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## GUIDANCE AND NAVIGATION

Approved: Milton B. Trageser Date: 2/6/64
Milton B. Trageser, Director
Apollo Guidance & Navigation Program

Approved: Roger B Woodbury Date: 2/10/64
Roger B. Woodbury, Deputy Director
Instrumentation Laboratory

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# A Class of Unified Explicit Methods For Steering Throttleable And Fixed-Thrust Rockets

by
George W. Cherry

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MIT INSTRUMENTATION LABORATORY
CAMBRIDGE 39, MASSACHUSETTS

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# A CLASS OF UNIFIED EXPLICIT METHODS FOR STEERING THROTTLEABLE AND FIXED-THRUST ROCKETS

George W. Cherry*

Massachusetts Institute of Technology,
Cambridge, Mass.

Abstract

This paper deals with the generation of a class of explicit guidance laws for computing rocket steering and throttling commands. The steering laws provide control of final components of the velocity vector, as well as, when it is appropriate, control of final coordinates of position. The viewpoint taken in the paper is that the commanded thrust vector can be computed in-flight as the explicit solution to a two-point boundary-value problem. Thus, the commanded thrust vector is found by a direct solution of the appropriate equations of motion subject to the initial boundary condition of the vehicle's instantaneous measured state and final boundary condition of the vehicle's desired state. Three goals motivate the synthesis of the guidance equations: 1), simplicity of the algorithms which must be programmed on the vehicle-borne computer; 2), fuel economy in traveling from the initial boundary condition to the final boundary condition; 3), independence of the steering laws from standard conditions and nominal trajectories. To illustrate the guidance method, the paper discusses three principal thrusting phases of a lunar reconnaissance and landing mission. Programming and simulation of the guidance laws for the lunar landing mission has shown them to achieve the three design goals in good measure.

## Introduction

Crucial to the success and safety of manned exploration of the moon is the problem of furnishing the spacecraft com-

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*Staff Member, Space Guidance Analysis Division, Instrumentation Laboratory.

puter with simple and explicit, economical and accurate, rocket-steering algorithms. We will discuss the need for explicit guidance in this paragraph. Consider the problem of guiding the spacecraft during the lunar landing. As the vehicle descends toward a chosen landing site, the events of new information, mishaps and other contingencies may intervene to change the boundary-value problem which the guidance system must solve. For example, when the altitude of the spacecraft becomes low enough, a small doppler radar may be used to improve the navigation system's estimate of important components of the vehicle's velocity vector. An updating of the spacecraft's velocity vector changes the initial boundary-point, the vehicle's instantaneous measured state. Thus, it is desirable to synthesize guidance equations which compute the control variable (the thrust vector) as an explicit function of the initial boundary conditions. For then, the thrust vector command can naturally, and almost instantaneously, reflect the appropriate steering response to the new initial boundary conditions. The boundary-value problem could also be changed by a re-specification of the final boundary conditions. Suppose that the astronaut observes the lunar terrain and perceives conditions at the specified landing point that make that site unattractive, or even dangerous. The astronaut may then specify a suitable landing site. His re-designation of the terminal position vector alters the final boundary conditions. Thus, it is desirable to compute the thrust vector command as an explicit function of the final boundary conditions. Consider, finally, the case of a mishap occurring sometime during the landing maneuver. Suppose that the spacecraft has an altitude of about 15,000 feet and a speed of about 1,000 feet/second, and that at this instant the throttleable landing engine fails. This failure necessitates the immediate staging of the landing engine, ignition of the ascent engine and guidance of the spacecraft to either a waiting orbit, or an immediate rendezvous with the parent spacecraft. The initial conditions for this "abort" ascent are far different from the initial conditions for the nominal ascent from the lunar surface following a normal lunar landing and surface exploration. Nevertheless, the same steering program used for the normal ascent can be used for the abort ascent-provided that the steering program is truly an explicit solution. Many other kinds of new information, contingencies and mishaps can be imagined. Most of them lead to a modification, perhaps a substantial modification, of the standard two-point boundary-value problems which are envisioned and planned for the nominal mission. It is hardly practical to try to anticipate each and every two-point boundary-value problem which might arise, and then equip the spacecraft computer with an ad hoc steering program for each possibility. It seems far more reasonable to find general explicit solutions to the two-point boundary-value problems of guided powered trajectories.

To possess the range of flexibility to which we alluded in the preceding paragraph, the guidance equations must be truly explicit. They must be direct solutions to the equations of motion subject to the initial boundary conditions of the vehicle's instantaneous measured state, and the final boundary conditions of the vehicle's required state. It is not satisfactory for the equations merely to give the appearance of being explicit. Some final-value control equations give the appearance of being explicit equations; however, they are precomputed solutions valid in only a small region about the nominal trajectory for which the equations were designed. Consider the following landing control equations in which the $\rho_i$'s are pre-computed and stored functions of time.

$$\alpha = \rho_1(\dot{x}_D - \dot{x}) + \rho_2(x_D - x) + \rho_3(\dot{y}_D - \dot{y}) + \rho_4(y_D - y)$$

$$T = \rho_5(\dot{x}_D - \dot{x}) + \rho_6(x_D - x) + \rho_7(\dot{y}_D - \dot{y}) + \rho_8(y_D - y)$$

The first equation yields the thrust angle, the second equation yields the thrust magnitude. The D-subscripted quantities are the desired final values of the components of position and velocity; their unsubscripted counterparts are the present values of the components of position and velocity. Thus, the equations for the control variables appear to express the thrust angle and thrust magnitude as explicit functions of the initial and final boundary conditions. Significant changes in the terminal components of position $x_D$ and $y_D$ would require a change in the time-varying coefficients $\rho_i$. Thus, there is an implicit dependence of $\alpha$ and T on $x_D$ and $y_D$ through the $\rho_i$'s. Because this dependence has not been stated explicitly, only small changes in $x_D$ and $y_D$ would be allowable. We might call the equations for $\alpha$ and T pseudo-explicit. A test for explicit steering equations is to ask whether it is possible to insert any values of initial and final boundary conditions into the equations and have them remain valid. If the answer is yes, then the equations are genuinely explicit. This criterion requires, of course, that the steering equations be direct solutions to the equations of motion.

For the mechanization of our guidance schemes, we assume a vehicle-borne digital computer and navigation system.

The navigation system determines for the computer the instantaneous position and velocity of the spacecraft, the initial "point" of the two-point boundary value problem. The final "point" is the terminal or burnout point, at which some or all coordinates of position and components of velocity are specified. These terminal conditions can be determined before launch and stored in the computer, computed in flight by subroutines in the computer, or placed into the computer by an astronaut.

## Orbital Plane-Control Guidance Law

The first guidance law we derive is a simple one. The more complicated guidance laws we present later are derived in essentially the same way.

One reason for treating the plane control guidance law first is that three-dimensional trajectory control problems can be analysed into two problems: a one-dimensional problem of establishing a vehicle in a desired plane, and a two-dimensional problem of establishing a desired position and velocity within that plane. By deriving the plane-control steering law at the outset, we may treat each subsequent three-dimensional problem as a two-dimensional problem.

The plane-control boundary value problem is illustrated in Figs. 1 and 2. The desired orbital plane passes through the center of the planet around which the powered maneuver is being performed. A coordinate axis, Y, normal to the desired plane, measures the deviation of the spacecraft from the plane. At any instant before rocket burnout, the spacecraft may have a displacement from the plane and component of velocity normal to the plane. By the terminal time, the rocket burnout time, the displacement from the plane and the component of velocity normal to the plane must be zero. We take the viewpoint that the origin of the computation time axis is located at the instant of the desired thrust vector computation. That is, when we compute the desired thrust vector, we take the viewpoint that we are just initiating the problem, that we have no concern with time past, and that $t = 0$. With this convention in mind, the statement of the boundary-value problem is the following: given any instantaneous values of deviation from, and velocity normal to, the plane, $y(0)$ and $\dot{y}(0)$ respectively, find the thrust angle regime $\alpha_Y$ which will ensure that $y(t)$ and $\dot{y}(t)$ will be zero by the terminal time $t = T$. Thus, the objective is to ensure that

$$y(T) = 0 \qquad (1)$$

$$\dot{y}(T) = 0 \qquad (2)$$

where T is the terminal time. Note that because the origin of the time-axis is located at the present instant, T is the time-to-go until the terminal boundary-point constraints are satisfied. Thus

$$T = T_{go} \tag{3}$$

We shall use the equivalent symbols in (3) interchangeably.

The differential equation of y motion is

$$\ddot{y} = a_T \sin \alpha_Y + \bar{g} \cdot \bar{y}_1 \tag{4}$$

where $\bar{y}_1$ is a unit vector normal to the plane. The last term on the right-hand side of (4) is the component of gravitational acceleration along the Y axis, $\bar{g}$ being the total gravitational acceleration vector. The gravitational acceleration normal to the desired plane is usually very small, is toward the desired plane, and approaches zero as the vehicle approaches the desired plane. This acceleration is of little dynamic consequence, and could be ignored. We shall include the effect of this acceleration in our guidance law in order to illustrate how such terms, which usually make the differential equations nonlinear, can be handled.

