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THE APOLLO RENDEZVOUS NAVIGATION FILTER THEORY, DESCRIPTION AND PERFORMANCE

Volume 1 of 2

by

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THE APOLLO RENDEZVOUS NAVIGATION FILTER THEORY, DESCRIPTION AND PERFORMANCE

ABSTRACT

The Apollo on-board rendezvous navigation filter represents the first application of Kalman filtering techniques to the solution of the rendezvous navigation problem - determination of the relative state of the rendezvousing vehicles. The recursive formulation of the Kalman filter, utilized in the Apollo on-board cislunar and orbital navigation systems to minimize errors in estimation of the inertial state of a single orbiting vehicle, is adapted to minimize errors in estimation of one orbiting vehicle state with respect to another. The optimum rendezvous navigation filter is developed as a standard against which to compare the actual Apollo filter, a sub-optimal formulation made necessary by limited guidance computer storage.

A new approach to orbit navigation is suggested since rendezvous navigation utilizing the optimum filter is shown to resolve individual state errors in addition to relative state errors.

The rendezvous radar angle bias estimation scheme, developed specifically for the Apollo mission, is described and its effectiveness in desensitizing a navigation system to sensor biases is evaluated.

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I. INTRODUCTION

The decision to apply recursive Kalman filtering techniques to the design of the Apollo rendezvous navigation system was made after studies of less complex techniques proved them to be incapable of meeting the exacting performance requirements of the Apollo mission. One such technique involved the measurement of a portion of the relative state (e.g., range and range rate) at a predetermined point in the transfer orbit where this data was adequate to allow an accurate calculation of the midcourse maneuver required to correct for trajectory dispersions. This method requires a specific type of transfer trajectory and could not possibly satisfy the requirement for rendezvous navigation in the wide variety of possible abort trajectories, or allow for desired flexibility in planning of the primary rendezvous mission profile.

Another technique studied, which was not as sensitive to the type of transfer trajectory, involved radar measurements of the complete relative state (range, range rate, angles and angle rates) at discrete intervals in the trajectory. The noisy data was smoothed using polynomial filtering techniques[4]. This scheme proved inadequate for providing the navigation accuracy required to make midcourse corrections at the long ranges (200 nm) involved in Apollo abort trajectories. The critical radar parameter was angle rate. The accuracy required in the knowledge of this parameter could not be provided either by direct measurement with state-of-the-art radars or by polynomial smoothing of radar angle data. Radar angle biases presented an additional problem not resolved with this technique.

It was at this point that serious consideration was given to utilization of the recursive Kalman filter to solve the rendezvous navigation problem. (It was noted that the cislunar navigation problem could be considered a rendezvous problem -- the moon being the target.) This filter was fully developed[1] for use in Apollo on-board cislunar navigation to the extent that most of the equations were already programmed for insertion into the Apollo guidance computer. Computer storage requirements were thus fairly well defined, so that it was not a difficult problem to verify that the Lunar Module guidance computer had sufficient storage to accommodate this type of filter. Also, there would conceivably be no prohibitive computer storage requirements to include this rendezvous filter in the command module guidance computer, since most of the cislunar navigation equations would be common to the rendezvous navigation equations. A simplified block diagram of the cislunar recursive navigation system is shown in Fig. I-1. After an initial state vector is uplinked to the on-board computer, this state vector is extrapolated to the time of the star-horizon angle measurement. The difference between the sextant measurement and the computed estimate of this angle (δQ) is fed to the navigation filter. A weighting vector ($\underline{\omega}$) is computed utilizing a state covariance matrix which is extrapolated to the measurement time (and updated, to take into account incorporation of a measurement, for use with subsequent measurements), a measurement geometry vector ($\underline{b}$) and preloaded sensor variance. The change ($\delta\hat{\underline{x}}$) applied to the current state estimate as a result of the external measurement is computed as:

$$\delta\hat{\underline{x}} = \begin{bmatrix} \delta\hat{\underline{r}} \\ \delta\hat{\underline{v}} \end{bmatrix} = \underline{\omega}\,\delta Q \tag{1}$$

where: $\delta\hat{\underline{r}}$ and $\delta\hat{\underline{v}}$ are position and velocity increments respectively.

The new state estimate ($\hat{\underline{R}}_n$, $\hat{\underline{V}}_n$) is then extrapolated to the time of the next measurement and the recursive process is repeated. (A detailed explanation of this system may be found in Ref. 3).

Adapting the system of Fig. I-1 to rendezvous navigation begins with consideration of the basic rendezvous navigation problem -- determination of the relative state between two orbiting vehicles, as opposed to determination of a single vehicle state in inertial space. Three fundamental changes then become apparent: (a) the target vehicle state must be included in the navigation system, (b) a sensor which measures all or some part of the relative state must be incorporated in the system (c) the navigation filter must be redesigned for minimization of a new cost function -- the relative state estimation error and possibly sensor bias estimation errors (if the rendezvous sensor should have prohibitive bias errors). Incorporation of these changes presents some fundamental problems to solve:

1. Should both vehicle states be updated after measurements or only one, and if one, which one?

   Inclusion of another vehicle state means there will necessarily be estimation errors associated with that state. Relative state measurements will be a function of both state errors, whereas in the cislunar system, very small uncertainties existed in the knowledge of star and planet positions.

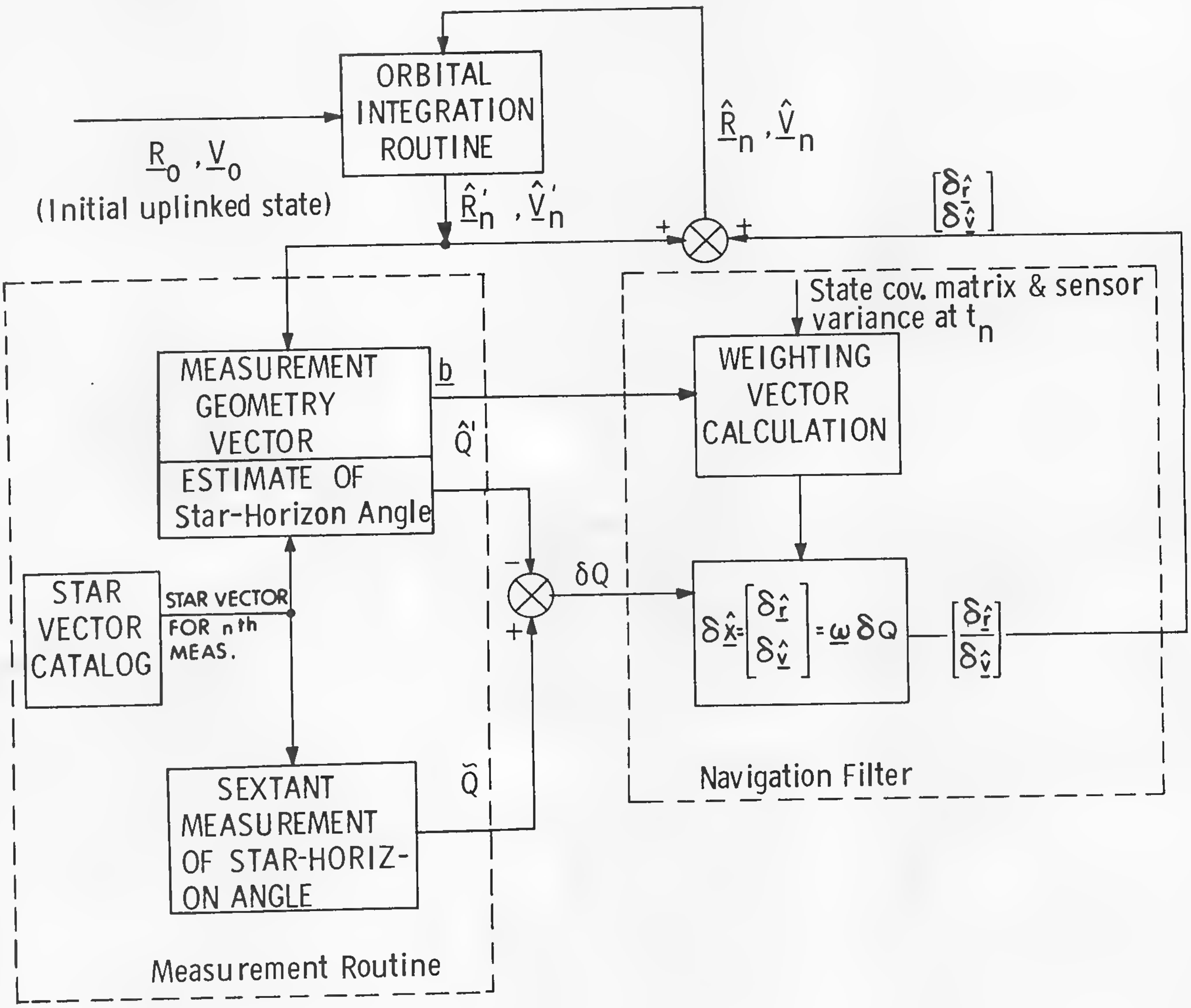


Fig. I-1 Cislunar Recursive Navigation System

2. How is the weighting vector ($\underline{\omega}$) of the navigation filter formulated?

   The recursive form of the optimum linear estimator (Eq. 1) should be preserved so that the problem reduces to determining $\underline{\omega}$ to minimize relative state errors for the alternatives mentioned in (1) above.

3. What environment factors should be modeled into the filter?

   Since the relative state is of prime importance, only those factors which create significant differential effects on the two vehicle states need consideration. For example, the cislunar case is essentially a three body problem whereas rendezvous needs only to consider a single central force field.

4. What, if any, sensor biases should be either estimated or just modeled into the filter?

   The inclusion of a rendezvous radar into the navigation system presented measurement bias errors of such magnitude to significantly affect navigation performance, whereas comparable bias errors do not exist in the cislunar sensor (sextant).

These problems are examined in detail in this paper. Section II develops the optimum rendezvous navigation filter. Section III contains a complete description of the present Apollo rendezvous navigation filter, showing how it differs from the optimum filter, and describing the effect these differences have on performance and required operational procedures. Section IV contains a more detailed description of the Apollo rendezvous radar bias estimation formulation, performance and alternate application. Section V presents a summary of rendezvous navigation performance on Apollo flights to date, with emphasis on particular portions of the missions which illustrate principles discussed in this paper.

II. OPTIMUM RENDEZVOUS NAVIGATION FILTER

A. Relative State Formulation

As shown in references (1, 2) and discussed in the introduction, the recursive form of the optimum linear estimator involves the periodic updating of the current best state estimate by the addition of a vector increment ($\delta\hat{\underline{x}}$) to the state, which is computed as a weighted difference between a measured portion of the state and the estimate of this measured parameter. (Eq. (1): $\delta\hat{\underline{x}} = \underline{\omega}\,\delta Q$). In rendezvous navigation, the state to be estimated is the relative state, i.e., the difference between the target vehicle and active vehicle states, and the measured parameter is all or some portion of this relative state. The general form of a rendezvous navigation system utilizing recursive Kalman filtering is represented in the flow diagram (Fig. II-1). Having established the

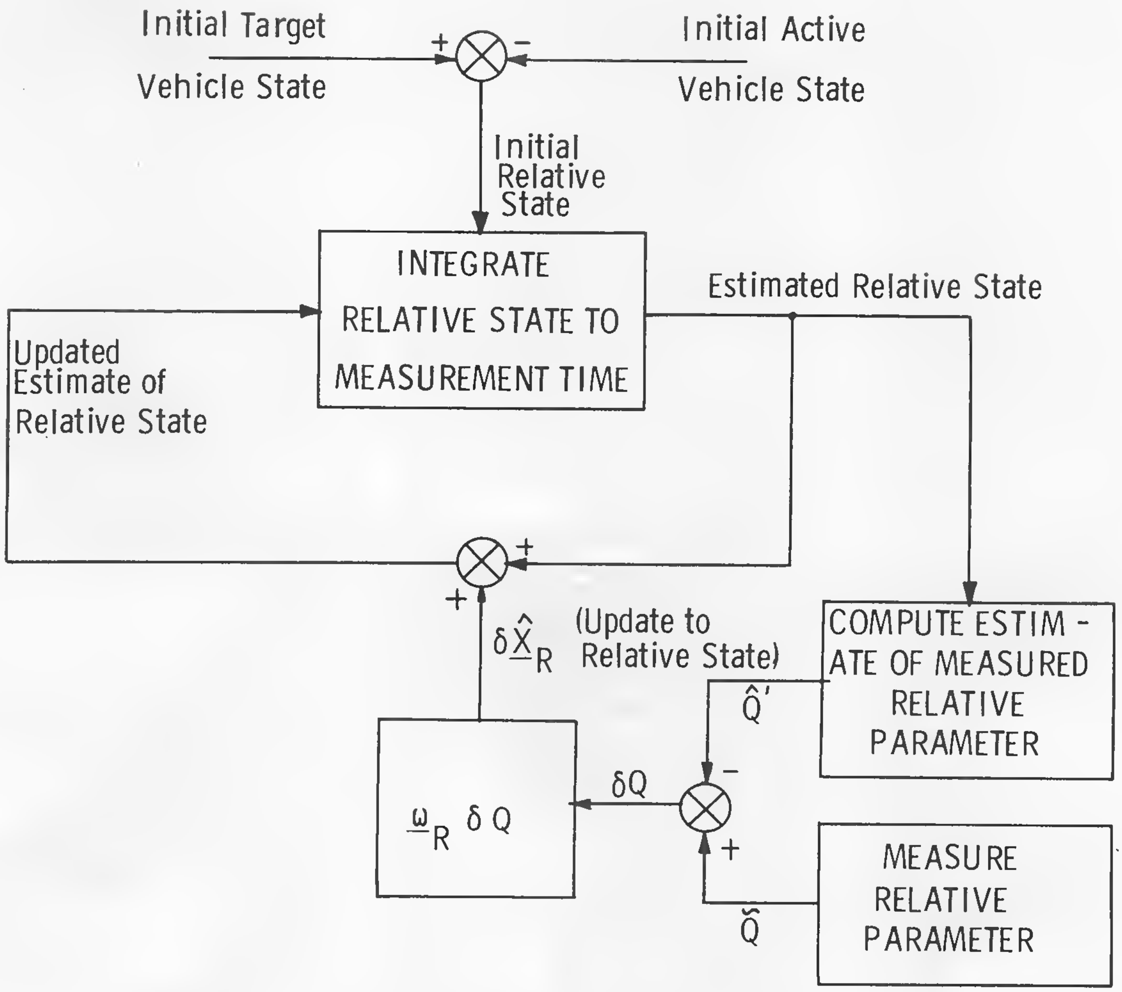


Fig. II-1 Rendezvous Navigation System Using Recursive Kalman Filter basic form of the navigation filter, the problem of determining the optimum rendezvous navigation filter reduces to the determination of the optimum weighting vector ($\underline{\omega}_R$) which minimizes the uncertainty in the estimate of the relative state between the active and target vehicles.

A.1 Derivation of Optimum Weighting Vector ($\underline{\omega}_R$)

As shown in Fig. II-1 the rendezvous navigation update equation is:

$$\delta \hat{\underline{x}}_R = \underline{\omega}_R \, \delta Q \qquad (2)$$

where:
- $\delta \hat{\underline{x}}_R$ = increment vector added to current state estimate to obtain new optimum estimate.
- δQ = $\tilde{Q} - \hat{Q}$ = measured relative parameter minus estimate of measured relative parameter
- $\underline{\omega}_R$ = weighting vector which minimizes error in relative state estimate

Eq. (2) is actually a special formulation of the general form of the optimum linear estimate of the relative state. This general form, derived after first linearizing the equations of motion by considering small perturbations about a reference trajectory[1] is given by:

$$\delta \hat{\underline{x}}_R = \delta \hat{\underline{x}}_R' + \underline{\omega}_R (\delta \tilde{Q} - \delta \hat{Q}') \qquad (3)$$

where:
- $\delta \hat{\underline{x}}_R$ = updated estimate of the deviation of the relative state from the reference relative state.
- $\delta \hat{\underline{x}}_R'$ = previous estimate of the deviation of the relative state from the reference state, extrapolated to current measurement time.
- $\underline{\omega}_R$ = weighting vector
- $\delta \tilde{Q}$ = measurement of the deviation of the measured parameter from the reference value of the measured parameter
- $\delta \hat{Q}'$ = current estimate of the deviation of the measured parameter from the reference value of the measured parameter
- $\delta \underline{x}_R = \begin{bmatrix} \delta \underline{r}_R \\ \delta \underline{v}_R \end{bmatrix}$ is a 6-dimensional deviation vector composed of the relative position and velocity deviation from reference

By the simple expedient of defining the reference trajectory as the current estimated trajectory, $\delta \hat{\underline{x}}_R'$ of Eq. (3) becomes the zero vector. This is achieved by applying $\delta \hat{\underline{x}}_R$ (which now becomes the update increment vector) to the current estimated relative state to form the new estimated relative state which is also the new reference trajectory. (The estimate of the deviation from the reference ($\delta \hat{\underline{x}}_R'$) is thus driven to zero at each update.) This approach preserves the assumption of linearity by assuring that perturbations from the reference are small. It is also noted that the quantity

$(\delta\tilde{Q} - \delta\hat{Q}')$ is equivalent to $(\tilde{Q} - \hat{Q}')$ since the same quantity (the reference value of the measured parameter) is subtracted from $\tilde{Q}$ and $\hat{Q}'$ to obtain $\delta\tilde{Q}$ and $\delta\hat{Q}'$ respectively. Thus it is seen that Eq. (2) is in fact equivalent to Eq. (3). For the derivation of $\underline{\omega}_R$ which follows, the more general form of the optimum linear estimator (Eq. (3)) will be used.

