BELLCOMM, INC.

SUBJECT: The Development and Implementation of the Cross Product Guidance Equations - Case 310

DATE: March 15, 1967

FROM: D. A. Corey

ABSTRACT

This memorandum discusses the development and implementation of the cross product steering law as employed by the Apollo Spacecraft guidance system. While the material itself is not new, information from several sources is consolidated and gaps filled in to provide the reader a straightforward and logical development of the steering law.

(NASA-CR-153717) THE DEVELOPMENT AND IMPLEMENTATION OF THE CROSS PRODUCT GUIDANCE EQUATIONS (Bellcomm, Inc.) 17 p

N79-72290

Unclas
00/17 12410

BELLCOMM, INC.

SUBJECT: The Development and Implementation of the Cross Product Guidance Equation - Case 310

DATE: March 15, 1967

FROM: D. A. Corey

MEMORANDUM FOR FILE

INTRODUCTION

The purpose of this memorandum is to describe rather completely, in one document, the development of the cross product steering law and to provide the reader with some insight into its operation. In addition, the implementation of the various equations is presented in a logical time sequence as it will be done in the Apollo spacecraft computer.

It should be mentioned that nothing in this document is represented as new or original. It's only contribution is that the material from several sources is combined into one document with a few gaps filled in to help the reader from one step to the next.

The Development of the Cross Product Steering Law

First, define several quantities.

$\underline{V}_r$ = the instantaneous velocity required at the current position in order to achieve the maneuver objective.

$\underline{V}$ = the current vehicle velocity vector.

$\underline{V}_g$ = the instantaneous velocity to be gained. It is further defined by

$$\underline{V}_g = \underline{V}_r - \underline{V}$$

$\underline{a}_T$ = the desired vehicle thrust acceleration vector

$|\underline{a}_T|$ = the available engine thrust acceleration magnitude.

Taking the first derivative of $\underline{V}_g$ with respect to time

$$\frac{d\underline{V}_g}{dt} = \dot{\underline{V}}_g = \frac{d\underline{V}_r}{dt} - \frac{d\underline{V}}{dt} \qquad (1)$$

but

$$\frac{d\underline{V}}{dt} = \underline{g} + \underline{a}_T \qquad (2)$$

where g is the gravitational acceleration. Substituting in equation 1 we get

$$\frac{d\underline{V}_g}{dt} = \frac{d\underline{V}_r}{dt} - (\underline{g} + \underline{a}_T) = \underline{b} - \underline{a}_T \qquad (3)$$

where

$$\underline{b} = \frac{d\underline{V}_r}{dt} - \underline{g} \qquad (4)$$

A convenient and efficient guidance law can be developed by recognizing that an effective way to drive all three components of $\underline{V}_g$ to zero simultaneously is simply to utilize the engine thrust acceleration of the vehicle to align the time rate of change of the $\underline{V}_g$ vector opposite to the $\underline{V}_g$ vector i.e.,

$$\dot{\underline{V}}_g \times \underline{V}_g = 0 \qquad (5)$$

Another possibility is to align the thrust acceleration directly along the $\underline{V}_g$ vector i.e.,

$$\underline{a}_T \times \underline{V}_g = 0 \qquad (6)$$

Both of these will drive V_g towards zero, and so will an $\underline{a}_T$ orientation lying between. This may be expressed as

$$\underline{a}_T \times \underline{V}_g = c\ \underline{b} \times \underline{V}_g \tag{7}$$

where c is a scalar factor. Equation 7 is essentially a linear combination of equations 5 and 6. It can be shown that equation 6 does represent a faster rate of decrease of $|\underline{V}_g|$ than does equation 5 at any given instant of time. However, gravity losses in a finite burn maneuver tend to nullify this advantage and non-zero values of c are found to be more efficient in terms of fuel economy. Note that when c=0 we have

$$\underline{a}_T \times \underline{V}_g = 0 \qquad \text{(equation 6)}$$

and when c=1, we have by rearrangement of 7

$$(\underline{a}_T - c\ \underline{b}) \times \underline{V}_g = 0 \tag{8}$$

from equation 3

$$(\underline{b} - \dot{\underline{V}}_g - c\ \underline{b}) \times \underline{V}_g = 0$$

$$\left[(1-c)\underline{b} - \dot{\underline{V}}_g\right] \times \underline{V}_g = 0$$

or

$$\left[(c-1)\underline{b} + \dot{\underline{V}}_g\right] \times \underline{V}_g = 0 \tag{9}$$