Equation (4) can be integrated, at least symbolically, between t = 0 and t = T, yielding

$$\dot{y}(T) - \dot{y}(0) = \int_0^T (a_T \sin \alpha_Y + \bar{g} \cdot \bar{y}_1)\, dt \tag{5}$$

Another integration yields

$$y(T) - y(0) - \dot{y}(0)\, T = \int_0^T \left[ \int_0^t (a_T \sin \alpha_Y + \bar{g} \cdot \bar{y}_1) ds \right] dt \tag{6}$$

Equations (5) and (6), with y (T) and $\dot{y}(T)$ replaced with zeros, are the two equations of constraint which our choice of $\alpha_Y(t)$ must satisfy. There are many choices of $\alpha_Y(t)$ which satisfy (5) and (6). We desire an $\alpha_Y(t)$ which moves the vehicle into the desired plane by the terminal time in a nearly optimum manner. What we mean by optimum, requires a discussion. We want to perform the plane control maneuver with as little rocket ΔV deflected from the X-Z plane as possible. Controlling the orbital plane requires a component of thrust acceleration parallel to the Y axis, reducing the thrust acceleration parallel to the X-Z plane. If we can perform the plane-control maneuver and maximize the integral of thrust acceleration remaining parallel to the X-Z plane, that is, maximize the $\Delta V$ parallel to the X-Z plane, we optimize the plane-control maneuver. By means of the calculus of variations, it can be shown that

$$\tan \alpha_Y(t) = C + Dt \tag{7}$$

prescribes the form of $\alpha_Y(t)$ which optimizes the plane-control maneuver.† The constants C and D must be chosen to satisfy (5) and (6). If we choose

$$\alpha_Y(t) = \tan^{-1}(C + Dt) \tag{8}$$

in accordance with the calculus of variation's prescription, we must then substitute this choice of $\alpha_Y(t)$ into Eqs. (5) and (6) and attempt to determine C and D to satisfy the two equations of constraint. The determination of C and D is not a simple task, and would involve disagreeable numerical interation procedures. We make an approximation to the prescription of the calculus of variations, which compromises, to an extent we discuss later, the optimality of the solution, but which allows us to obtain a closed-form solution to the boundary value problem and satisfy exactly Eqs. (1, 2, 5 and 6). We postulate that

$$\sin \alpha_Y = A + Bt - (\bar{g} \cdot \bar{y}_1)/a_T \tag{9}$$

We note that the second term in Eq. (9) is small since we divide the small component of gravitational acceleration along the Y axis by the total thrust acceleration of the rocket engine. For moderate angles, the sine and tangent of an angle are nearly equal. Consequently, Eq. (9) is quite close to satisfying the calculus of variations prescription. Substituting Eq. (9) into the differential equation of y motion, Eq. (4) yields

$$\ddot{y} = Aa_T + Ba_T t \tag{10}$$

It should be evident that the motivation for defining $\sin \alpha_Y$ as in Eq. (9) was to cancel the gravitational acceleration from Eq. (4) and produce the simple linear differential Eq. (10). The next step in the derivation of the plane-control steering law is to integrate Eq. (10) to obtain the two equations of

† For a constant $\bar{g} \cdot \bar{y}_1$ or zero $\bar{g} \cdot \bar{y}_1$ case, which is approximately the case we have.

constraint which A and B must satisfy. Note that A and B multiply linearly independent functions of time. (This is true of course if $a_T$ is not identically zero.) We must determine A and B so that the right-hand side of Eq. (9) can be evaluated to yield the sine of the required thrust angle. We may look upon the determination of A and B, as the problem of determining how much $a_T$ and how much $a_T t$ must be added to $-(\bar{g} \cdot \bar{y}_1)$ to give the required total allocation of thrust acceleration along the Y axis. Since $Aa_T$ and $Ba_T t$ are linearly independent, we see that A and B constitute two independent control parameters. It is appropriate that we should have two independent control parameters since we have two equations of constraint to satisfy.

Integrating Eq. (10) between the present time t = 0 and t yields

$$\dot{y}(t) - \dot{y}(0) = A\int_0^t a_T(s)\,ds + B\int_0^t a_T(s)\,s\,ds \qquad (11)$$

We have introduced the dummy variable of intergration s to avoid confusion between the upper limit of integration and the variable of integration. Substituting T for t in (11) yields

$$\dot{y}(T) - \dot{y}(0) = A\int_0^T a_T(s)\,ds + B\int_0^T a_T(s)\,s\,ds \qquad (12)$$

Integrating (11) between t = 0 and t = T yields

$$y(T) - y(0) - \dot{y}(0)T = A\int_0^T f(t)\,dt + B\int_0^T g(t)\,dt \qquad (13)$$

where

$$f(t) = \int_0^t a_T(s)\,ds \qquad (14)$$

$$g(t) = \int_0^t a_T(s)\,s\,ds \qquad (15)$$

Now the solution to the boundary value problem is near at hand. Substituting y(T) = 0 and $\dot{y}(T) = 0$ into (12) and (13) produces the two equations of constraint which A and B must satisfy. These equations constitute a pair of simultaneous linear algebraic equations in the two unknowns, A and B. Solving for A and B is consequently simple. Obtaining A and B from (12) and (13), we then compute the desired thrust angle $\alpha_Y$ from Eq. (9).

The reader, no doubt, wonders how we evaluate the integrals which are the coefficients of A and B in Eqs. (12) and (13). The solutions for A and B from Eqs. (12) and (13) are in terms of the left-hand sides of these equations and the coefficients of A and B. If we assume the thrust acceleration $a_T$ is constant, the evaluation of the integrals is trivial. But we are interested in obtaining the solution for large thrust, chemical rocket engines. Such engines typically have constant thrust and mass flow, and consequently, linearly decreasing mass and increasing thrust acceleration. A mathematical expression for the thrust acceleration of such an engine is

$$a_T(t) = V_e/(\tau - t) \tag{16}$$

where

$$\tau = V_e/a_T(0) \tag{17}$$

(The reader who is not familiar with this expression for the thrust acceleration of a constant thrust, constant mass flow engine may turn to the Appendix for a derivation. He will also find there a simple scheme for smoothing the thrust acceleration $a_T$ and estimating the effective exhaust velocity $V_e$.)

By using the expression for the thrust acceleration given in Eq. (16), we can evaluate all the integrals in Eqs. (12) and (13) and frame A and B in closed-form expressions involving only known or measurable quantities. The evaluation of the integrals yields

$$a_{11} = \int_0^{T_{go}} a_T(t)\, dt = - V_e \ln(1 - T_{go}/\tau) = \Delta V \tag{18}$$

$$a_{12} = \int_0^{T_{go}} a_T(t)\, t\, dt = a_{11}\tau - V_e T_{go} \tag{19}$$

$$a_{21} = \int_0^{T_{go}} f(t)\, dt = - a_{12} + a_{11} T_{go} \tag{20}$$

$$a_{22} = \int_0^{T_{go}} g(t)\, dt = a_{21}\, \tau - 0.5 V_e\, T_{go}^2 \qquad (21)$$

With the symbols we have just introduced, we reexpress the equations of constraint (12) and (13) as

$$\dot{y}(T) - \dot{y}(0) = a_{11} A + a_{12} B \qquad (22)$$

$$y(T) - y(0) - \dot{y}(0)\, T_{go} = a_{21} A + a_{22} B \qquad (23)$$

In terms of the following symbols

$$\Delta = a_{11} a_{22} - a_{12} a_{21} \qquad (24)$$

$$b_{11} = a_{22}/\Delta \qquad (25)$$

$$b_{12} = - a_{12}/\Delta \qquad (26)$$

$$b_{21} = - a_{21}/\Delta \qquad (27)$$

$$b_{22} = a_{11}/\Delta \qquad (28)$$

we may finally formulate the expressions for A and B

$$A = b_{11}\left[\dot{y}(T) - \dot{y}(0)\right] + b_{12}\left[y(T) - y(0) - \dot{y}(0)\, T_{go}\right] \qquad (29)$$

$$B = b_{21}\left[\dot{y}(T) - \dot{y}(0)\right] + b_{21}\left[y(T) - y(0) - \dot{y}(0)\, T_{go}\right] \qquad (30)$$

To summarize our results, we describe the sequence of events which occur in the guidance computer.

1) The position vector, velocity vector and gravity acceleration vector $\overline{R}$, $\overline{V}$ and $\overline{g}$, respectively, are determined by the navigation subsystem.

2) The thrust acceleration magnitude and effective exhaust velocity $a_T$ and $V_e$, respectively, are estimated by the filtering process described in the Appendix.

3) The displacement from, and velocity normal to, the desired orbital plane, $y(0)$ and $\dot{y}(0)$, respectively, are computed by forming the dot product of $\overline{R}$ and $\overline{V}$ with $\overline{y}_1$.

4) The $a_{ij}$'s are computed from the expressions on the right-hand side of Eqs. (18-21).

5) The $b_{ij}$'s are computed from Eqs. (25-28).

6) The quantities A and B are computed from (29) and (30), it being understood, of course, that $\dot{y}(T)$ and $y(T)$ are identically zero in these equations.

7) Finally, the thrust vector orientation command is computed so that the following component of thrust acceleration lies along the Y axis

$$a_T \sin \alpha_Y = a_T(A + Bt) - \bar{g} \cdot \bar{y}_1 \qquad (31)$$

In Eq. (31), t is the time that has elapsed since the calculation of A and B. Since A and B are nearly constant (they change only because of thrust perturbations, navigation sensor errors, autopilot errors, etc.), Eq. (31) can be used for many seconds without recomputation of A and B. Thus, it is possible to establish major and minor computation loops. During the major computation loop, A and B are computed. During the much faster minor computation loop, the required component of thrust acceleration along $\bar{y}_1$ is computed from Eq. (31).

The $b_{ij}$'s in Eqs. (29) and (30) are the sensitivities of the y component of the commanded thrust acceleration to the y velocity error and effective y displacement error. The quantity $\Delta$ is the determinant of Eqs. (22) and (23), the system of equations which we solved for A and B. The determinant $\Delta$ approaches zero as $T_{go}$ approaches zero, causing the $b_{ij}$'s to increase without bound. This infinite sensitivity to out-of-plane position and velocity components, when $T_{go}$ approaches zero, reflects the fact that as the time remaining to make corrections becomes vanishingly small, the control action required to correct nonvanishing errors becomes excessively large. Therefore, recomputation of A and B must be avoided during the last few seconds of powered flight. Only the minor loop computation is performed when $T_{go}$ is very small.

Additional insight may be gained into the definition of $\sin \alpha_Y$ in Eq. (9). The second term in Eq. (31) represents a small component of commanded thrust acceleration which cancels the gravitational acceleration along the Y axis [We canceled the gravitational acceleration to obtain the tractable differential equation Eq. (10).] The first component on the right-hand side of (31) represents the additional thrust acceleration required along the Y axis to satisfy the terminal boundary-point constraint.

Summarizing the solution for the orbital plane-control problem: we have found an exact, noniterative procedure for solving the boundary-value problem; we have not followed exactly the prescription of the calculus of variations for obtaining an optimum solution; but simulations have shown that the mass of propellant used with our solution is negligibly different from that used by a rigorously optimum solution.

## Radial Position and Velocity Vector Control

### Preliminary Discussion

Having developed a guidance law for establishing the vehicle in the desired plane, we next develop guidance equations for obtaining desired values of the vehicle's velocity vector and radial position within the plane. We shall break this remaining problem, which is a problem in the plane and hence two-dimensional, into two one-dimensional problems. The first one-dimensional problem is controlling the vehicle's component of velocity and displacement along the radius vector. The second one-dimensional problem is controlling the vehicle's component of velocity normal to the radius vector, i. e., the vehicle's horizontal component of velocity. Controlling the vehicle's radial position and its radial and horizontal components of velocity, is sufficient to control the size and shape of the attained orbit. It should be pointed out, that the equations we now develop, can perform more capable guidance roles than merely attaining an orbit of specified size and shape. The equations can be used to obtain a specified velocity vector at a controlled radius, and an uncontrolled - but predictable - planetary central angle measured from some radial reference line. Thus, the equations can be used to insert a spacecraft into a coasting trajectory which impacts a given target vector. To play this role, the equations must be used in conjunction with a subroutine which computes the velocity vector which is required at the burnout position to establish the coasting trajectory to the impact point.

