BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

B70 06074

SUBJECT: Linear Stability Analysis of LM Rate-of-Descent Guidance Equations Case 310

DATE: June 25, 1970

FROM: J. A. Sorensen

ABSTRACT

Large oscillations occurred in the commanded thrust during the P66 (final) phase of LM descent in the Apollo 11 and 12 flights. A linear stability analysis of the governing guidance equations shows that the problem was partially due to a smaller in-flight value of the descent engine time constant than was originally assumed. Possible solutions are obtained by lowering the values of two guidance constants which affect the roots of the system characteristic equation.

(NASA-CR-113075) LINEAR STABILITY ANALYSIS OF LM RATE-OF-DESCENT GUIDANCE EQUATIONS (Bellcomm, Inc.) 20 p

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MEMORANDUM FOR FILE

INTRODUCTION

It was observed from the Apollo 11 and 12 Descent Engine Control Assembly data that large commanded-thrust oscillations occurred during the P66 (rate-of-descent - ROD) phase of the flights. During Apollo 12 descent, these oscillations reached peak-to-peak values of 2000 lb, and the commanded oscillations continued after the engine was shut off.

A study was made of the basic stability of the guidance equations governing P66. The equations were examined by placing them and the associated dynamics into a set of linear, constant first-order difference equations. The degree of stability was determined by observing the root locations of the resulting characteristic equation on the Z-plane. (Z-plane analysis is briefly reviewed in Appendix A.) Various parameters in the equations were varied to determine their effect upon stability.

THE SYSTEM CHARACTERISTIC EQUATIONS

The linearized difference equations governing P66 are derived in Appendix B, and are summarized in a matrix equation:

$$\begin{bmatrix} a_{c_{n-1}} \\ \Delta\dot{H}_{n+1} \\ \Delta a_n \\ \Delta a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -C_1 & -C_2 \\ 0 & 0 & 0 & 1 \\ -d & K_3 & -C_1 d & -e \end{bmatrix} \begin{bmatrix} a_{c_{n-2}} \\ \Delta\dot{H}_n \\ \Delta a_{n-1} \\ \Delta a_n \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ K_3 & -K_3 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} . \quad (1)$$

Here, a_c is the commanded acceleration, $\Delta\dot{H}$ is the desired minus the actual value of vertical rate, and Δa is the throttle-command acceleration increment issued to the engine. The terms u and w are the astronaut input and random driving disturbance, respectively. The coefficients C_1, C_2, d, e, and K_3 are defined in Appendix B.

Equations (1) can be described in the form

$$\vec{x}_{n+1} = F\,\vec{x}_n + G\vec{u}_n. \tag{2}$$

The characteristic equation is

$$|zI - F| = 0, \tag{3}$$

where I is the identity matrix. From (1), this is

$$\begin{aligned} z^4 &+ (e-2)\,z^3 + (1-2e+C_1d+C_2K_3)\,z^2 \\ &+ (e-2C_1d+C_1K_3-C_2K_3+d)\,z + (C_1d-C_1K_3+K_3-d) = 0. \end{aligned} \tag{4}$$

ANALYSIS RESULTS

The effects of the erasable guidance gain (Lag/TAUROD), the engine time constant τ_e, the computational delay Δ_d, and the fixed-memory constant τ_{th} on the roots of Eq. (4) were parametrically examined. Each of these terms is used in deriving Eqs. (1) or Eqs. (B.19) of Appendix B. Previously used nominal values for the various equation parameters have been approximately:

TAUROD = 1.5 sec,

Lag/TAUROD = 0.4133,

Δ_d = 0.3 sec,

τ_{th} = 0.2 sec,

τ_e = 0.3 sec.

First, each of these terms was held constant except Lag/TAUROD which was varied from -1.5 to +1.5 in steps of 0.05. Stable portions of the resulting root loci of Eq. (4) are shown on the Z-plane of Fig. 1. The system is stable for Lag/TAUROD varied from -0.6 to +1.0.

The "best" value of Lag/TAUROD is defined here to be that which causes the larger magnitude of the roots on loci coming out from the origin to equal the magnitude of each root on loci approaching the origin as Lag/TAUROD is increased. This corresponds to these roots having equal damping (see Appendix A). For this first case, the best value was Lag/TAUROD is 0.4, i.e., nearly the nominal value stated above and used on Apollo 11 and 12. It produces a maximum root magnitude of about 0.5.

The telemetry data indicated that the engine time constants τ_e were more like 0.075 sec rather than 0.3 sec so the effect of this value was next examined. Again, Lag/TAUROD was varied with the stable portions of the root loci presented in the Z-plane of Fig. 2. For stability, Lag/TAUROD ranged from about -0.8 to +0.55. The best value was found to be lowered to +0.15. For Lag/TAUROD = 0.4, the root location along the negative real axis has a magnitude of 0.83. (i.e., approaching instability.)

