 0. \tag{4.7}$$

Choose symplectic polar coordinates (I, θ) via the relation $p = \sqrt{2I}\cos(\theta), q = \sqrt{2I}\sin(\theta)$, and define $\hat{H}(I, \theta) = H(p, q)$. In particular,

$$\hat{H}(I, \theta) = F(\sqrt{2I}) + \omega_\delta(\sqrt{2I})G(\theta).$$

Under this symplectic change of coordinates the Cauchy-Riemann equations become

$$I_s - 2I\theta_t + 2I\hat{H}_I(I, \theta) = 0, \tag{4.8}$$

$$\theta_s + \frac{1}{2I}I_t + \frac{1}{2I}\hat{H}_\theta(I, \theta) = 0. \tag{4.9}$$

If we restrict x to the annulus $\mathbb{A}_{2\delta} = \{x \in \mathbb{D}^2 : 2\delta \leq |x| \leq 1 - 2\delta\}$, the particular choice of H described above yields the following equations

$$\begin{aligned} I_s - 2I\theta_t + \sqrt{2I}f(\sqrt{2I}) &= 0, \\ \theta_s + \frac{1}{2I}I_t + \frac{1}{2I}g(\theta) &= 0, \end{aligned}$$

where $g = G'$. Before giving a general result for braid classes for which $\mathbf{x}$ is a single-strand rotation we employ the above model to get insight into the Floer homology of $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$.

Floer homology of the annulus

The formal calculation in the above example indicates the Floer homology of a the relative braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ described above. We will now prove a theorem about the Floer homology of a larger set of braid classes of which the above example is a special case. To do this we will employ the Floer homology of a annulus.

In the example above to method to compute the Floer homology of the given braid class is to continue to a special system for which explicit knowledge of the space of bounded solutions gives the Floer complex. Note that we cannot continue to the case of contracted loops as is done in Floer homology of symplectic manifolds [**29**]. The reason is that the free strands $\mathbf{x}$ in $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ are not contractible. By employing the result about composing braids with full twist we can relate the Floer homology of certain relative braid class to a situation in which the free strands are contractible.

As for the 2-disc we can consider an annulus $\mathbb{A} = \mathbb{A}_\delta$. The boundary orientation is the canonical Stokes orientation and the orientation form on $\partial\mathbb{A}$ is given by $\lambda = i_{\mathbf{n}}\omega$.Hamiltonians H satisfy the hypotheses:

(a1) $H \in C^2(\mathbb{R} \times \mathbb{R}^2; R)$;
(a2) $H(t+1, x) = H(t, x)$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^2$;
(a3) $H(t, x) = 0$ for all $x \in \partial\mathbb{A}$ and all $t \in \mathbb{R}$.

This class of Hamiltonians is denoted by $\mathcal{H}(\mathbb{A})$. We will consider Floer homology of the annulus in the case that H has prescribed behavior on $\partial\mathbb{A}$.

(a4+) $i_{X_H}\lambda > 0$ on $\partial\mathbb{A}$;
(a4-) $i_{X_H}\lambda < 0$ on $\partial\mathbb{A}$.

The class of Hamiltonians that satisfy (a1)-(a3), (a4+) is denoted by $\mathcal{H}^+$ and those satisfying (a1)-(a3), (a4-) are denoted by $\mathcal{H}^-$. For Hamiltonians in $\mathcal{H}^+$ the boundary orientation induced by X_H is coherent with the canonical orientation of $\partial\mathbb{A}$, and for Hamiltonians in $\mathcal{H}^-$ the boundary orientation induced by X_H is opposite to the canonical orientation of $\partial\mathbb{A}$.

