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Project Apollo

DETERMINATION OF LEM LANDING
SITE INERTIAL COORDINATES
BY CSM LANDMARK TYPE SIGHTINGS


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## SUMMARY

A method for computing the three inertial components of a previously unknown lunar landing site is presented. The computation of the landing site vector can be simply computed onboard the CSM while it is in orbit about the moon prior to LEM separation for descent. The computation requires that two unit vectors, separated by a few minutes in time, be measured with an optical instrument to the landing site and with respect to the IMU. The method is also extended to include the computation of the expected errors in the landing site position. These errors are dependent on the expected CSM errors in position and velocity as well as on the error in the sighting instrument. An example is presented to illustrate the computation.

## INTRODUCTION

For the Apollo lunar landing mission the landing site coordinates are expected to be selected well in advance in order to insure adherence to the many operational constraints imposed on the mission. However, the operational flexibility exists in the onboard guidance computer program to allow the navigator, while in lunar orbit, to select a new landing site or to update the preselected site. Due to the large uncertainties that exist in the current estimates of the lunar radius, it is anticipated that an update of the landing site altitude alone would substantially increase mission success.

This note will present a method for determining the landing site vector in inertial coordinates, during the orbit phase of the Apollo Lunar Landing Mission, using the onboard optics as they are presently planned.

This note will also include as part of the method the derivation of a set of equations from which the expected errors of the landing site vector may be calculated. These errors may be computed either on the earth, or onboard in the event that the expected errors are found to be a necessary input for the LEM descent guidance scheme. An example case is presented for illustration.

### Determination of the Landing Site Vector

At a time, $t_o$, prior to the lunar landing, and while the CSM and LEM are both in orbit about the moon, the navigator sights the optical instrument on a distinguishable feature which is located on the lunar surface in the general area where the landing is to occur. When the mark button is pressed, a unit vector, $\hat{e}_o$ , from the spacecraft to the landing site is obtained. This unit vector is determined to within the sighting accuracy by the two angles, $\alpha_0$ and $\delta_0$, which are measured by the sighting instrument with respect to the Inertial Measurement Unit (IMU); see figure 1. The unit vector, $\hat{\rho}_0$, is given by,

$$\hat{\rho}_0 = \hat{i} \cos\delta_0 \cos\alpha_0 + \hat{j} \cos\delta_0 \sin\alpha_0 + \hat{k} \sin\delta_0 \tag{1}$$

The unit vectors $\hat{i}$ , $\hat{j}$ , $\hat{k}$ , are along the selenocentric (inertial) coordinate axes. As presented here, it is assumed that the IMU is aligned along $\hat{i}$ , $\hat{j}$ , $\hat{k}$ , so that $\alpha$ and $\delta$ refer to right ascension and declination. There is no loss of generality in making this assumption.

The navigator continues to track the landing site through a period of a few minutes. At time $t_1$, the navigator again presses the mark button, determining in the same manner the unit vector,

$$\hat{\rho}_1 = \hat{i} \cos\delta_1 \cos\alpha_1 + \hat{j} \cos\delta_0 \sin\alpha_0 + \hat{k} \sin\delta_0 \tag{2}$$

Since the moon is rotating very slowly, the following relation holds to a good approximation,

$$\underline{\ell} = \underline{r}_0 + \rho_0 \hat{\rho}_0 = \underline{r}_1 + \rho_1 \hat{\rho}_1 \tag{3}$$

as can be seen from figure (1).

In order to determine the landing site vector, $\underline{\ell}$, it is necessary that the magnitude of the vector $\underline{\rho}_0$ or $\underline{\rho}_1$ be known. There is no way of determining the magnitudes $\rho_0$ or $\rho_1$ by measurement. However, knowing $\hat{\rho}_0$ and $\hat{\rho}_1$ , and the position vectors $\underline{r}_0$ and $\underline{r}_1$ , $\rho_0$ or $\rho_1$ may be calculated. This may be shown as follows: From equation (3),

$$\underline{c} = \underline{r}_1 - \underline{r}_0 = \rho_0 \hat{\rho}_0 - \rho_1 \hat{\rho}_1 \tag{4}$$