$\delta\tilde{Q}$ and $\delta\hat{Q}'$ are given by the following equations:

$$\begin{aligned} \delta\tilde{Q} &= \underline{b}_R^T \, \delta\underline{x}_R + \underline{b}_\beta^T \, \underline{\beta} + \alpha \\ \delta\hat{Q}' &= \underline{b}_R^T \, \delta\hat{\underline{x}}'_R + \underline{b}_\beta^T \, \hat{\underline{\beta}}' \end{aligned} \tag{4}$$

where: $\delta\underline{x}_R$ = true deviation in relative state from reference state.

$\underline{\beta}$ is an $n \times 1$ vector representing the actual biases in the measuring instrument

$\hat{\underline{\beta}}'$ is the estimate of biases in the measurements*

α is the random error in the measurement (assumed to be white noise)

the "b" vectors ($\underline{b}_R$, $\underline{b}_\beta$) are the partial derivatives of the measured quantity with respect to the state or measurement bias. (e.g. $\underline{b}_R = \partial Q/\partial\underline{x}_R$ so that δQ due to a state deviation is given by $\delta Q = \underline{b}_R^T \, \delta\underline{x}_R$.)

Substituting Eq. (4) into Eq. (3) and using the following definitions for the error in estimation of the relative state and error in estimate of the measurement bias respectively,

$$\begin{aligned} \underline{e}_R &= \begin{bmatrix} \underline{e}_{RP} \\ \underline{e}_{RV} \end{bmatrix} = \delta\hat{\underline{x}}_R - \delta\underline{x}_R \\ \underline{e}_\beta &= \hat{\underline{\beta}} - \underline{\beta} \end{aligned} \tag{5}$$

where: $\underline{e}_{RP}$ = relative position estimation error
$\underline{e}_{RV}$ = relative velocity estimation error

the relative state estimation error after measurement incorporation is found by subtracting $\delta\underline{x}_R$ from each side of Eq. (3) to yield:

$$\underline{e}_R = \underline{e}'_R - \underline{\omega}_R \left[\underline{b}_R^T \, \underline{e}'_R + \underline{b}_\beta^T \, \underline{e}'_\beta + \alpha \right] \tag{6}$$

*$\hat{\underline{\beta}}'$ takes on non-zero values only if bias estimation is included in the filter formulation. In the present development, only the relative state is estimated so that $\hat{\underline{\beta}}' = \underline{0}$. (Bias estimation is covered in Section IV.)

where: $\underline{e}'_R$ = previous relative state estimation error extrapolated to the measurement time.

The covariance matrix of relative state estimation errors, defined by:

$$E_R = \overline{\underline{e}_R \, \underline{e}_R^T}$$

becomes after measurement incorporation (using Eq. (6)):

$$E_R = E'_R - \underline{\omega}_R \left[\underline{b}_R^T E'_R + \underline{b}_\beta^T GR\beta'^T \right] - \left[E'_R \underline{b}_R + GR\beta' \underline{b}_\beta \right] \underline{\omega}_R^T \qquad (7)$$

$$+ \underline{\omega}_R \left[\underline{b}_R^T E'_R \underline{b}_R + \underline{b}_\beta^T E'_\beta \underline{b}_\beta + 2 \underline{b}_R^T GR\beta' \underline{b}_\beta + \overline{\alpha^2} \right] \underline{\omega}_R^T$$

where: $GR\beta = \overline{\underline{e}_R \, \underline{e}_\beta^T}$ and $E_\beta = \overline{\underline{e}_\beta \, \underline{e}_\beta^T}$

In order to determine the value of $\underline{\omega}_R$ which minimizes the mean squared error in relative state estimation (the trace of E_R), a variation in Eq. (7) is taken and $\delta\underline{\omega}_R$ is solved for. By setting $\delta\underline{\omega}_R$ to zero and solving for $\underline{\omega}_R$, we have (derivation in Appendix B. 1):

$$\underline{\omega}_R = \left[E'_R \underline{b}_R + GR\beta' \underline{b}_\beta \right] / A \qquad (8)$$

$$\text{where: } A = \underline{b}_R^T E'_R \underline{b}_R + \underline{b}_\beta^T E'_\beta \underline{b}_\beta + 2 \underline{b}_R^T GR\beta' \underline{b}_\beta + \overline{\alpha^2}$$

Eq (8) is the optimum weighting vector for updating the relative state between the two vehicles at a given measurement incorporation.

A. 2 Mechanizing a Filter Using the Relative State Formulation

The optimum rendezvous filter is now completely defined. Eq. (2) ($\delta\hat{\underline{x}}_R = \underline{\omega}_R \, \delta Q$) defines the update increment at measurement points and Eq. (8) defines the value of $\underline{\omega}_R$ to be used in order to minimize the mean squared relative position and velocity errors. In order to compute $\underline{\omega}_R$ at each measurement update, it is necessary to have current values of E_R, E_β, $GR\beta$ and the geometry vectors. The geometry vectors are computed from the current estimated relative state. The computation of E_R, E_β, and $GR\beta$ is performed in the following sequence: From a-priori values, these matrices are extrapolated to the first measurement time. $\underline{\omega}_R$ is computed and E_R and $GR\beta$ are updated using $\underline{\omega}_R$ and, with E_β, are extrapolated to the next measurement time.

a. a-priori values

$(E_R)_0$ = initial covariance matrix of relative state estimation errors (derived for example from monte carlo simulations of that part of the mission prior to rendezvous).

$(E_\beta)_0$ = initial covariance matrix of measurement biases (derived from statistical studies of actual radar performance)

$(GR\beta)_0$ = zero matrix (no correlation between relative state errors and measurement biases before using rendezvous sensor).

b. extrapolation of matrices from time = t_n to time = t_{n+1}

$$E'_R = \phi_{n+1,n} E_R (t_n) \phi_{n+1,n}{}^T$$

$$E'_\beta = \phi_{\beta\, n+1,n} E_\beta (t_n) \phi_{\beta\, n+1,n}{}^T$$

$$GR\beta' = \phi_{n+1,n} GR\beta (t_n) \phi_{\beta\, n+1,n}{}^T$$

(Note: prime indicates value of variable at time = t_{n+1})

where: ϕ is the transition matrix defined by:

$$\dot{\phi} = F \phi, \ \phi_0 = \text{identity matrix,}$$

for a state error defined by:

$$\dot{\underline{e}}_R = F \underline{e}_R$$

c. matrix update after measurement incorporation

E_R is updated using Eq. (7)

$$GR\beta = (I - \underline{\omega}_R \underline{b}_R{}^T) GR\beta' - \underline{\omega}_R \underline{b}_\beta{}^T E'_\beta$$

The optimum filter just derived would serve satisfactorily as a rendezvous navigation filter provided explicit equations of motion for the relative state exist. This, in general, is not the case. Only in rendezvous missions involving small relative ranges and short transfers (typically less than 90 degrees) may the expression for differential gravity (gravity acceleration acting on target vehicle minus gravity acceleration acting on active vehicle) be accurately linearized so that the relative state may be extrapolated without precise knowledge of the vehicle inertial state.[5, 6] For this case, components of the relative state may be measured with respect to any convenient coordinate system and the filter developed above utilized to minimize relative state errors.

In the general case, where linearization of differential gravity does not provide satisfactory accuracy, the relative state may not be extrapolated directly, but is instead formulated as the difference between the target and active vehicle inertial states, which are extrapolated by utilizing standard techniques for integrating the equations of motion. Because of this necessity for maintaining current estimates of two inertial state vectors, a simple update of the relative state as specified by the optimum filter is no longer adequate. The optimum change to the relative state derived above must be accomplished by making appropriate changes to the separate vehicle state vectors. In addition, the relative state may no longer be measured with respect to any arbitrary coordinate frame, but must be referenced to a known inertial frame. (This reference is provided by a stable platform aligned to known inertial directions.)

There is only one constraint when applying navigation updates to each vehicle state: the resultant relative state update must be that specified by Eqs. (2) and (8). Thus, defining $\delta\hat{\underline{x}}_A$ and $\delta\hat{\underline{x}}_T$ as the active and target vehicle state updates, this constraint may be stated:

$$\delta\hat{\underline{x}}_T - \delta\hat{\underline{x}}_A \equiv \delta\hat{\underline{x}}_R = \underline{\omega}_R \, \delta Q \tag{9}$$

Rewriting Eq. (9) using the optimum recursive form for each vehicle update we have

$$\underline{\omega}_T \, \delta Q - \underline{\omega}_A \, \delta Q = \underline{\omega}_R \, \delta Q \tag{9a}$$

or

$$\boxed{\underline{\omega}_T - \underline{\omega}_A = \underline{\omega}_R} \tag{10}$$

Eq. (10) states that any weighting vectors may be used to update each vehicle state recursively and the relative state will be updated optimally as long as the difference in the two weighting vectors equals the optimum relative weighting vector. Thus, by the use of Eq. (10) and Eq. (8) for $\underline{\omega}_R$, a more general form of the optimum rendezvous navigation filter has been formulated.

If $\underline{\omega}_T$ and $\underline{\omega}_A$ are chosen arbitrarily to satisfy Eq. (10), the relative state will indeed be updated optimally at each measurement incorporation, but the onboard estimates of each individual vehicle state will be continually degraded. Rather than making an arbitrary choice of $\underline{\omega}_T$ and $\underline{\omega}_A$, an investigation can be made to determine if some optimum value of these weighting vectors can be found which minimizes or at least contains the estimation errors of each vehicle state.

B. Optimizing Individual State Updates

The general form of the optimum recursive update for the active and target vehicle states respectively is given by:

$$\delta\hat{\underline{x}}_A = \delta\hat{\underline{x}}'_A + \underline{\omega}_A \left[\delta\tilde{Q} - \delta\hat{Q}' \right] = \delta\hat{\underline{x}}'_A + \underline{\omega}_A \, \delta Q \tag{11a}$$

$$\delta\hat{\underline{x}}_T = \delta\hat{\underline{x}}'_T + \underline{\omega}_T \left[\delta\tilde{Q} - \delta\hat{Q}' \right] = \delta\hat{\underline{x}}'_T + \underline{\omega}_T \, \delta Q \tag{11b}$$

$\delta\breve{Q}$ and $\delta\hat{Q}'$ are given by:

$$\delta\breve{Q} = \underline{b}_A^T \, \delta\underline{x}_A + \underline{b}_T^T \, \delta\underline{x}_T + \underline{b}_\beta^T \, \underline{\beta} + \alpha$$

$$\delta\hat{Q}' = \underline{b}_A^T \, \delta\hat{\underline{x}}_A + \underline{b}_T^T \, \delta\hat{\underline{x}}_T + \underline{b}_\beta^T \, \hat{\underline{\beta}}' \quad (12)$$

where the "b" vectors are given by: $\underline{b}_A = \frac{\partial Q}{\partial \underline{x}_A}$, $\underline{b}_T = \frac{\partial Q}{\partial \underline{x}_T}$, $\underline{b}_\beta = \frac{\partial Q}{\partial \underline{\beta}}$

and the other quantities are the same as those defined in Eq. (4).

Substituting Eq. (12) into Eq. (11) and defining active and target vehicle state errors by:

$$\underline{e}_A = \delta\hat{\underline{x}}_A - \delta\underline{x}_A \ , \quad \underline{e}_T = \delta\hat{\underline{x}}_T - \delta\underline{x}_T$$

Eq. (11) becomes

$$\underline{e}_A = \underline{e}_A' - \underline{\omega}_A \left[\underline{b}_A^T \, \underline{e}_A' + \underline{b}_T^T \, \underline{e}_T' + \underline{b}_\beta^T \, \underline{e}_\beta' + \alpha \right] \quad (13a)$$

$$\underline{e}_T = \underline{e}_T' - \underline{\omega}_T \left[\underline{b}_A^T \, \underline{e}_A' + \underline{b}_T^T \, \underline{e}_T' + \underline{b}_\beta^T \, \underline{e}_\beta' + \alpha \right] \quad (13b)$$

Before utilizing Eqs. (13a) and (13b) to derive optimum values of $\underline{\omega}_A$ and $\underline{\omega}_T$, the optimum relative weighting vector ($\underline{\omega}_R$) will be derived for the general case where both vehicle states are updated. Defining the relative state error by:

$$\underline{e}_R = \delta\hat{\underline{x}}_R - \delta\underline{x}_R$$

and since: $\delta\hat{\underline{x}}_R = \delta\hat{\underline{x}}_T - \delta\hat{\underline{x}}_A$ and $\delta\underline{x}_R = \delta\underline{x}_T - \delta\underline{x}_A$

we have: $$\underline{e}_R = \underline{e}_T - \underline{e}_A \quad (14)$$

Thus, from Eqs. (13a) and (13b) we can write:

$$\underline{e}_R = \underline{e}_R' - \underline{\omega}_R \left[\underline{b}_A^T \, \underline{e}_A' + \underline{b}_T^T \, \underline{e}_T' + \underline{b}_\beta^T \, \underline{e}_\beta' + \alpha \right] \quad (15)$$

where: $$\underline{\omega}_R = \underline{\omega}_T - \underline{\omega}_A$$

Comparing Eq. (15) to Eq. (6), the two expressions are equivalent if we note that $\underline{b}_T = - \underline{b}_A = \underline{b}_R$. That $\underline{b}_A$ does equal the negative of $\underline{b}_T$ can be seen from the fact that $\underline{b}$ vectors represent a change in the measured quantity due to a change in a vehicle state. Since relative parameters are being measured, a change in the target vehicle state ($\delta \underline{x}_T$) produces a change in the measured relative quantity which is the same as an opposite change ($-\delta \underline{x}_T$) in the active vehicle state. $\underline{b}_R$ is equivalent to $\underline{b}_T$ since the relative state was defined as target state-active state (Fig. II-1).

Forming the covariance matrix of relative state uncertainties ($E_R = \overline{\underline{e}_R \underline{e}_R^T}$) and using variational calculus to determine the optimum $\underline{\omega}_R$ which minimizes relative state errors as done above, we obtain the general expression for the optimum relative state error weighting vector $\underline{\omega}_R$. (derivation in Appendix B.2)

$$\underline{\omega}_R = -\left[E_A' \underline{b}_A - E_T' b_T - G'^T \underline{b}_A + G' \underline{b}_T + GA\beta' \underline{b}_\beta - GT\beta' \underline{b}_\beta \right] / A \tag{16}$$

$$A = \underline{b}_A^T E_A' \underline{b}_A + \underline{b}_T^T E_T' \underline{b}_T + \underline{b}_\beta^T E_\beta' \underline{b}_\beta + 2\, \underline{b}_A^T G' \underline{b}_T + 2\, b_A^T GA\beta' \underline{b}_\beta + 2\, b_T^T GT\beta' \underline{b}_\beta + \overline{\alpha^2}$$

where: $E_A = \overline{\underline{e}_A \underline{e}_A^T}$, $E_T = \overline{\underline{e}_T \underline{e}_T^T}$, $E_\beta = \overline{\underline{e}_\beta \underline{e}_\beta^T}$

$G = \overline{\underline{e}_A \underline{e}_T^T}$, $GA\beta = \overline{\underline{e}_A \underline{e}_\beta^T}$, $GT\beta = \overline{\underline{e}_T \underline{e}_\beta^T}$

It is easily seen that Eq. (16) reduces to Eq. (8) by realizing that:

$$\underline{b}_T = -\underline{b}_A = \underline{b}_R, \quad GR\beta = GT\beta - GA\beta$$

and from Eq. (14)

$$E_R = E_A + E_T - G - G^T \tag{17}$$

For the individual vehicle state updates given by Eqs. (11a) and (11b) which utilize measurements of the relative state, expressions will be now derived for the values of $\underline{\omega}_A$ and $\underline{\omega}_T$ which minimize the errors in estimation of the active vehicle state and target vehicle state respectively. In other words, errors in each inertial state estimate will be minimized without regard to the resulting relative state error.

Using Eq. (13) the expressions for the covariance matrix of active and target vehicle inertial state estimation errors after measurement incorporation are:

$$\begin{aligned} E_A = {} & (I - \underline{\omega}_A \underline{b}_A^T) E_A' (I - \underline{b}_A \underline{\omega}_A^T) + \underline{\omega}_A (\underline{b}_T^T E_T' \underline{b}_T + \underline{b}_\beta^T E_\beta' \underline{b}_\beta + \underline{b}_T^T GT\beta' \underline{b}_\beta \\ & + \underline{b}_\beta^T GT\beta'^T \underline{b}_T + \overline{\alpha^2}) \underline{\omega}_A^T - (I - \underline{\omega}_A \underline{b}_A^T)(G' \underline{b}_T \underline{\omega}_A^T + GA\beta' \underline{b}_\beta \underline{\omega}_A^T) \\ & - (\underline{\omega}_A \underline{b}_T^T G'^T + \underline{\omega}_A \underline{b}_\beta^T GA\beta'^T)(I - \underline{b}_A \underline{\omega}_A^T) \end{aligned} \tag{18}$$

$$E_T = (I - \underline{\omega}_T \underline{b}_T^T) E_T' (I - \underline{b}_T \underline{\omega}_T^T) + \underline{\omega}_T (\underline{b}_A^T E_A' \underline{b}_A + \underline{b}_\beta^T E_\beta' \underline{b}_\beta + b_A^T GA\beta' \underline{b}_\beta$$

$$+ \underline{b}_\beta^T GA\beta'^T \underline{b}_A + \overline{\alpha^2}) \underline{\omega}_T^T - (I - \underline{\omega}_T \underline{b}_T^T) (G'^T \underline{b}_A^T \underline{\omega}_T^T + GT\beta' \underline{b}_\beta \underline{\omega}_T^T)$$

$$- (\underline{\omega}_T \underline{b}_A^T G' + \underline{\omega}_T \underline{b}_\beta^T GT\beta'^T) (I - \underline{b}_T \underline{\omega}_T^T) \quad (19)$$

Variational calculus is again used to determine the optimum $\underline{\omega}_A$ and $\underline{\omega}_T$ for the minimization of inertial state estimation errors of both the active and target vehicle, with the following results (derived in Appendix B.3)

$$\underline{\omega}_A = (E_A' \underline{b}_A + G' \underline{b}_T + GA\beta' \underline{b}_\beta)/A \quad (20a)$$

$$\underline{\omega}_T = (E_T' \underline{b}_T + G'^T \underline{b}_A + GT\beta' \underline{b}_\beta)/A \quad (20b)$$

where:
$$A = \underline{b}_A^T E_A' \underline{b}_A + \underline{b}_T^T E_T' \underline{b}_T + \underline{b}_\beta^T E_\beta' \underline{b}_\beta + 2 \underline{b}_A^T G' \underline{b}_T + 2 \underline{b}_A^T GA\beta' \underline{b}_\beta + 2 \underline{b}_T^T GT\beta' \underline{b}_\beta + \overline{\alpha^2} \quad (20c)$$

Examination of $\underline{\omega}_A$ and $\underline{\omega}_T$ in Eq. (20) yields an extremely significant result. It is seen that the difference $(\underline{\omega}_T - \underline{\omega}_A)$ is precisely the optimum relative weighting vector $(\underline{\omega}_R)$ given in Eq. (16)! This is just the constraint (specified by Eq. (10)) that is required of $\underline{\omega}_T$ and $\underline{\omega}_A$ in order to yield an optimum relative state update. Consequently, there is no reason to arbitrarily choose $\underline{\omega}_T$ and $\underline{\omega}_A$ to satisfy Eq. (10). They can be chosen to optimally update each individual vehicle state. Thus, we may state this important conclusion: The optimum inertial updates of both the active and target vehicle states, using a relative state measurement, also yields the optimum relative state update.