THEOREM 4.8. *For the pairs $(J,H) \in \mathcal{J}^+ \times \mathcal{H}^+(\mathbb{A})$ the Floer homology $FH_*(\mathbb{A};J,H)$ is denoted by $FH_*(\mathbb{A};\mathcal{H}^+)$ and there is a natural isomorphism*

$$FH_k(\mathbb{A};\mathcal{H}^+) \cong H_{k+1}(\mathbb{A}) = \begin{cases} \mathbb{Z}_2 & \text{for } k=-1,0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for the pairs $(J,H) \in \mathcal{J}^+ \times \mathcal{H}^-(\mathbb{A})$ the Floer homology $FH_*(\mathbb{A};J,H)$ is denoted by $FH_*(\mathbb{A};\mathcal{H}^-)$ and there is a natural isomorphism

$$FH_k(\mathbb{A};\mathcal{H}^-) \cong H_k(\mathbb{A}) = \begin{cases} \mathbb{Z}_2 & \text{for } k=0,1 \\ 0 & \text{otherwise,} \end{cases}$$

where H_* denotes the singular homology with coefficients in $\mathbb{Z}_2$.

PROOF. Let us start with Hamiltonians in the class $\mathcal{H}^+$. Consider $\mathbb{A} = \mathbb{A}_\delta$ and choose $H = F + \omega G$, with $F(r) = \frac{1}{2}\left(r-\frac{1}{2}\right)^2 - \frac{1}{2}\left(\delta-\frac{1}{2}\right)^2$ and $G(\theta) = \varepsilon\cos(\theta)$. Using the symplectic polar coordinates we obtain that

$$\nabla_g \hat{H}(I,\theta) = \begin{pmatrix} 2I - \frac{1}{2}\sqrt{2I} + \varepsilon\sqrt{2I}\omega'(\sqrt{2I})\cos(\theta) \\ -\frac{\varepsilon}{2I}\omega(\sqrt{2I})\sin(\theta) \end{pmatrix}.$$

For $\frac{1}{2}\delta^2 \le I \le 2\delta^2$ and for $\frac{1}{2}(1-2\delta)^2 \le I \le \frac{1}{2}(1-\delta)^2$ it holds that $|\sqrt{2I}-\frac{1}{2}| \ge \frac{1}{2} - 2\delta$ and thus if we choose $\varepsilon < \frac{1}{4\delta} - 1$ all zeroes of $\nabla_g\hat{H}$ lie in the annulus set $\mathbb{A}_{2\delta} \subset \mathbb{A}_\delta$. The zeroes of $\nabla_g\hat{H}$ are found at $I = \frac{1}{8}$ and $\theta = 0,\pi$, which are both non-degenerate critical points. Linearization yields

$$d\nabla_g\hat{H}(1/8,0) = \begin{pmatrix} 1 & 0 \\ 0 & -4\varepsilon \end{pmatrix}, \quad d\nabla_g\hat{H}(1/8,\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 4\varepsilon \end{pmatrix},$$

an index-1 saddle point and a minimum (index-0). For $\tau \le 1$ the Conley-Zehnder indices of the associated symplectic paths defined by $\Psi_t = J_0 d\nabla_g\hat{H}\Psi$ are given by $\mu_\sigma(\Psi,\tau) = 0,1$. Therefore the index $\mu_\tau(I,\theta) = -\mu_\sigma(\Psi,\tau) = 0,-1$ for (I,θ) equal to $(\frac{1}{8},0)$ and $(\frac{1}{8},\pi)$ respectively.

Next consider Hamiltonians of the form τH and the associated Cauchy-Riemann equations are $u_s + J_0 u_t + \tau\nabla H(u) = 0$. Rescale $\tau s \to s$, $\tau t \to t$ and $u(s/\tau,t/\tau) \to u(s,t)$, then the satisfies the Cauchy-Riemann equations in (4.7) again with periodicity $u(s,t+\tau) = u(s,t)$. The 1-periodic solutions of the Cauchy-Riemann equations with τH are transformed to τ-periodic solutions of (4.7). Note that if τ is sufficiently small then all τ-periodic solutions of the stationary Cauchy-Riemann equations (4.7) are independent of of t and thus critical points of H.