This case was further examined by determining what effect a decrease in the computational delay time Δ_d had on stability. Values of 0, 0.1 and 0.2 sec were studied with results presented in Table 1. It is seen that decreasing the delay time shifts both the range of stability of Lag/TAUROD and the best value in the negative direction. Less than 0.1 sec delay causes the Lag/TAUROD = 0.4133 point to become unstable. The actual root loci shapes did not significantly vary with the changed delay time.

I conclude, therefore, that the apparent oscillation problem of Apollo 11 and 12 was due to a system with weakly dampened roots exciting by some external driving force. The root shift was probably due to a combination of the decreased engine time constant, a decreased computational delay time, and various nonlinear effects.

One suggested solution to this problem has been to change the guidance equation constant τ_{th} from 0.2 sec. to 0. This possibility was examined by again varying Lag/TAUROD, and the resulting root loci are plotted in Fig. 3. The stability range is from -0.8 to +1.3. The best value of Lag/TAUROD is 0.2 when the delay time Δ_d is 0.3 sec. A decrease in Δ_d again shifted the range of stable values of Lag/TAUROD negatively. However, the range is wide enough such that no potential problem appears. The dashed line of Fig. 3 indicates a small change in the locus shape for a decrease in Δ_d. The change in τ_{th} definitely improves the stability of the system.

CONCLUSIONS

The oscillatory character of the P66 throttle command was apparently due to the actual value of the LM descent engine time constant being smaller than that assumed. This, coupled with a decreased computational lag time, causes a root of the system characteristic equation to approach the unit circle (and produce marginal stability) for the nominal gains previously used.

The problem can be eliminated by changing the parameter τ_{th} and the gain (Lag/TAUROD) to smaller values. Actual magnitudes should be based on the handling quality of the spacecraft.

T.B. Hoekstra for

2014-JAS-ksc J. A. Sorensen

APPENDIX A

Z-TRANSFORM METHOD

The Laplace transform of a delayed impulse $A\delta(t-T)$ is given as

$$L\left\{A\delta(t-T)\right\} = Ae^{-Ts} \tag{A.1}$$

In a system containing many delayed pulses (like a sampled data system), it is convenient (Ref. 1) to approximate the pulses as impulses (ideal pulses) and make the change of variables $z = e^{Ts}$ or $s = (1/T) \ln z$. The Laplace transform of a sampled signal with this change of variable is called the Z-transform. Thus, the Z-transform of the delayed impulse $A\delta(t-T)$ is

$$Z\left\{A\delta(t-T)\right\} = z^{-1}A \tag{A.2}$$

Suppose the transformed function (i.e., impulse equation) is expressed as a ratio of polynomials in z, that is $Q(z)/P(z)$, where each polynomial has positive exponents in z and the degree of $P(z)$ is greater than the degree of $Q(z)$, and $P(z)$ has no repeated roots. Then, by partial fraction expansion,

$$\frac{Q(z)}{P(z)} = \frac{K_1}{z-Z_1} + \frac{K_2}{z-Z_2} + \ . \ . \ . + \frac{K_n}{z-Z_n} \ . \tag{A.3}$$

$$\frac{K_i}{z-Z_i} = K_i \ \{z^{-1} + Z_i z^{-2} + \ . \ . \ . + Z_i^{n} z^{-(n+1)} + \ . \ . \ .\} \ . \tag{A.4}$$

Because the inverse transform of z to a negative power is a delayed impulse, Eq. (A.4) has the inverse transform

$$\mathcal{Z}^{-1}\left\{\frac{K_i}{z-Z_i}\right\} = K_i\ \{\delta(t-T) + Z_i\delta(t-2T) + \ldots + Z_i^{\,n}\delta(t-(n+1)T) + \ldots\} \tag{A.5}$$

This is a train of ideal pulses of strength $K_i Z_i^{\,n}$, where n is the pulse number associated with the pulse occurring at (n+1)T seconds.

From Eq. (A.5), the stability requirement is seen to be $|Z_i| < 1$. This also can be seen by noting that the change of variable $z = e^{sT}$ maps the entire left half of the s-plane into the interior of the circle $|z| = 1$. The imaginary axis maps onto the unit circle and the right half plane maps into the exterior of this circle.