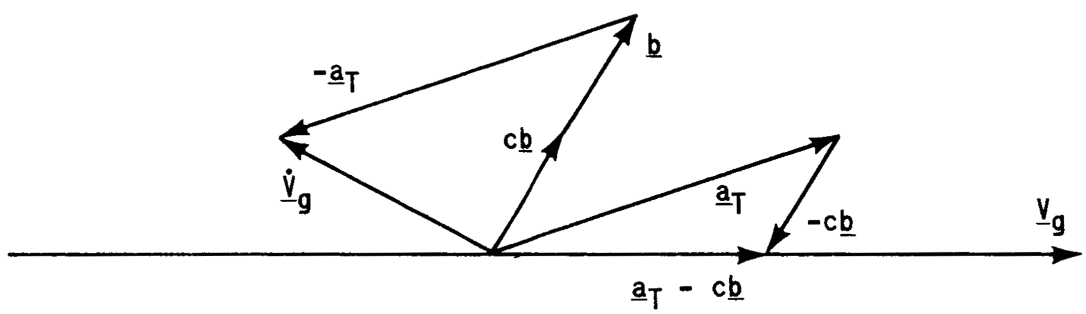
substituting c=1 we get

$$\dot{V}_g \times V_g = 0 \qquad (5)$$

which is just equation 5.

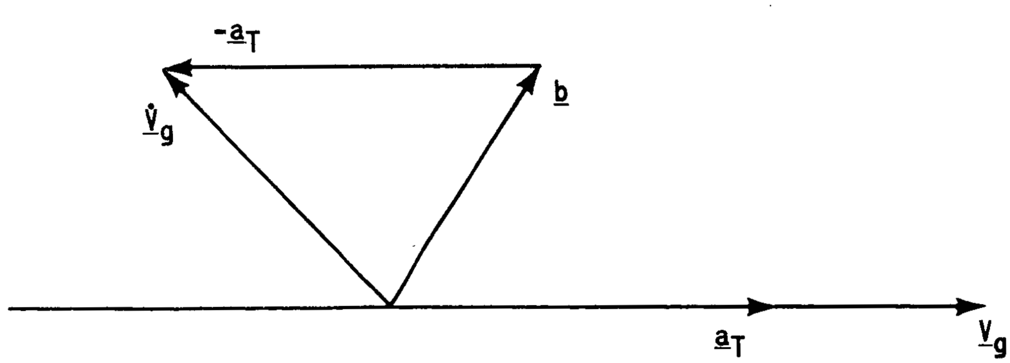
In order to get a feeling for the phenomenom here, consider the following vector diagrams

FIGURE 1



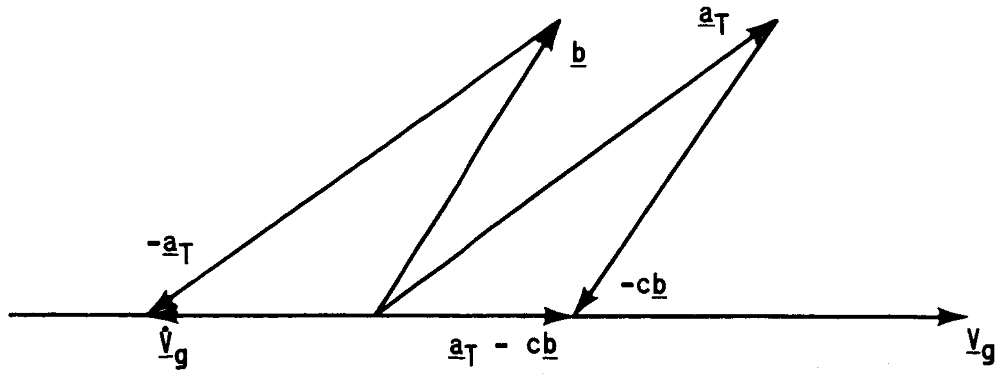
note that $(\underline{a}_T - cb) \times V_g$ is indeed zero. The case for c=0 and c=1 appear as