If we linearize around t-independent solutions of (4.7) then $\frac{d}{ds}+d\nabla H(u(s))$ is Fremholm and thus also

$$\bar{\partial}_{K,\Delta}=\frac{\partial}{\partial s}+J\frac{\partial}{\partial t}+K,$$

with $K=d\nabla H(u(s))$, is Fredholm, see [**63**]. We claim that if τ is sufficiently small then all contractible τ-periodic bounded solutions $u(s,t+\tau)=u(s,t)$ of (4.7) are t-independent, i.e. solutions of the equation $u_s=-\nabla H(u)$. Let us sketch the argument following [**63**]. Assume by contradiction that there exists a sequence of $\tau_n\to 0$ and bounded solutions u_n of Equation (4.7). If we embed $\mathbb{A}$ into the 2-disc $\mathbb{D}^2$ we can use the compact results for the 2-disc. One can assume without loss of generality that $u_n\in\mathcal{M}^{J_0,H}(\mathbb{A};\tau)=\mathcal{M}^{J_0,\tau H}(\mathbb{A})$. Following the proof in [**63**] we conclude that for $\tau>0$ small all solutions in $\mathcal{M}$ are t-independent. The system (J,H)can be continued to $(J_0,\tau H)$ for which we know $\mathcal{M}^{J_0,\tau H}(\mathbb{A})$ explicitly via $u_s+\tau\nabla H=0$ and gives the desired homology.

As for Hamiltonians in $\mathcal{H}^-$ we choose $F(r)=-\frac{1}{2}l(r-\frac{1}{2}\boldsymbol{r})^2+\frac{1}{2}\left(\delta-\frac{1}{2}\right)^2$. The proof is identical to the previous case except for the indices of the stationary points. Here we have that $\mu_\tau(I,\theta)=-\mu_\sigma(\Psi,\tau)=1,0$ for (I,θ) equal to $(\frac{1}{8},0)$ and $(\frac{1}{8},\pi)$ respectively, which gives the homology indicated above. □

Floer homology for single-strand cyclic braid classes

We apply the results in the previous subsection to compute the Floer homology of family of cyclic braid classes $[\![\mathbf{x}\text{ rel }\mathbf{y}]\!]$. The skeletons $\mathbf{y}$ consist of two braid components $\mathbf{y}^1$ and $\mathbf{y}^2$. The first component can be described as follows. In complex notation $p+iq$ we have

$$\mathbf{y}^1=\left\{r_1e^{\frac{2\pi n}{m}it},r_1e^{\frac{2\pi}{m}i(nt-1)},\cdots,r_1e^{\frac{2\pi}{m}i(nt-m+1)}\right\},$$

where $0<r_1<1$. In the braid group $\mathcal{B}_m$ the braid $\mathbf{y}^1$ is represented by the word $\beta^1=(\sigma_1\cdots\sigma_{m-1})^n$, $m\geq 2$, and $n\in\mathbb{Z}$. In a similar fashion we define the braid $\mathbf{y}^2$:

$$\mathbf{y}^2=\left\{r_2e^{\frac{2\pi n'}{m'}it},r_2e^{\frac{2\pi}{m'}i(n't-1)},\cdots,r_2e^{\frac{2\pi}{m'}i(n't-m'+1)}\right\},$$

where $0<r_1<r_2<1$. In the braid group $\mathcal{B}_{m'}$ the braid $\mathbf{y}^2$ is represented by the word $\beta^2=(\sigma_1\cdots\sigma_{m'-1})^{n'}$, $m'\geq 2$, and $n'\in\mathbb{Z}$. In order to describe the relative braid class $[\![\mathbf{x}\text{ rel }\mathbf{y}]\!]$ with the skeleton defined above we consider a single strand braid $\mathbf{x}=\{x^1(t)\}$ with

$$x^1(t)=re^{2\pi git}$$

where $0<r_1<r<r_2<1$ and $g\in\mathbb{Z}$. We now consider two cases for which $\mathbf{x}$ rel $\mathbf{y}$ is a representative.

