MASSACHUSETTS INSTITUTE OF TECHNOLOGY

APOLLO

GUIDANCE, NAVIGATION AND CONTROL

Approved: G. L. Silver Date: 10-26-70
G. L. SILVER, DIRECTOR, SYSTEMS TEST
APOLLO GUIDANCE AND NAVIGATION PROGRAM

Approved: N. E. Sears Date: 10-26-70
N. E. SEARS, DIRECTOR, G&N SYSTEMS
APOLLO GUIDANCE AND NAVIGATION PROGRAM

Approved: David G. Hoag Date: 30 Oct 70
D. G. HOAG, DIRECTOR
APOLLO GUIDANCE AND NAVIGATION PROGRAM

Approved: Ralph R. Ragan Date: 30 Oct 70
R. R. RAGAN, DEPUTY DIRECTOR
CHARLES STARK DRAPER LABORATORY

Submitted for presentation at the 4th IFAC Symposium on Automatic Control in Space, 1971.

R-676

CLOSED-LOOP CONTROL OF STOCHASTIC NONLINEAR SYSTEMS

by

G. T. Schmidt

OCTOBER 1970

MIT CHARLES STARK DRAPER LABORATORY
CAMBRIDGE, MASSACHUSETTS, 02139

ACKNOWLEDGEMENT

This report was prepared under DSR Project 55-23890, sponsored by the Manned Spacecraft Center of the National Aeronautics and Space Administration through Contract NAS 9-4065.

Published by the Charles Stark Draper Laboratory of the Massachusetts Institute of Technology.
Printed in Cambridge, Massachusetts, U. S. A., 1970

R-676

Closed-Loop Control of Stochastic Nonlinear Systems

Abstract

A new approach for optimum control of stochastic nonlinear systems is developed from practical engineering assumptions. Systems amenable to this new approach include optimum guidance and navigation systems for space and terrestrial vehicles, optimum closed-loop process controllers, and optimum controllers for systems with unknown parameters. The classical quadratic synthesis approach – optimization of a deterministic cost and perturbation estimation and control about that solution – is shown to give 24% more cost and 97% more mean-squared terminal error than the combined optimization approach presented for a sample problem involving control of a first-order system with an unknown time constant. Furthermore, the optimum controller automatically designs the best controller to minimize the effects of the unknown parameter without artificial augmentation of the cost function as is done in the sensitivity theory approach. The solution is obtained by expansion of the cost function in a power series around a deterministic trajectory with the assumption of linear perturbation estimation and control about that trajectory. Optimization of the expanded cost function gives necessary conditions dependent on the covariance matrices and the deterministic portion of the cost. When the necessary conditions are solved, a set of open-loop controls, perturbation controller gains, and perturbation estimator gains are obtained that can be precomputed and implemented into the system.

by Dr. G. T. Schmidt

October 1970

Closed-Loop Control of Stochastic Nonlinear Systems

by

George T. Schmidt

Charles Stark Draper Laboratory

Massachusetts Institute of Technology

Cambridge, Massachusetts

1. Introduction

This paper considers the closed-loop control of systems with unknown parameters. Since linear systems with unknown parameters may be considered nonlinear systems, the solution offered in this paper effectively treats treats a much wider class of problem – control of stochastic nonlinear systems. Problems in this category include optimum guidance and navigation systems for space and terrestrial vehicles and optimum closed-loop process controllers. The example used to illustrate the control technique, however, involves only unknown parameters. Many systems have characteristics that are either unknown or highly variable. The control system designer must take this into account in order to achieve satisfactory results.

There are two ways of approaching the problem which have been found useful. First, it is possible to study the effect of these unknown changes on system performance and to try to design a controller so these effects are tolerable. This is called the sensitivity approach[2]. Second, if it is possible to make continuous measurements of system behavior and determine the dynamical characteristics, the controller parameters can then be adjusted based on these measurements. This is called the adaptive approach[3, 4].

The solution offered here lies somewhere in between these two approaches. The technique developed can handle a priori statistical information about the unknown parameters and does not require an artificial augmentation of the cost to cause the controller to consider the unknown parameters. The dimension is the number of state variables and unknown parameters. The controller is partially adaptive in the sense that the unknown quantities are estimated and control action taken. However, the gains used are determined from nominal values of the parameters and nominal values of their statistics rather than basing the gains on the present-observed quantities. Given an infinitely fast computing machine, this could be done but is impractical at the present time.