The equations of constraint; i e., the final boundary conditions which the guidance method must satisfy are

$$R(T) = R_D \tag{32}$$

$$\dot{R}(T) = \dot{R}_D \tag{33}$$

$$V_H(T) = V_{HD} \tag{34}$$

We may specify the magnitude of the terminal value of specific angular momentum rather then the terminal value of the horizontal component of velocity Thus, we can replace Eq. (34) with

$$h(T) = h_D = R_D V_{HD} \tag{35}$$

We now have a boundary-value problem involving the prespecification of the terminal values of three independent quantities. We need, therefore, at our disposal, three independent control quantities which we must regulate to satisfy Eqs. (32, 33 and 35). Our approach will be to solve the radius and radial rate control problem; i. e., satisfy (32) and (33), in the the same manner we solved the plane-control problem. (Remember that in the plane-control problem, we had two independent control parameters A and B which we used to satisfy the terminal constraints $\dot{y}(T) = 0$ and $y(T) = 0$.) We shall have remaining then the requirement to satisfy the final boundary condition (35). Since we have at our disposal the time of powered flight remaining at any instant, we shall use this variable as our third control quantity; i. e., $T_{go}$ is chosen to ensure that equation (35) is satisfied.

<u>The Radius and Radial Rate Control Law</u>

The differential equation of radial motion is

$$\frac{d^2R}{dt^2} = (-\mu/R^2 + V_H^2/R) + a_T \sin\alpha_R \tag{36}$$

Figure 3 defines the angle $\alpha_R$. Note that the parenthesized term is simply the sum of attractive gravitational acceleration and repulsive centrifugal acceleration. The boundary-value problem is: Given any instantaneous values of radius and radial rate $R(0)$ and $\dot{R}(0)$, respectively, find the required thrust angle regime $\alpha_R(t)$ such that Eqs. (32) and (33) are satisfied. The reader may already suspect that we will solve the radius and radial rate control problem by specifying that

$$\sin\alpha_R = -(-\mu/R^2 + V_H^2/R)/a_T + C + Dt \tag{37}$$

Substituting this expression for $\sin\alpha_R$ into equation (36) yields the following linear differential equation.

$$\frac{d^2R}{dt^2} = Ca_T + Da_T t \tag{38}$$

Noting the similarity between this equation and Eq (10) for the orbital plane control problem, we immediately write the solution for C and D

$$C = b_{11}\left[\dot{R}_D - \dot{R}(0)\right] + b_{12}\left[R_D - R(0) - \dot{R}(0)\,T_{go}\right] \qquad (39)$$

$$D = b_{21}\left[\dot{R}_D - \dot{R}(0)\right] + b_{22}\left[R_D - R(0) - \dot{R}(0)\,T_{go}\right] \qquad (40)$$

Rewriting Eq. (37) as

$$a_T \sin\alpha_R = -(-\mu/R^2 + V_H^2/R) + a_T(C + Dt) \qquad (41)$$

we note that the component of thrust acceleration required along the radius vector is composed of two parts. One part cancels the sum of gravitational and centrifugal acceleration. The other part represents the additional thrust acceleration required along the radius vector to ensure that the final radius and radial rate boundary conditions are satisfied. (What we have done here is analogous to the atmospheric entry procedure of cancelling the effective gravitational force with vehicle lift in order to obtain an integrable differential equation from which landing range can be predicted.)

Note that the $b_{ij}$'s used to solve the orbital plane-control problem are also used to solve the radius and radial-rate control problem.

Simulation of this control law has shown it to be remarkably close to optimum. In fact, it has frequently solved the typical boundary-value problems of space flight with less fuel than a steepest descent numerical optimization program.

## The Specific Angular Momentum Control Law

We proceed now to develop the method which is used to control the specific angular momentum. Remember that in the development of both the plane-control steering law and the radius and radial rate control steering law, we left open for our later disposal the matter of how to choose $T_{go}$. At any given instant, we can choose $T_{go}$ from an interval of permissible values. For each $T_{go}$ chosen from this interval, we can compute, by means of the previously derived laws, thrust angle regimes which satisfy the plane and radius and radial rate final boundary conditions. To make more lucid the discussion which follows, we shall consider controlling final specific angular momentum when no plane change is required.

For the problem of planar guidance, the left-hand end of the interval of permissible $T_{go}$'s is set by the limited ratio of the component of thrust acceleration along the radius vector to the total thrust accleration. The absolute value of this ratio must not exceed one which is equivalent to stating that the absolute value of $\sin \alpha_R$ must not exeed one. There is a converse way of stating this limitation: for any given $T_{go}$, there is a limit to the change in radius or radial rate which can be accomplished.

The right-hand end of the interval of permissible $T_{go}$'s is set, mathematically, by the requirement that $T_{go}$ be less than the burnup time of the spacecraft. The burnup time of the spacecraft is $\tau$. Note that the argument of the natural logarithm in Eq. (18) is negative, if $T_{go}$ exceeds $\tau$. Of course, the practical right-hand end of the interval is to the left of $\tau$.

We must assume that the boundary-value problems to be solved are such that the desired specific angular momentum can be attained by choice of a $T_{go}$ from within the interval of permissible $T_{go}$'s. Tests could be incorporated into the computer program to provide an alternative action if too short a $T_{go}$ is chosen (which would result in an arcsine error) or too long a $T_{go}$ (which would result in a logarithm error). In practice, the boundary-value problems most often encountered offer no difficulty in respect to the limited range of $T_{go}$. In fact, if a $T_{go}$ outside the permissible interval appears to be required to attain the specified final value of angular momentum, then the boundary-value problem is very probably unfeasible for some reason other than the limitation of the guidance laws.

The differential equation for the magnitude of the specific angular momentum is

$$dh/dt = Ra_T \cos \alpha_R \tag{42}$$

The principle of determining $T_{go}$ from this equation is the following: first, integrate the differential equation from the current time $t = 0$ to the terminal time $t = T$; and second, solve the resulting equation for $T_{go}$. The result of integrating the differential equation is

$$h(T) - h(0) = \int_0^T (Ra_T \cos \alpha_R)\, dt \tag{43}$$

Recalling that a final boundary condition is

$$h(T) = h_D \tag{44}$$

and that

$$T = T_{go} \tag{45}$$

and denoting h(0) hereafter as

$$h(0) = h_0 \tag{46}$$

Eq. (43) becomes

$$h_D - h_0 = \int_0^{T_{go}} (Ra_T \cos \alpha_R)\, dt \tag{47}$$

We may recall that the derivation of the radius and radial rate control law was made simple by the definition of $\sin \alpha_R$ given in Eq. (37). The computation of the parenthesized term in Eq. (37) offers no difficulty. (Vector position and velocity are available from the navigation system.) But the presence of this term is now awkward because $\cos \alpha_R$ appears under an integral, an integral which we must evaluate in order to solve for $T_{go}$. Recalling that

$$\cos \alpha_R = \sqrt{1 - \sin^2 \alpha_R} \tag{48}$$

and that $\sin \alpha_R$ is defined as in Eq. (37), we see that the integrand in Eq. (47) is very complicated indeed. It is not surprising that the difficulties which we suppressed while synthesizing the radius and radial rate control equations, should now reappear in another form and greatly magnified. We proceed for the time-being by evading the difficult integration in Eq. (47).

Let us express the specific torque, the right-hand side of Eq. (42), as the difference of two terms

$$Ra_T \cos \alpha_R = R_D a_T - M_c \tag{49}$$

The term $R_D a_T$ is easily integrated. The second specific torque term $M_c$ is a correction term. The purpose of writing the specific torque, as in Eq. (49), is to permit a partial integration of the differential equation for angular momentum. This differential equation is now

$$dh/dt = R_D a_T - M_c \tag{50}$$

Integrating Eq. (50) between t = 0, the current time, and t = T, the terminal time, yields

$$h_D - h_0 = - R_D v_e \ln(1 - T_{go}/\tau) - \int_0^T M_c \, dt \tag{51}$$

We define $h_c$ as follows

$$h_c = \int_0^T M_c \, dt \tag{52}$$

and solve Eq. (51) for the $T_{go}$ which appears in the argument of the natural logarithm.

$$T_{go} = \tau \left\{ 1 - \exp\left[ -(h_D - h_0 + h_c)/R_D v_e \right] \right\} \tag{53}$$

It is convenient to rewrite the exponential function in Eq. (53) as the product of a exponential factor containing the <u>unknown</u> correction term $h_c$ and an exponential factor containing the <u>known</u> angular momentum to-be gained $(h_D - h_0)$ The result is

$$T_{go} = \tau \left\{ 1 - \exp\left[ -(h_D - h_0)/R_D v_e \right] \cdot \exp(-h_c/R_D v_e) \right\} \tag{54}$$

Since the first exponential factor in Eq. (54) can be directly and simply computed, the problem of computing $T_{go}$ is now focused on estimating the second exponential factor, $\exp(-h_c/R_D v_e)$. This latter exponential expression is a correction factor which appears in Eq. (54) because we derived Eq. (54) by assuming that the specific torque was equal to $R_D a_T$ plus a correction term $M_c$. The term $h_c$ arose from integration of $M_c$.

The direct computation of $\exp(-h_c/R_D v_e)$ is just as difficult as the evaluation of the right-hand side of (43), the equation containing the difficult integration which we tried to avoid by introducing $M_c$ and $h_c$. However, an indirect estimate of $h_c$ is convenient, and the ensuing paragraphs deal with an indirect approach.