Take the dot product of (4) with $\hat{\rho}_1$,

$$\underline{c} \cdot \hat{\rho}_1 = \rho_0 (\hat{\rho}_0 \cdot \hat{\rho}_1) - \rho_1 \tag{5}$$

Take the dot product of (4) with $\hat{\rho}_0$,

$$\underline{c} \cdot \hat{\rho}_0 = \rho_0 - (\hat{\rho}_0 \cdot \hat{\rho}_1) \rho_1 \tag{6}$$

multiply (5) by $-\hat{\rho}_0 \cdot \hat{\rho}_1$ and add the result to (6),

$$\underline{c} \cdot \hat{\rho}_0 - (\hat{\rho}_0 \cdot \hat{\rho}_1)\, \underline{c} \cdot \hat{\rho}_1 = \rho_0 \left[1 - (\hat{\rho}_0 \cdot \hat{\rho}_1)^2\right]$$

or by simplification,

$$\rho_0 = \frac{\underline{c} \cdot (\hat{\rho}_0 - \hat{\rho}_1 \cos\phi)}{\sin^2\phi} \tag{7}$$

where, $\cos\phi = \hat{\rho}_0 \cdot \hat{\rho}_1$

$$\sin^2\phi = 1 - (\hat{\rho}_0 \cdot \hat{\rho}_1)^2$$

With $\rho_0$ determined by (7), the landing site vector may be determined by combining (7) and (3)

$$\underline{\ell} = \underline{r}_0 + \frac{\underline{c}}{\sin^2\phi} \cdot (\hat{\rho}_0 - \hat{\rho}_1 \cos\phi)\, \hat{\rho}_0 \tag{8}$$

In summary, all that is required to compute the landing site vector $\underline{\ell}$ are the spacecraft position vectors at $t_o$ and $t_1$ and the measured quantities $\alpha_o$, $\delta_o$, $\alpha_1$, and $\delta_1$.

### Determination of the Estimation Errors in the Landing Site Vector

Let the actual position of the landing site be given by,

$$\underline{\ell}^* = \underline{r}_o^* + \underline{\rho}_o^* \tag{9}$$

where $\underline{r}_o^*$ and $\underline{\rho}_o^*$ are the actual position vectors of the spacecraft and the landing site with respect to the spacecraft at time $t_o$. Now let equation (3) represent the similar relation between the estimated landing site position, $\underline{\ell}$, the estimated position vector, $\underline{r}_o$, and the estimated vector, $\underline{\rho}_o$, from spacecraft to landing site.

That is,

$$\underline{\ell} = \underline{r}_o + \underline{\rho}_o \tag{10}$$

where $\underline{\rho}_o = \rho_o \hat{\rho}_o$

The error in the estimate of $\underline{\ell}$ is now found by subtracting (9) from (10),

$$\underline{\ell} - \underline{\ell}^* = \underline{r}_o - \underline{r}_o^* + \underline{\rho}_o - \underline{\rho}_o^*$$

or

$$\delta\underline{\ell} = \delta\underline{r}_o + \delta\underline{\rho}_o \tag{11}$$

The covariance matrix of estimation errors in the landing site vector may be found by taking the expected value of (11) times its transpose.

$$E_L = \mathcal{E}(\delta \underline{\ell}\, \delta \underline{\ell}^T)$$

$$= \mathcal{E}(\delta \underline{r}_o\, \delta \underline{r}_o^T) + \mathcal{E}(\delta \underline{\rho}_o\, \delta \underline{\rho}_o^T) + \mathcal{E}(\delta \underline{\rho}_o\, \delta \underline{r}_o^T) + \mathcal{E}(\delta \underline{r}_o\, \delta \underline{\rho}_o^T) \qquad (12)$$

The term $\mathcal{E}(\delta \underline{r}_o \delta \underline{r}_o^T)$ is the covariance matrix, $E_{r_o}$, of estimation errors in spacecraft position at $t_o$. Define,

$$\Sigma_{\rho_o} = \mathcal{E}(\delta \underline{\rho}_o\, \delta \underline{\rho}_o^T) \qquad (13)$$

$$\Sigma_{\rho_o r_o} = \mathcal{E}(\delta \underline{\rho}_o\, \delta \underline{r}_o^T) \qquad (14)$$