The approach is based on using practical engineering assumptions to achieve a solution to the control problem. The system is assumed nonlinear and subject to independent white noise. Some nonlinear measurements corrupted by white noise are available and are related to the state of the system. It is desired to minimize the expected value of a cost function that measures the performance of the system. The first practical assumption made in Section 2 is that a controller can be built that will keep the actual state vector near a pre-planned value during the operation of the system so that the expected value of the first-order state deviations is zero. Second, the assumption is made that the controller that keeps these perturbations small is a linear function of the best estimate of these deviations. Third, the best estimate is to be obtained from a linear filter. The cost function is then expanded in a power series around the pre-planned trajectory. Because the deviations are held to first-order, the expansion is correct to second-order. Then, in taking the expected value, first-order terms in the expansion are zero and the expected value of second-order terms are covariance matrices. Thus, the cost function is actually evaluated in terms of a deterministic part due to the pre-planned trajectory and calculatable covariance matrices due to the statistical effects.

The cost, once evaluated, is to be minimized, subject to the constraining differential equations. In Section 3 the calculus-of-variations approach is used to determine the necessary conditions for optimality. It is first shown that the optimal linear filter is a Kalman filter used to estimate the deviations. Second, the optimal perturbation controller is identical in form to that obtained by quadratic synthesis[5]. The third and most important result shows that the necessary conditions defining the pre-planned or deterministic trajectory specify the trajectory as a function of the covariance matrices as well as of the deterministic part of the cost. This latter result is different from the quadratic synthesis approach which picks the pre-planned trajectory on deterministic criteria alone and then uses perturbation estimation and control to follow it. The combined optimization procedure defined in this paper gives a set of necessary conditions that can be straightforwardly applied in practical problems to design the best trajectory considering the statistical nature of the problem.

Section 4 presents the design of a controller for a first-order system with an unknown time constant. For the criteria used, the quadratic synthesis approach would give 24.2% more cost and 97% more mean-squared terminal error over the combined optimization procedure. It is shown that this procedure automatically designs the best controller to minimize the effects of the unknown time constant.

2. Transformation of the Performance Index

Consider a stochastic nonlinear system subject to independent zero-mean white noise $\underline{n}$.

$$\underline{\dot{x}}^a = \underline{f}\,(\underline{x}^a,\ \underline{u}^a,\ t) + \underline{n}(t) \tag{2-1}$$

Continuous measurements are available, subject to independent zero-mean white noise $\underline{v}$.

$$\underline{m}^a = \underline{h}\,(\underline{x}^a,\ \underline{u}^a,\ t) + \underline{v}(t) \tag{2-2}$$

Explicit control over the state and the measurements is allowed through $\underline{u}^a$. It is desired to minimize the expected value of a cost function of the form

$$< J_1 > \ = \ < J\,(\underline{x}^a,\ \underline{u}^a,\ t) > \tag{2-3}$$

Define a system of identical dynamics to that of Eq. 2-1 except for the white noise

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t), \qquad \underline{x}(0) = \langle \underline{x}^a(0) \rangle \tag{2-4}$$

and let

$$\delta\underline{x} = \underline{x}^a - \underline{x} \tag{2-5}$$

$$\delta\underline{u} = \underline{u}^a - \underline{u} \tag{2-6}$$

Assuming continuous first and second derivatives of J with respect to $\underline{x}$ and $\underline{u}$ exist, the cost Eq. 2-3 can be expressed in an infinite series around a cost associated with the noise-free dynamics Eq. 2-4 where the partial derivatives in the expansion would be evaluated on the noise-free dynamics. In general, an infinite number of terms must be considered to adequately represent the cost function. It will, therefore, be specified that there exist a suitable control law that makes the system with noise approximate the noise-free dynamics; i.e., a controller that guarantees that a first-order representation of $\delta\underline{x}$ is valid where Eq. 2-1 is linearized to give

$$\delta\dot{\underline{x}} = \underline{f}_{\underline{x}}\, \delta\underline{x} + \underline{f}_{\underline{u}}\, \delta\underline{u} + \underline{n} \tag{2-7}$$

or

$$\delta\dot{\underline{x}} = F\, \delta\underline{x} + G\, \delta\underline{u} + \underline{n} \tag{2-8}$$

Representation of $\delta\underline{x}$ to first-order retains J_1 correct to second-order

$$\begin{aligned} \langle J_1 \rangle = {} & J(\underline{x}, \underline{u}, t) + \langle \underline{J}_{\underline{x}}\, \delta\underline{x} + \underline{J}_{\underline{u}}\, \delta\underline{u} \\ & + 0.5\, \delta\underline{x}^T \underline{J}_{xx}\, \delta\underline{x} + 0.5\, \delta\underline{u}^T \underline{J}_{uu}\, \delta\underline{u} \\ & + 0.5\, \delta\underline{u}^T \underline{J}_{ux}\, \delta\underline{x} + 0.5\, \delta\underline{x}^T \underline{J}_{xu}\, \delta\underline{u} \rangle \end{aligned} \tag{2-9}$$

This equation is valid for any control system that has the ability to exert tight control such that the effects of noise can be overcome. In the presence of noise this surely requires feedback. Thus, "small" noise is not explicitly assumed, but, rather, the existence of a suitable perturbation controller that exerts "reasonable" values of $\delta \underline{u}$ in keeping $\delta \underline{x}$ small. It should be noted that, for those states which are controllable, their perturbations are controllable through Eq. 2-8. For uncontrollable states, their perturbations are also uncontrollable, so that their deviations must remain small for Eq. 2-9 to be a valid representation of the cost.