If the Z_i in (A.5) is positive, real, and less than one, the terms in (A.5) will monotonically decrease as n becomes large. For Z_i negative and real, the terms will alternate in sign. Complex values of $Z_i (=Z_R + jZ_I)$ will always occur in conjugate pairs (for a realizable system), and the inverse transform of

$$K\{z^{-1} + (Z_R + jZ_I)z^{-2} + \ldots + (Z_R + jZ_I)^n \cdot z^{-(n+1)} + \ldots\}$$

$$+ \bar{K}\{z^{-1} + (Z_R - jZ_I)z^{-2} + \ldots + (Z_R - jZ_I)^n \cdot z^{-(n+1)} + \ldots\}$$

(where $\bar{K}$ is the complex conjugate of K) is the expression

$$2|K|\{\cos(\angle K)\delta(t-T) + |Z_i|\cos(\angle K + |Z_i|)\ \delta(t-2T) + \ldots$$

$$+ |Z_i|^n\cos(\angle K + n|\angle Z_i|)\delta(t-(n+1)T) + \ldots\} \tag{A.6}$$

where $\angle K$ indicates the phase angle of the complex quantity K. This is an ideal pulse train multiplied by a damped cosine wave. The frequency of the cosine wave is seen to depend on the angle of Z_i -- the larger the angle of Z_i, the greater the frequency of oscillation.

The responses for different locations of the roots of $P(z) = 0$ can be thought of as the response to a unit impulse at $t = 0$. A summary of these responses appears in Fig. 4. If $Q(z)/P(z)$ represents the transfer function of a system, the equation $P(z) = 0$ is its characteristic equation.

Consider a system represented by the linear, first-order matrix difference equation

$$\vec{x}_{n+1} = F\vec{x}_n + G\vec{u}_n \qquad (A.7)$$

Here, $\vec{x}_n$ in the state at time nT, $\vec{x}_{n+1}$ is the state at $(n+1)T$, $\vec{u}_n$ is the system input, F is the system matrix, and G is the distribution matrix. The system characteristic equation is

$$|zI - F| = 0 \qquad A.8)$$

where I is the identity matrix.

APPENDIX B

DERIVATION OF THE DIFFERENCE EQUATIONS

The following equations govern the rate of descent during P66. They are taken from pages 5.3 - 96, 5.3 - 97, and 5.3 - 114 of Ref. 2 and are stated approximately as they appear. Definitions of individual terms are found in Ref. 2.

Measured Rate-of-Descent

$$\dot{H} = \text{UNIT}\ [\underline{r}(\text{PIPTIME})_p] \cdot [\underline{v}(\text{PIPTIME})_p + \Delta\underline{v}(\Delta T)_p + (\underline{g}_p - \underline{v}_{BIASP})\Delta T] \qquad (B.1)$$

Commanded Rate-of-Descent

$$\dot{H}_D = \dot{H}_D + \text{RODCOUNT} \cdot \text{RODSCALE} \qquad (B.2)$$

Commanded Acceleration

$$|AFC|_{raw} = \frac{\dfrac{\dot{H}_D - \dot{H}}{\text{TAUROD}} + |g_p|}{\cos(\xi)} \qquad (B.3)$$

$$|AF| = \frac{|\Delta\underline{P}ipa(\Delta t_{comp})_p + \underline{V}_{BIASP}|}{\Delta t_{comp}} \qquad (B.4)$$

$$|AFC|_c = \frac{\text{Lag}}{\text{TAUROD}} \left[\frac{|g_p|}{\cos(\xi)} - |AF| - \frac{\delta f_p}{\text{MASS}} \right] \qquad (B.5)$$

$$|AFC|_{new} = |AFC|_c + |AFC|_{raw} \tag{B.6}$$

$$s = |AFC|_{new} \text{ limited between } ACC_{max} \text{ and } ACC_{min} \tag{B.7}$$

Corrected Measured Thrust

$$\tilde{f} = |AF| \cdot m + \delta f_p \tag{B.8}$$

Commanded Thrust

$$f = m \cdot s \tag{B.9}$$

Throttle Command

$$\Delta f_{th} = f - \tilde{f} \tag{B.10}$$

Measurement Correction for Next Cycle

$$\delta f_p = \Delta f_{th} \left[\frac{t_{fc} - t_n}{\Delta t} + \frac{\tau_{th}}{\Delta t} + \frac{|\Delta f_{th}|/FRATE}{2\Delta t} \right] \tag{B.11}$$

To linearize these equations the following assumptions are made:

a. Gravity is constant so g_p can be ignored.

b. Vehicle mass (m and MASS above) is constant so that acceleration terms can be worked with directly.

c. Descent is vertical so $\cos(\xi) = 1$.

d. The difference between the actual and measured $\dot{H}$ is due to some random driving force w. The $\underline{v}_{BIASP}$ term is dropped.

e. The throttle command increment Δf_{th} is small so that the final term in Eq. (B.11) can be dropped.

f. The time lag between when the PIPAs (accelerometers) are read, and the throttle command is issued (represented by $(t_{fc} - t_n)$ in Eq. (B.11) is constant. The command sequence occurs every computational period of $\Delta t = 1$ sec.