FIGURE 2



c = 0

FIGURE 3



$c = 1$

Other values of c may, of course, be used but values of 0, .5, and 1.0 are most often considered. For Translunar Injection, a value of c=1.0 is planned.

It is possible to derive an expression for $\underline{a}_T$ directly as follows. Starting with equation 7

$$\underline{a}_T \times \underline{V}_g = c\ \underline{b} \times \underline{V}_g \tag{7}$$

cross both sides with $\underline{V}_g$

$$\underline{V}_g \times (\underline{a}_T \times \underline{V}_g) = \underline{V}_g \times (c\ \underline{b} \times \underline{V}_g)$$

using the following identity

$$\underline{V}_1 \times (\underline{V}_2 \times \underline{V}_3) = \underline{V}_2(\underline{V}_1 \cdot \underline{V}_3) - \underline{V}_3(\underline{V}_1 \cdot V_2)$$

we obtain

$$\underline{a}_T(\underline{V}_g \cdot \underline{V}_g) - \underline{V}_g(\underline{V}_g \cdot \underline{a}_T) = c\underline{b}(\underline{V}_g \cdot \underline{V}_g) - \underline{V}_g(\underline{V}_g \cdot c\underline{b})$$

or

$$\underline{a}_T|\underline{V}_g|^2 - \underline{V}_g(\underline{V}_g \cdot \underline{a}_T) = c\underline{b}\,|\underline{V}_g|^2 - \underline{V}_g(\underline{V}_g \cdot c\underline{b})$$

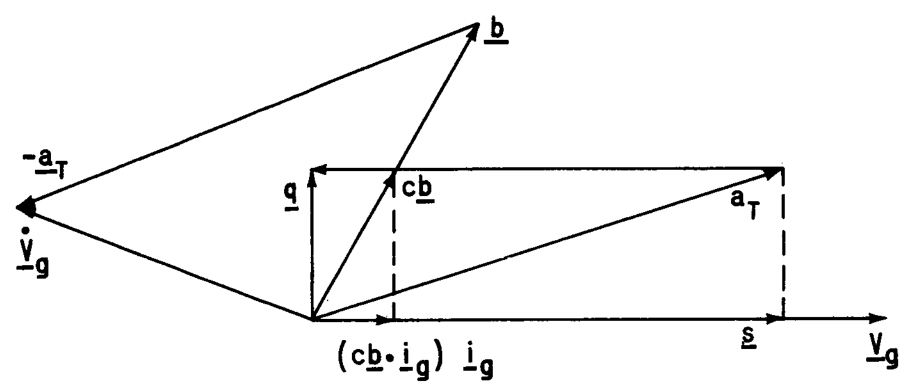
rearranging and dividing by $|\underline{V}_g|^2$ yields

$$\underline{a}_T = c\underline{b} - (\underline{i}_g \cdot c\underline{b})\,\underline{i}_g + (\underline{i}_g \cdot \underline{a}_T)\,\underline{i}_g \tag{10}$$

where

$$\underline{i}_g = \frac{\underline{V}_g}{|\underline{V}_g|} \tag{11}$$

To evaluate the term $(\underline{i}_g \cdot \underline{a}_T)$ we appeal to the geometry of figure 1, reproduced here with some additions.



the quantity $(\underline{i}_g \cdot \underline{a}_T)$ is simply the component of $\underline{a}_T$ lying along $\underline{V}_g$. Let

$$\underline{s} = (\underline{i}_g \cdot \underline{a}_T)\,\underline{i}_g$$

Let q be the component of $\underline{a}_T$ perpendicular to $\underline{V}_g$.