At this point two practical constraints are imposed which then provide an elegant solution to this control problem. They are:

(1) The control perturbation to be applied is a linear function of an estimate of the state perturbation

$$\delta \underline{u} = - C \, \delta \hat{\underline{x}} \tag{2-10}$$

where the gains C depend on the noise-free system and are to be determined in some optimal way. It will be seen that, when the gains are picked in an optimal manner, they are independent of any uncontrollable states, but the control does depend on those states through the estimates of them. Furthermore, it is assumed that $\delta \underline{u}$ can be applied exactly, although the method of analysis to be used can be easily extended to the case where this is not true.

(2) The estimate $\delta \hat{\underline{x}}$ is to be obtained from an unbiased linear estimator that has the property

$$< \underline{e}(t) > = 0 \tag{2-11}$$

where the error in the estimate is defined as

$$\underline{e} = \delta \hat{\underline{x}} - \delta \underline{x} \tag{2-12}$$

and the form of the perturbation estimator is specified as

$$\delta\dot{\hat{\underline{x}}} = F\ \delta\hat{\underline{x}} + G\ \delta\underline{u} + K\ (\delta\underline{m} - \underline{h}_x\ \delta\hat{\underline{x}} - \underline{h}_u\ \delta\underline{u}) \tag{2-13}$$

with K to be determined in an optimal fashion.

With the constraint of Eq. 2-13, the initial conditions,

$$< \delta\hat{\underline{x}}(0) > \ = \ 0 \tag{2-14}$$

$$< \delta\underline{x}(0) > \ = \ 0 \tag{2-15}$$

the linearized measurements,

$$\delta\underline{m} = \underline{h}_x\ \delta\underline{x} + \underline{h}_u\ \delta\underline{u} + \underline{v} \tag{2-16}$$

and the perfect knowledge of $\delta\underline{u}$; Eq. 2-8, 2-10, and 2-13 yield for all time:

$$< \delta\hat{\underline{x}}(t) > \ = \ 0 \tag{2-17}$$

$$< \delta\underline{x}(t) > \ = \ 0 \tag{2-18}$$

$$< \delta\underline{u}(t) > \ = \ 0 \tag{2-19}$$

Using these last two conditions in taking the expected value of the cost function Eq. 2-9 results in the elimination of the expected values of $\delta\underline{x}$ and $\delta\underline{u}$; then using the general relationship for any $\underline{y}$, $\underline{w}$, and V,

$$\underline{y}^T V\ \underline{w} = tr\ (V\ \underline{w}\ \underline{y}^T) \tag{2-20}$$

Eq. 2-9 becomes

$$\langle J_1 \rangle = J(\underline{x}, \underline{u}, t) + 0.5 \, tr\,(\underline{J_{xx}} \langle \delta\underline{x}\, \delta\underline{x}^T \rangle)$$
$$+0.5 \, tr\,(\underline{J_{uu}} \langle \delta\underline{u}\, \delta\underline{u}^T \rangle)$$
$$+0.5 \, tr\,(\underline{J_{ux}} \langle \delta\underline{x}\, \delta\underline{u}^T \rangle)$$
$$+0.5 \, tr\,(\underline{J_{xu}} \langle \delta\underline{u}\, \delta\underline{x}^T \rangle) \tag{2-21}$$

Now, using the control law $\delta\underline{u} = -C \; \delta\hat{\underline{x}}$ and defining

$$E = \langle \underline{e}\, \underline{e}^T \rangle = \text{cov. of the estimation error} \tag{2-22}$$

$$\hat{X} = \langle \delta\hat{\underline{x}}\, \delta\hat{\underline{x}}^T \rangle = \text{cov. of the estimate} \tag{2-23}$$

$$Z = \langle \underline{e}\, \delta\hat{\underline{x}}^T \rangle = \text{cross-cov. of the error and the estimate} \tag{2-24}$$

$$X = \langle \delta\underline{x}\, \delta\underline{x}^T \rangle = \text{cov. of the actual state deviation} \tag{2-25}$$

where, from $\underline{e} = \delta\hat{\underline{x}} - \delta\underline{x}$ and Eq. 2-22 -- 2-25

$$X = E + \hat{X} - Z - Z^T \tag{2-26}$$

then Eq. 2-21 becomes

$$\langle J_1 \rangle = J(\underline{x}, \underline{u}, t) + 0.5 \, tr \left[\underline{J_{xx}} (E + \hat{X} - Z - Z^T) \right]$$
$$+ 0.5 \, tr \left[\underline{J_{uu}} \, C \hat{X} C^T \right] - 0.5 \, tr \left[\underline{J_{ux}} (\hat{X} - Z) \, C^T \right]$$
$$- 0.5 \, tr \left[\underline{J_{xu}} \, C \, (\hat{X} - Z^T) \right] \tag{2-27}$$