C. Mechanization of Optimum Filter

The complete set of equations necessary for the mechanization of the optimum rendezvous navigation filter (i.e. computation of $\underline{\omega}_A$ and $\underline{\omega}_T$) follow:

a. a-priori values

$(E_A)_0$, $(E_T)_0$ = initial covariance matrix of active and target vehicle state estimation errors

(E_β) = covariance matrix of measurement biases

$(G)_0 = (GA\beta)_0 = (GT\beta)_0$ = ZERO matrix

b. extrapolation of matrices from time = t_n to time = t_{n+1} (prime indicates extrapolated value at t_{n+1})

$$E_A' = \phi_{A\ n+1,n} E_A(t_n) \phi_{A\ n+1,n}^T, \quad E_T' = \phi_{T\ n+1,n} E_T(t_n) \phi_{T\ n+1,n}^T$$

$$E_\beta' = \phi_{\beta\ n+1,n} E_\beta(t_n) \phi_{\beta\ n+1,n}^T, \quad G' = \phi_{A\ n+1,n} G(t_n) \phi_{T\ n+1,n}^T$$

$$GA\beta' = \phi_{A\ n+1,n} GA\beta(t_n) \phi_{\beta\ n+1,n}^T, \quad GT\beta' = \phi_{T\ n+1,n} GT\beta(t_n) \phi_{\beta\ n+1,n}^T \tag{21}$$

where: ϕ is the state transition matrix appropriate to the state being extrapolated.

c. matrix update after measurement incorporation

E_A and E_T: use Eqs. (18) and (19)

$$G = (I - \underline{\omega}_A \underline{b}_A^T) G' (I - \underline{b}_T \underline{\omega}_T^T) + \underline{\omega}_A (\underline{b}_T^T G'^T \underline{b}_A + \underline{b}_\beta^T E_\beta' \underline{b}_\beta + \underline{b}_\beta GA\beta'^T \underline{b}_A$$
$$+ \underline{b}_T^T GT\beta' \underline{b}_\beta + \overline{\alpha^2}) \underline{\omega}_T^T - (I - \underline{\omega}_A \underline{b}_A^T)(E_A' \underline{b}_A \underline{\omega}_T^T + GA\beta' \underline{b}_\beta \underline{\omega}_T^T)$$
$$- (\underline{\omega}_A \underline{b}_T^T E_T' + \underline{\omega}_A \underline{b}_\beta^T GT\beta'^T)(I - \underline{b}_T \underline{\omega}_T^T) \tag{22}$$

$$GA\beta = (I - \underline{\omega}_A \underline{b}_A^T) GA\beta' - \underline{\omega}_A (\underline{b}_T^T GT\beta' + \underline{b}_\beta^T E_\beta') \tag{23}$$

$$GT\beta = (I - \underline{\omega}_T \underline{b}_T^T) GT\beta' - \underline{\omega}_T (\underline{b}_A^T GA\beta' + \underline{b}_\beta^T E_\beta') \tag{24}$$

The mechanization of this filter in a rendezvous navigation system is depicted in Fig. II-2.

This optimum rendezvous filter (Fig. II-2) is completely general and requires no explicit extrapolation of the relative state with its associated approximations. Instead, the inertial states of the active and target vehicle are extrapolated to the measurement time and these two states yield the estimated relative state and the associated estimated relative state parameters. Each vehicle state is updated by an increment given by: $\delta\hat{\underline{x}}_A = \underline{\omega}_A\ \delta Q$, $\delta\hat{\underline{x}}_T = \underline{\omega}_T\ \delta Q$, where $\underline{\omega}_A$ and $\underline{\omega}_T$ are given in Eq. (20). (NOTE: These update equations use the form which results when the reference trajectory is defined as the current estimated trajectory as discussed previously.)

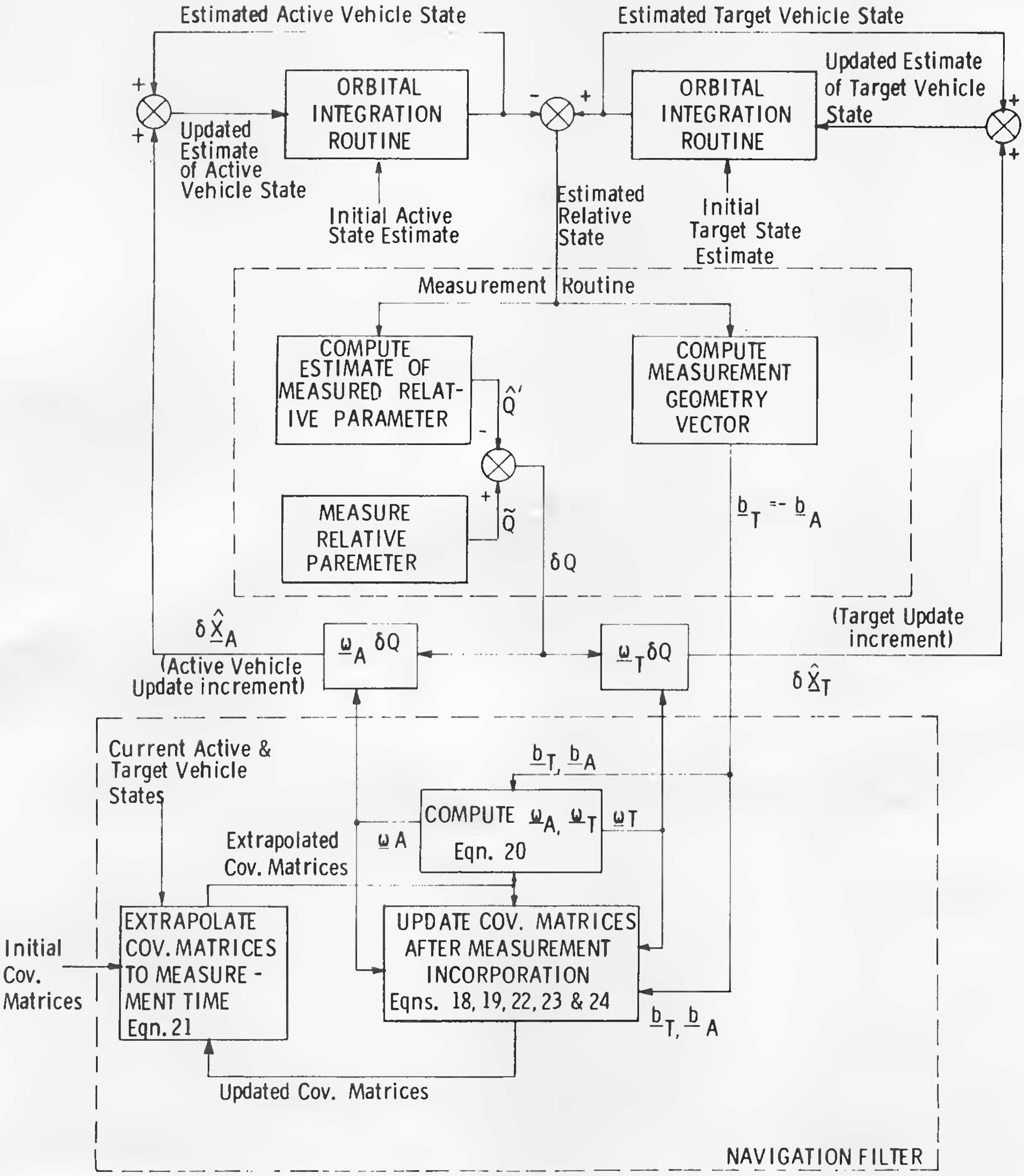


Fig. II-2 Optimum Rendezvous Navigation System

Some special comments should be made about the error term ($\underline{e}_\beta$) in Eq. (13). This is a general term which may represent any systematic measurement error (e.g. radar angle biases, stable platform misalignment, etc.). If this error contribution behaves significantly different from the estimated vehicle state errors and an accurate model of its behavior is known, it can be estimated along with the vehicle states. Under these special conditions, a new $\underline{\omega}_R$ may be formulated which minimizes both the relative state errors and bias estimation error, thus providing superior performance to the filter developed above. The simple expedient of including $\underline{\beta}$ in the estimated state and updating the estimate of this measurement bias yields the optimum filter in this case (this procedure is discussed in detail in Section IV).

Certain modifications to the optimum filter will now be discussed in order to provide additional background for the discussion of the Apollo filter (Section III). One modification is to update only one vehicle state (e.g. the active state). For this case we have:

$$
\begin{aligned}
\underline{\omega}_T &= \underline{0}\ , \text{ so that } \delta\hat{\underline{x}}_T = \underline{0} \\
\underline{\omega}_A &= -\underline{\omega}_R \text{ (from Eq. (10))} \\
\delta\hat{\underline{x}}_A &= -\delta\hat{\underline{x}}_R \text{ (from Eq. (9))}
\end{aligned}
\qquad (25)
$$

where: $\underline{\omega}_R$ is the optimum relative weighting vector given in Eq. (8) (or more generally in Eq. (16)).

This single state update (SSU) filter is identical in every respect to that developed originally for the optimum relative state update ($\hat{\delta}\underline{x}_R$) except that this optimum increment is applied to a single inertial state to achieve the required optimum relative state after a measurement incorporation. Although the SSU filter yields the optimum relative state update at each measurement incorporation, it does not achieve the minimum relative state error throughout the entire navigation period. This is because the individual inertial states are not optimally updated as specified in Eq. (20). Consequently, the inertial state estimates are not optimum and this inertial estimation error manifests itself in a larger relative error when the states are extrapolated after measurement incorporation. Even if, after a measurement incorporation, the relative state error is driven to zero, each individual state will have an estimation error equal to the state error of the target vehicle. Depending on the magnitude of this error (which has been growing since the start of navigation), the relative error will increase accordingly during an extrapolation period. The optimum filter, however, reduces the inertial state errors in addition to the relative error, so that the relative error buildup during extrapolation is minimized. Of course, if the target vehicle state errors are zero, the SSU filter is equivalent to the optimum filter.

By neglecting to model measurement bias errors ($\underline{e}_\beta$) (or by including them in the estimated state) and updating only the active vehicle, Eq. (8) reduces to:

$$\underline{\omega}_A = E_R' \, \underline{b}_A \Big/ \left(\underline{b}_A^T E_R' \, \underline{b}_A + \overline{\alpha^2}\right) \qquad (26)$$

(noting that $\underline{\omega}_A = -\underline{\omega}_R$ (Eq. (25)) and $\underline{b}_A = -\underline{b}_R$

Eq. (26) forms the basis for the Apollo filter which is discussed in Section III.

D. Optimum Filter Performance Results

In order to evaluate the optimum rendezvous navigation filter performance in a realistic rendezvous sequence, simulations were run of the Apollo 12 rendezvous profile (Fig. II-3) with the Apollo filter replaced by the optimum filter. The nominal tracking schedule was followed and the standard Apollo rendezvous radar model was used with measurements of range, range rate and angles measuring line of sight direction. Initial conditions are summarized in Fig. II-4. (Note that quantities are included for a linearized statistical analysis to provide rms results and also a 'single run' simulation to provide results for one particular mission.)

The initial LM (active vehicle) state errors (and covariance matrix) represent nominally expected errors after injection off the lunar surface. The CSM (target vehicle) state errors, however, are not the nominal ground tracking errors but were chosen specifically since they are known to cause seriously degraded performance when the Apollo filter is utilized. The reason is that this particular distribution of state errors, although not particularly large in magnitude initially, grows exceedingly fast as the CSM state is integrated in its orbit with no navigational updates. Consequently, since in the nominal Apollo mission the LM state is the only one updated, both vehicle inertial state errors grow tremendously (as mentioned in the previous discussion) so that the relative state errors eventually diverge also. This behavior is illustrated in Figs. II-5 and II-6 which plot relative state errors and Figs. II-7 - II-10 which plot each vehicles' inertial state estimation errors, in a simulation which utilized the Apollo filter. (NOTE: Nominal operation of the Apollo filter requires reinitialization of the filter covariance matrix at specified times as indicated by (R) in these figures. This procedure is not necessary with the optimum filter. A full discussion of this procedure is contained in Section III.)

When the optimum filter is utilized (with both states updated) in exactly the same simulation, the resulting relative and inertial state errors are as shown in Figs. II-11 - II-16. The improvement is striking. Not only are the relative errors vastly decreased, but the inertial errors are also decreased appreciably!

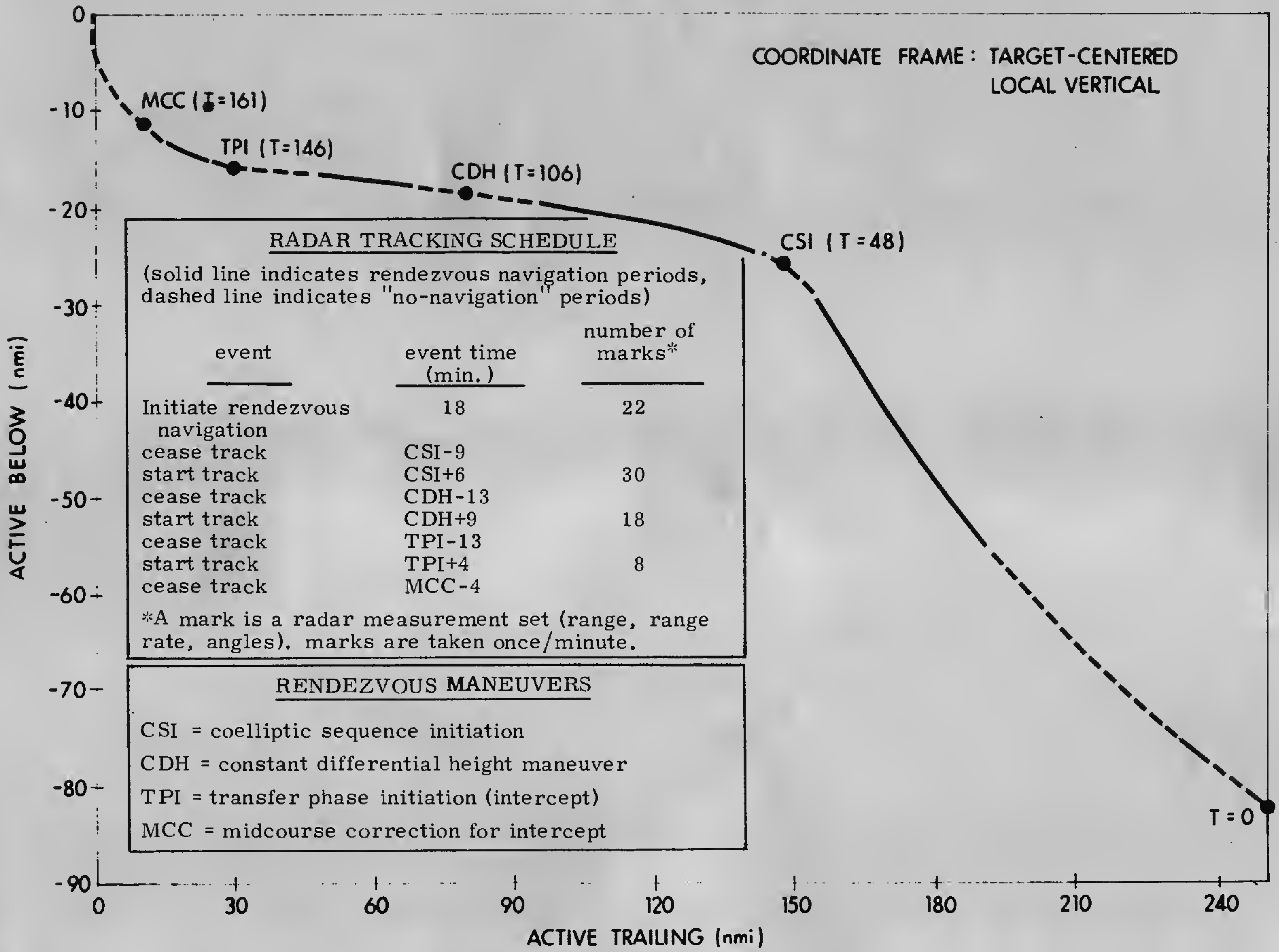


Fig. II-3 Relative Motion Plot - Apollo 12 Rendezvous Mission

● INITIAL VEHICLE STATES (Lunar centered inertial frame)*

ACTIVE: position (ft) = 6.45E05, -5.1E06, -2.57E06

velocity (fps) = -5485, -469, -535 (Note: E06 = 10^6)

TARGET: position (ft) = 9.28E05, -5.3E06, -2.66E06

velocity (fps) = 5281, 849, 149

● ERRORS: The following errors represent nominal Apollo 12 lunar mission errors with the following exceptions:

(a) The target vehicle (CM) state errors are not nominal ground track errors. They were chosen specifically as non-representative ground track errors which produce inordinately large errors as the trajectory progresses.