Hence

$$\underline{s} = \underline{a}_T - \underline{q}$$

and $$|\underline{s}| = (\underline{i}_g \cdot \underline{a}_T) = (|\underline{a}_T|^2 - \underline{q} \cdot \underline{q})^{1/2}$$

The vector $\underline{q}$ is defined by noting that the steering law forces the component of $\underline{a}_T$ perpendicular to $\underline{V}_g$ to be equal to the component of $c\underline{b}$ perpendicular to $\underline{V}_g$. Hence

$$\underline{q} = c\underline{b} - (c\underline{b} \cdot \underline{i}_g)\,\underline{i}_g \tag{12}$$

Substituting in equation 10 yields

$$\underline{a}_T = \underline{q} + (|\underline{a}_T|^2 - \underline{q} \cdot \underline{q})^{1/2}\,\underline{i}_g \tag{13}$$

Actually it is not necessary to generate $\underline{a}_T$ directly in order to develop the vehicle steering commands. Instead, the angular rate of rotation of the vehicle is chosen to be proportional to the small angular error, θ_{error}. Once more appealing to a vector diagram, figure 5, let $\underline{a}_{T(actual)}$ be the actual current thrust acceleration vector. Define a vector $\underline{m}$ by

$$\underline{m} = c\underline{b} - \underline{a}_{T(actual)} \tag{14}$$

Recall from equation 8, (multiplied by -1)

$$(c\underline{b} - \underline{a}_T) \times \underline{V}_g = 0$$

the vector $(c\underline{b} - \underline{a}_T)$ should lie opposite to $\underline{V}_g$. The angle between the vectors $(c\underline{b} - \underline{a}_T)$ and $\underline{m}$ is the angular error which should be zero in order for the steering law to be satisfied. Call this angle θ_{error}.