Then (12) becomes,

$$E_L = E_{r_o} + \Sigma_{\rho_o} + \Sigma_{\rho_o r_o} + \Sigma^T_{\rho_o r_o} \qquad (15)$$

It is shown in Appendix A that equation (15) reduces to,

$$\begin{aligned} E_L = E_{r_o} &+ M[D - E_{r_o}] + [D^T - E_{r_o}^T]M^T \\ &+ M[E_{r_1} + E_{r_o} - D - D^T] \\ &+ NRN^T \end{aligned} \qquad (16)$$

where $E_{r_1}$ is the covariance matrix of estimation errors in spacecraft position at time $t_1$. The other quantities in (16) are:

$$M = \frac{\hat{\rho}_0 \hat{\rho}_0^T \left[ I - \hat{\rho}_1 \hat{\rho}_1^T \right]}{1 - (\hat{\rho}_0 \cdot \hat{\rho}_1)^2} \tag{17}$$

$$N = \left[ \frac{\partial M}{\partial \delta_0} \underline{c}, \frac{\partial M}{\partial \alpha_0} \underline{c}, \frac{\partial M}{\partial \delta_1} \underline{c}, \frac{\partial M}{\partial \alpha_1} \underline{c} \right] \tag{18}$$

$$R = \frac{\sigma^2}{2} \, DIAG \left[ 1, \sec^2 \delta_0, 1 \sec^2 \delta_1 \right] \tag{19}$$

where $\sigma$ is in the standard deviation of the sighting instrument.

$$D = \varphi_{11} E_{r_0} + \varphi_{12} E_{r_0 v_0}^T \tag{20}$$

where $E_{r_0 v_0}^T$ is the cross correlation between velocity and position estimation errors at time $t_0$. The matrices $\varphi_{11}$ and $\varphi_{12}$ are submatrices of the six by six transition matrix for Keplerian motion; for example, see reference 1.

The matrix $E_{r_1}$ may be computed from

$$E_{r_1} = \left[ \varphi_{11} E_{r_0} + \varphi_{12} E_{r_0 v_0}^T \right] \varphi_{11}^T + \left[ \varphi_{11} E_{r_0 v_0} + \varphi_{12} E_{v_0} \right] \varphi_{12}^T \tag{21}$$

where $E_{v_0}$ is the covariance matrix of estimation errors in spacecraft velocity at time $t_0$.

In summary, all that is required to compute the estimation errors in the landing site vector is the covariance matrix of estimation errors at $t_o$; the measured quantities $\alpha_0$, $\delta_0$, $\alpha_1$, $\delta_1$; and the standard deviation $\sigma$ of the sighting instrument.

## A Numerical Example

The equations to compute the landing site vector and its errors were programmed on a digital computer. Inputs to the program were taken from the results of a completely independent program. The following quantities were given to the program:

Ephemeris Time (ET) = 1969 yr, 260 day, 77 hrs, 10 mins, 12.2 secs

$t_o$ = 0.703125 hrs from ET

$t_1$ = 0.765625 hrs from ET

At $t_o$, $\underline{r}_0 = (-934.952,\ 370.206,\ 183.861)$ n.mi.

$\alpha_0 = 21.3439^\circ$

$\delta_0 = 14.4697^\circ$

At $t_1$, $\underline{r}_1 = (-837.079,\ 527.752,\ 252.152)$ n.mi.