The following definitions are made:

$$K_1 \triangleq \left[\frac{t_{fc} - t_n + \tau_{th}}{\Delta t}\right]$$

$$K_2 \triangleq \text{Lag/TAUROD}$$

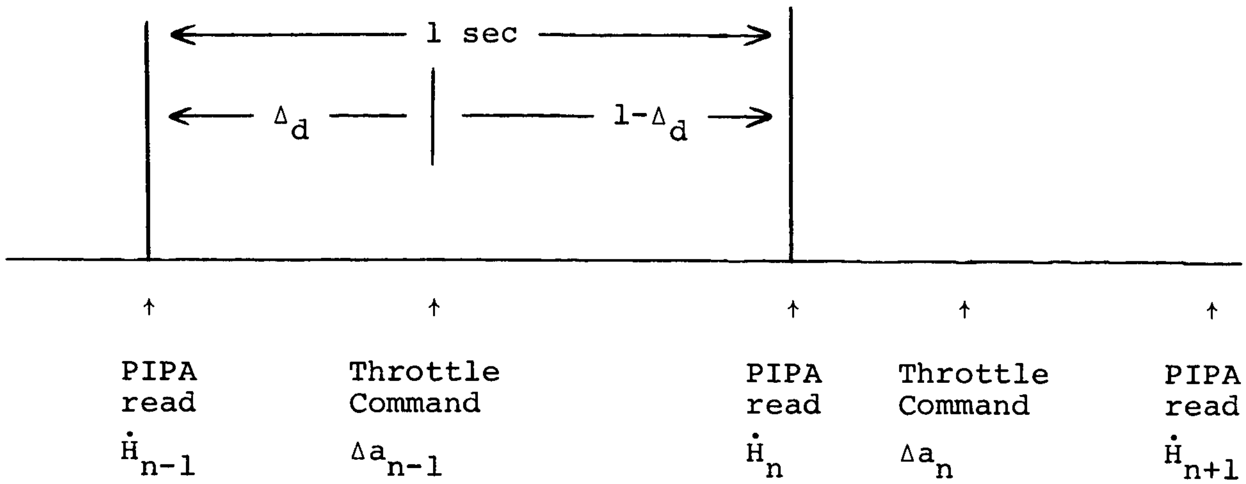
$$K_3 \triangleq \text{1/TAUROD}$$

$$\Delta a = \Delta f_{th}/m$$

The command acceleration at time n is

$$a_{c_n} = a_{c_{n-1}} + \Delta a_n \tag{B.12}$$

where the subscripts n and n-1 refer to the sample times which are one second apart. The computational process can be represented by the time line:



The throttle command is issued Δ_d sec. after the PIPAs are read. The throttle command voltage to the engine changes in a constant ramped manner rather than as a step change due to Δa. This also produces a variable lag dependent upon Δa's magnitude but is ignored here. The engine has the dynamic delay represented by the transfer function $1/(\tau_e s + 1)$. Therefore, the change in acceleration due to a step change Δa at time zero is

$$\Delta a_t = \Delta a(1 - e^{-t/\tau_e}) \qquad \text{(B.13)}$$

Actually, the entire past Δa sequence must be accounted for in determining $\dot{H}$. However, it is assumed here that τ_e is small enough so that the average acceleration from n to n+1 is

$$a_{n+1} = a_{c_{n-2}} + \Delta a_{n-1} \int_{1-\Delta d}^{2-\Delta d} (1-e^{-t/\tau_e}) dt + \Delta a_n \int_0^{1-\Delta d} (1-e^{-t/\tau_e}) dt,$$

$$= a_{c_{n-2}} + \Delta a_{n-1} \left[1 \pm \tau_e \left(e^{(\Delta d-2)/\tau_e} - e^{(\Delta d-1)/\tau_e} \right) \right]$$

$$+ \Delta a_n \left[1 - \Delta d + \tau_e (e^{(\Delta d-1)/\tau_e} - 1) \right] ,$$