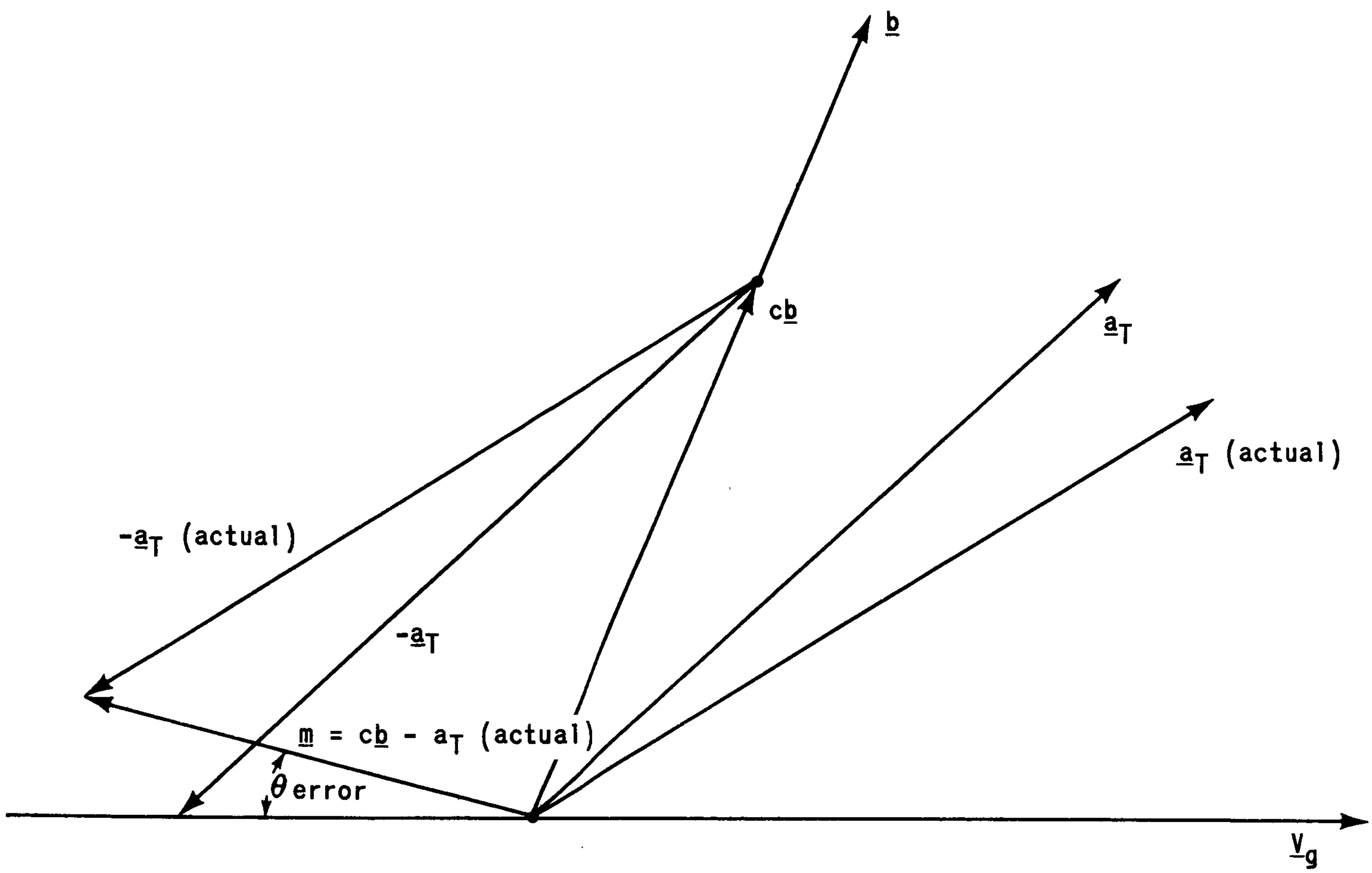


FIGURE 5

Thus

$$\sin(\theta_{error}) = \left| \frac{\underline{V}_g \times \underline{m}}{|\underline{V}_g|\ |\underline{m}|} \right| \tag{15}$$

Assuming θ_{error} is small we set

$$\theta_{error} = \sin(\theta_{error})$$

The desired rate of rotation is then

$$\underline{\omega}_c = K \frac{\underline{V}_g \times \underline{m}}{|\underline{V}_g|\ |\underline{m}|} \tag{16}$$

where K is a constant of proportionality selected to work well with the autopilot. Summarizing, $\underline{\omega}_c$ has a magnitude proportional to the small angular difference between the vector $\underline{m}$ and $c\underline{b} - \underline{a}_T$ and at the same time defines the direction to rotate the vehicle to null the error. The result is a vector rate command in stable member coordinates. Note that when θ_{error} is zero, $\underline{a}_{T(actual)}$ and $\underline{a}_T$ also coincide.

In the discussion so far, nothing has been said concerning the methods of obtaining the $\underline{V}_r$ or $\underline{b}$ vectors. This will be discussed here. The definition of $\underline{V}_r$ is dependent on the objectives of the particular maneuver. In the case of the Translunar Injection maneuver, $\underline{V}_r$ is defined as that velocity, which if attained instantaneously would cause the vehicle to pass through a specified target position at a specified time. Determination of this $\underline{V}_r$ involves a solution of Lambert's problem, i.e., given $\underline{r}(t)$, $\underline{r}(t + t_f)$, and t_f, solve for $\underline{V}(t)$.

where $\underline{r}(t)$ is the present position vector

$\underline{r}(t + t_f)$ is the desired position vector* at time $t + t_f$

t is the current time

t_f is the desired time of flight between the two positions

and $\underline{V}(t)$ is the required velocity at the current time. The concept of determining $\underline{V}_r$ as the instantaneous velocity required is used in all maneuvers involving cross product steering. Only the particular criteria for determining $\underline{V}_r$ varies according to the particular maneuver objectives.