$\alpha_1 = -85.3062^\circ$

$\delta_1 = -40.1314^\circ$

With these quantities given, the inertial components of the landmark were found from equation (8) to be,

$$\underline{\ell} = (-5.02893,\ 2.50418,\ .936535) \times 10^6 \text{ ft.}$$

In order to compute the estimation errors in the landing site vector, additional statistical quantities were also input as follows:

$$Er_o = \begin{bmatrix} 1.96819 & .840991 & -.0841973 \\ & 4.55638 & -.4947417 \\ (SYM) & & 5.475946 \end{bmatrix} \times 10^5 ft^2$$

$$Ev_o = \begin{bmatrix} .751570 & .0559336 & .212788 \\ & 1.491713 & -.325594 \\ (SYM) & & 8.173442 \end{bmatrix} \frac{ft^2}{sec^2}$$

The root-mean-square position and velocity errors computed from the traces of $Er_o$ and $Ev_o$ are found to be,

$$RMSPOS(t_o) = 1095 \text{ ft}$$

$$RMSVEL(t_o) = 3.23 \text{ ft/sec}$$

$$Er_ov_o = \begin{bmatrix} .580603 & 2.06510 & -.417946 \\ .349543 & 3.88734 & -6.429013 \\ 1.748310 & -6.83694 & 17.69110 \end{bmatrix} \times 10^2 \frac{ft^2}{sec}$$

The sighting accuracy of the instrument was chosen to be

$$\sigma = .001 \text{ rad}$$

Using these inputs, the landing site covariance matrix was found from equation (16) to be,

$$E_L = \begin{bmatrix} 4.59429 & 2.88405 & -1.90845 \\ & 5.75483 & -1.48731 \\ (SYM) & & 6.28671 \end{bmatrix} \times 10^5 ft^2$$

The root-mean-square landing site error is computed from the trace of $E_L$ and is found to be,

$$RMSLS = 1290.0 \text{ ft}$$

If $E_L$ is transformed into the selenographic polar coordinate system as in reference 2, the root-mean-square errors in latitude, longitude, and vertical directions are given by,

$$RMSLAT = 854.6 \text{ ft}$$
$$RMSLON = 800.2 \text{ ft}$$
$$RMSVERT = 544.4 \text{ ft}$$

In reference 3 it is seen that the predicted uncertainties of known lunar landmarks vary from 220 meters to 910 meters in the horizontal and from 730 meters to 800 meters in the vertical. The uncertainties in the vertical are possibly even larger. Based on results from the Ranger series, the Jet Propulsion Lab has recently decreased its estimate of the lunar radius from 1738.0 KM to 1735 KM.

The results of this example show that these uncertainties can be significantly improved during the Apollo Lunar Landing Mission.

## CONCLUDING REMARKS

The analytic description of a method for determining an unknown lunar landing site vector during the lunar orbit phase of the Apollo mission is presented. The method requires the measurement using an onboard optical instrument of two unit vectors to the landing site with respect to the IMU. The method includes the computation of the landing site errors, which are dependent upon the CSM position and velocity errors, as well as the error in the sighting instrument. An example was presented to illustrate the computation. The results of this example show that by the use of onboard sightings on the landing site, the errors in the components of the landing site vector are determined to be substantially smaller than predicted uncertainties for lunar landmarks.

REFERENCES

1. Bond, V.R., "An Analytical Formulation of the Conic State Transition Matrix Using Battin's Auxiliary Conic Variable" MSC Internal Note (to be published).

2. Bond, V. R.; "Apollo Navigational Accuracy in Lunar Orbit Including Landmark Updating" MSC Internal Note No. 66-EG-3, Jan 4, 1966.

3. "Positional Uncertainties in Lunar Landmarks" MSC Internal Note No. 65-ET-2, Jan 5, 1965.

## APPENDIX A

### The Covariance Matrix of Estimation Errors in the Landing Site Vector

Consider the equation (15) for the covariance matrix of estimation errors in the landing site vector,

$$E_L = E_{r_0} + \Sigma_{\rho_0} + \Sigma_{\rho_0 r_0} + \Sigma_{\rho_0 r_0}^T \tag{A1}$$

where $\Sigma_{\rho_0}$ and $\Sigma_{\rho_0 r_0}$ are given by (13) and (14). In order to evaluate $\Sigma_{\rho_0}$ and $\Sigma_{\rho_0 r_0}$ it is expedient to find an expression for $\underline{\rho}_0$ that is easily differentiated. Using (7),