The original expected value of the cost function has now been evaluated in terms of a deterministic part $J(\underline{x}, \underline{u}, t)$ and second moments. This cost is to be minimized, subject to the differential constraints on $\underline{x}$, and the covariance matrices must also obey differential equations. They are[1]

$$\dot{E} = (F - KM)\,E + E\,(F - KM)^T + KUK^T + Q \tag{2-28}$$

$$\dot{\hat{X}} = (F - GC)\,\hat{X} + \hat{X}\,(F - GC)^T - KMZ - ZM^T K^T + KUK^T \tag{2-29}$$

$$\dot{Z} = (F - KM)\,Z + Z\,(F - GC)^T - EM^T K^T + KUK^T \tag{2-30}$$

with given initial conditions and

$$M = \underline{h}_{\underline{x}} \tag{2-31}$$

$$Q\,\delta(t - t') = \langle \underline{n}(t)\ \underline{n}(t')^T \rangle \tag{2-32}$$

$$U\,\delta(t - t') = \langle \underline{v}(t)\ \underline{v}(t')^T \rangle \tag{2-33}$$

The optimization problem is to minimize Eq. 2-27, subject to Eq. 2-28, 2-29, 2-30, and 2-4, by finding the optimal control $\underline{u}$, the optimal linear feedback controller gains C, and the optimal linear filter gains K. The original statistical measure of performance is reflected in the cost by the appearance of covariance matrices.

3. The Necessary Conditions

The derivation of the necessary conditions for optimality proceeds in the usual calculus-of-variations approach. First, for convenience, assume the original cost function was to be minimized over a fixed time and was of the form

$$\langle J_1 \rangle = \langle k\left[\underline{x}^a(T)\right] + \int_0^T L\,(\underline{x}^a, \underline{u}^a, t)\,dt \rangle \tag{3-1}$$

and define

$$S(T) = k_{\underline{xx}} \tag{3-2}$$

$$A(\underline{x}, \underline{u}, t) = L_{\underline{xx}} \quad (3\text{-}3)$$

$$B(\underline{x}, \underline{u}, t) = L_{\underline{uu}} \quad (3\text{-}4)$$

$$N(\underline{x}, \underline{u}, t) = L_{\underline{xu}} \quad (3\text{-}5)$$

Then, Eq. 2-27 becomes

$$\begin{aligned} < J_1 > &= k\left[\underline{x}(T)\right] + \int_0^T L(\underline{x}, \underline{u}, t)\, dt \\ &+ 0.5\, \mathrm{tr}\left\{S(T)\left[E(T) + \hat{X}(T) - Z(T) - Z(T)^T\right]\right\} \\ &+ 0.5\, \mathrm{tr}\left[\int_0^T (AE + A\hat{X} - AZ - AZ^T + BC\hat{X}C^T \right. \\ &\left. - NC\hat{X} + NCZ^T - N^T\hat{X}C^T + N^TZC^T)\, dt\right] \end{aligned} \quad (3\text{-}6)$$

It can be shown[1] that with $\delta Z(0) = \delta\hat{X}(0) = \delta E(0) = 0$ and an assumption that the initial error and the estimate are uncorrelated ($Z(0) = 0$), by choice of

$$K = EM^T U^{-1} \quad (3\text{-}7)$$

then

$$Z(t) = 0 = \delta E(t) = \delta\hat{X}(t) = \delta Z(t) \quad (3\text{-}8)$$

$$\delta < J_1 > = 0 \quad (3\text{-}9)$$

The cost is optimized (stationary) with respect to changes in K. Furthermore, this choice of K for the optimal linear filter results in the estimate and the error in the estimate being orthogonal for all time. This K corresponds to the Kalman filter and the cost function now reduces to

$$\langle J_1 \rangle = k\left[\underline{x}(T)\right] + \int_0^T L(\underline{x}, \underline{u}, t)\, dt$$

$$+ 0.5\, \mathrm{tr} \left[S(T)\, E(T)\right] + 0.5\, \mathrm{tr} \left[S(T)\, \hat{X}(T)\right]$$

$$+ 0.5\, \mathrm{tr} \left[\int_0^T (AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T)\, dt\right] \tag{3-10}$$

subject to

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \tag{3-11}$$

$$\dot{E} = FE + EF^T + Q - EM^T U^{-1} ME \tag{3-12}$$

$$\dot{\hat{X}} = (F - GC)\hat{X} + \hat{X}(F - GC)^T + EM^T U^{-1} ME \tag{3-13}$$