Since the bracketed quantities are constants, a_{n+1} can be rewritten as

$$a_{n+1} = a_{c_{n-2}} + C_1 \Delta a_{n-1} + C_2 \Delta a_n \; . \tag{B.14}$$

Equations (B.1) and (B.2) can be rewritten as

$$\dot{H}_{n+1} = \dot{H}_n + a_{n+1} + w, \tag{B.15}$$

$$\dot{H}_{D_{n+1}} = \dot{H}_{D_n} + u. \tag{B.16}$$

These can be combined by defining

$$\Delta\dot{H} \triangleq \dot{H}_D - \dot{H},$$

so that

$$\Delta\dot{H}_{n+1} = \dot{H}_n + u - a_{n+1} - w. \tag{B.17}$$

From Eqs. (B.3) through (B.11), the commanded acceleration increment at time n+1 is

$$\Delta a_{n+1} = K_3 \Delta\dot{H}_{n+1} + K_2(-a_{n+1} - K_1 \Delta a_n) - a_{n+1} - K_1 \Delta a_n \tag{B.18}$$

Equation (B.14) is substituted into Eqs. (B.17) and (B.18), and these are used with Eq. (B.12) to produce the difference equations:

$$\begin{bmatrix} a_{c_{n-1}} \\ \Delta\dot{H}_{n+1} \\ \Delta a_n \\ \Delta a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -C_1 & -C_2 \\ 0 & 0 & 0 & 1 \\ -d & K_3 & -C_1 d & -e \end{bmatrix} \begin{bmatrix} a_{c_{n-2}} \\ \Delta\dot{H}_n \\ \Delta a_{n-1} \\ \Delta a_n \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ K_3 & -K_3 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}. \quad (B.19)$$

Here,

$$d = 1 + K_2 + K_3,$$

and

$$e = C_2(1 + K_2 + K_3) + K_1(K_2+1).$$

REFERENCES

1. Tou, Julius T., Digital and Sampled Data Control Systems, McGraw-Hill, New York, 1959.

2. Levine, G. M., Guidance System Operations Plan for Manned LM Earth Orbital and Lunar Missions Using Program Luminary 1C (LM 131 Rev. 1), Section 5 Guidance Equations (Rev 8), MIT/CSDL, Cambridge, Massachusetts, April 1970.

Table 1. Effect on stability of decreasing the time delay between PIPA readings and throttle command during P66. The engine time constant is 0.075 sec. and τ_{th} is 0.2 sec.

Time delay Δ_d, sec.	Range of stable Lag/TAUROD	Best value of Lag/TAUROD	Magnitude of largest root when Lag/TAUROD equals 0.4.
0.3	-0.75 to +0.50	+0.15	0.838
0.2	-0.80 to +0.45	+0.05	0.902
0.1	-0.85 to 0.40	0	0.967
0	-0.95 to 0.35	-0.05	1.033