$$\underline{\rho}_0 = \rho_0 \hat{\rho}_0 = \left[\frac{\underline{c} \cdot (\hat{\rho}_0 - \hat{\rho}_1 \cos\phi)}{\sin^2\phi}\right] \hat{\rho}_0$$

using $\cos\phi = \hat{\rho}_0 \cdot \hat{\rho}_1$ AND $\sin^2\phi = 1 - (\hat{\rho}_0 \cdot \hat{\rho}_1)^2$,

$$\underline{\rho}_0 = \hat{\rho}_0 \left[\hat{\rho}_0 - (\hat{\rho}_0 \cdot \hat{\rho}_1)\hat{\rho}_1\right]^T \left[1 - (\hat{\rho}_0 \cdot \hat{\rho}_1)^2\right]^{-1} \underline{c}$$

$$= \hat{\rho}_0 \hat{\rho}_0^T \left[I - \hat{\rho}_1 \hat{\rho}_1^T\right]\left[1 - (\hat{\rho}_0 \cdot \hat{\rho}_1)^2\right]^{-1} \underline{c}$$

or by defining,

$$M = \frac{\hat{\rho}_0 \hat{\rho}_0^T \left[I - \hat{\rho}_1 \hat{\rho}_1^T\right]}{1 - (\hat{\rho}_0 \cdot \hat{\rho}_1)^2} \tag{A2}$$

The expression for $\underline{\rho}_0$ becomes,

$$\underline{\rho}_0 = M\underline{c} \tag{A3}$$

Now the first order deviations to $\underline{\rho}_0$ become,

$$\delta\underline{\rho}_0 = \delta M\underline{c} + M\delta\underline{c} \tag{A4}$$

where $$\delta\underline{c} = \delta r_1 - \delta r_0$$

But $$M = M(\delta_0, \alpha_0, \delta_1, \alpha_1) \tag{A5}$$

so, $$\delta M = \frac{\partial M}{\partial \delta_0}\delta\delta_0 + \frac{\partial M}{\partial \alpha_0}\delta\alpha_0 + \frac{\partial M}{\partial \delta_1}\delta\delta_1 + \frac{\partial M}{\partial \alpha_1}\delta\alpha_1 \tag{A6}$$

Now define the vectors,

$$\underline{n}_1 = \frac{\partial M}{\partial \delta_0}\underline{c} \quad , \quad \underline{n}_2 = \frac{\partial M}{\partial \alpha_0}\underline{c}$$

$$\underline{n}_3 = \frac{\partial M}{\partial \delta_1}\underline{c} \quad , \quad \underline{n}_4 = \frac{\partial M}{\partial \alpha_1}\underline{c}$$

And (A4) may be written,

$$\delta\underline{\rho}_0 = N\delta\underline{\beta} + M\delta\underline{c} \tag{A7}$$

where $$N = [\underline{n}_1, \underline{n}_2, \underline{n}_3, \underline{n}_4] \tag{A8}$$

and, $$\delta \underline{\beta} = \begin{pmatrix} \delta \delta_0 \\ \delta \alpha_0 \\ \delta \delta_1 \\ \delta \alpha_1 \end{pmatrix} \tag{A9}$$

Now consider equation (14), using (A7) and $\delta \underline{c} = \delta \underline{r}_1 - \delta \underline{r}_0$ ,

$$\begin{aligned} \Sigma_{\rho_0 r_0} &= \mathcal{E}(\delta \underline{\rho}_0 \delta \underline{r}_0^T) \\ &= N\, \mathcal{E}(\delta \underline{\beta} \delta \underline{r}_0^T) + M\, \mathcal{E}(\delta \underline{r}_1 \delta \underline{r}_0^T) \\ &\quad - M\, \mathcal{E}(\delta \underline{r}_0 \delta \underline{r}_0^T) \end{aligned} \tag{A10}$$