The derivation of the necessary conditions for optimality now proceeds in the usual calculus-of-variations approach. Adjoin to the cost the constraints ($\dot{\underline{x}}$, $\dot{E}$, and $\dot{\hat{X}}$) by means of arbitrary multipliers ($\underline{p}$, 0.5P, 0.5S) and define a Hamiltonian

$$H = L + \underline{p}^T \underline{f} + 0.5\, \mathrm{tr}\,(P\dot{E}) + 0.5\, \mathrm{tr}\,(S\dot{\hat{X}})$$

$$+ 0.5\, \mathrm{tr}\,(AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T) \tag{3-14}$$

The adjoint variables must satisfy

$$\dot{\underline{p}} = -H_{\underline{x}}^T \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \tag{3-15}$$

$$\dot{P} = -2H_E \quad , \quad P(T) = S(T) \tag{3-16}$$

$$\dot{S} = -2H_{\hat{X}} \quad , \quad S(T) = k_{\underline{xx}} \tag{3-17}$$

The optimal control parameters ($\underline{u}$ and C) are determined from

$$H_{\underline{u}} = 0 \qquad (3\text{-}18)$$

$$H_C = 0 \qquad (3\text{-}19)$$

Using Eq. 3-17 first, results in

$$\dot{S} = -(F - GC)^T S - S(F - GC) + NC + C^T N^T - C^T BC - A \qquad (3\text{-}20)$$

Similarly application of Eq. 3-16 yields

$$\dot{P} = -(F - EM^T U^{-1} M)^T P - P(F - EM^T U^{-1} M)$$
$$-M^T U^{-1} MES - SEM^T U^{-1} M - \mathcal{A} \qquad (3\text{-}21)$$

Application of Eq. 3-19 yields for arbitrary $\hat{X}$

$$C = B^{-1}(G^T S + N^T) \qquad (3\text{-}22)$$

and substituting into Eq. 3-20 gives

$$\dot{S} = -F^T S - SF + (G^T S + N^T)^T B^{-1}(G^T S + N^T) - A \qquad (3\text{-}23)$$

The feedback-controller gains C are identical to those that would be obtained by using quadratic synthesis around a given reference trajectory. However, application of Eq. 3-18 and 3-15 shows quite clearly that the noise-free system must be chosen to include the effects of the stochastic nature of the problem:

$$H_{\underline{u}} = 0 = L_{\underline{u}} + \underline{p}^T G + 0.5\left[\mathrm{tr}(P\dot{E})\right]_{\underline{u}} + 0.5\left[\mathrm{tr}(S\dot{\hat{X}})\right]_{\underline{u}}$$
$$+ 0.5\left[\mathrm{tr}(AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T)\right]_{\underline{u}} \qquad (3\text{-}24)$$

$$\dot{\underline{p}} = -H_{\underline{x}}^T = -L_{\underline{x}}^T - F^T\underline{p} - 0.5\left[\text{tr}\,(\dot{P}\dot{E})\right]_{\underline{x}}^T - 0.5\left[\text{tr}\,(S\dot{\hat{X}})\right]_{\underline{x}}^T$$

$$-0.5\left[\text{tr}\,(AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T)\right]_{\underline{x}}^T \quad (3\text{-}25)$$

Only for the case of a linear system with linear measurements, noises independent of the state and control, and quadratic cost are the terms involving the derivatives of traces equal to zero, and in that case the noise-free trajectory may be designed without regard for the statistics. This section has shown that, under practical engineering constraints of linear perturbation estimation and feedback control, the overall optimization procedure results in a set of necessary conditions that can be straightforwardly applied in practical design problems.

Finally, the end result of the optimization program will be an optimal control history $\underline{u}(t)$, an optimal trajectory $\underline{x}(t)$, a set of feedback controller gains C(t), and a set of estimator gains K(t). All of these quantities can be calculated a priori and implemented into the system. In Reference 1 some special cases are considered together with a comparison of the work most closely related to this approach[6].