(b) Radar biases and inertial platform misalignment errors were assumed zero for simulations in order to isolate state vector error performance of the filters.

Error Source	Statistical Analysis	"Single run" Simulation (see note below)
Active Vehicle (LM) State Errors (inertial frame)*	E_A: LM Covariance Matrix (ft^2, fps^2) $\begin{bmatrix} 9.83E06, & -2.83E06, & -1.35E06, & 3.12E02, & 5.0E03, & 3.31E03 \\ & 1.18E07, & -2.05E06, & 7.36E03, & 3.37E03, & 2.91E03 \\ & & 1.46E07, & 2.79E03, & 1.81E03, & -6.03E02 \\ & & & 1.62E01, & 3.23E00, & 1.62E00 \\ \text{Symmetrical} & & & & 1.75E01, & -4.1E-01 \\ & & & & & 1.8E01 \end{bmatrix}$	Position error (ft) 8464, 1673, 1201 Velocity error (fps) 0.16, 5.64, 3.5
Target Vehicle (CM) State Errors (inertial frame)*	E_T: CM Covariance Matrix (ft^2, fps^2) (Diagonal Matrix) Diagonal Terms: 2.5E07, 1.6E07, 9E06, (position) 25, 16, 9 (velocity)	Position error (ft) 5000, 4000, 3000 Velocity error (fps) 5, 4, 3
Measurement Errors Radar range Radar range rate Radar angles	Error Variances [1/3 % of true range]^2 [1.3/3 % of true range rate]^2 (1 milliradian)^2/angle	1 σ errors in random number generator 1/3 % of true range 1.3/3 % of true range rate 1 milliradian

● FILTER LOADS

Covariance Matrix {Opt Filter, Apollo	E_A and E_T (above) Diagonal Matrix: Initial Diagonals = 10,000, 10,000, 10,000 (ft^2, fps^2) 10, 10, 10 Reinitialized Values = 2000, 2000, 2000, 2, 2, 2	Same
Sensor Variances {range, range rate, angles	[1/3 % of estimated range]^2 [1.3/3 % of estimated range rate]^2 (1 milliradian)^2/angle	Same

NOTE: "Single run" simulation refers to a simulation of a mission in which only a single set of errors is used (i.e., one actual mission), as opposed to the statistical analysis which produces RMS results for an ensemble of missions.

* – Inertial frame is Mean Besselian Frame (1969).

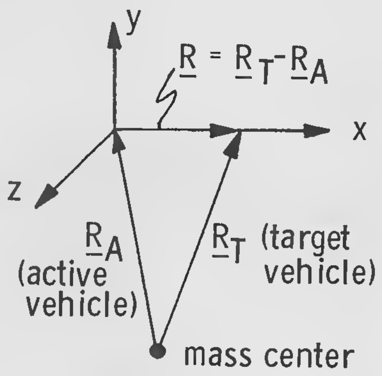
Fig. II-4 Simulation Initial Conditions

LEGEND FOR RENDEZVOUS NAVIGATION PLOTS

STATISTICAL UNCERTAINTY refers to 1σ (RMS) results
ACTUAL refers to "single run" results
B : rendezvous maneuver
R : Apollo filter covariance matrix reinitialization
Solid line in plots (no symbol notation) represents error magnitudes

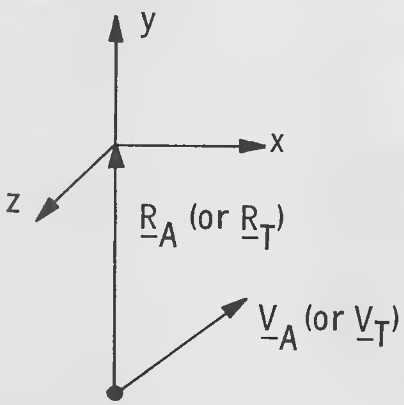
- COORDINATE FRAME DEFINITION

a. Relative error plots



symbol	coordinate	definition
×	x	UNIT ($\underline{R}$)
+	y	$\underline{z} \times \underline{x}$
*	z	UNIT ($\underline{R}_T \times \underline{R}_A$)

b. Inertial error plots



symbol	coordinate	definition
×	x	$\underline{y} \times \underline{x}$
+	y	UNIT ($\underline{R}_A$)
*	z	UNIT ($\underline{V}_A \times \underline{R}_A$)

Active vehicle inertial errors in active vehicle local vertical frame
Passive vehicle inertial errors in passive vehicle local vertical frame

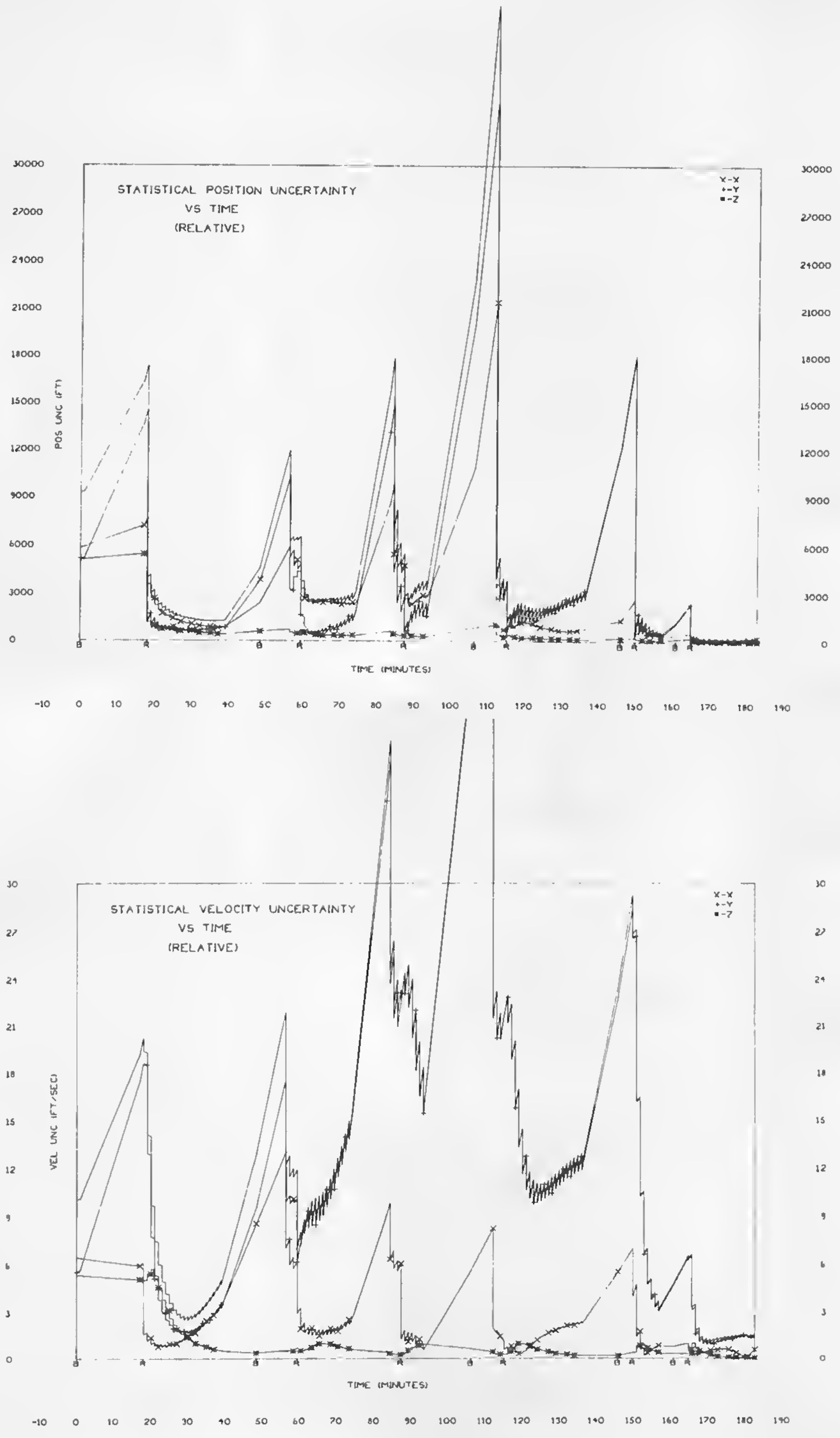


Fig. II-5 RMS Relative Position and Velocity Uncertainties (Off-Nominal Apollo 12 Rendezvous with Apollo Filter)

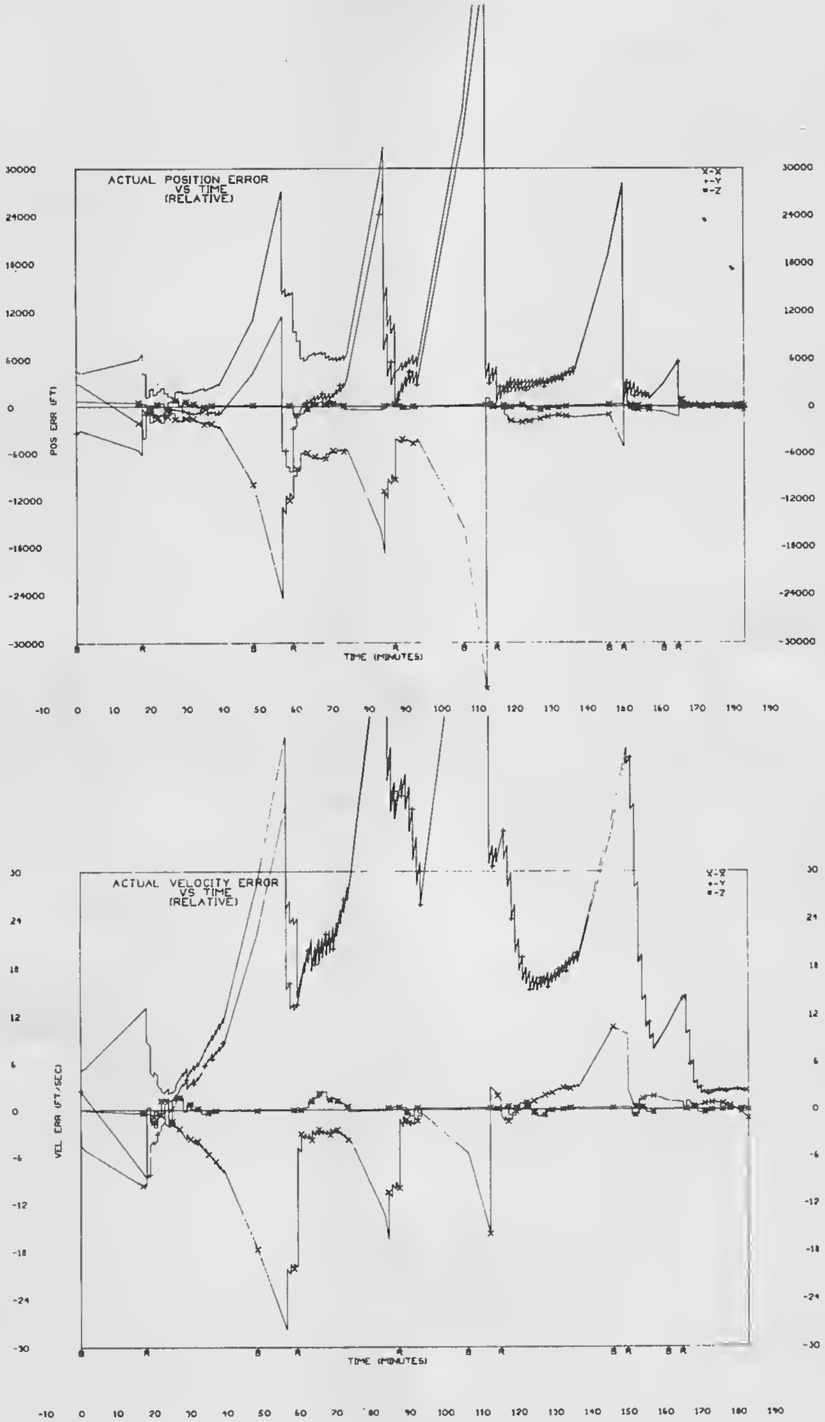


Fig. II-6 "Single Run" Relative Position and Velocity Errors (Off-Nominal Apollo 12 Rendezvous with Apollo Filter)

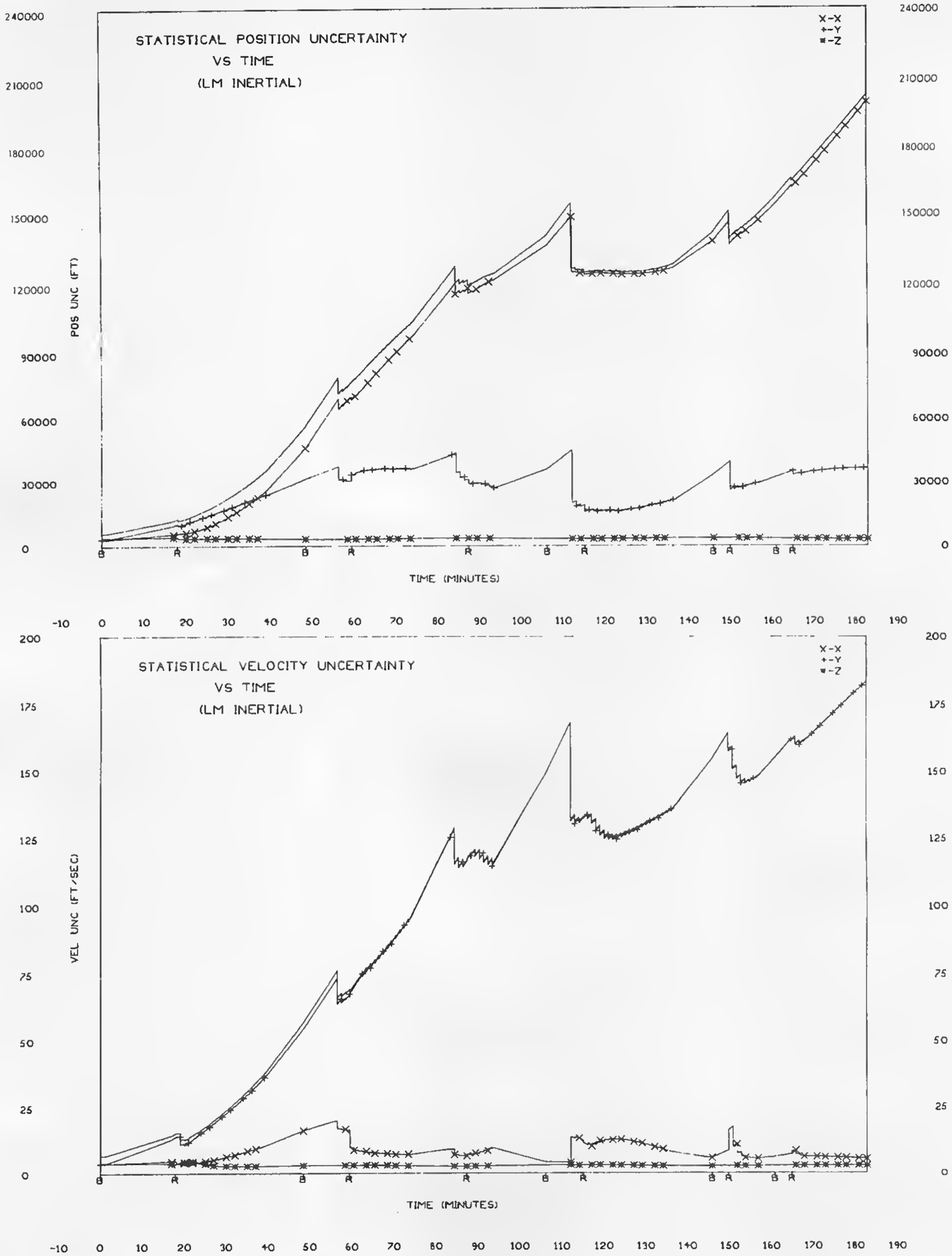


Fig. II-7 RMS Active Vehicle Inertial State Uncertainties (Off-Nominal Apollo 12 Rendezvous with Apollo Filter)