The case $\frac{n}{m}<g<\frac{n'}{m'}$. The relative braid class $[\![\mathbf{x}\text{ rel }\mathbf{y}]\!]$ is a proper braid class since the inequalities are strict. We have

LEMMA 4.9. *The Floer homology is given by*

$$FH_k([\![\mathbf{x}\text{ rel }\mathbf{y}]\!],\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \textit{for } k=-2g-1,-2g\\ 0 & \textit{otherwise.}\end{cases}$$

The Poincaré polynomial is given by $P_t([\![\mathbf{x}\text{ rel }\mathbf{y}]\!])=t^{-2g-1}+t^{-2g}$.

PROOF. Since $FH_*([\![\mathbf{x} \text{ rel } \mathbf{y}]\!], \mathbb{Z}_2)$ is independent of the representative we consider the class $[x]$ rel $\mathbf{y}$ with $\mathbf{x}$ and $\mathbf{y}$ as defined above. Apply $-g$ full twists to $\mathbf{x}$ rel $\mathbf{y}$: $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \overline{S}^{-g}(\mathbf{x}, \mathbf{y})$. Then by Theorem 4.5

$$FH_k([\![\hat{\mathbf{x}} \text{ rel } \hat{\mathbf{y}}]\!]) \cong FH_{k+2g}([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]). \tag{4.10}$$

We now compute the homology $FH_k([\![\hat{\mathbf{x}} \text{ rel } \hat{\mathbf{y}}]\!])$ using Theorem 4.8. The free strand $\hat{\mathbf{x}}$ in $\hat{\mathbf{x}}$ rel $\hat{\mathbf{y}}$ is unlinked with the y^1 and can be deformed to the constant strand $x^1(t) = r$. Consider an explicit Hamiltonian $H(x) = F(|x|) + \omega(|x|)G(\arg(x))$. Choose F such that $F(r_1) = F(r_2) = 0$ and

$$\frac{f(r_1)}{2\pi r_1} = \frac{n}{m} - g < 0, \quad \text{and} \quad \frac{f(r_2)}{2\pi r_2} = \frac{n'}{m'} - g > 0.$$

Clearly $\mathbf{y} \in \mathrm{Crit}_H$ and the circles $|x| = r_1$ and $|x| = r_2$ are invariant for the Hamiltonian vector field X_H. Therefore $\mathcal{M}^{J_0,H}([\hat{\mathbf{x}}] \text{ rel } \mathbf{y}) = \mathcal{M}^{J_0,H}(\mathbb{A})$. It holds that $H \in \mathcal{H}^+$ and by Theorem 4.8 it follows that $FH_{-1}([\hat{\mathbf{x}}] \text{ rel } \hat{\mathbf{y}}) \cong FH_{-1}(\mathbb{A}; \mathcal{H}^+) = \mathbb{Z}_2$ and $FH_0([\hat{\mathbf{x}}] \text{ rel } \hat{\mathbf{y}}) \cong FH_0(\mathbb{A}; \mathcal{H}^+) = \mathbb{Z}_2$. This proves, using (4.10), that $FH_{-2g-1}([\mathbf{x}] \text{ rel } \mathbf{y}) = \mathbb{Z}_2$ and $FH_{-2g}([\mathbf{x}] \text{ rel } \mathbf{y}) = \mathbb{Z}_2$ which completes the proof. □

The case $\frac{n}{m} > g > \frac{n'}{m'}$. The relative braid class $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ with the reversed inequalities is also a proper braid class. We have

LEMMA 4.10. *The Floer homology is given by*

$$FH_k([\![\mathbf{x} \text{ rel } \mathbf{y}]\!], \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = -2g, -2g+1 \\ 0 & \text{otherwise.} \end{cases}$$

The Poincaré polynomial is given by $P_t([\![\mathbf{x} \text{ rel } \mathbf{y}]\!]) = t^{-2g} + t^{-2g+1}$.