4. Example. Closed-Loop Control of a First-Order System With Unknown Time Constant

As an illustration of the new control technique, a closed-loop controller will be designed for the stochastic first-order system

$$\dot{y}^a = -b^a y^a + u^a + n \quad (4\text{-}1)$$

The inverse-time constant b^a is assumed to be an unknown constant picked from a Gaussian distribution with mean b. This unknown parameter is considered to be another state variable so that the augmented state vector is of dimension 2 and obeys

$$\dot{\underline{x}}^a = \begin{bmatrix} \dot{y}^a \\ \\ \dot{b}^a \end{bmatrix} = \begin{bmatrix} \dot{x}_1^a \\ \\ \dot{x}_2^a \end{bmatrix} = \begin{bmatrix} -x_2^a \, x_1^a \\ \\ 0 \end{bmatrix} + \begin{bmatrix} u^a \\ \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ \\ 0 \end{bmatrix} \qquad (4\text{-}2)$$

The noise-free system obeys

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 \, x_1 \\ \\ 0 \end{bmatrix} + \begin{bmatrix} u \\ \\ 0 \end{bmatrix} \qquad (4\text{-}3)$$

with the assumed initial conditions

$$x_2(0) = \langle b^a \rangle = b \qquad (4\text{-}4)$$

$$x_1(0) = \langle x_1^a(0) \rangle = 0 \qquad (4\text{-}5)$$

Furthermore, it is assumed that the expected value of y^a at the terminal time is specified as

$$x_1(T) = \langle y^a(T) \rangle = 1 \qquad (4\text{-}6)$$

The matrices F and G are

$$F = \begin{bmatrix} -x_2 & -x_1 \\ \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \qquad (4\text{-}7)$$

Linear measurements of y^a corrupted by white noise are available to the controller

$$m^a = y^a + v \tag{4-8}$$

then

$$M = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4-9}$$

The stochastic cost function to be minimized is

$$< J_1 > = 0.5 \quad < \int_0^T (u^a)^2 \, dt > \tag{4-10}$$

Taking the expected value with the assumption of perturbation estimation and control results in

$$< J_1 > = 0.5 \int_0^T u^2 \, dt + 0.5 \, tr \left[\int_0^T B C \hat{X} C^T \, dt \right] \tag{4-11}$$

where

$$B = L_{uu} = 1 \tag{4-12}$$

From Eq. 4-11 it is clear that no penalty would be attached to deviations in $x_1^a(T)$ away from specified nominal $x_1(T)$. Thus, the cost is augmented to weight terminal mean-squared deviations in the perturbation controller

$$< J_1 > = 0.5 \int_0^T u^2 \, dt + 0.5 \, tr \left[\int_0^T B C \hat{X} C^T \, dt \right] + 0.5 \, tr \left\{ S(T) \left[E(T) + \hat{X}(T) \right] \right\} \tag{4-13}$$

The numerical values used in the solution to this problem were $T = 10$, $b = 1$, $U = 1$,

$$E(0) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \tag{4-14}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{4-15}$$

and

$$S(T) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \tag{4-16}$$

The necessary conditions were solved numerically, using a first-order method[1]. The results of this combined optimization approach will be compared with the quadratic-synthesis approach. In this latter case the nominal trajectory is determined from the same necessary conditions with the exception that the adjoint variables are determined without regard to the statistics so that

$$\dot{\underline{p}} = -F^T \underline{p} \tag{4-17}$$

as a result of minimizing

$$0.5 \int_0^T u^2 \, dt \tag{4-18}$$

without the covariance terms. The time constant is being identified.

The optimal deterministic control signals are shown in Figure 1. The quadratic-synthesis approach results in a control $u = 2 \exp(t - 10)$, minimizing the energy integral Eq. 4-18 with a value of 1.00. The combined optimization approach yields a value of 1.31 for the energy integral. However, the quadratic-synthesis approach yields a value of 1.51 for the remaining matrix terms in the cost Eq. 4-13 as opposed to 0.71 for the combined optimization. The total average cost is thus 2.51 versus 2.02; the quadratic synthesis approach actually costs 24.2% more. Such a substantial improvement in performance in a more practical problem would be significant.

The difference in cost between the two approaches is due primarily to the performance in minimizing the mean-squared deviation in the state at the terminal time. Figure 2 and 3 show the differences between the two cases in this respect, 1.36 versus 0.69. Figure 4 gives the covariances for the inverse-time constant (4.42 versus 3.37). Note that the estimation of the inverse time constant is poorer in the combined optimization case. This is because the control system tends to minimize the sensitivity to the unknown parameter.

This last statement can be better understood from Figure 5. The final value of x_1 can be written as

$$x_1(T) = -b \int_0^T x_1 \, dt + \int_0^T u \, dt \qquad (4\text{-}19)$$

Clearly, variations in $x_1(T)$ with respect to changes in b are minimized, if the area under the x_1 versus t curve is minimized. The combined optimization procedure attempts to do just that, as is shown in Figure 5, completely automatically as opposed to the sensitivity-theory design approach to problems of this type.