The strategy of the approach is the following: suppose that we have an estimate for $h_c$, call it $\tilde{h}_c$, (we could start with $\tilde{h}_c = 0$), and that we use $\tilde{h}_c$ in Eq. (54) to obtain an estimate of $T_{go}$, call it $\tilde{T}_{go}$. Now, suppose that we use this estimate for $T_{go}$ to determine the thrust angle regime which satisfies the radius and radial rate boundary conditions. Suppose further that we fly the spacecraft (hypothetically) for $T_{go}$ seconds according to the calculated thrust angle regime, and then observe the resultant final specific angular momentum $h_f$. If $h_f$ is not equal to $h_D$ then it is evident that $\tilde{h}_c$ should be modified to obtain a better estimate of $h_c$. A logical equation for computing an improved $\tilde{h}_c$ is the following.

$$\tilde{h}_{c,\,n+1} = h_D - h_{f,\,n} + \tilde{h}_{c,\,n} \tag{55}$$

With the improved estimate of $h_c$ obtained from Eq (55), a new estimate of $T_{go}$ can be obtained from Eq. (54) and then a new thrust angle regime computed from the radius and radial rate control equations. After another hypothetical flight of the spacecraft and observation of its terminal specific angular momentum, a still better estimate of $h_c$ can be obtained from Eq. (55). Although the following development is along the lines summarized here, the computation of $\tilde{h}_c$ is not as direct as shown in Eq. (55). The following development is organized to take advantage of the near linearity of a certain dynamical quantity which is a function of the specific angular momentum. An important feature of the following treatment is that no matter how many times the estimates of $\tilde{h}_c$ and $\tilde{T}_{go}$ are iterated and improved, only one evaluation of an exponential function is required.

It is possible to present a strong analytic argument, and provide empirical data to verify, that the following dynamical quantity is very nearly a linear function of time.

$$H\left[h(t)\right] = \exp\left[-h(t)/R_D v_e\right] \tag{56}$$

A plot of H for an ascent from the surface of the moon into a circular orbit is given in Fig. 4.

As a result of the definition of H and the laws of arithmetic for exponents, we have the following useful identities

$$H(x)\,H(y) = H(x + y) \tag{57}$$

$$H(x)/H(y) = H(x - y) \tag{58}$$

As an example of this notation, Eq. (54) can be rewritten as

$$T_{go} = \tau\left[1 - H(h_D - h_0)\,H(h_c)\right] \tag{59}$$

or as

$$T_{go} = \tau\left[1 - H(h_D)\,H(-h_0)\,H(h_c)\right] \tag{60}$$

Another example of the notation is

$$H(\tilde{h}_{c,\,n+1}) = H(h_D - h_0 + \tilde{h}_{c,\,n})/H(h_{f,\,n} - h_0) \tag{61}$$

where we have used Eq. (55) as well as Eq. (58). Equation 61 is a key formula in the procedure for computing time-to-go. Equation 61 is the expression used to compute an improved estimate of $H(h_c)$, given a previous estimate and its corresponding $H(h_{f,n} - h_0)$. We can always produce a suitable first estimate $\tilde{h}_{c,1}$ of $h_c$, since the final estimate of $h_c$ obtained from iterating Eq. (61) is independent of the first estimate of $h_c$. The problem then is to find, given any $H(\tilde{h}_{c,n})$, the actual $H(h_{f,n} - h_0)$ which would result if the spacecraft were flown with the corresponding $\tilde{T}_{go,n}$; for then better estimates of $H(h_c)$ and $T_{go}$ could be provided by use of first Eq. (61) and then Eq. (60). Equation 60 tells us that having possession of $H(\tilde{h}_c)$ is equivalent to possessing $\tilde{T}_{go}$; and Eq. (61) tells us that having possession of $H(\tilde{h}_c)$ and the $H(h_f - h_0)$ which would result if the vehicle were flown with the $\tilde{T}_{go}$ corresponding to $H(\tilde{h}_c)$, is sufficient to produce a better $H(\tilde{h}_c)$. We need, therefore, only to provide the link for computing $H(h_f - h_0)$ from $T_{go}$ in order to complete the iterative chain. This link is forged on the basis of the near time linearity of $H[h(t)]$.

The function H is a composite function of time. To emphasize H's time dependence, we suppress the reference to the intermediate variable h as follows.

$$Q(t) = H[h(t)] \tag{62}$$

Because of the near linearity of Q(t), it is reasonable and quite accurate to represent Q(t) by an abruptly truncated Taylor's series, as follows

$$Q(t) = Q(0) + \dot{Q}(0)t + \ddot{Q}(0)t^2/2 \tag{63}$$

The coefficients of t and $t^2$ in Eq. (63) are found by differentiating Eq. (56) with respect to t. The Taylor's series coefficients are

$$Q(0) = H(h_0) \tag{64}$$

$$\dot{Q}(0) = - H(h_0)\dot{h}_0/R_D v_e \tag{65}$$

$$\ddot{Q}(0) = + H(h_0)(\dot{h}_0^2/R_D^2 v_e^2 - \ddot{h}_0/R_D v_e) \tag{66}$$

as may be easily verified. Defining

$$\eta_0 = 1 \tag{67}$$

$$\eta_1 = - \dot{h}_0/R_D v_e \tag{68}$$

$$\eta_2 = \dot{h}_0^2/R_D^2 v_e^2 - \ddot{h}_0/R_D v_e \tag{69}$$

and using Eq. (62), we rewrite Eq. (63) as

$$H[h(t)] = H(h_0)(\eta_0 + \eta_1 t + \eta_2 t^2/2) \tag{70}$$

(Expressions for the first and second derivatives of specific angular momentum and the variables defined in Eqs. (67-69) are derived in Appendix B.) From Eq. (70), we obtain the final link for completing the iteration chain. Dividing each side of Eq. (70) by $H(h_0)$, and replacing t with $\tilde{T}_{go,n}$ yields

$$H(h_{f,n} - h_0) = \eta_0 + \eta_1 \tilde{T}_{go,n} + \eta_2 \tilde{T}_{go,n}^2/2 \tag{71}$$

Summary of Method For Controlling Final Velocity Vector and Radial Position

We now summarize the guidance law derived in this section. We will assume that the final values of radius, radial rate and specific angular momentum are somehow specified. If the guidance problem is to insert a one stage vehicle into a circular orbit around the moon, the appropriate specified final boundary conditions are

$$R_D = \text{desired orbital radius} \tag{72}$$

$$\dot{R}_D = 0 \tag{73}$$

$$h_D = \sqrt{R_D \mu_{MOON}} \tag{74}$$

If the guidance problem is a soft landing on the surface of the moon without any longitudinal range constraint, then the equations of this section are applicable, and the appropriate specified final boundary conditions are

$$R_D = \text{radius of the moon} \tag{75}$$

$$\dot{R}_D = 0 \tag{76}$$

$$h_D = 0 \tag{77}$$

(When the actual touchdown point is constrained-as, for example, in the case of landing next to a lunar logistics vehicle or a radio beacon or transponder-the equations of the next section are more appropriate )

We assume that the position vector and velocity vector of the vehicle are available from the navigation equations. Appendix C gives an algorithm for computing present position and velocity from accelerometer outputs, and initial position and velocity. The present radial position, vehicle radial rate and specific angular momentum are obtained from operations on the current vehicle position and velocity vectors, as shown below.

$$R(0) = \left|\overline{R}(0)\right| \tag{78}$$

$$h_0 = h(0) = \left|\overline{R}(0) \times \overline{V}(0)\right| \tag{79}$$

$$\dot{R}(0) = \overline{R}(0) \cdot \overline{V}(0)/R(0) \tag{80}$$

We now will give a summary of the computation steps in the vehicle-borne computer program. The purpose of the program is, of course, to compute the thrust angle profile and $T_{go}$ required to attain some set of desired final boundary conditions [such as Eqs. (72-74) or Eqs. (75-77)] given some initial set of boundary conditions, Eqs. (78-80).

The first step, done at the beginning of the powered flight regime, is to compute and store away the H function of $h_D$.

$$H(h_D) = \exp(-h_D/R_D v_e) \tag{81}$$

Every few seconds, or so, after the engine is ignited, a new thrust angle regime and $T_{go}$ are computed. Thus, thrust angle regime and $T_{go}$ errors resulting from the effects of thrust magnitude perturbations, autopilot execution errors and approximations in the gravity model, are compensated by a closed loop control in which current boundary conditions are the input. The calculations performed are the following. Calculate the H function of $-h_0$.

$$H(-h_0) = \exp(h_0/R_D v_e) \tag{82}$$

Then estimate the current time-to-go by decrementing the last $T_{go}$ by the time that has elapsed since its calculation. (If this is 1st time-to-go calculation, choose any reasonable value.) Next, calculate an estimate for the H function of the correction term $h_c$.

$$H(\tilde{h}_c) = (1 - \tilde{T}_{go}/\tau)/H(h_D)\,H(-h_0) \tag{83}$$

(We use a tilde over the trial variables $T_{go}$ and $h_c$; but we will not always use an iteration subscript. It is to be understood that these variables are only approximate until we have cycled the equations a few times to obtain accurate values.) The next step in the calculation is to compute the thrust angle regime required to satisfy the final radius and radial rate boundary conditions. The approximate time-to-go, $\tilde{T}_{go}$, is used at this time. (A better estimate is not yet available.) The equations which must be calculated are the following: first, Eqs. (18-21); second, Eqs. (24-28); and third, Eqs. (39-41). At this point in the calculations, we have a thrust angle regime which would satisfy the final radius and radial rate boundary conditions. However, we have chosen an approximate time-to-go which may not succeed in satisfying the final boundary condition on the specific angular momentum. The next calculation is equivalent to finding out how much specific angular momentum would be gained with the calculated thrust angle regime and the approximate time-to-go, $\tilde{T}_{go}$.

$$H(h_{f,n} - h_0) = \eta_0 + \eta_1 \tilde{T}_{go} + \eta_2 \tilde{T}_{go}^2 / 2 \qquad (84)$$

The next step is equivalent to comparing the actual increase in specific angular momentum with the increase desired, and then incrementing or decrementing the correction term so that the boundary condition on specific angular momentum is better satisfied.

$$H(\tilde{h}_{c,n+1}) = H(h_D) H(-h_0) H(\tilde{h}_{c,n}) / H(h_{f,n} - h_0) \qquad (85)$$

It is illuminating to rewrite Eq. (85) as follows

$$H(\tilde{h}_{c,n+1}) = H(h_D - h_{f,n} + \tilde{h}_{c,n}) \qquad (86)$$

Equation (86) tells us that the new value of the correction term, $\tilde{h}_{c,n+1}$, is obtained by adding to the old value the amount by which the final boundary condition on specific angular momentum was missed. Our next computation step is to compute a better estimate of time-to-go from the latest estimate of $h_c$.

$$T_{go} = \tau \left[ 1 - H(h_D) H(-h_0) H(h_c) \right] \qquad (87)$$

A new thrust angle regime, corresponding to the new estimate of time-to-go, can now be determined, and the whole process we described can be repeated.