PROOF. The proof is identical to the proof of Lemma 4.9. Because the inequalities are reversed we construct a Hamiltonian such that

$$\frac{f(r_1)}{2\pi r_1} = \frac{n}{m} - g > 0, \quad \text{and} \quad \frac{f(r_2)}{2\pi r_2} = \frac{n'}{m'} - g < 0.$$

This yields a Hamiltonian in $\mathcal{H}^-$ and we therefore repeat the above argument using the homology $FH_*(\mathbb{A}; \mathcal{H}^-)$, which proves the lemma. □

REMARK 4.11. If we define the Floer homology for braid class with respect to the conjugate equations by reflecting the time $s \to -s$, the homology in the Lemmas 4.9 and 4.10 becomes

$$FH_k([\![\mathbf{x} \text{ rel } \mathbf{y}]\!], \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 2g, 2g+1 \\ 0 & \text{otherwise,} \end{cases}$$

when $\frac{n}{m} < g < \frac{n'}{m'}$, and

$$FH_k([\![\mathbf{x} \text{ rel } \mathbf{y}]\!], \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 2g-1, 2g \\ 0 & \text{otherwise,} \end{cases}$$

when $\frac{n}{m} > g > \frac{n'}{m'}$. This agrees with the calculations in [**33**] for positive relative braid classes.

Applications to disc maps

Let $\Psi = x(1;\cdot) : \mathbb{D}^2 \to \mathbb{D}^2$ be the time-1 map for a Hamiltonian system $x_t = X_H$ on $(\mathbb{D}^2, \omega_0)$ with $H \in \mathcal{H}(\mathbb{D}^2)$. Then Ψ is an area and orientation preserving map (diffeomorphism) of the 2-disc $\mathbb{D}^2$.

THEOREM 4.12. *Let $\mathbf{y}' \in [\mathbf{y}]$, with $\mathbf{y}$ as described above with $\frac{n}{m} \neq \frac{n'}{m'}$. Assume that $\mathbf{y}' \in \mathrm{Crit}_H$, then for any $g \in \mathbb{Z}$ such that*

$$\frac{n}{m} < g < \frac{n'}{m'}, \quad \text{or} \quad \frac{n}{m} > g > \frac{n'}{m'},$$

the associated time-1 map Ψ has distinct fixed points.

PROOF. The existence of a stationary relative braid follows from Theorem 4.1 since by Lemmas 4.9 and 4.10 the Floer homology of $[\![\mathbf{x} \text{ rel } \mathbf{y}]\!]$ is non-trivial. A stationary strand x yields a fixed point for Ψ. □

As a direct consequence of Theorem 4.12 we obtain the following corollary.

COROLLARY 4.13. *Let $\mathbf{y}' \in [\mathbf{y}]$, with $\mathbf{y}$ as described above with $\frac{n}{m} \neq \frac{n'}{m'}$. Assume that $\mathbf{y}' \in \mathrm{Crit}_H$, then for any $g \in \mathbb{Z}$ and $k \in N$ such that*

$$\frac{n}{m} < \frac{g}{k} < \frac{n'}{m'}, \quad \text{or} \quad \frac{n}{m} > \frac{g}{k} > \frac{n'}{m'},$$

the associated time-1 map Ψ has distinct k-periodic points, i.e. $\Psi^k(x) = x$ (see Figure 4.1 below).

PROOF. Consider the Hamiltonain kH, then the time-1 map associated with Hamiltonian system $x_t = X_{kH}$ is equal to Ψ^k. Applying Theorem 4.12 gives the desired result. □

REMARK 4.14. As pointed out in Section 4.1 we conjecture that the Floer homology implies the existence of at least two fixed points. This agrees with Figure 4.1.

TO BE DRAWN

Figure 4.1: The k-periodic points circle around the skeletal points and the k-periodic points occur in pairs, i.e. saddle-like and elliptic points (figure is taken from J. José and E. Saletan, Classical Dynamics).

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