The first term of (A10) vanishes if it assumed that the measurement errors and trajectory errors are uncorrelated,

$$\mathcal{E}(\delta \underline{\beta} \delta \underline{r}_0^T) = 0 \tag{A11}$$

Since, $$\begin{pmatrix} \delta \underline{r}_1 \\ \delta \underline{v}_1 \end{pmatrix} = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{pmatrix} \delta \underline{r}_0 \\ \delta \underline{v}_0 \end{pmatrix} \tag{A12}$$

as shown in reference 1, the second term in (A10) becomes,

$$\begin{aligned} \mathcal{E}(\delta \underline{r}_1 \delta \underline{r}_0^T) &= \varphi_{11}\, \mathcal{E}(\delta \underline{r}_0 \delta \underline{r}_0^T) + \varphi_{12}\, \mathcal{E}(\delta \underline{v}_0 \delta \underline{r}_0^T) \\ &= \varphi_{11} E_{r_0} + \varphi_{12} E_{v_0 r_0} \end{aligned} \tag{A13}$$

where $E v_o r_o$ is the cross correlation between velocity and position estimation errors.

Equation (A10) finally becomes,

$$\Sigma_{\rho_o r_o} = M[\varphi_{11} E_{r_o} + \varphi_{12} E_{v_o r_o}] - M E_{r_o} \tag{A14}$$

Now consider equation (13), using (A7)

$$\begin{aligned}\Sigma_{\rho_o} &= \mathcal{E}(\delta\underline{\rho}_o \delta\underline{\rho}_o^T) \\ &= \mathcal{E}\left[(N\delta\underline{\beta} + M\delta\underline{c})(\delta\underline{\beta}^T N^T + \delta\underline{c}^T M^T)\right]\end{aligned}$$

$$\begin{aligned}\Sigma_{\rho_o} = {} & N\mathcal{E}(\delta\underline{\beta}\,\delta\underline{\beta}^T)N^T + M\mathcal{E}(\delta\underline{c}\,\delta\underline{c}^T)M^T \\ & + M\mathcal{E}(\delta\underline{c}\,\delta\underline{\beta}^T)N^T + N\mathcal{E}(\delta\underline{\beta}\,\delta\underline{c}^T)M^T\end{aligned} \tag{A15}$$

The third and fourth terms vanish by invoking the assumption that measurement and instrument errors are uncorrelated, that is,

$$\mathcal{E}(\delta\underline{c}\,\delta\underline{\beta}^T) = \mathcal{E}(\delta\underline{r}_1\,\delta\underline{\beta}^T) - \mathcal{E}(\delta\underline{r}_o\,\delta\underline{\beta}^T) = 0$$

Define the matrix

$$R = \mathcal{E}(\delta\underline{\beta}\,\delta\underline{\beta}^T) \tag{A16}$$

which can be shown using reference 2 to be

$$R = \frac{\sigma^2}{2} DIAG(1, \sec^2\delta_o, 1, \sec^2\delta_1) \tag{A17}$$

Using $\delta \underline{c} = \delta \underline{r}_1 - \delta \underline{r}_0$ equation (A15) becomes,

$$\Sigma_{\rho_0} = M \{ \mathcal{E}(\delta \underline{r}_1 \delta \underline{r}_1^T) + \mathcal{E}(\delta \underline{r}_0 \delta \underline{r}_0^T) - \mathcal{E}(\delta \underline{r}_0 \delta \underline{r}_1^T) - \mathcal{E}(\delta \underline{r}_1 \delta \underline{r}_0^T) \} M^T + NRN^T$$

All of the terms in the last equation have been previously defined, so finally equation (13) becomes

$$\Sigma_{\rho_0} = M \{ E_{r_1} + E_{r_0} - (E_{r_0}^T \varphi_{11}^T + E_{v_0 r_0}^T \varphi_{12}^T) - (\varphi_{11} E_{r_0} + \varphi_{12} E_{v_0 r_0}) \} M^T + NRN^T \tag{A18}$$

Using the definition (20) for the matrix D, along with equations (A14) and (A18) the equation (A15) becomes

$$E_L = E_{r_0} + M[D - E_{r_0}] + [D^T - E_{r_0}^T] M^T + M[E_{r_1} + E_{r_0} - D - D^T] M^T + NRN^T \tag{A19}$$