Simulations have shown that this computation process converges very rapidly. In fact, except for the very first calculation when only a crude estimate of time-to-go is available, the process converges in one or two passes through the calculations. The time-to-go estimate is only fairly accurate early in flight when the time-to-go is large; but the calculation becomes more and more accurate as $T_{go}$ shrinks. A more accurate estimate may be obtained for large time-to-go by including higher order derivatives in the expansion of Q(t) in Eq. (63). A more accurate time-to-go prediction is not necessary for the guidance process; but it might be useful as a basis for monitoring and decision-making. For example, a precision time-to-go prediction early in a powered flight maneuver could warn an astronaut that he is coming dangerously close to depleting his estimated fuel supply.

Another possible use of the accurate long-range time-to-go prediction is in the optimization of engine ignition time. For example, imagine a spacecraft approaching a planet on a high-energy trajectory. Suppose the mission objective is insertion into a reconnaissance orbit around the planet. By human or computer monitoring of the predicted time-to-go for the maneuver, the engine ignition time could be chosen when the time required for the orbital insertion passes through a minimum.

The fuel economy of the guidance laws presented in this section is, in general, very good. In fact, this guidance method has, on several occasions, surpassed the results obtained from a steepest descent numerical optimization program.

## Guidance Law For Throttleable Rockets

### The Rendezvous Problem

We shall now derive a guidance law, suitable for continuously throttleable rockets, which can steer a spacecraft to a specified state vector at a specified time. Thus, we solve the rendezvous problem for continuously throttleable rockets.

In cases where the terminal time can be left open, as in the lunar landing problem, we have a certain freedom in the choice of time-to-go. The next section of this paper deals with the calculation of time-to-go when the terminal time is unconstrained. For the purposes of the present section, we regard the terminal time as fixed.

The mathematical statement of the rendezvous problem is: given the state vector of a vehicle at $t = t_0$

$$\bar{R}(t_0) = x(t_0)\bar{i} + y(t_0)\bar{j} + z(t_0)\bar{k} \tag{88}$$

$$\bar{V}(t_0) = \dot{x}(t_0)\bar{i} + \dot{y}(t_0)\bar{j} + \dot{z}(t_0)\bar{k} \tag{89}$$

and assuming that the vehicle is subject to the equations of motion of a body undergoing thrust in a central force field in vacuum

$$\frac{d\dot{x}}{dt} = -\mu x/R^3 + a_{Tx} \tag{90}$$

$$\frac{d\dot{y}}{dt} = -\mu y/R^3 + a_{Ty} \tag{91}$$

$$\frac{d\dot{z}}{dt} = -\mu z/R^3 + a_{Tz} \tag{92}$$

find a control vector for $t_0 \leq t \leq T$

$$\bar{a}_T(t) = a_{Tx}(t)\bar{i} + a_{Ty}(t)\bar{j} + a_{Tz}(t)\bar{k} \tag{93}$$

such that the vehicle will have a desired state by t = T

$$\bar{R}(T) = x_D\bar{i} + y_D\bar{j} + z_D\bar{k} \tag{94}$$

$$\bar{V}(T) = x_D\bar{i} + y_D\bar{j} + z_D\bar{k} \tag{95}$$

The strategy of the solution is the same as in the first section of the paper. Define the components of the thrust acceleration vector as follows

$$a_{Tx} = \mu x/R^3 + C_1 f(t) + C_2 g(t) \tag{96}$$

$$a_{Ty} = \mu y/R^3 + C_3 m(t) + C_4 n(t) \tag{97}$$

$$a_{Tz} = \mu z/R^3 + C_5 p(t) + C_6 q(t) \tag{98}$$

The functions, f(t), g(t) m(t), n(t), p(t), q(t), are specified functions of time. We require only that each pair of these functions in each of Eqs. (96-98) be linearly independent. The actual choice of the functions is predicated on the basis of engineering goals such as simplicity, fuel optimization, etc. We shall particularize these functions later, and obtain a useful steering law. For the time-being, we continue in a general way.

Substitution of Eqs. (96-98) into Eqs. (90-92) yields

$$\frac{d\dot{x}}{dt} = C_1 f(t) + C_2 g(t) \tag{99}$$

$$\frac{d\dot{y}}{dt} = C_3 m(t) + C_4 n(t) \tag{100}$$

$$\frac{d\dot{z}}{dt} = C_5 p(t) + C_6 q(t) \tag{101}$$

We note that Eqs. (99-101) are linear and decoupled, a tremendous advantage in our derivation because we must find expressions for the $C_i$'s in terms of the errors in the boundary conditions and the time-to-go. Note that the price of obtaining convenient tractable Eqs. (99-101) is a constraint between the three expressions in Eqs. (96-98). However, this constraint is readily satisfied: we merely calculate the right-sides of Eqs. (96-98) and then command

$$a_T = \sqrt{a_{Tx}^2 + a_{Ty}^2 + a_{Tz}^2} \tag{102}$$

The throttleable engine solves, so to speak, the constraint between Eqs. (96-98). We must now calculate the constants in Eqs. (96-98) by integration and solution of Eqs. (99 101). We shall demonstrate the method of solution for the $C_i$'s by carrying out the derivation for the x coordinate only. The $C_i$'s for the other coordinates are solved in the same manner.

We use Eq. (99) to obtain two linear algebraic equations in $C_1$ and $C_2$. The two equations are precisely the equations of constraint demanded by the initial and final boundary conditions, Eqs. (88-89) and Eqs. (94-95). Integration of Eq. (99) between $t = t_0$ and variable time t yields

$$\dot{x}(t) - \dot{x}(t_0) = C_1 \int_{t_0}^{t} f(s)ds + C_2 \int_{t_0}^{t} g(s)ds \tag{103}$$

Substitution of the terminal time T for t yields

$$\dot{x}(T) - \dot{x}(t_0) = C_1 \int_{t_0}^{T} f(t)dt + C_2 \int_{t_0}^{T} g(t)dt \tag{104}$$

Integration of Eq. (103) between $t = t_0$ and $t = T$ yields

$$x(T) - x(t_0) - x(t_0)T_{go} = C_1 \int_{t_0}^{T} \left[ \int_{t_0}^{t} f(s)ds \right] dt + C_2 \int_{t_0}^{T} \left[ \int_{t_0}^{t} g(s)ds \right] dt \tag{105}$$

where

$$T_{go} = T - t_0 \tag{106}$$

Now, assuming that f(t) and g(t) are integrable functions, the coefficients of $C_1$ and $C_2$ in Eq. (104) and Eq. (105) are merely algebraic functions of T and $t_0$; they may therefore be evaluated. Furthermore, the left-hand sides of Eqs. (104-105) are easily computed; for, x(t) and $\dot{x}(t)$ at the terminal time T are specified, and x(t) and $\dot{x}(t)$ at the current time $t_0$ are measured. Therefore, $C_1$ and $C_2$ can be solved from Eqs. (104-105) in terms of the boundary conditions and T and $t_0$. In a corresponding manner, we may develop expressions for the $C_i$'s in the other axes. Now, in possession of all the $C_i$'s, we compute the components of the required thrust acceleration vector from Eqs. (96-98). We are now in full possession of the solution to the boundary-value problem; for we have an algorithm for Eq. (93) which satisfies Eq. (88-89), Eqs. (90-92) and Eqs. (94-95).

We shall now choose specific functions for f(t), g(t), m(t), n(t), p(t) and q(t); evaluate the integrals in Eqs. (104-105) and the similar equations for the y and z axes; solve for the $C_i$'s and produce expressions for the three components of the solution thrust acceleration vector. If we choose

$$f(t) = m(t) = p(t) = 1 \tag{107}$$

$$g(t) = n(t) = q(t) = t - t_0 \tag{108}$$

and, if the magnitudes of the components of the gravitational acceleration along the coordinate axes change only slightly, (as when the spacecraft travels through a small planetary angle, and changes altitude by a small amount compared to the planetary radius), then it may be shown that the steering law developed by using these functions minimizes the integral of the square of the thrust acceleration. Substituting for f(t) and g(t) in Eqs. (104-105) in accordance with the definitions in Eqs. (107-108) yields

$$C_1 = (4/T_{go})(\dot{x}_D - \dot{x}) + (6/T_{go}^2)(x_D - x - \dot{x}_D T_{go}) \tag{109}$$

$$C_2 = (-6/T_{go})(\dot{x}_D - \dot{x}) + (-12/T_{go}^3)(x_D - x - \dot{x}_D T_{go}) \tag{110}$$

and, of course, from Eq. (96), we have

$$a_{Tx} = \mu x/R^3 + C_1 + C_2(t - t_0) \tag{111}$$

with similar equations for the y and z components of the thrust acceleration vector.

Summary of Equations and Order of Computation

In this section, we describe a few considerations for programming the equations. In particular, we describe order of computation, handling of computation lags and avoidance of excessive commands when time-to-go is very small. The computation steps are:

1) Calculate the lastest spacecraft position and velocity vectors, $R(t_0)$ and $V(t_0)$.

2) Calculate the time-to-go

$$T_{go} = T - t_0 \tag{112}$$

3) Calculate the $C_i$'s by first calculating

$$e_{11} = 4/T_{go} \tag{113}$$

$$e_{12} = 6/T_{go}^2 \tag{114}$$

$$e_{21} = -6/T_{go}^2 \tag{115}$$

$$e_{22} = -12/T_{go}^3 \tag{116}$$

and then

$$C_1 = e_{11}(\dot{x}_D - \dot{x}) + e_{12}(x_D - x - \dot{x}_D T_{go}) \tag{117}$$

$$C_2 = e_{21}(\dot{x}_D - \dot{x}) + e_{22}(x_D - x - \dot{x}_D T_{go}) \tag{118}$$

$$C_3 = e_{11}(\dot{y}_D - \dot{y}) + e_{12}(y_D - y - \dot{y}_D T_{go}) \tag{119}$$

$$C_4 = e_{21}(\dot{y}_D - \dot{y}) + e_{22}(y_D - y - \dot{y}_D T_{go}) \tag{120}$$

$$C_5 = e_{11}(\dot{z}_D - \dot{z}) + e_{12}(z_D - z - \dot{z}_D T_{go}) \tag{121}$$

$$C_6 = e_{21}(\dot{z}_D - \dot{z}) + e_{22}(z_D - z - \dot{z}_D T_{go}) \tag{122}$$

4) Calculate the components of the desired thrust acceleration vector

$$a_{Tx}(t) = \mu x(t)/R(t)^3 + C_1 + C_2(t - t_0) \tag{123}$$

$$a_{Ty}(t) = \mu y(t)/R(t)^3 + C_3 + C_4(t - t_0) \quad (124)$$

$$a_{Tz}(t) = \mu z(t)/R(t)^3 + C_5 + C_6(t - t_0) \quad (125)$$

From Eqs. (123-125), the magnitude and direction of the desired thrust acceleration vectory may be computed. On the basis of the magnitude of the thrust acceleration vector a command for the engine throttle servo is computed.