The $\underline{b}$ vector ($\underline{b} = \dot{\underline{V}}_r - \underline{g}$) similarly depends on the maneuver objectives. It is generally possible to derive analytic expressions for $\underline{b}$ but in practice this is not done. Rather, $\dot{\underline{V}}_r$ is determined using first differences. Thus

$$\underline{b} = \frac{\underline{V}_r(t) - \underline{V}_r(t - \Delta t)}{\Delta t} + \frac{\mu}{|\underline{r}(t)|^3}\underline{r}(t) \qquad (17)$$

where Δt is a small increment in time usually equivalent to the interval between consecutive guidance computations.

Implementation of Cross Product Steering

This section is intended to carry the reader step by step through the implementation of the cross product steering scheme as planned for Apollo flights.

1. Begin with a current estimate of the vehicle state vector, (position, velocity and time) and with the necessary guidance parameters for the maneuver, e.g., for TLI, - the time of engine ignition t_{ig}, the target position vector $\underline{r}(t_{ig} + t_f)$ and the desired

---

*$\underline{r}(t + t_f)$, of course, lies on a two body conic passing through $\underline{r}(t)$ and $\underline{r}(t + t_f)$ and in that sense is a pseudo aim point.

time of flight between ignition time and the target position, t_f.

2. Propagate the current state vector forward to engine ignition time including an estimate of the central body gravity vector.

3. Form the $\underline{V}_r(t_{ig})$ vector valid for time t_{ig}.

4. Propagate the ignition time state vector forward (free fall) to obtain the state vector valid for time $t_{ig} + \Delta t$, where Δt is the interval between computations. Obtain $\underline{V}_r(t_{ig} + \Delta t)$.

5. Obtain the initial $\underline{b}$ vector by

$$\underline{b} = \frac{\underline{V}_r(t_{ig} + \Delta t) - \underline{V}_r(t_{ig})}{\Delta t} - \underline{g}(t_{ig})$$

where $\underline{g}(t_{ig})$ is the central body gravity vector computed in step 2 using the average G equations presented in step 12.

6. Make an estimate of the magnitude of the initial thrust acceleration by

$$a_T = \frac{\text{Thrust x } 32.17405}{\text{Vehicle Weight}}$$

7. Determine the velocity to be gained at the time of ignition by

$$\underline{V}_g(t_{ig}) = \underline{V}_r(t_{ig}) - \underline{V}(t_{ig})$$

8. Determine the desired orientation of the thrust acceleration vector at ignition time by

$$\underline{i}_T = \text{UNIT}\left[\underline{q} + (a_T{}^2 - \underline{q} \cdot \underline{q})^{1/2}\ \underline{i}_g\right]$$

where

$$\underline{q} = c\underline{b} - (\underline{i}_g \cdot c\underline{b})\ \underline{i}_g$$

and

$$\underline{i}_g = \frac{\underline{V}_g}{|\underline{V}_g|}$$

9. Align the vehicle thrust axis along $\underline{i}_T$.

10. At the prespecified ignition time, turn on the engine.

11. Δt seconds after ignition, the guidance computations are entered again as they will be every Δt seconds until engine cutoff. Δt will generally have a value of 2 seconds.

12. The position and velocity estimates are updated to the current time using the Average-G equations and the incremental velocity information. The thrust acceleration comes from the Pulsed Integrating Pendulous Accelerometers (PIPA) in the form of velocity increments ($\Delta\underline{V}$) over the interval of time Δt. If $\underline{r}_{n-1}$ and $\underline{V}_{n-1}$ are the position and velocity estimates at the beginning of the n-th computational cycle, then $\underline{r}_n$ and $\underline{V}_n$ are computed from

$$\underline{r}_n = \underline{r}_{n-1} + t(\underline{V}_{n-1} + \underline{g}_{n-1}\frac{\Delta t}{2} + \frac{\Delta\underline{V}}{2})$$

$$V_n = \underline{V}_{n-1} + \frac{\underline{g}_{n-1} + \underline{g}_n}{2}\Delta t + \Delta\underline{V}$$