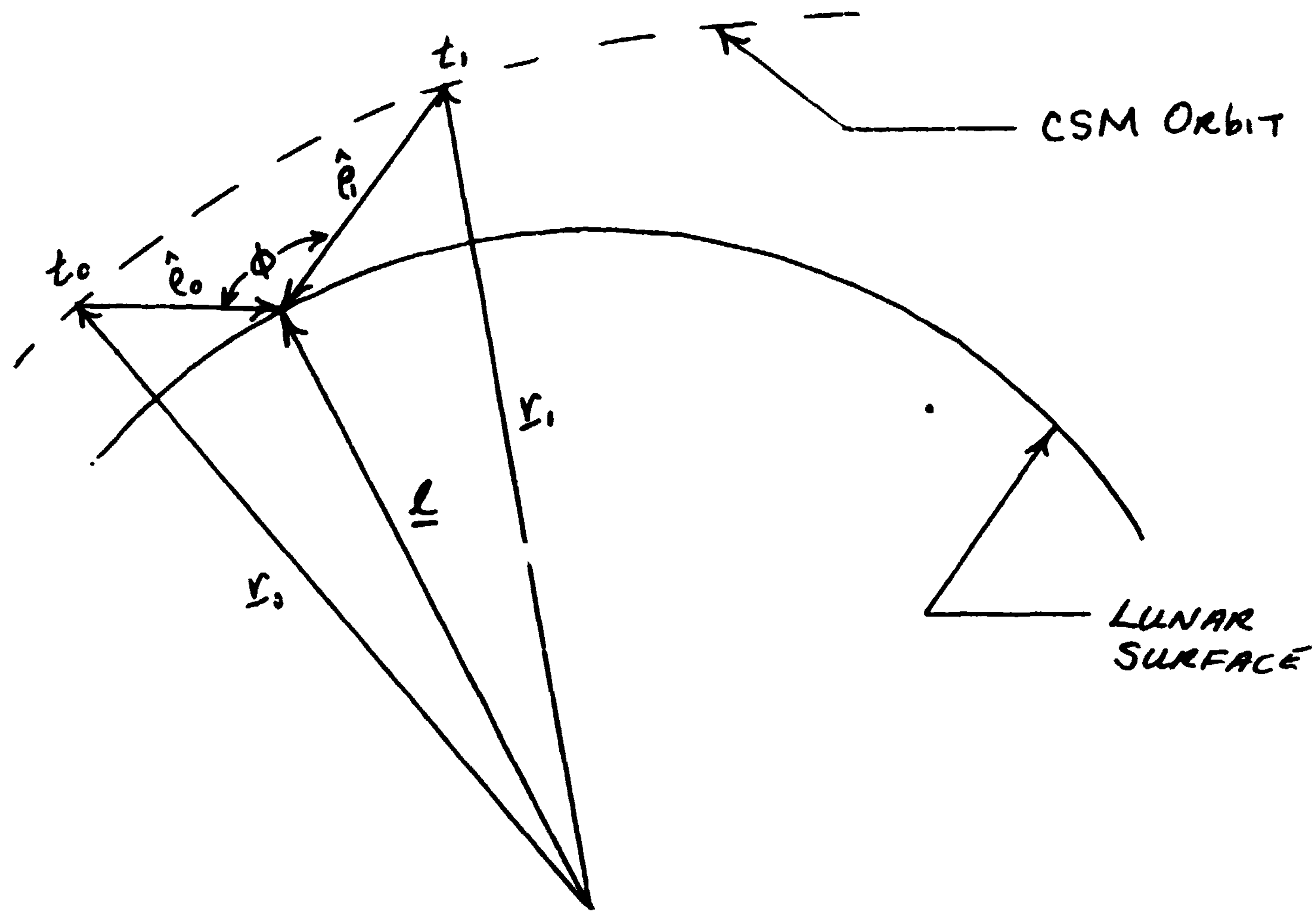


Figure 1a.- The geometry that defines the relations between landing site vector, $\underline{\ell}$ ; the measured unit vectors $\hat{\rho}_0$ and $\hat{\rho}_1$ ; and the spacecraft position vectors at $t_o$ and $t_1$.