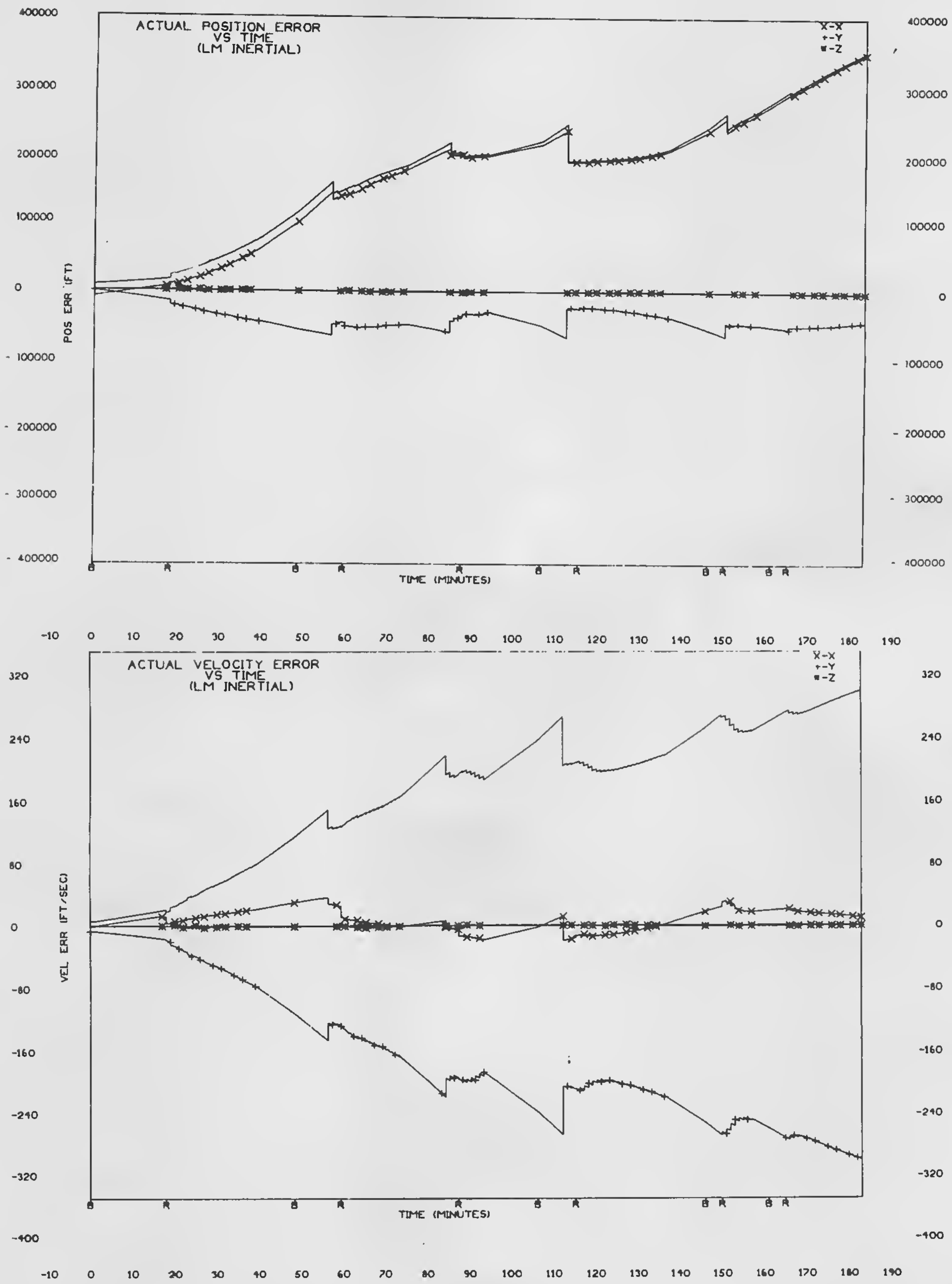


Fig. II-8 "Single Run" Active Vehicle Inertial State Errors (Off-Nominal Apollo 12 Rendezvous with Apollo Filter)

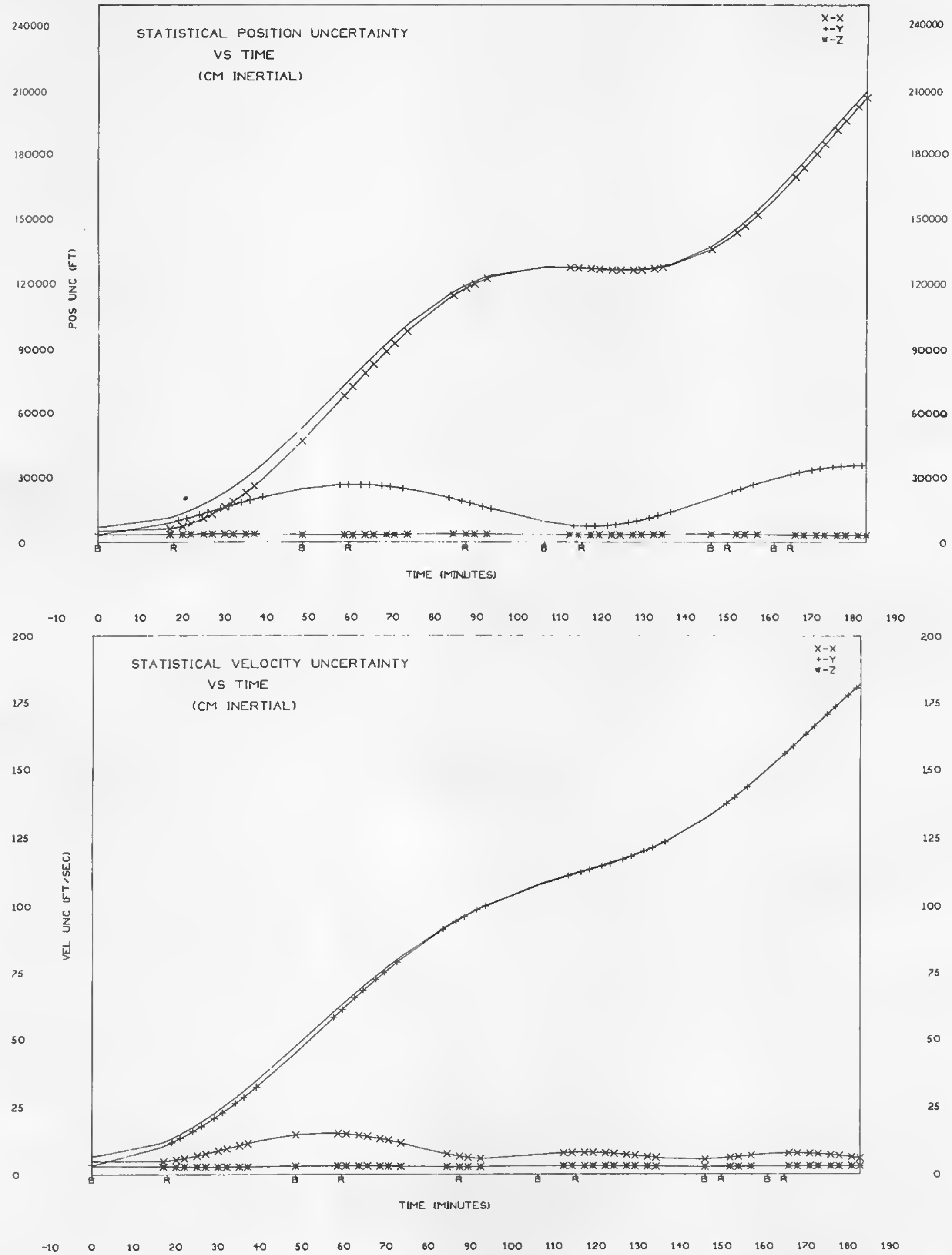


Fig. II-9 RMS Target Vehicle Inertial State Uncertainties (Off-Nominal Apollo 12 Rendezvous with Apollo Filter)

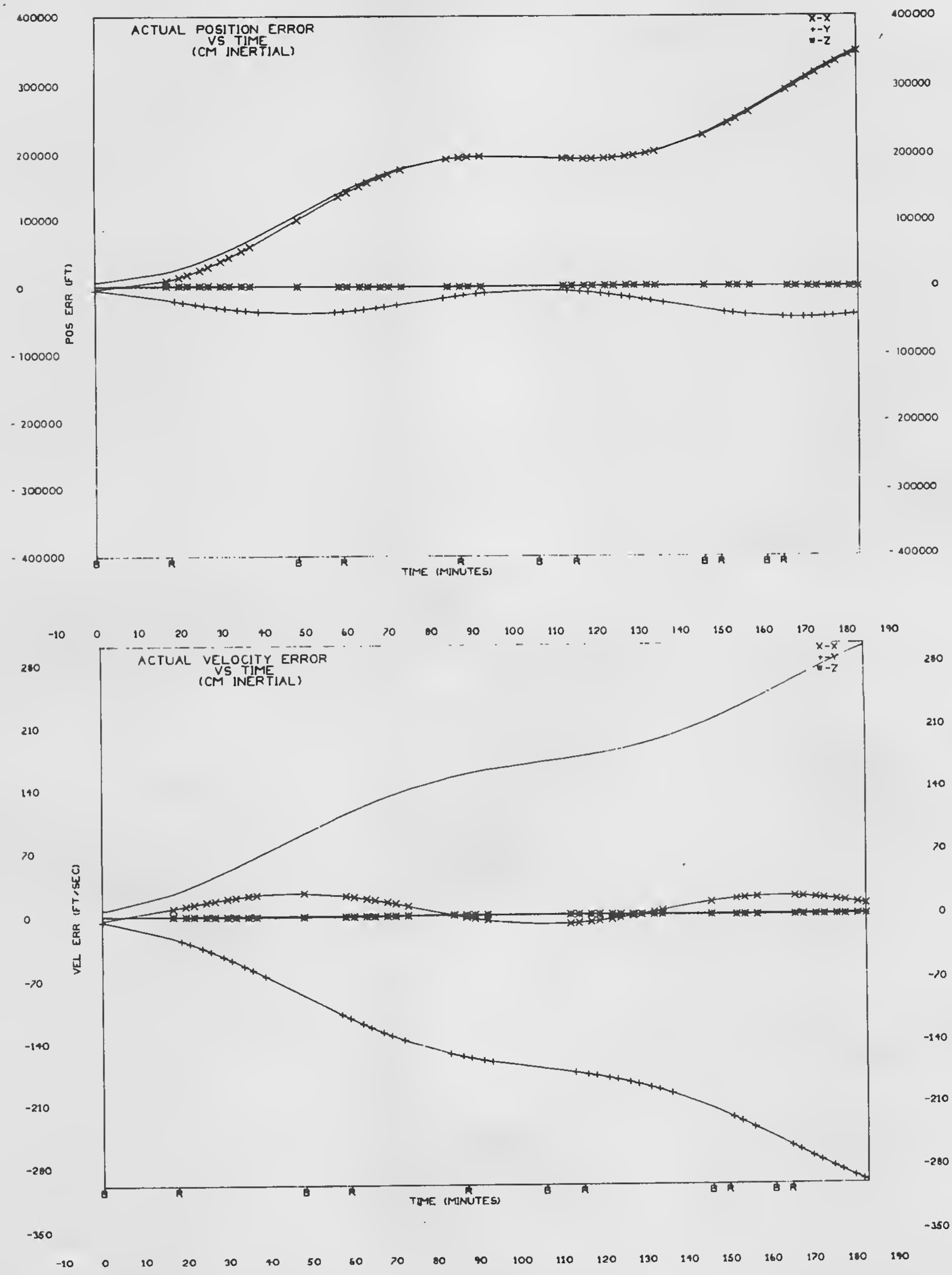


Fig. II-10 "Single Run" Target Vehicle Inertial State Errors (Off-Nominal Apollo 12 Rendezvous with Apollo Filter)

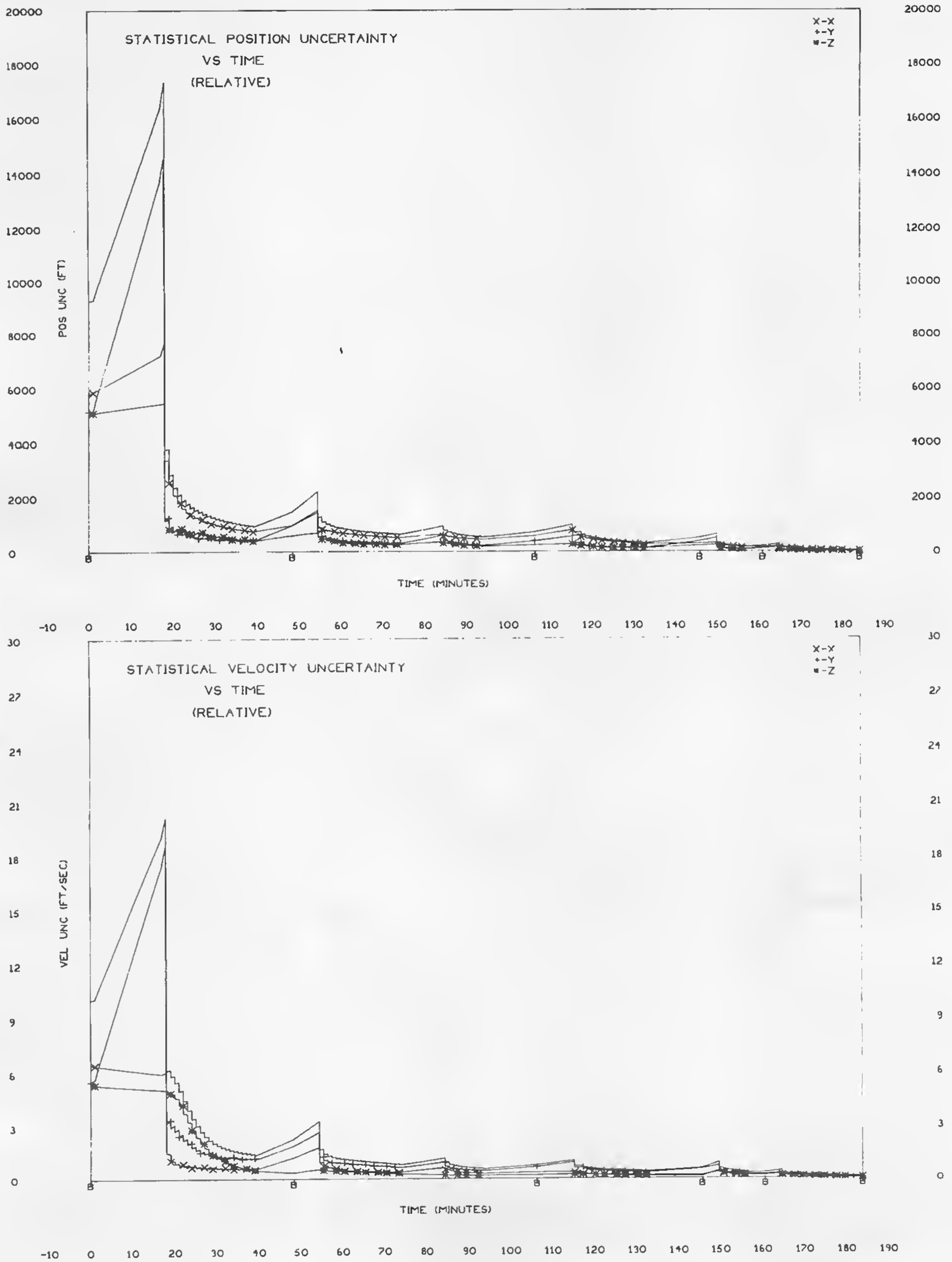


Fig. II-11 RMS Relative Position and Velocity Uncertainties (Off-Nominal Apollo 12 Rendezvous with Optimum Filter)

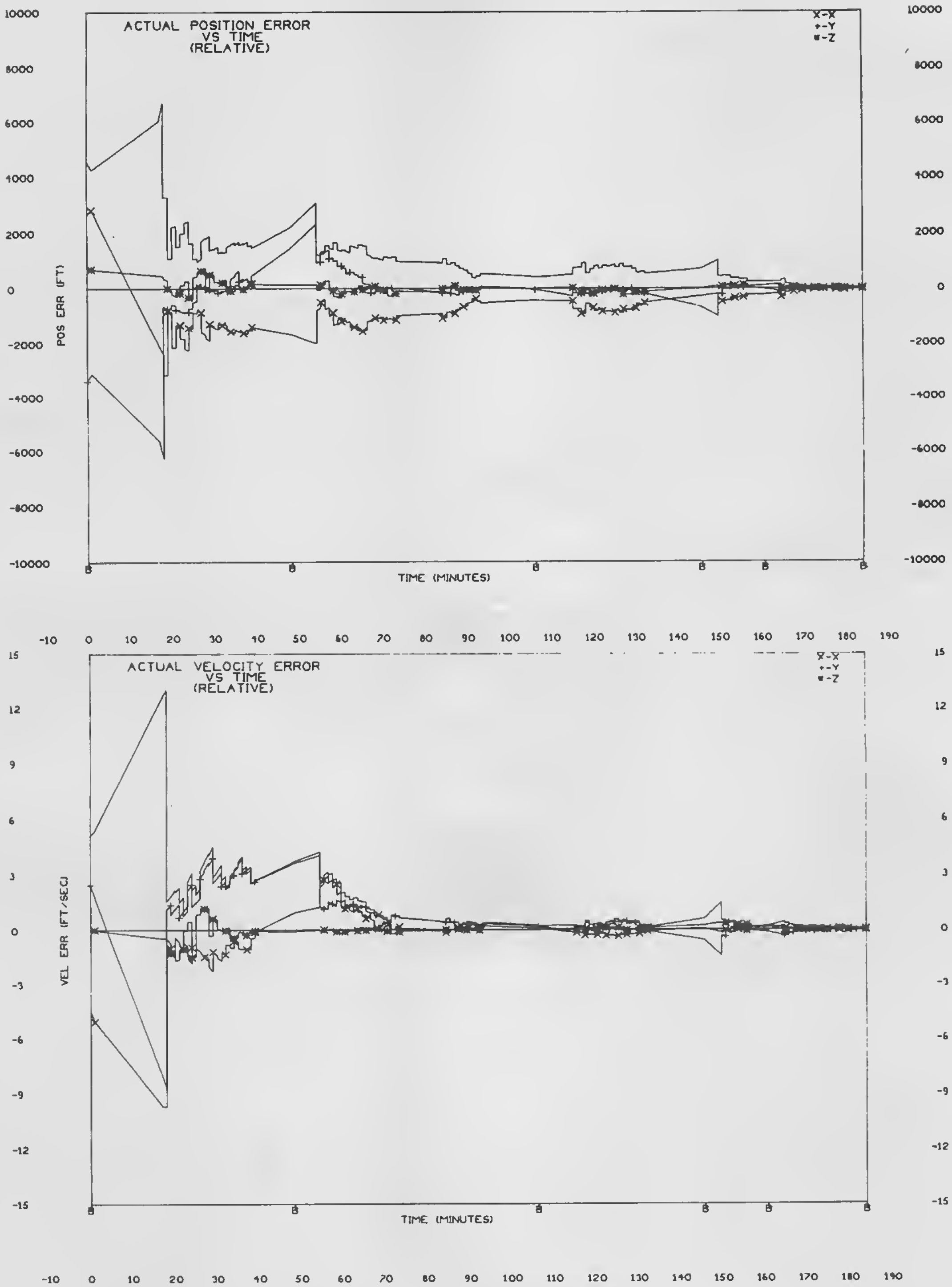


Fig. II-12 "Single Run" Relative Position and Velocity Errors (Off-Nominal Apollo 12 Rendezvous with Optimum Filter)