$$\underline{g}_n = -\frac{\mu}{|\underline{r}_n|^2}\left\{\underline{U}_{r_n} + \frac{3}{2} J_2 \left(\frac{r_{eq}}{|r_n|}\right)^2 \left[(1-5\cos^2\phi)\underline{U}_{r_n} + 2\cos\phi\ \underline{U}_z\right]\right\}$$

where $\cos\phi = \underline{U}_{r_n} \cdot \underline{U}_z$. The vectors $\underline{U}_{r_n}$ and $\underline{U}_z$ are unit vectors in the direction of $\underline{r}_n$ and the polar axis of the central body (earth) respectively, μ is the gravitational constant of the central body, r_{eq} is the equatorial radius of the central body, and J_2 is the second harmonic coefficient of the body's potential function (for earth, $J_2 = 1.0823067 \times 10^{-3}$).

13. t_f is decremented by Δt, i.e.,

$$t_{f_n} = t_{f_{n-1}} - \Delta t \quad .$$

14. $\underline{V}_{r_n}$ is redetermined. In general, this can be done every computational cycle. Where the Lambert problem routine is used to determine $\underline{V}_r$, as in TLI, a new $\underline{V}_r$ is calculated only every other cycle in order to avoid going through that rather lengthy process every cycle. Thus, t_f in step 13 is actually decremented only every other cycle and then $t_{f_n} = t_{f_{n-2}} - 2\Delta t$

15. Compute a new $\underline{V}_{g_n}$. If a new $\underline{V}_{r_n}$ was computed then

$$\underline{V}_{g_n} = \underline{V}_{r_n} - \underline{V}_n$$

If a new $\underline{V}_{r_n}$ was not computed then

$$\underline{V}_{g_n} = \underline{V}_{g_{n-1}} + b_{n-1}\,\Delta t - \Delta\underline{V}$$

16. Compute $\underline{b}_n$. If $\underline{V}_{r_n}$ was computed this cycle then

$$\underline{b}_n = \frac{\underline{V}_{r_n} - \underline{V}_{r_{n-2}}}{2\Delta t} - \underline{g}_n$$

If $\underline{V}_{r_n}$ was not computed this cycle then b_n will be given by

$$b_n = \underline{b}_{n-1}$$

17. The command for the attitude control system is generated by

$$\underline{\omega}_c = k \frac{\underline{V}_g \times \Delta \underline{m}}{|\underline{V}_g| \, |\Delta m|}$$

where $\underline{\omega}_c$ = commanded attitude rate to CSM attitude control system, (digital autopilot)

$$\Delta \underline{m} = c\underline{b} \, \Delta t - \Delta \underline{V}$$

$\Delta \underline{V}$ is the accumulated velocity change since the previous computation.

18. Compute the estimated time-to-go until engine cutoff, T_{go}, by

$$T_{go} = k_1 \frac{\underline{V}_g \cdot \underline{V}_g}{|\underline{V}_g \cdot \Delta \underline{m}|} \Delta t + \Delta t_{tail\text{-}off}$$

where

$$k_1 = 1 - \frac{1}{2} \underline{V}_g /(32.17405 \, I_{sp})$$

I_{sp} = specified specific impulse of the engine.

$\Delta t_{tail-off}$ = the duration of a burn at full or maximum thrust equivalent to the tail-off impulse after the engine-off signal is issued.

19. Repeat steps 12 through 18 until such time as $T_{go} < 4$. At that time the engine-off signal is scheduled to be sent in T_{go} seconds and $\underline{\omega}_c$ is set to $\underline{0}$.

20. Updating the position and velocity estimate is continued until the engine is off.

D. A. Corey

2012-DAC-jdc D. A. Corey

Attachment
References

REFERENCES

1. Guidance System Operations Plan AS-278, Volume 1, CM GNCS Operations, MIT Instrumentation Lab Report R-547, October 1966.

2. Powered Flight Phases of CSM-Analytical Description and Mechanization of Steering Equations and the Derivation of CDU Commands, MIT/IL SGA Memo #13-64 (Revision 1), E. M. Copps, Jr., May 28, 1964.