5. Summary

This paper has presented a new technique for the control of stochastic nonlinear systems. For the sample problem considered, the procedure was seen to offer substantial improvements in system performance as compared to the quadratic synthesis approach. Certainly the main disadvantage of the procedure lies in the fact that it is only appropriate in situations where the reference-trajectory concept is valid. One situation where this is true is in atmospheric-entry problems where the reference-trajectory concept is well-established and this technique has been applied to that particular problem[1]. Other possible applications might include, for example, optimal guidance and navigation policies for space and terrestrial vehicles and optimum closed-loop process controllers. The extension of the theory to discrete systems represents a straightforward, but not necessarily trivial task.

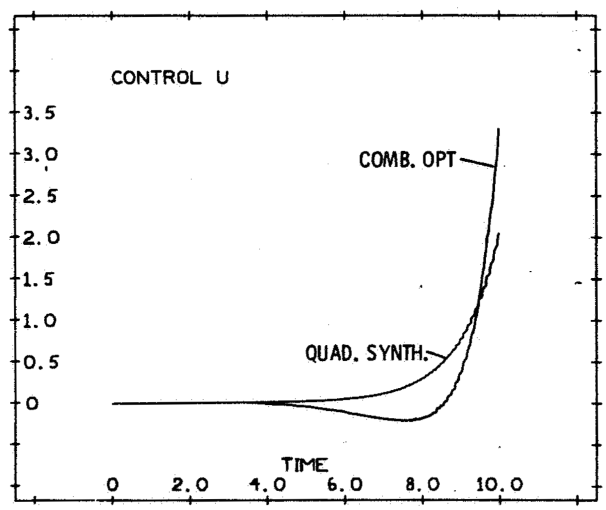

Figure 1 Optimal Control Input

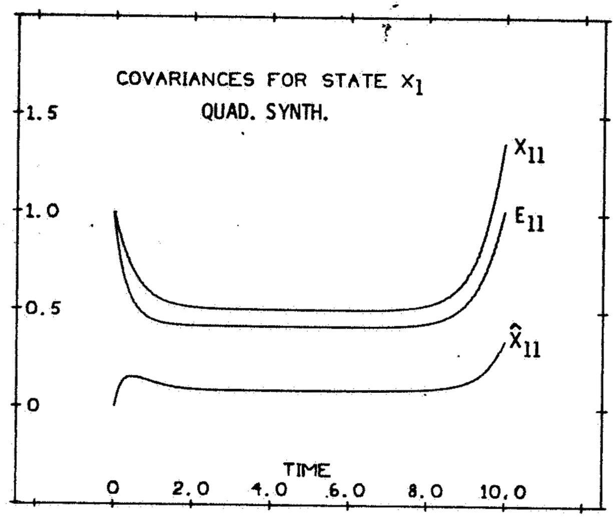

Figure 2 Covariances for x_1 - Quadratic Synthesis

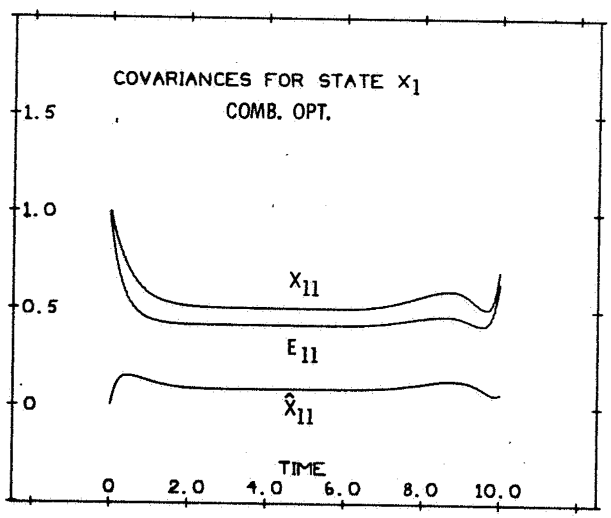

Figure 3 Covariances for x_1 - Combined Optimization

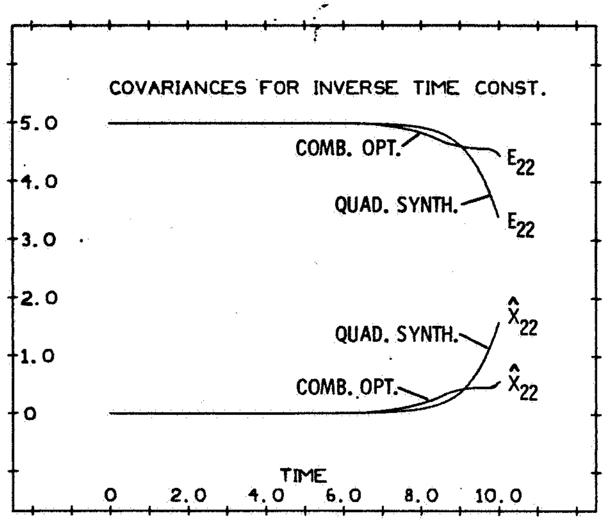

Figure 4 Covariances for Inverse Time Constant

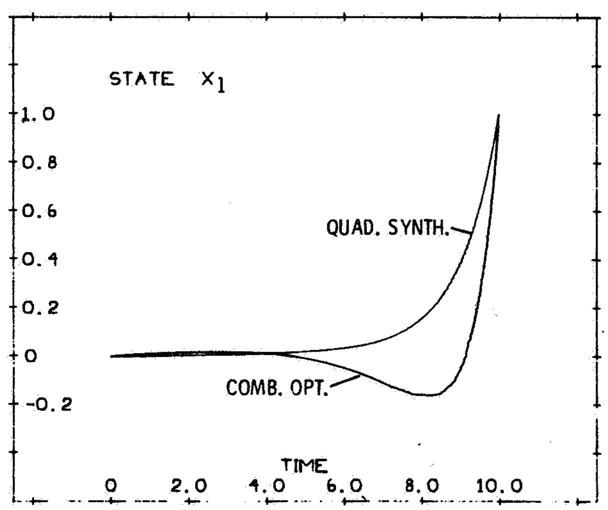

Figure 5 Optimum Trajectory for x_1

References

1. Schmidt, George T., "A New Technique for Identification and Control of Systems With Unknown Parameters", Sc.D. Thesis, M.I.T., Cambridge, Mass., 1970. (Draper Lab. Rpt. T-542)

2. Kahne, S., "Low Sensitivity Design of Optimal Linear Control Systems", IEEE Trans. on Aero. and Elec. Sys., Vol AES-4, No. 3, pp. 374-379, 1968.

3. Aoki, M., Optimization of Stochastic Systems, Academic Press, New York, 1966.

4. Swoder, D., Optimal Adaptive Control Systems, Academic Press, New York, 1966.

5. Bryson, A. and Ho, Y-C, Applied Optimal Control, Blaisdell, Waltham, Mass., 1969.