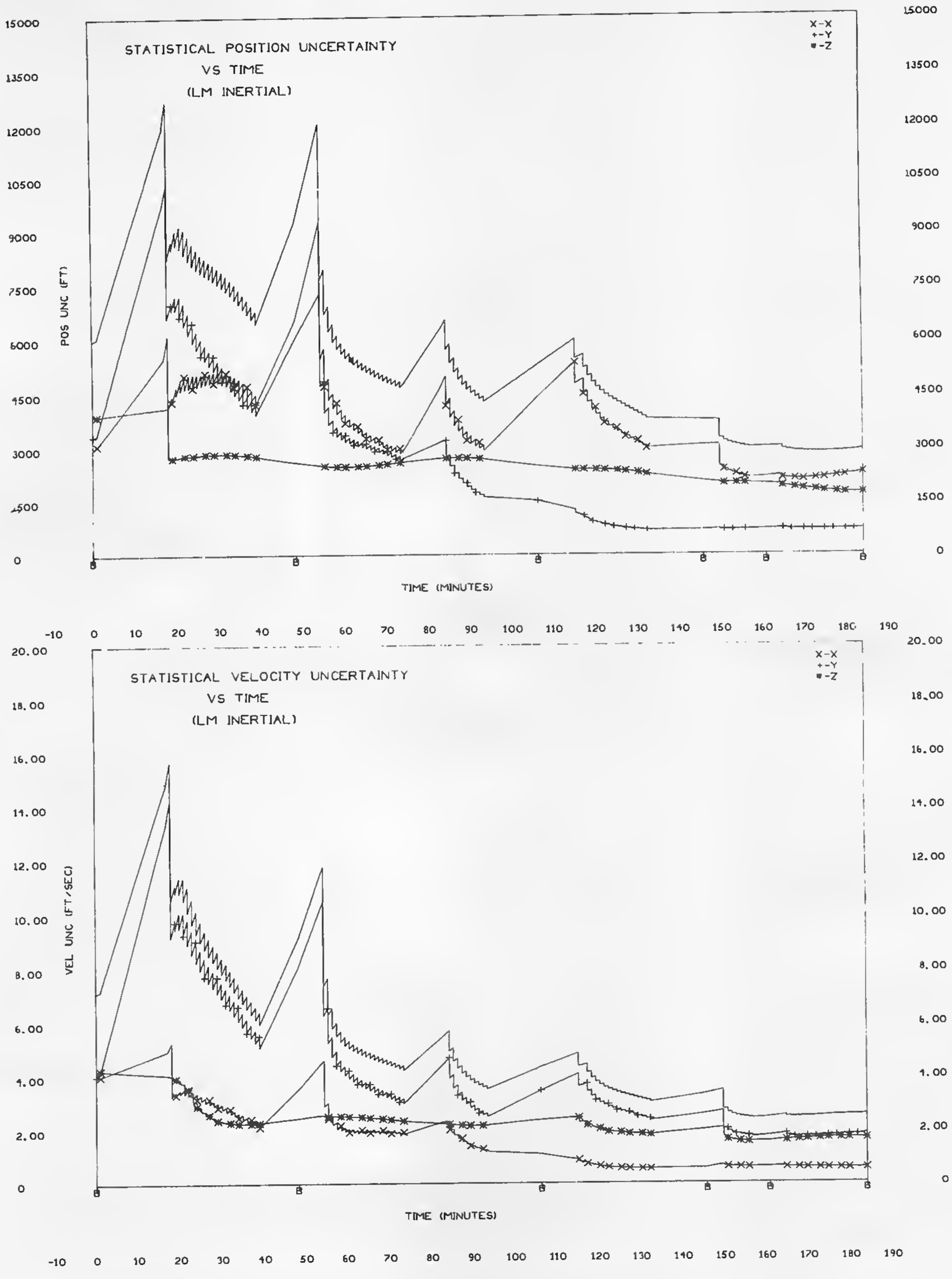


Fig. II-13 RMS Active Vehicle Inertial State Uncertainties (Off-Nominal Apollo 12 Rendezvous with Optimum Filter)

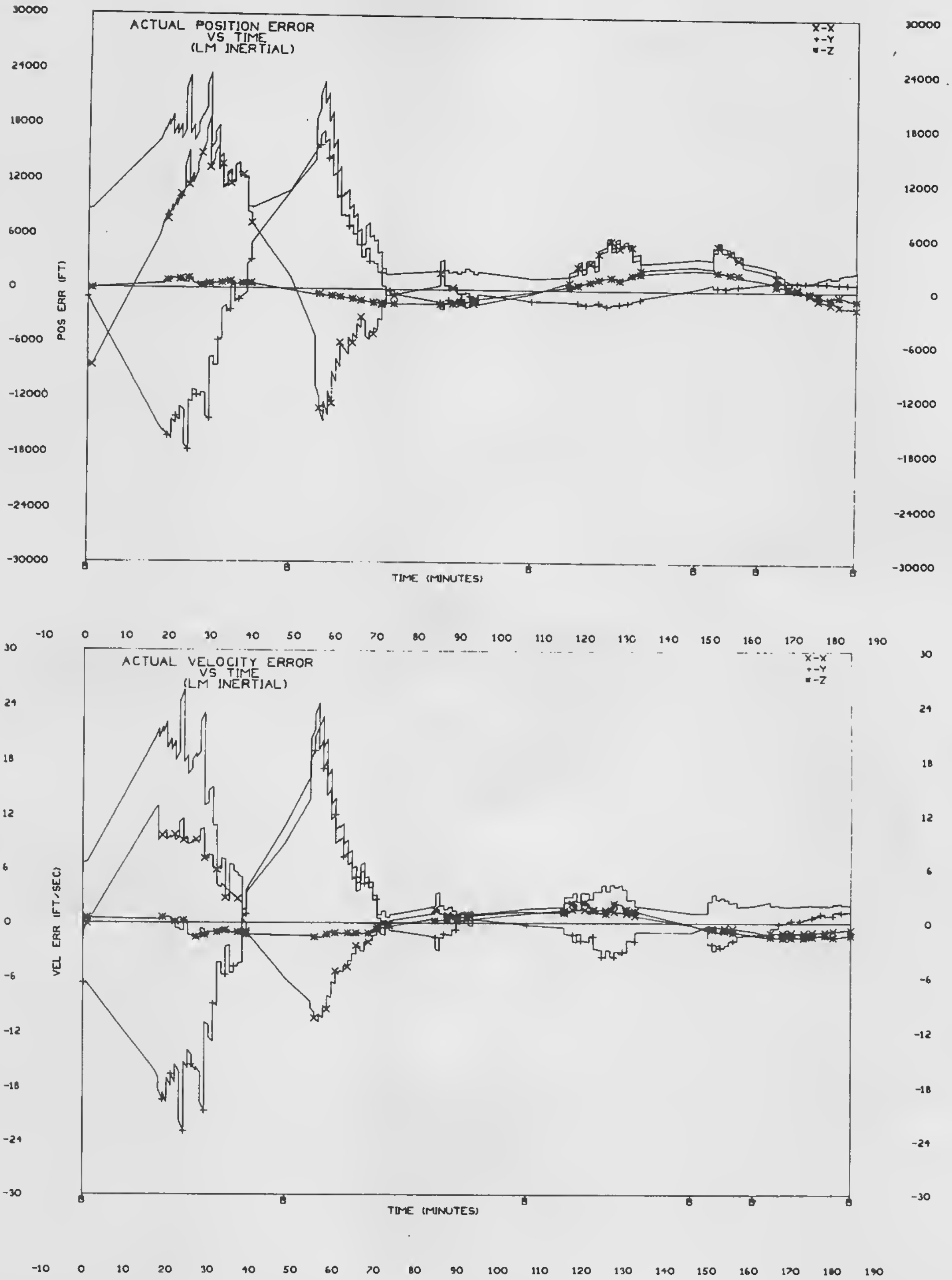


Fig. II-14 "Single Run" Active Vehicle Inertial State Errors (Off-Nominal Apollo 12 Rendezvous with Optimum Filter)

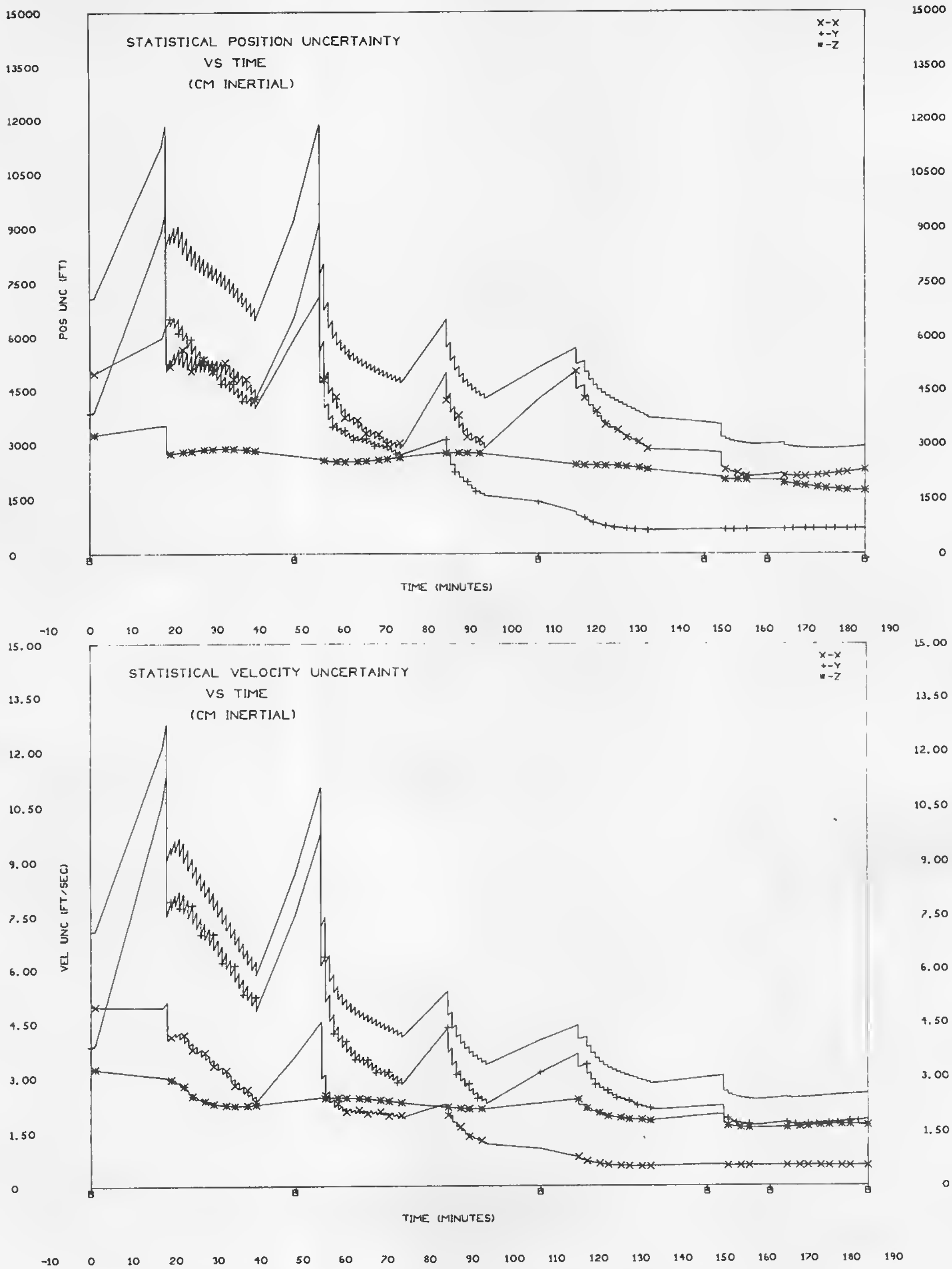


Fig. II-15 RMS Target Vehicle Inertial State Uncertainties (Off-Nominal Apollo 12 Rendezvous with Optimum Filter)

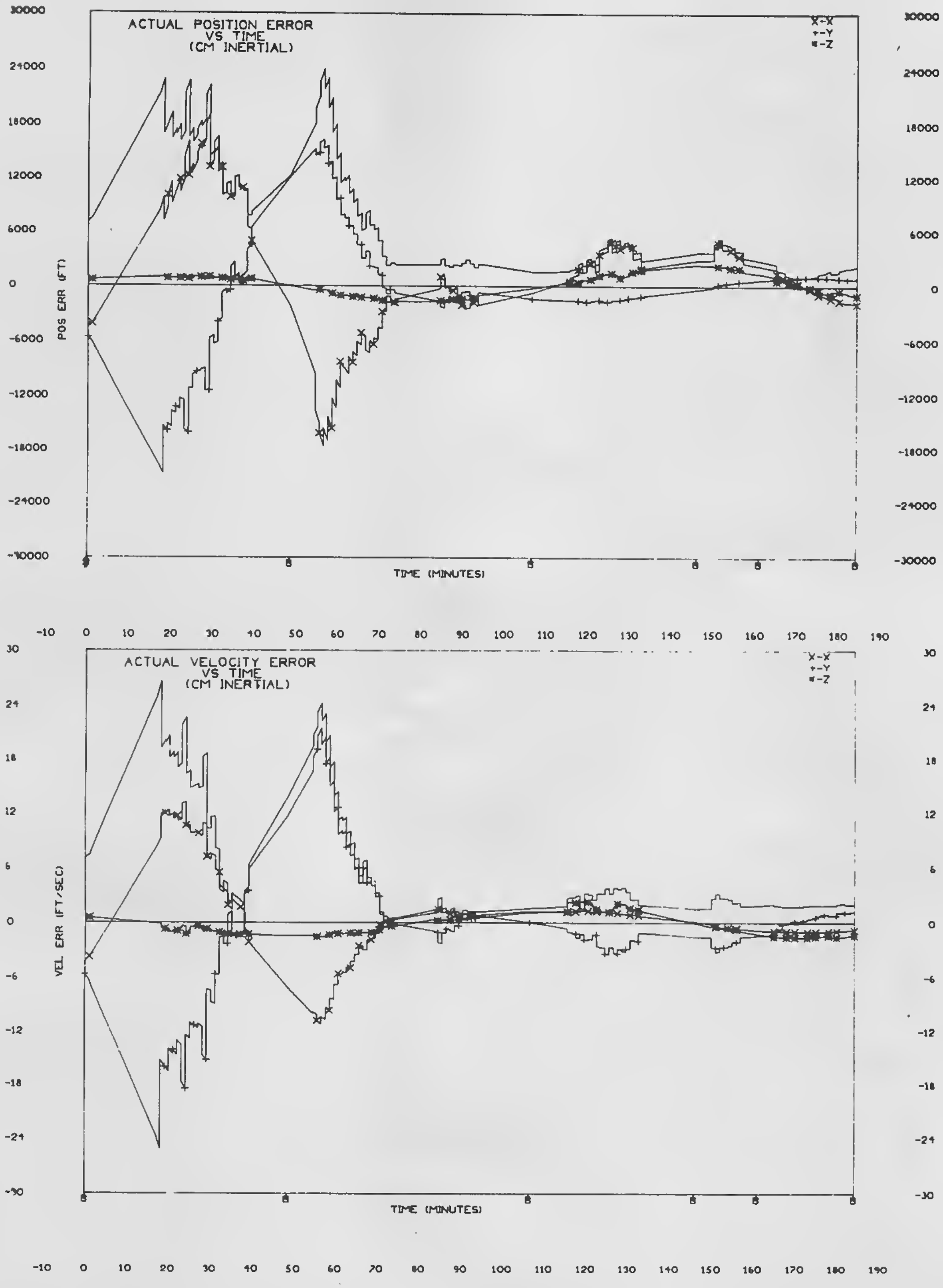


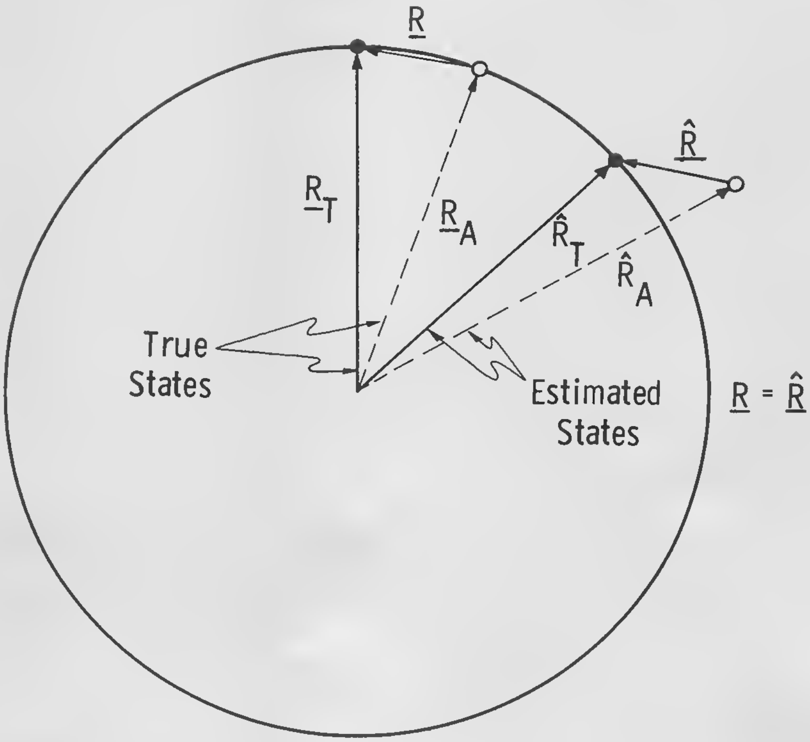
Fig. II-16 "Single Run" Target Vehicle Inertial State Uncertainties (Off-Nominal Apollo 12 Rendezvous with Optimum Filter)

Maintaining small errors in estimation of each vehicle's inertial state in a rendezvous mission (where it is generally assumed that control of relative state estimation errors is sufficient to solve the rendezvous problem) is important for two reasons. The first reason, discussed previously, is to assure a small buildup of estimation errors in the navigated relative state during extrapolation periods. A large buildup has two deleterious effects on performance: (a) relative state errors are maintained at a higher level during navigation periods, (b) since operational considerations (e.g. attitude maneuver, maneuver solution comparisons, maneuver targeting) require navigation to be terminated an appreciable time prior to initiation of a rendezvous maneuver, a large relative state error buildup during this extrapolation period causes large maneuver targeting errors since the accuracy of the ΔV solution depends on the relative state errors at maneuver initiation time.