Equations (123-125) are written in a form which emphasizes that although the $C_i$'s are computed for the values of the state vector and time-to-go at a particular instant, $t = t_0$; the components of the thrust acceleration vector may be computed from Eqs. (123-125) for times $t > t_0$ without recalculation of the $C_i$'s. The form of Eqs. (123-125) makes compensation of lags in the guidance system simple. By setting t in Eqs. (123-125) to the time of transmission of the commands, or even to a later time to compensate for autopilot and throttle servo lags, the commands may be appropriately timed. Of course, the effects of control subsystem execution errors finally necessitate computation of fresh $C_i$'s. Note that the coefficients in Eqs. ( 113-116 ) "blow up" as time-to-go approaches zero Excessive commands are avoided for vanishingly small time-to-go by use of Eqs. (123-125) without recalculation of the quantities in Eqs. (113-122)

## The Landing Problem

The primary guidance distinction between the rendezvous and landing problem is that for the latter problem, the achievement of the final boundary conditions need not occur at any particular time. Consequently, we have the liberty of choosing time-to-go from within an interval of physically feasible values. In this section, we shall discuss a method of choosing time-to-go which simulation has proven very useful. Our discussion will also provide a great deal of insight into the behaviour of the throttleable rocket steering law.

Imagine that the vehicle is at the phase of its free-fall trajectory where it is desirable to ignite the descent engine for the landing maneuver. We may imagine that the spacecraft is in a very low altitude circular orbit, or at the perilune of a transfer trajectory from a higher altitude orbit. Notice, from Eqs. (113-125) that the magnitudes of the components of the thrust acceleration vector are functions of time-to-go. Thus, as initial time-to go is varied, both the magnitude and direction of the initial thrust vector change Since the thrust magnitude of the spacecraft is limited, we shall be particularly interested in the behaviour of the initial thrust magnitude as a function of the initial choice of time-to-go. Figure 5 is a typical plot of initial thrust magnitude as a function of time-to-go.

Since the specified landing site for this data was about 200 miles downrange from the engine ignition point, the very short times-to-go are highly unrealistic. We shall discuss these cases, however, for they afford a great deal of insight into the nature of the steering law. We may notice in Fig. 5 that the thrust magnitude increases rapidly toward infinity as time-to-go decreases below 140 seconds. For such very short times-to-go, the vehicle must initially be accelerated toward the landing site to speed its arrival there. For very, very short time-to-go, this acceleration toward the landing site must be immense. (We have not indicated the thrust vector orientation in the figures, but we shall discuss the thrust direction, as we discuss Fig. 5.) As the vehicle moves rapidly toward the site, this prodigious and ever-growing thrust vector must be rotated at a very high angular rate to point finally away from the landing site, causing deceleration and shrinking of the velocity vectory just before the landing site is reached. Obviously, such trajectories are very uneconomical since the vehicle must be accelerated and then decelerated. Furthermore, they are usually quite unrealizable since the thrust magnitudes called for eventually, if not immediately, exceed the engine output limit. Point (1) on the plot, though initially yielding a thrust magnitude within the capacity of the engine, would be the type of trajectory we have just discussed. Therefore, the time-to-go corresponding to this point should not be chosen.

The trajectory corresponding to point (3) of Fig. 5 is very peculiar, not to say ridiculous. Specification of too short a time-to-go caused the situation in trajectory (1) where the vehicle is initially accelerated toward the site in order to speed its arrival and in the terminal seconds tremendously decelerated to kill its arrival speed. Trajectory (3) is the opposite side of the coin: how to spend a very long time in traveling from the initial boundary conditions to the final boundary conditions. Where and how does the spacecraft spend all this time? It climbs at first instead of descending, although continuously decelerating its speed, flies over and past the landing site, continues to decelerate, stops, heads back toward the landing site, passes the site again, decelerates, stops, and finally, approaches and stops at the landing site. Obviously, trajectory (3) is to be avoided from the viewpoint of fuel expenditure, as well as, astronaut sanity.

The trajectory corresponding to point (2) on the plot is workable and, fortunately, quite reasonable. For this trajectory, the range from the spacecraft to the landing site monotonically shrinks to zero as the vehicle is continually decelerated. For a certain band of initial ranges from the landing site, the trajectory steered closely approximates an optimum landing.

Even point (2) on the plot in Fig. 5 is unsuitable if the spacecraft is very close to the landing site when the engine is ignited. If the vehicle is very close to the landing site, and its speed is high, the decelerative thrust must eventually be very large; for the spacecraft must lose its speed before the site is reached. Thus, if the range-to-go is short when the engine is ignited, the guidance system may call for a growing thrust magnitude. This is not a fault of the guidance method so much as a result of the basic dynamics of the boundary-value problem. For example, if the site is approached very closely before engine ignition, it may be quite impossible to slow up sufficiently before reaching the landing site no matter what steering law is used. On the other hand, the engine should not be ignited when the vehicle is excessively far from the landing site; for then, the guidance law will call for a rapidly decaying thrust magnitude, operate the engine at a less efficient lower thrust setting, take a long time to land, and use, as the result of all these effects, a great deal of fuel.

Figure 6 is a plot of $\Delta V$ required to achieve the final boundary conditions versus initial range-to-go. The points for this plot were generated by repeatedly flying a spacecraft from some initial velocity vector and altitude to some terminal velocity vector and altitude, varying, for each flight, only the range-to-go at engine ignition. For each case, the time-to-go was initially chosen to make the initial thrust acceleration magnitude maximum. The time-to-go during each flight was then simply decremented by the time elapsed since engine ignition. Of course, time-to-go corresponding to point (2) of Fig. 5 was chosen in each case. A Newton-Raphson technique was chosen to solve the following expression

$$a_{TMAX} = a_T(T_{go})$$

for the initial time-to-go which yields an initial maximum thrust acceleration. Points (1 and 3) are easily distinguished from the desired solution (2) because of the difference in the sign of the derivative of thrust with respect to time-to-go at these points. This derivative must be computed for the Newton-Raphson procedure, and also serves as a check that the correct root (2) has been obtained. The vertical dotted line in Fig. 6 indicates a critical range-to-go for engine ignition. If the vehicle approaches the landing site closer than this range before the engine is ignited, then the guidance program commands a thrust magnitude profile which would require a higher than maximum thrust at some point. If the engine is ignited before the critical range-to-go is reached, then the thrust magnitude subsequently decays. It is seen in Fig. 6 that the fuel optimum point for engine ignition is the critical range-to-go; that is, the spacecraft should come as close to the landing site as possible before engine ignition. On the other hand, choosing a standard engine ignition point of 12.2 degrees range-to-go incurs a $\Delta V$ penalty of only 17 ft/sec and provides an engine igntion "window" of about 8 seconds.

## Appendices

### Appendix A: Development of Thrust Acceleration and Estimation of Effective Exhaust Velocity

In this appendix, we derive an expression for thrust acceleration, $a_T(t)$, which was very useful for the development of the algorithms in the text of the paper. We also suggest a simple and accurate means of smoothing the thrust acceleration, $a_T(t)$. Simulation of the guidance laws developed in the paper was undertaken with random noise added to the measured thrust acceleration. Although the attainment of the desired terminal boundary conditions was not seriously affected, the steering commands tended to be undesirably erratic. (One source of the noise is quantization, which is exacerbated if the specified force units are integrating accelerometers since in that event their outputs must be differentiated.) We also show how the vacuum specific impulse $I_{SP}$, an important parameter in many of our guidance laws, can be estimated on board the spacecraft. We remind the reader that

$$V_e = g I_{SP} \tag{A1}$$

The fundamental scalar law of motion of a rocket-propelled vehicle is

$$m\, a_T = V_e \dot{m} \tag{A2}$$

where

$$\dot{m} > 0 \tag{A3}$$

If the massflow $\dot{m}$ of the rocket is constant

$$m = m_0 - \dot{m}t \tag{A4}$$

and

$$a_T = V_e\dot{m}/(m_0 - \dot{m}t) \tag{A5}$$

Dividing numerator and denominator of the right-hand side of (A5) by $\dot{m}$, we obtain

$$a_T = V_e/(m_0/\dot{m} - t) \tag{A6}$$

It is undesirable to have the original mass of the vehicle in our expression for $a_T$ because that quantity is difficult to measure. Also, although we assume $\dot{m}$ is constant, we do not wish to use its magnitude because of the difficulty of measuring the massflow.. The offensive quotient can be eliminated from (A6) by appealing to (A2). Evaluating (A2) for $t = 0$, we obtain

$$m_0 a_T(0) = V_e\dot{m} \tag{A7}$$

from which we obtain

$$m_0/\dot{m} = V_e/a_T(0) \tag{A8}$$

We now define

$$\tau = V_e/a_T(0) = m_0/\dot{m} \tag{A9}$$

so that (A6) becomes

$$a_T = V_e/(\tau - t) \tag{A10}$$

The quantity $\tau$ is the "burnup time", the time required for a vehicle that is all fuel to consume itself completely. Naturally, we must always have

$$T_{go} < \tau \tag{A11}$$

It is seen from Eq. (A10) that $a_T(t)$ is inversely proportional to a linear function of time. Consequently, the reciprocal of $a_T(t)$ is a linear function of time. Therefore, a convenient quantity to smooth is

$$1/a_T = \gamma + \beta t \tag{A12}$$

Since the reciprocal of $a_T$ is required for (A9), there is no extra division incurred by forming $1/a_T$ after taking each accelerometer reading. The procedure then is to fit the reciprocals of the $a_T$ measurements to a straight line, i.e., estimate the parameters $\gamma$ and $\beta$ from a multitude of $1/a_T$ measurements.

Since

$$\beta = -1/V_e \tag{A13}$$

we can estimate $V_e$ by taking the negative reciprocal of the slope of the fitted straight line.