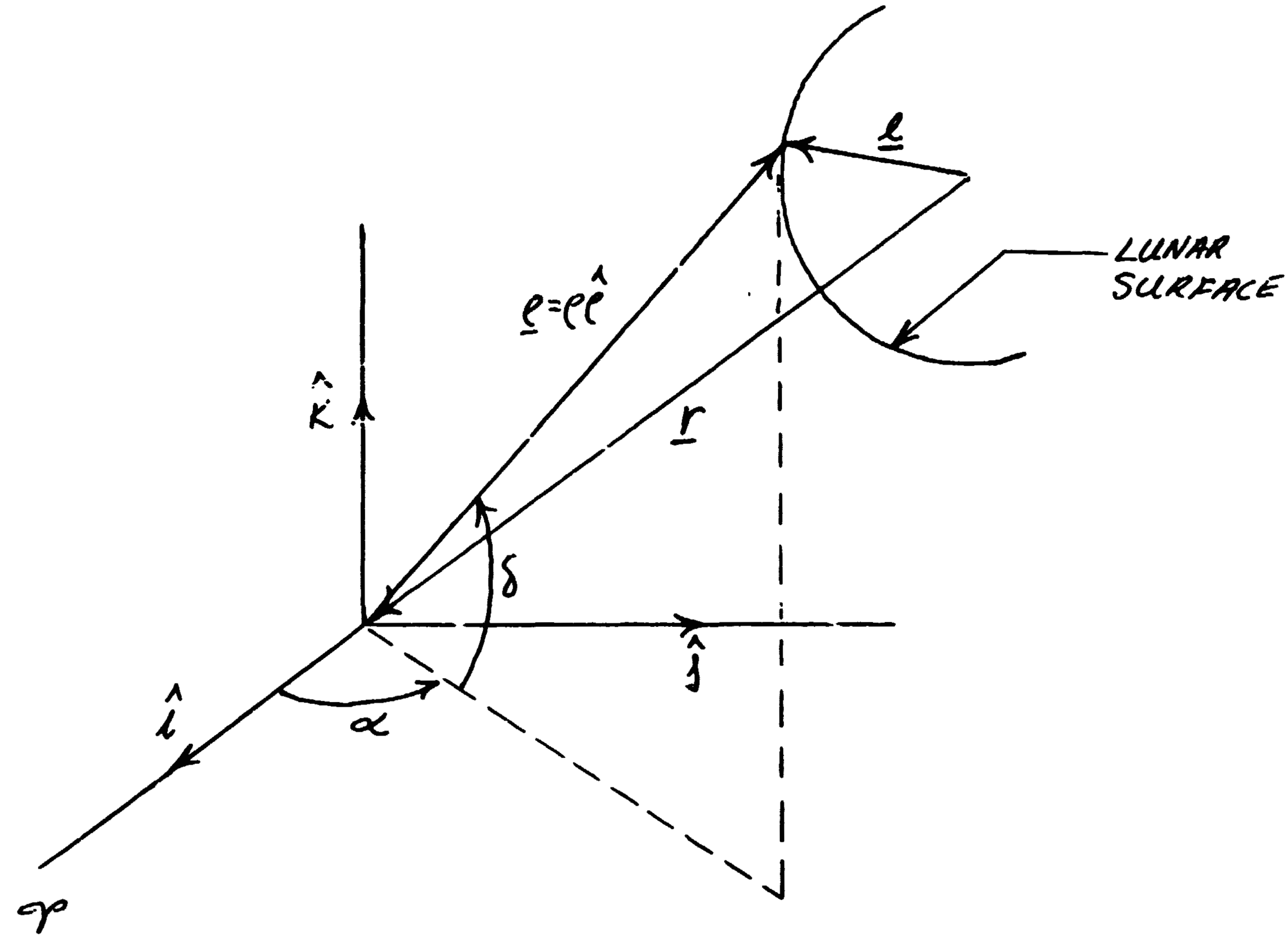


Figure 1 (b).- The geometry that defines a unit vector $\hat{\rho}$ .