The second reason also involves rendezvous maneuver targeting accuracy, but is fundamentally independent of relative state estimation errors. Even if the maneuver could be initiated immediately following a navigation period or if the relative state errors were identically zero, large errors in knowledge of each vehicle's inertial state cause large errors in rendezvous maneuver solutions. This effect is illustrated in Fig. II-17. A large downrange error (central angle) in the inertial state of the target vehicle is assumed, but the estimated relative state vector is identical to the true relative state vector in inertial space. (This results in an altitude and downrange error in the knowledge of the active vehicle inertial state.) It is quite evident in a situation like the one depicted in Fig. II-17 that a rendezvous maneuver solution computed from the estimated states will differ appreciably from that required with utilization of the true vehicle states.

As a result of the improved rendezvous navigation with the optimum filter, the accuracy of the onboard targeted maneuvers (the objective of rendezvous navigation) is significantly improved as shown in Fig. II-18.

As discussed in the development of the optimum filter, the mechanization of this filter onboard requires certain a priori statistical data in order to initialize the filter. Among these are two complete covariance matrices which theoretically should represent precisely the expected mean squared estimation errors and all correlations of estimation errors for the active and target vehicle states. (For the above simulations this was in fact done, the actual error vectors being generated from the initial active and target vehicle covariance matrices utilized in the filter.) As a practical matter, this could prove extremely difficult. Sufficient simulations would have to be run before confidence could be placed in these matrices. Then, if the mission is altered significantly, these matrices could be worthless. In order to gain some insight into how critical it is to initialize the filter with precise covariance matrices, the above simulations were rerun utilizing simple diagonal covariance matrices as



$\underline{R}_T$ = Target position vector

$\underline{R}_A$ = Active vehicle position vector

$\underline{R}$ = Relative state vector = $\underline{R}_T - \underline{R}_A$

Fig. II-17 Effect of Inertial State Errors on Rendezvous Targeting

RENDEZVOUS NAVIGATION FILTER	RENDEZVOUS MANEUVER	MANEUVER UNCERTAINTY* (fps)	
		STATISTICAL ANALYSIS (1σ)	"SINGLE RUN" SIMULATION
APOLLO	CSI	5.5	11.8
	CDH	26.5	28.0
	TPI	35.3	44.5
	MCC	5.1	15.7
OPTIMUM	CSI	0.82	0.03
	CDH	0.25	0.024
	TPI	0.85	1.1
	MCC	0.46	0.75

* Maneuver uncertainty is defined as the magnitude of $(\Delta\hat{\underline{V}} - \Delta\underline{V}_{True})$,

where: $\Delta\hat{\underline{V}}$ = estimated maneuver solution

$\Delta\underline{V}_{True}$ = maneuver solution required if true states were known exactly.

Fig. II-18 Rendezvous Maneuver Uncertainties initial filter loads. These were as follows:

E_A: Position Diagonals = 20,000, 20,000, 20,000 (ft)
Velocity Diagonals = 20, 20, 20 (fps)

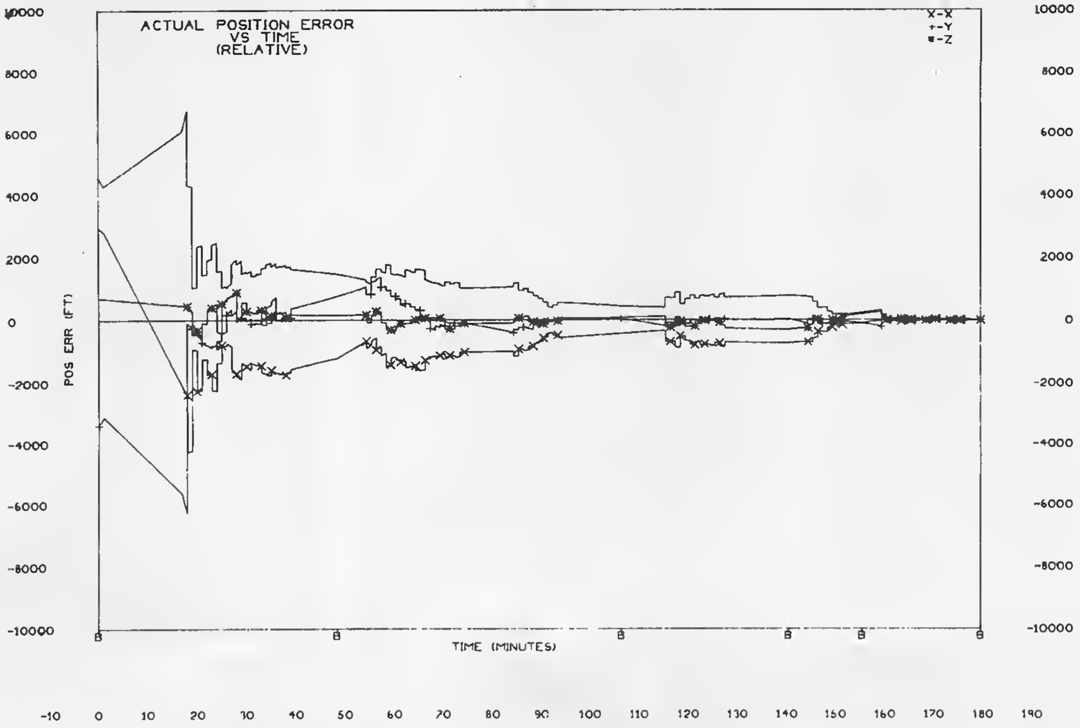
E_T: Position Diagonals = 5,000, 5,000, 5,000 (ft)
Velocity Diagonals = 5, 5, 5 (fps)

Identical "single run" state vector errors and sensor errors were those used in the previous simulation. The resulting "single run" relative state errors and inertial state errors are plotted in Figs. II-19 - II-21. It can be seen that the initial covariance matrix loads are indeed not critical. The values selected for E_A and E_T were purposely selected quite different from each other and were not meant to even correspond to the "single run" error vector magnitudes. The values chosen for E_T do, however, correspond somewhat in magnitude to the target vehicle "single run" state error (Fig. II-4). To see if this was significant, the values for E_A and E_T were just reversed and the simulation rerun. The appropriate results are shown in Figs. II-22 - II-24. These results are quite similar to the others, further emphasizing the insensitivity of the filter to initial covariance matrix loads. The major effect of using non-optimum initial covariance matrices is to insert an initial transient which degrades performance during early periods of rendezvous navigation. Eventually, the filter covariance matrices become a function solely of the sensor measurement errors and trajectory geometry, independent of the initial values. These few simulations are by no means conclusive, but they do indicate that selection of initial covariance matrices should present no particularly difficult problem.

E. Application of Optimum Rendezvous Filter for Orbit Navigation

The performance of the optimum filter in the above rendezvous simulations reveals a result of appreciable significance -- this filter is extremely effective in reducing inertial state estimation errors. The performance with respect to relative state errors is excellent as expected, but it might be expected that the best the filter could do with inertial errors is to contain them within some reasonable limits or at best obtain a slight reduction, when one considers that measurements are made of the relative state only. The key point, however, is that this relative state is measured with respect to known inertial coordinates provided by the stable platform. Thus, subject to navigation errors and platform misalignment errors, the direction of the relative state vector is defined in inertial space.

It is this definition of the relative state vector in inertial space which allows the filter to solve each vehicle inertial state. If the relative state trajectory of two vehicles orbiting in a central force field is completely specified inertially, the separate vehicle inertial states are uniquely specified. Specification of the relative state at a



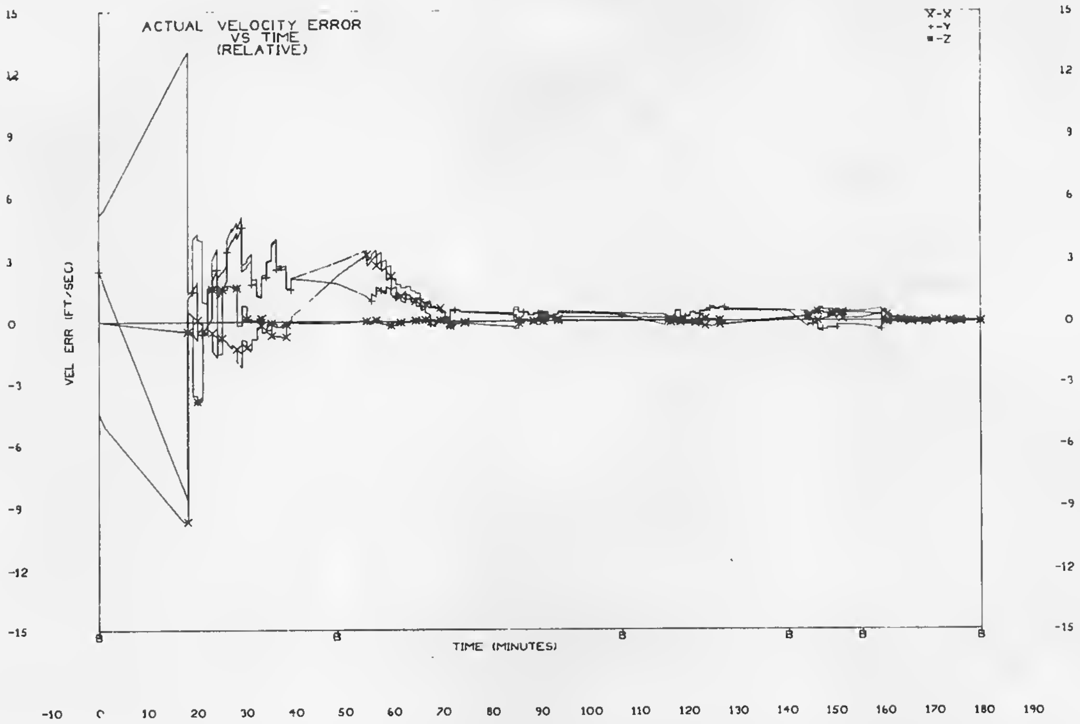


Fig. II-19 "Single Run" Relative Position and Velocity Errors (Off-Nominal Apollo 12 Rendezvous - Optimum Filter Using Diagonal Cov Matrices: E_A = 20,000 ft^2, 20 fps^2, E_T = 5,000 ft^2 5 fps^2)

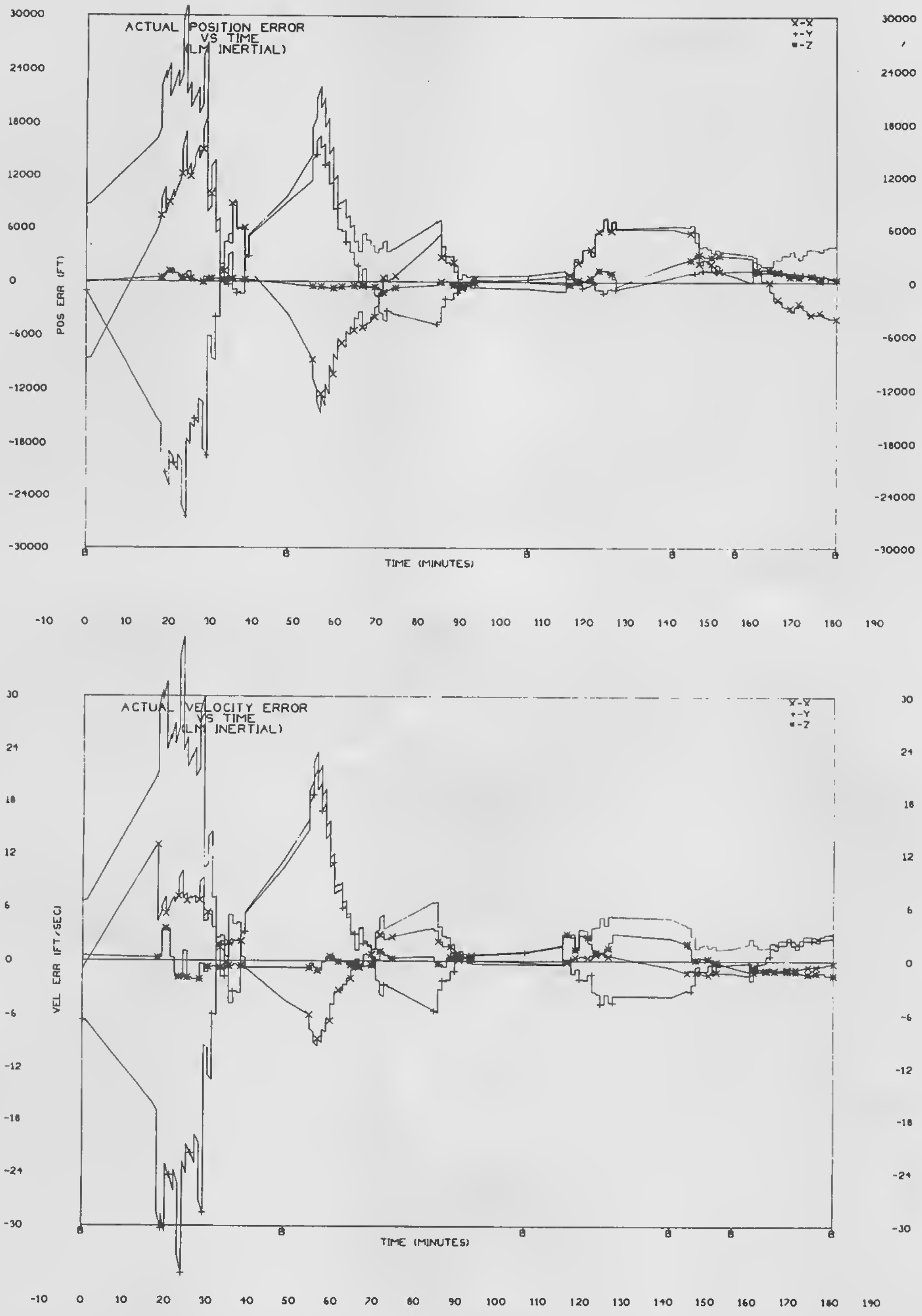
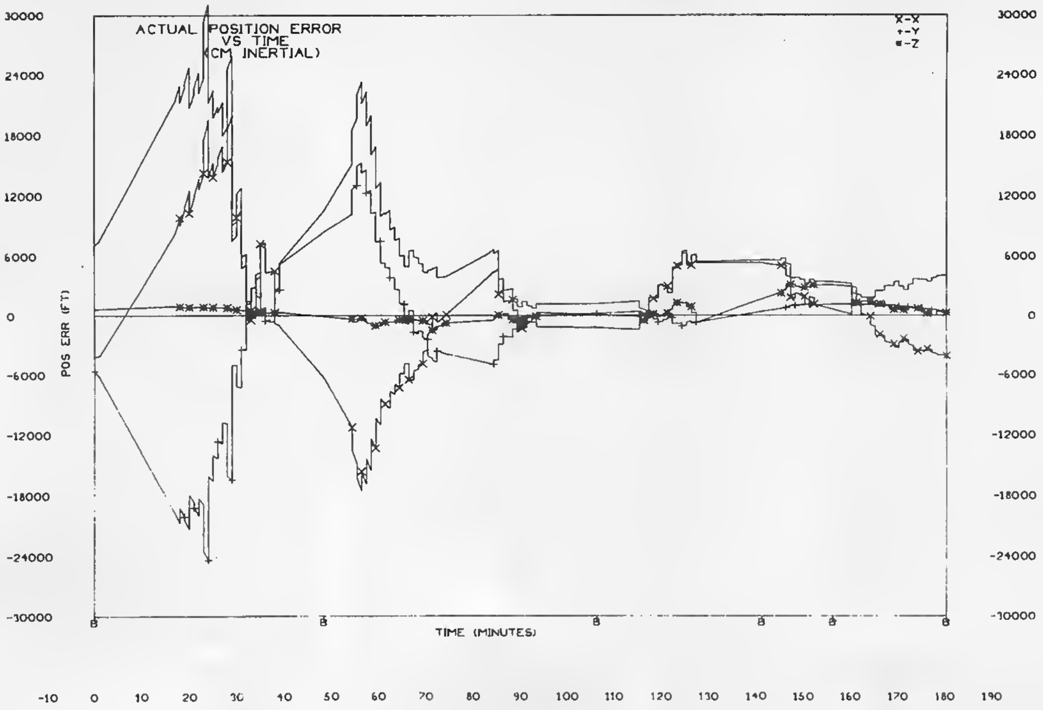


Fig. II-20 "Single Run" Active Vehicle Inertial State Errors (Off-Nominal Apollo 12 Rendezvous - Optimum Filter Using Diagonal Cov Matrices: E_A = 20,000 ft^2, 20 fps^2, E_T = 5,000 ft^2, 5 fps^2)