## Appendix B: Formulae for First and Second Derivatives of Specific Angular Momentum

To evaluate $\eta_1$ and $\eta_2$ of Eqs (68-69), we require a formula for the second derivative of specific angular momentum. This is developed below

$$dh/dt = Ra_T \cos \alpha_R \tag{B1}$$

$$d^2h/dt^2 = \dot{R}a_T \cos \alpha_R + Ra_T^2 \cos \alpha_R/V_e - Ra_T(\sin \alpha_R)\alpha_R \tag{B2}$$

We see in Eq. (B2) that we require the first derivative of $\alpha_R$, which is derived next. We recall that

$$\sin \alpha_R = -(-\mu/R^2 + h^2/R^3)/a_T + C + Dt \tag{B3}$$

Taking the first derivative of each side of (B3) yields

$$(\cos \alpha_R)\dot{\alpha}_R = (-\mu/R^2 + h^2/R^3)/V_e - (+2\mu\dot{R}/R^3 - 3\dot{R}h^2/R^4 + 2h\dot{h}/R^3)/a_T + C \tag{B4}$$

Also

$$\cos \alpha_R = \sqrt{1 - \sin^2 \alpha_R} \tag{B5}$$

An expression for $\dot{\alpha}_R$ can be found from Eqs (B4-B5)

It is evident that we can develop higher order derivatives of specific angular momentum, and thus include higher order derivatives of Q(t) in Eq. (63) It is advisable to include higher-order derivatives of Q(t) in Eq. (63); for, in cases where the thrust vector rotates from one side of the local horizontal to the other side during the powered flight, the representation in Eq. (63) may be inaccurate , and even, under some conditions, diverge.

## Appendix C: Equations for Updating Position and Velocity

The following equations for updating the vehicle position and velocity vectors were developed by Dr. James Potter of the MIT Instrumentation Laboratory. The purpose of these equations is to compute the current state (position and velocity) of the spacecraft, given the last computed state and the $\overline{\Delta V}$ which is measured between the time of computation of the last state and the current time. We assume that $\overline{\Delta V}$ is the output of the three integrating accelerometers of an inertial measuring unit. Suppose that the current time is $t_0$ and that the time elapsed since the calculation of the last state is $\Delta t$. Then, the last state is

$$\bar{R}(t_0 - \Delta t), \quad \bar{V}(t_0 - \Delta t)$$

which is a quantity available in storage. Also available is $\bar{G}(t_0 - \Delta t)\Delta t/2$ which, of course, may be computed, the first time, from $\bar{R}(t_0 - \Delta t)$ and the gravity model. Then, the equations for updating the state to the present time, $t_0$, are:

$$\bar{R}(t_0) = \bar{R}(t_0 - \Delta t) + \bar{V}(t_0 - \Delta t)\Delta t + \bar{G}(t_0 - \Delta t)(\Delta t)^2/2 + \overline{\Delta V}\,\Delta t/2 \tag{C1}$$

$$R^2 = \bar{R}(t_0) \cdot \bar{R}(t_0) \tag{C2}$$

$$\bar{G}(t_0)\Delta t/2 = -(\mu/R \cdot R^2)\,\bar{R}(t_0)\Delta t/2 \tag{C3}$$

$$\bar{V}(t_0) = \bar{V}(t_0 - \Delta t) + \overline{\Delta V} + \bar{G}(t_0 - \Delta t)\Delta t/2 + \bar{G}(t_0)\Delta t/2 \tag{C4}$$

These efficient equations have been called the "average G equations", because the current velocity vector is computed using an average of the old and new gravity vectors.

It has sometimes been said that implicit steering equations are computationally convenient because they avoid the necessity of computing the gravitational acceleration and integrating the outputs of accelerometers to obtain position. The equations derived in this paper require, of course, explicit knowledge of position and velocity. The "average G" equations illustrate how little computational burden is actually incurred by requiring explicit knowledge of $\overline{G}$, $\overline{R}$ and $\overline{V}$.