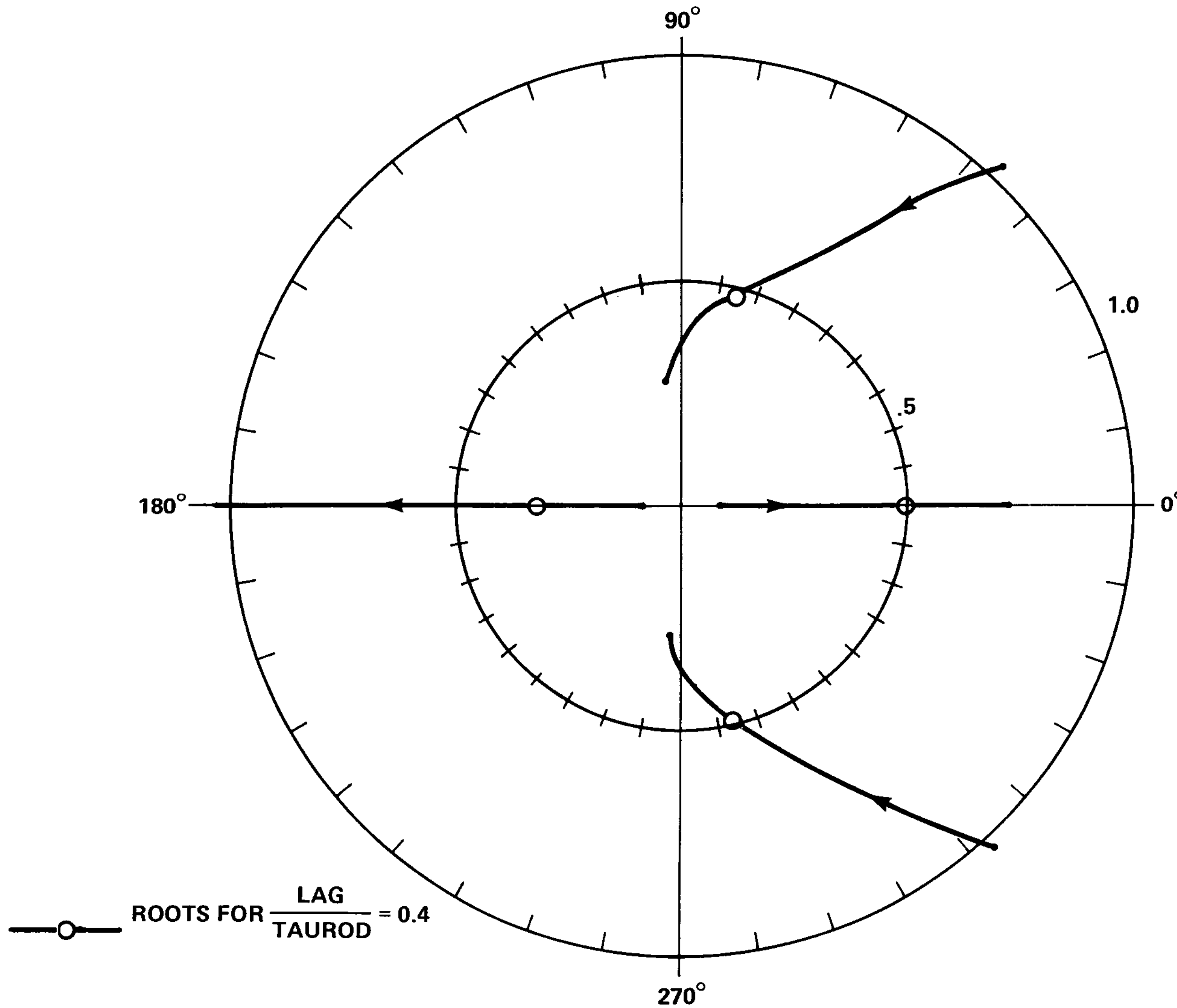


FIGURE 1 - Z-PLANE REPRESENTATION OF P66 GUIDANCE EQUATION ROOT LOCUS FOR GAIN (LAG/TAUROD) VARIED FROM –0.6 TO +1.0. ENGINE TIME CONSTANT IS 0.3 SEC. PARAMETER τ_{th} IS 0.2 SEC.