6. Denham, W., "Choosing the Nominal Path for a Dynamic System With Random Forcing Functions to Optimize the Statistical Performance", TR 449, Div. Eng. and Applied Physics, Harvard University, Cambridge, Mass., 1964.

DISTRIBUTION LIST

N. Sears
D. Hoag
R. Ragan
A. Laats
R. Crisp
J. Gilmore
E. Grace
R. Battin
T. Edelbaum
L. Quagliata
G. Ogletree
C. Gray (2)
G. Cherry
G. Levine
P. Kachmar
P. Philliou
C. Pu
D. Keene
R. Schlundt
R. Stengel
M. Womble
G. Silver
P. Maybeck
M. Athans
D. Gustafson
P. Felleman
T. Lawton (MSC)
R. O'Donnell (KSC)
R. Larson
S. Copps
M. Hamilton
J. Nevins
T. Fitzgibbon
E.C. Hall
J.H. Laning
J. Deyst
J. Potter
W. Wrigley
D. Fraser (2)
G. Schmidt (50)
A. Klumpp
T. Chien
W. McFarland
J. Deckert
J. Speyer
K. Fertig
R. Weatherbee
Apollo Library (2)
CSDL/TDC (10)
W. Vander Velde

External:

NASA/RASPO (1)

AC Electronics (3)

Kollsman (2)

Raytheon (2)

MSC: (21&1R)

National Aeronautics and Space Administration
Manned Spacecraft Center
Houston, Texas 77058
ATTN: Apollo Document Control Group (BM 86) (18&1R)
M. Holley (2)
T. Gibson (1)

KSC: (1R)

National Aeronautics and Space Administration
J.F. Kennedy Space Center
J.F. Kennedy Space Center, Florida 32899
ATTN: Technical Document Control Office

LRC: (2)

National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia
ATTN: Mr. A.T. Mattson

GA: (3&1R)

Grumman Aerospace Corporation
Data Operations and Services, Plant 25
Bethpage, Long Island, New York
ATTN: Mr. E. Stern

NAR: (8&1R)

North American Rockwell, Inc.
Space Division
12214 Lakewood Boulevard
Downey, California 90241
ATTN: CSM Data Management
D/096-402 AE99

NAR RASPO: (1)

NASA Resident Apollo Spacecraft Program Office
North American Rockwell, Inc.
Space Division
12214 Lakewood Boulevard
Downey, California 90241

GE: (1)

General Electric Company
Apollo Systems
P.O. Box 2500
Daytona Beach, Florida 32015
ATTN: E.P. Padgett, Jr./Unit 509