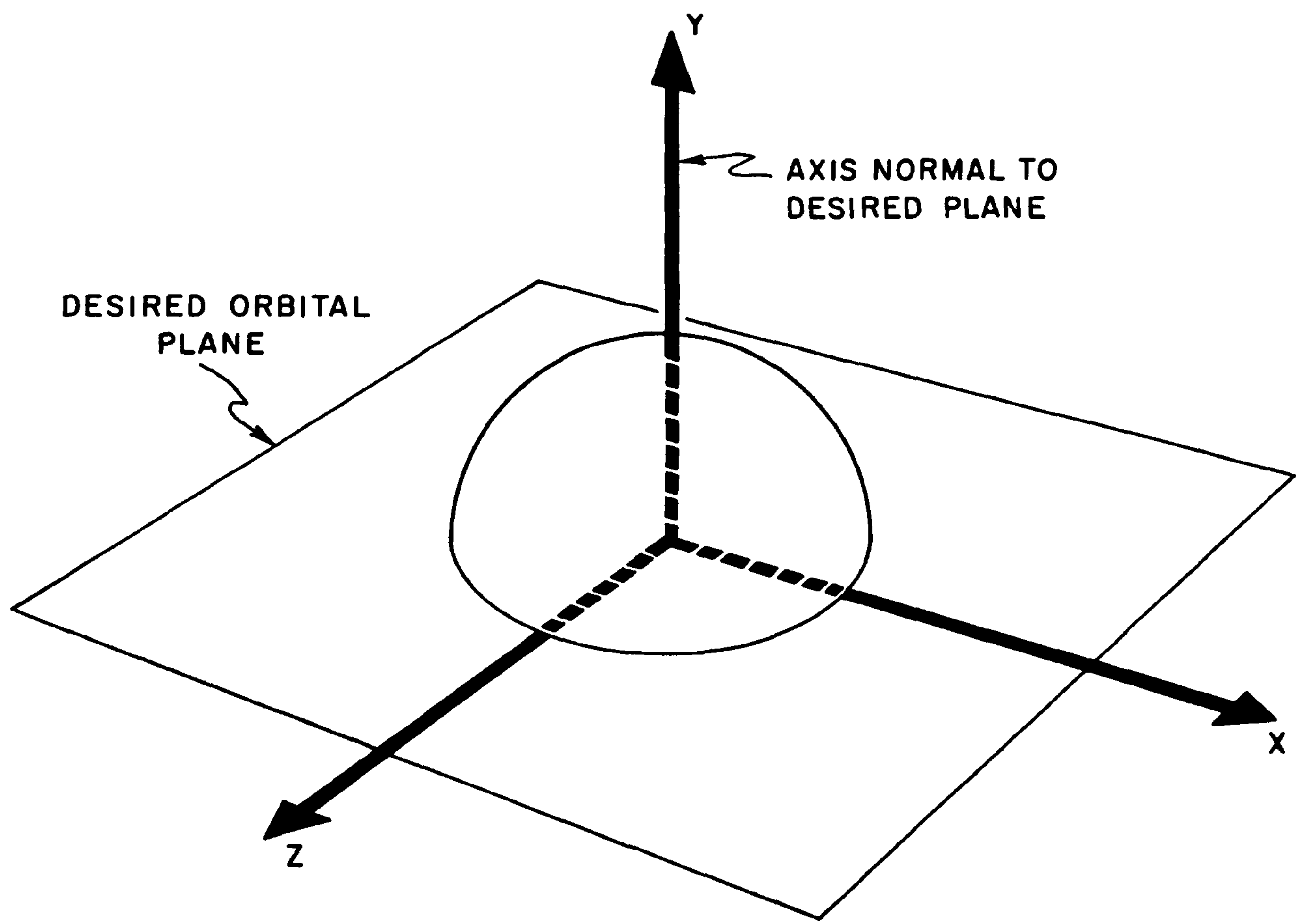


Fig. 1 Coordinate System For Plane Control Boundary-Value Problem

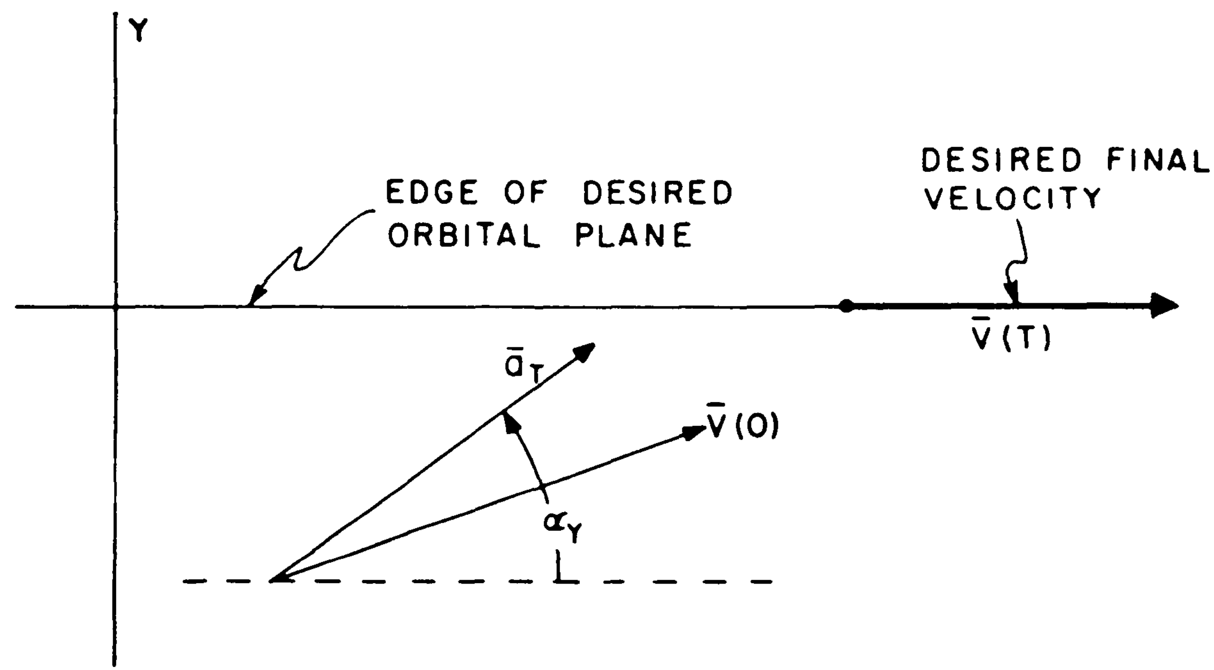


Fig. 2 Definition of $\alpha_Y$ For Plane Control Boundary-Value Problem

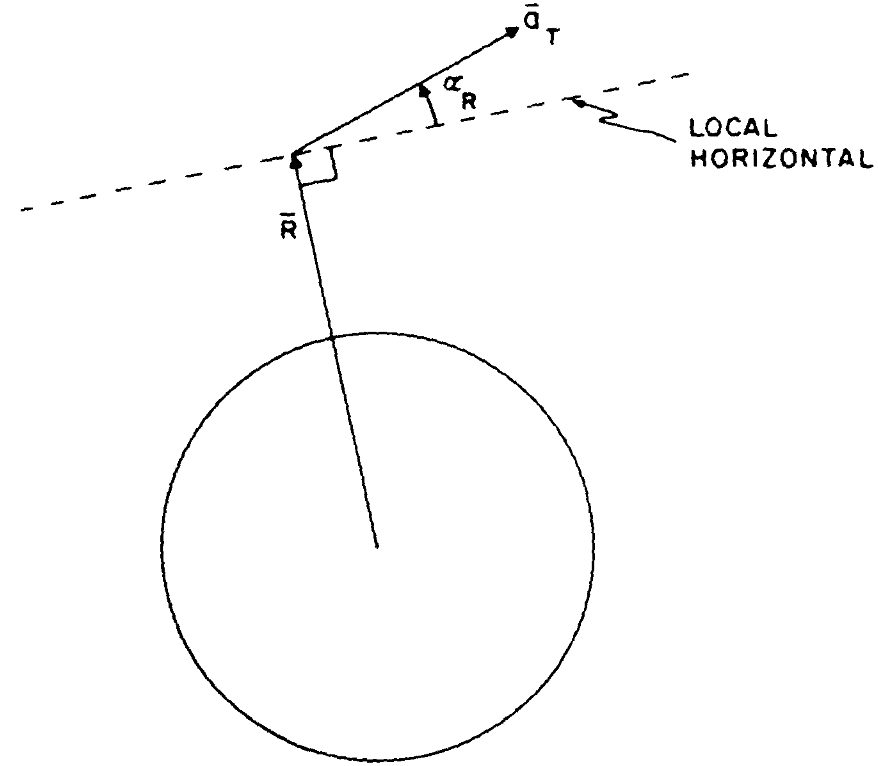


Fig. 3 Definition of $\alpha_R$ For the Radius and Radial Rate Boundary-Value Problem

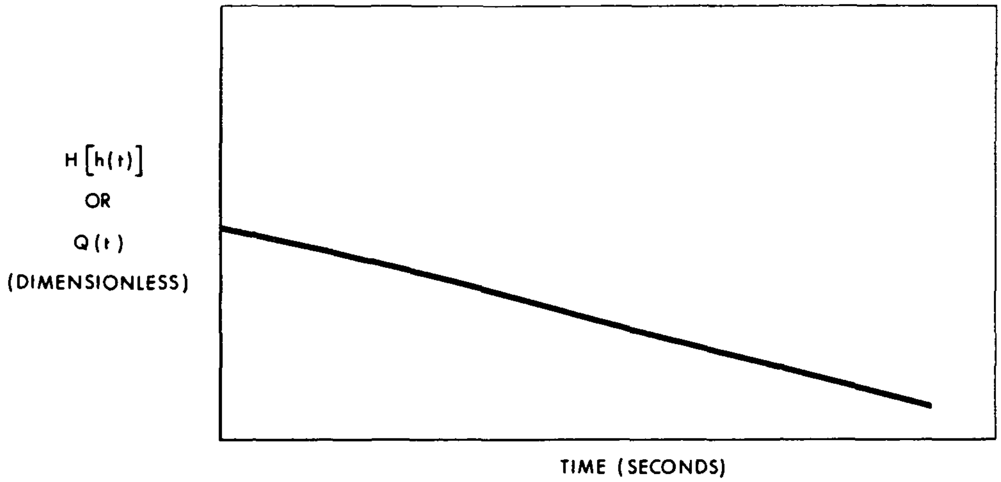


Fig. 4 Graph of Dimensionless Variable Q(t) or H h(t) Versus Time

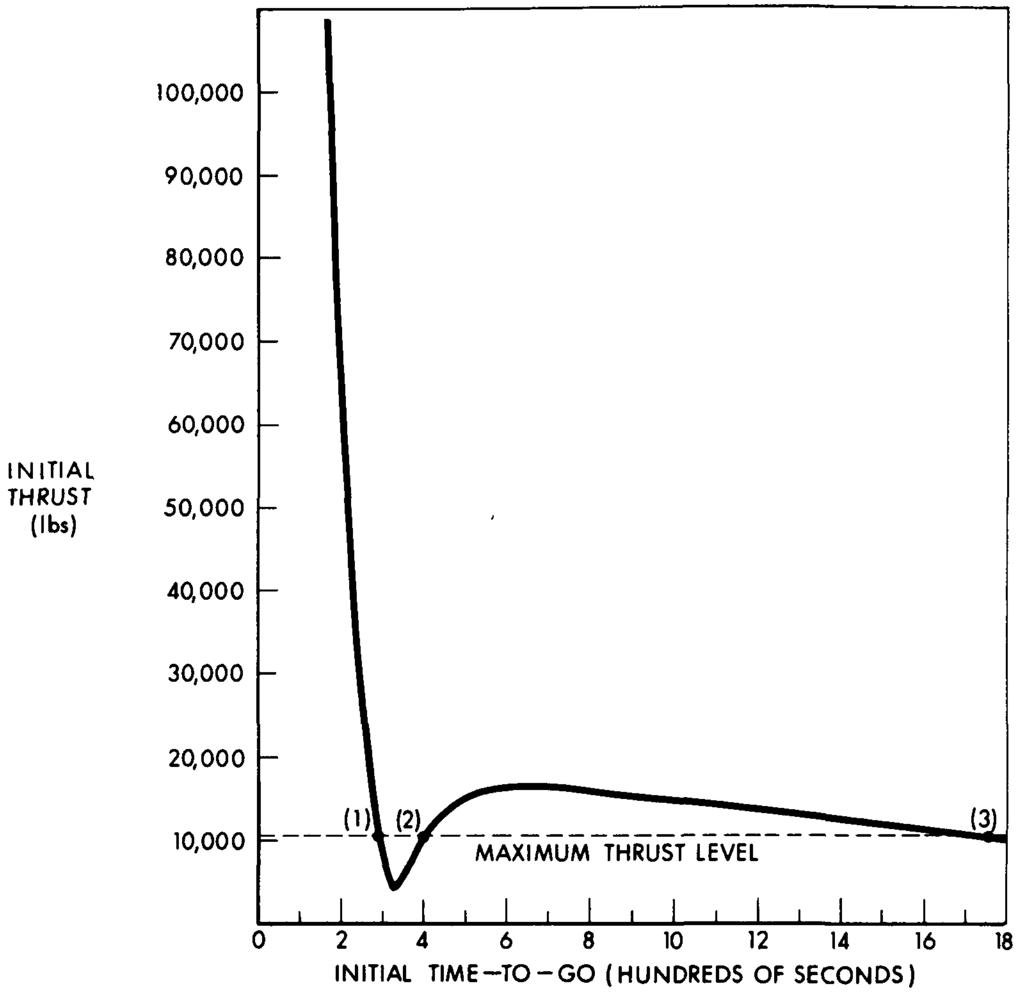


Fig. 5 Initial Thrust Versus Initial $T_{go}$ For Lunar Landing

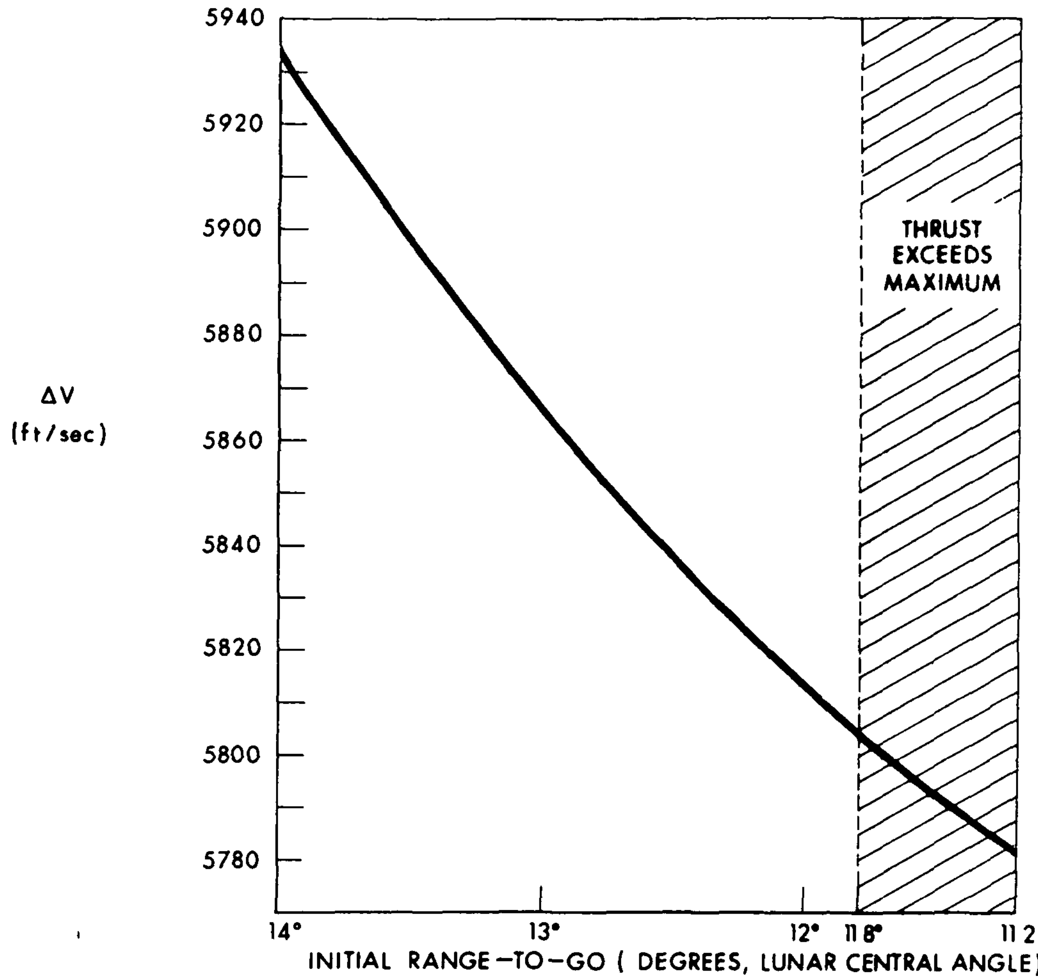


Fig. 6 ΔV Versus Initial Range-to-Go For Lunar Landing

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