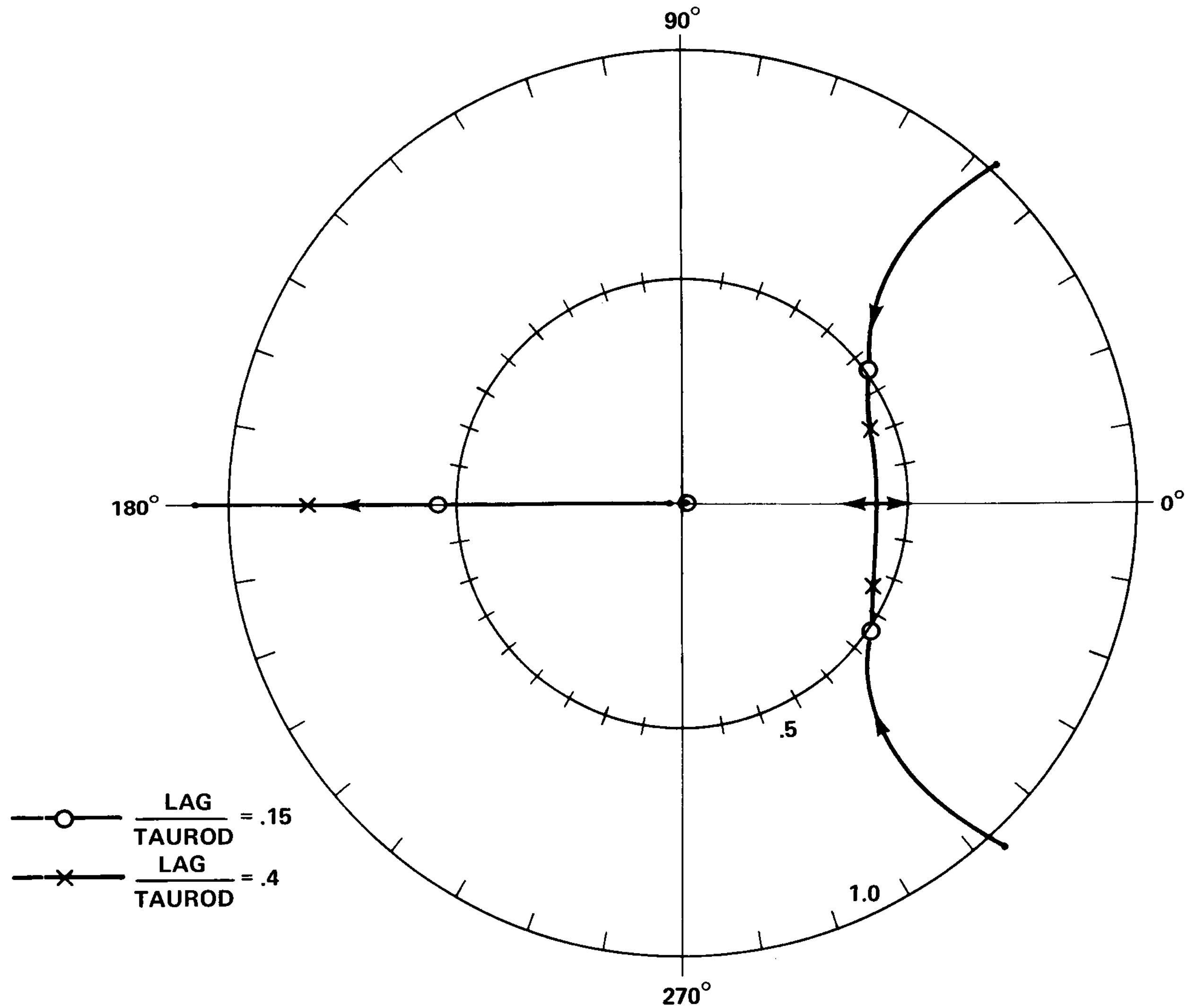


FIGURE 2 - Z-PLANE ROOT LOCUS FOR GAIN (LAG/TAUROD) VARIED FROM −0.8 TO +0.6
ENGINE TIME CONSTANT IS 0.075 SEC. PARAMETER τ_{th} is 0.2 SEC.

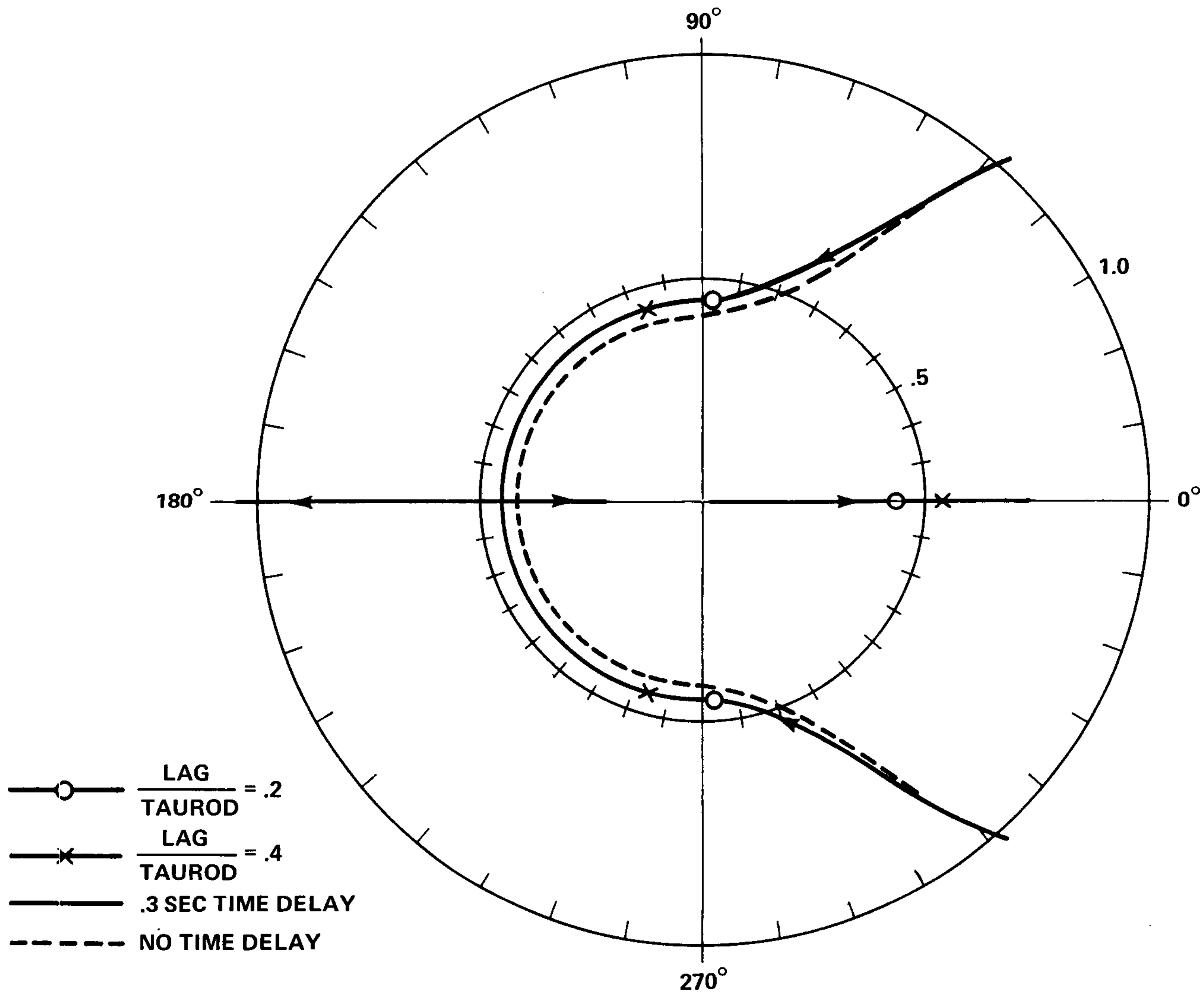


FIGURE 3 - Z-PLANE ROOT LOCUS FOR GAIN (LAG/TAUROD) VARIED FROM –0.8 TO +1.3. ENGINE TIME-CONSTANT IS 0.075 SEC. PARAMETER τ_{th} IS ZERO. TIME DELAY BETWEEN PIPA READING AND THROTTLE COMMAND IS 0.3 SEC. AND ZERO.

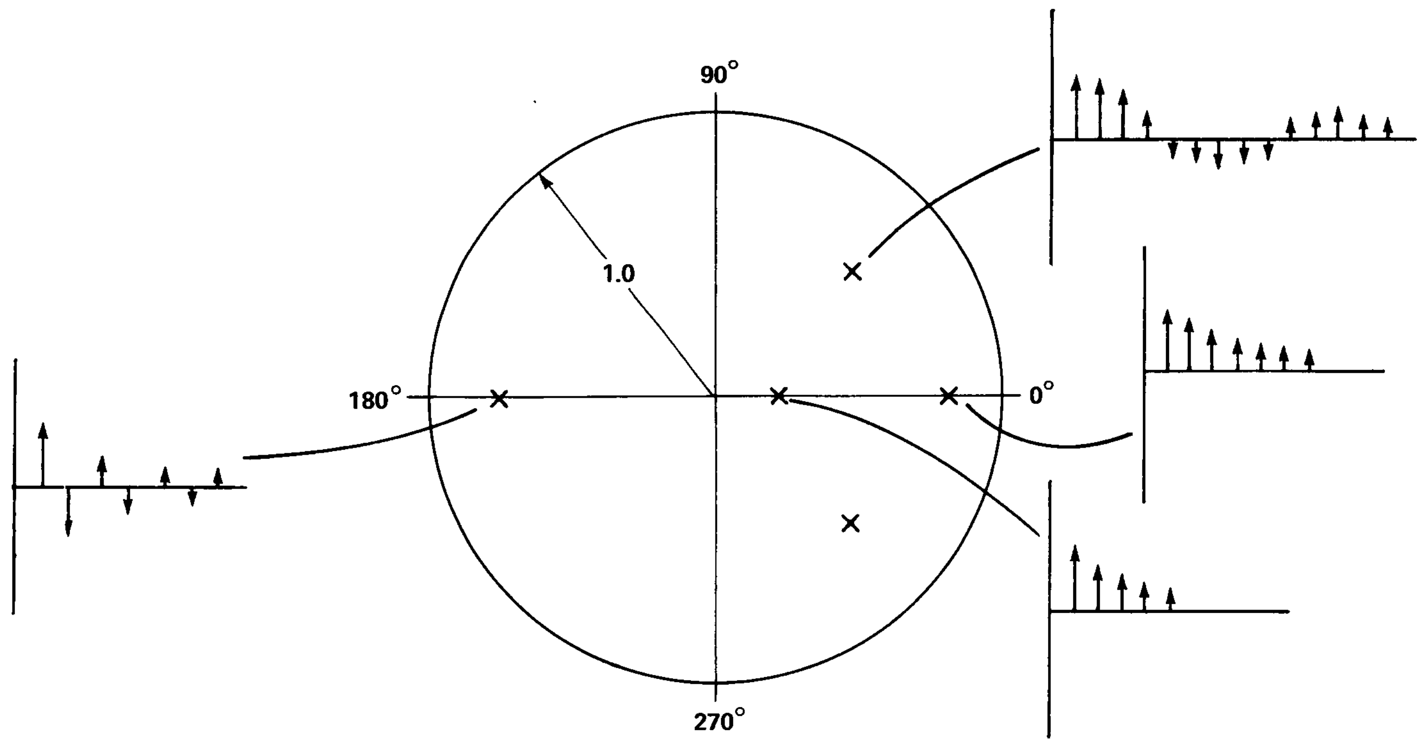


FIGURE 4 - SUMMARY OF IMPULSE RESPONSE FOR VARIOUS ROOT LOCATIONS IN THE Z-PLANE