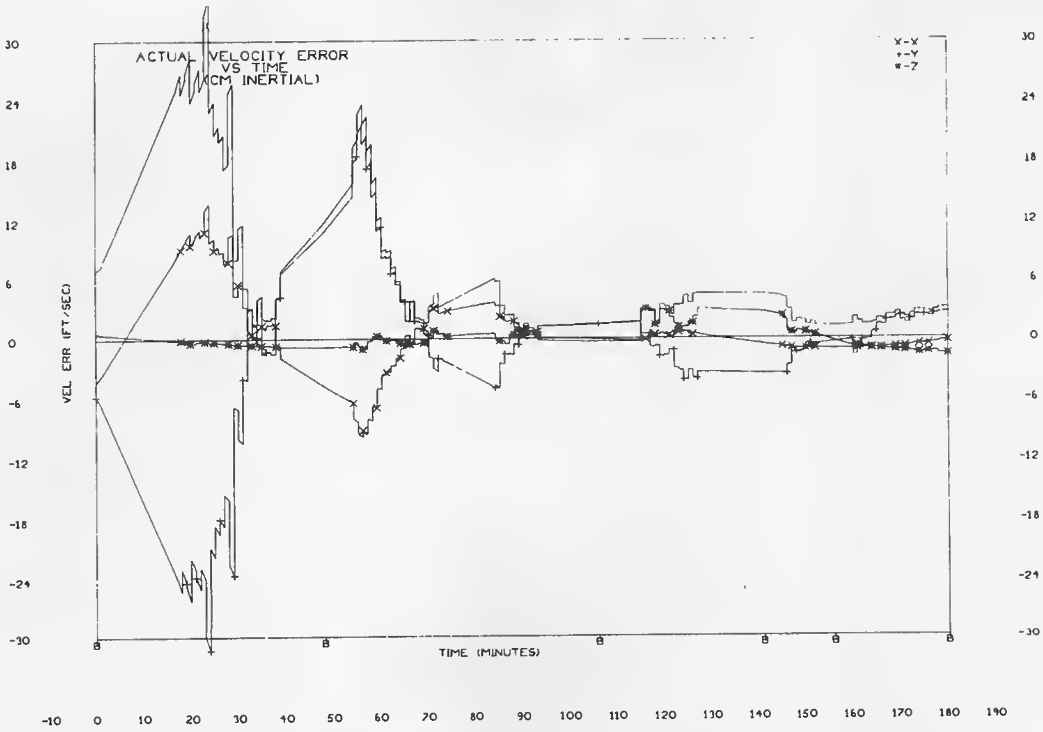
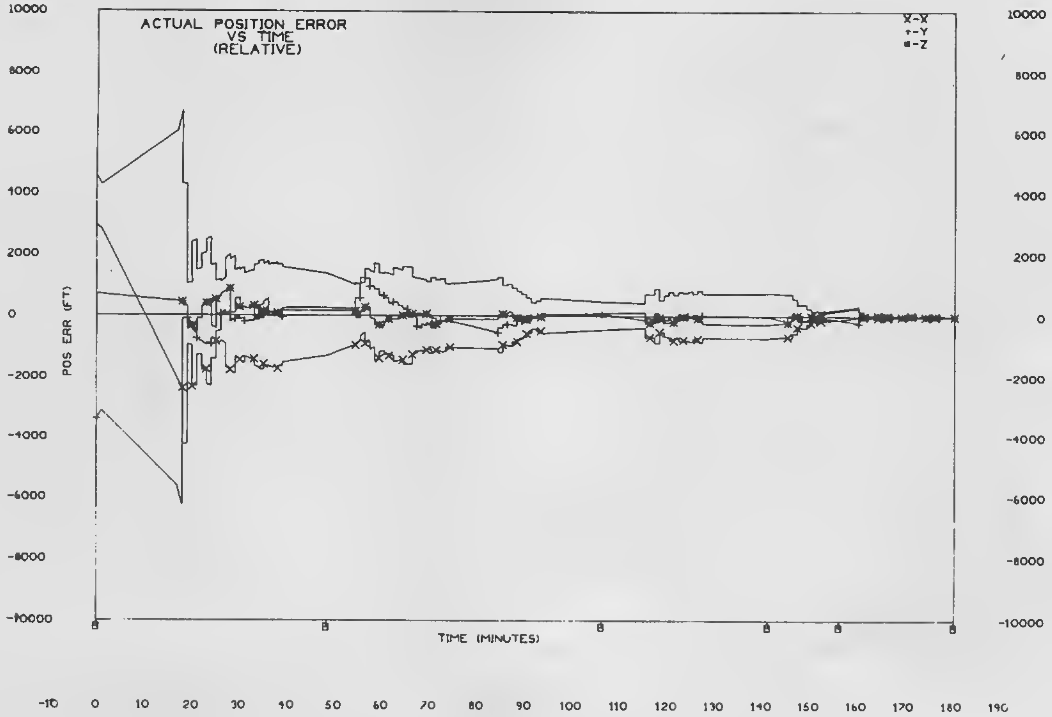


Fig. II-21 "Single Run" Target Vehicle Inertial State Errors (Off-Nominal Apollo 12 Rendezvous - Optimum Filter Using Diagonal Cov Matrices: E_A = 20,000 ft^2, 20 fps^2, E_T = 5,000 ft^2, 5 fps^2)



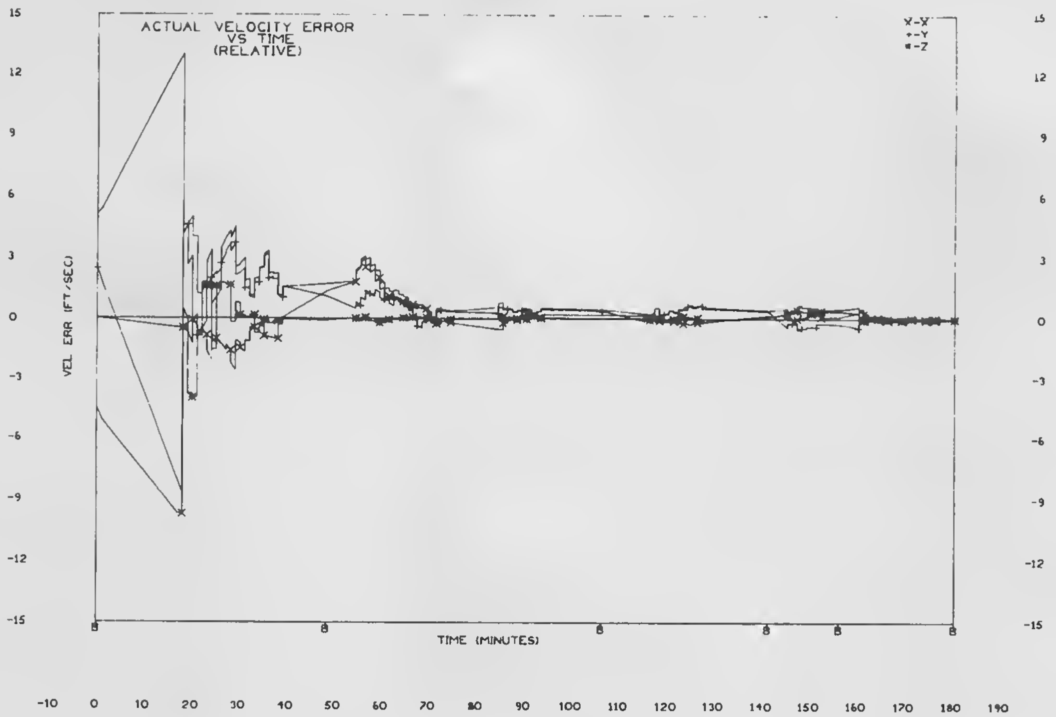


Fig. II-22 "Single Run" Relative Position and Velocity Errors (Off-Nominal Apollo 12 Rendezvous - Optimum Filter Using Diagonal Cov Matrices: E_A = 5,000 ft^2, 5 fps^2, E_T = 20,000 ft^2, 20 fps^2)

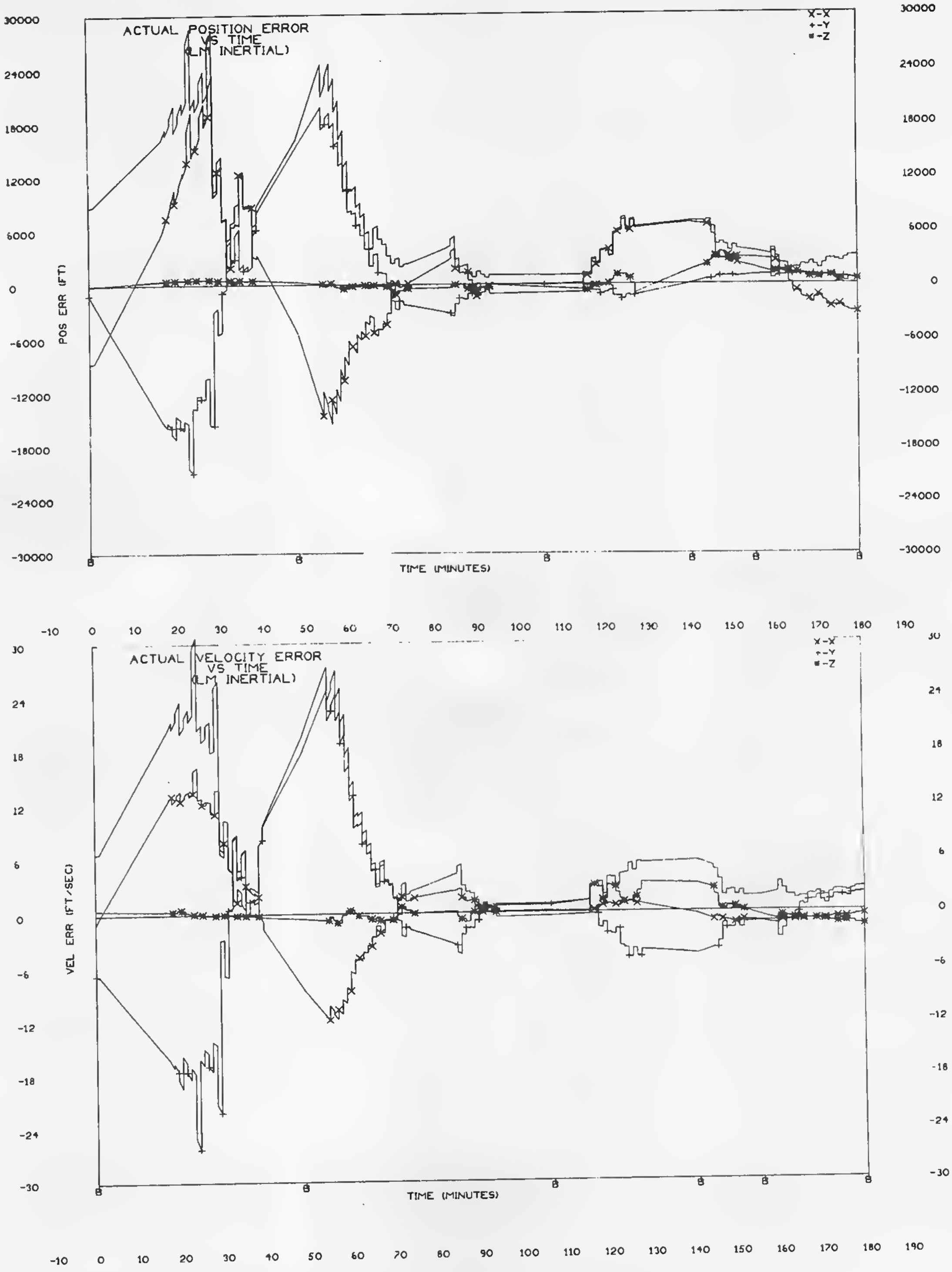


Fig. II-23 "Single Run" Active Vehicle Inertial State Errors (Off-Nominal Apollo 12 Rendezvous - Optimum Filter Using Diagonal Cov Matrices: E_A = 5,000 ft^2, 5 fps^2, E_T = 20,000 ft^2, 20 fps^2)

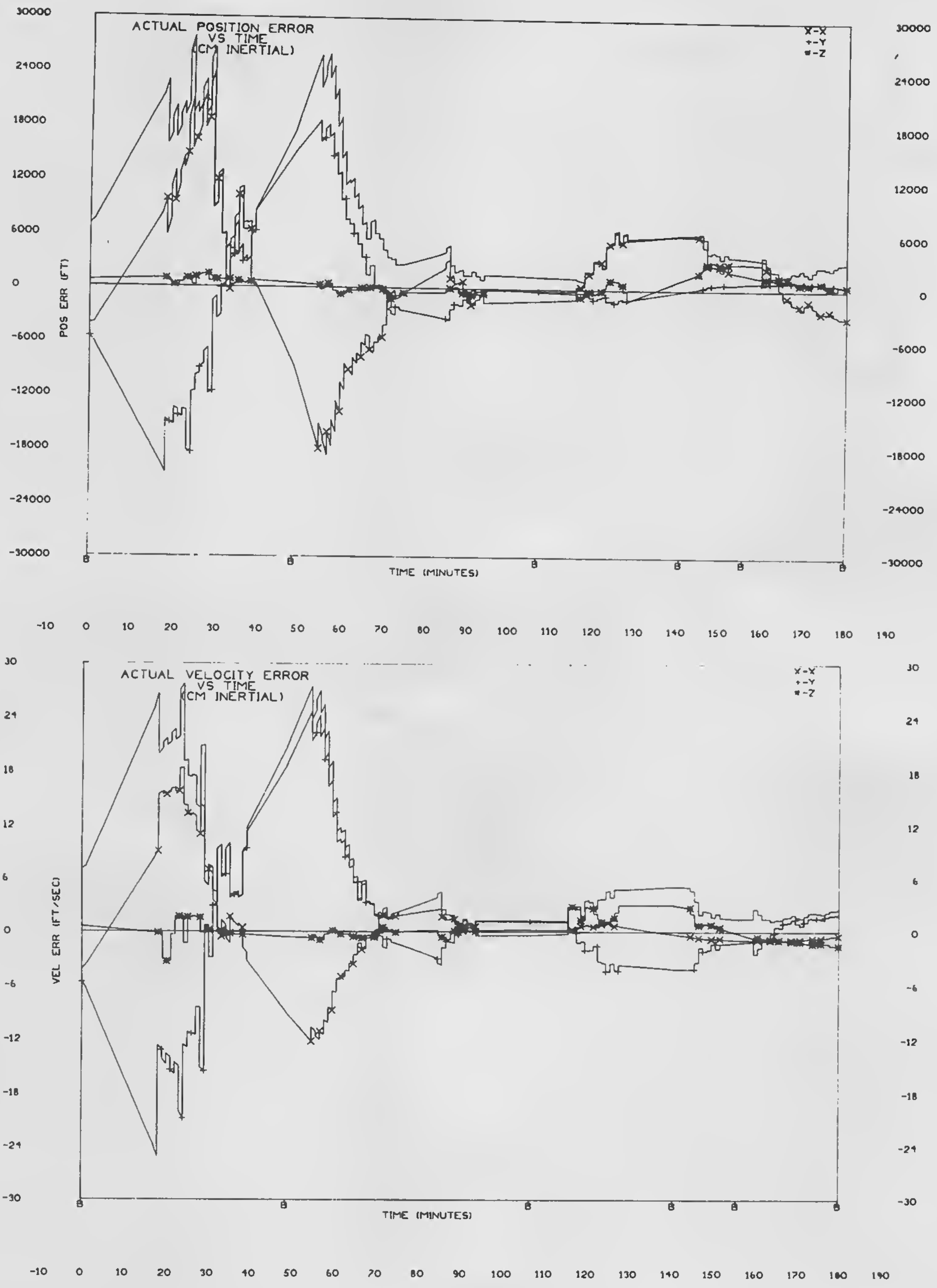


Fig. II-24 "Single Run" Target Vehicle Inertial State Errors (Off-Nominal Apollo 12 Rendezvous - Optimum Filter Using Diagonal Cov Matrices: E_A = 5,000 ft^2, 5 fps^2, E_T = 20,000 ft^2, 20 fps^2)

single point in the trajectory will not uniquely define the two inertial states. It is necessary to take advantage of the fact that the relative state extrapolates differently for different inertial states, in order to determine the particular inertial states which produce the measured relative trajectory. Depending on the relative geometry, this process may be fairly rapid or may require long periods of rendezvous navigation before differences in the measured and estimated relative state are large enough to indicate errors in inertial state estimates. Thus, one would expect to resolve inertial state errors of two vehicles at a large relative range more rapidly than if the two vehicles were orbiting at close range. In the former case, inertial state estimation errors cause the relative state estimation errors to increase more rapidly through extrapolation than in the latter case, thus providing the information needed by the filter to resolve inertial errors from relative measurements.

This effect is graphically illustrated in Figs. II-13 - II-16. The navigation updates of the inertial states are extremely effective at the start of the rendezvous mission and gradually diminish until they are virtually non-existant in the terminal phase. The relative range is approximately 200 nm when navigation begins (time = 18 min in the figures) and is down to 15 nm at a time of 160 min.

The ability of the optimum rendezvous navigation filter to resolve individual state estimation errors by utilizing relative measurements of another orbiting vehicle could have a significant impact in the field of orbit navigation. The rendezvous phase of a mission may also be utilized to satisfy orbit navigation requirements. This could eliminate the need to make some sort of inertial measurements (horizon sensing, e.g.) or the requirement of using ground tracking stations in order to achieve an accurate onboard inertial state estimate. Even in a mission which does not include a rendezvous, one might envision the primary vehicle ejecting a small satellite (possibly an inflatable radar reflecting sphere) and performing rendezvous navigation on the satellite to determine its own orbit before performing some orbital maneuvers. The satellite's orbit could be designed to produce a relative trajectory whose geometry is most sensitive to inertial state errors in order to rapidly generate a relative error and thus maximize the filter's ability to resolve the primary vehicle inertial state error.

E.1 Simulation of Orbit Navigation Scheme

This unique approach to orbit navigation was tested in a simulation in which no particular attempt was made to optimally select the relative trajectory between the primary vehicle and its satellite. The pertinent initial conditions for this simulation and the resulting relative motion plot are shown in Fig. II-25. (Sensor errors are the same as given in Fig. II-4.) The primary vehicle was placed in a 57 nm circular lunar orbit and a satellite was ejected at T = 0 with a horizontal velocity increment of 30 fps and a radial velocity increment of 30 fps. These values were more or less arbitrarily chosen to achieve a reasonable relative range after a short time. Rendezvous navigation utilizing the optimum filter was initiated 50 minutes after ejection.

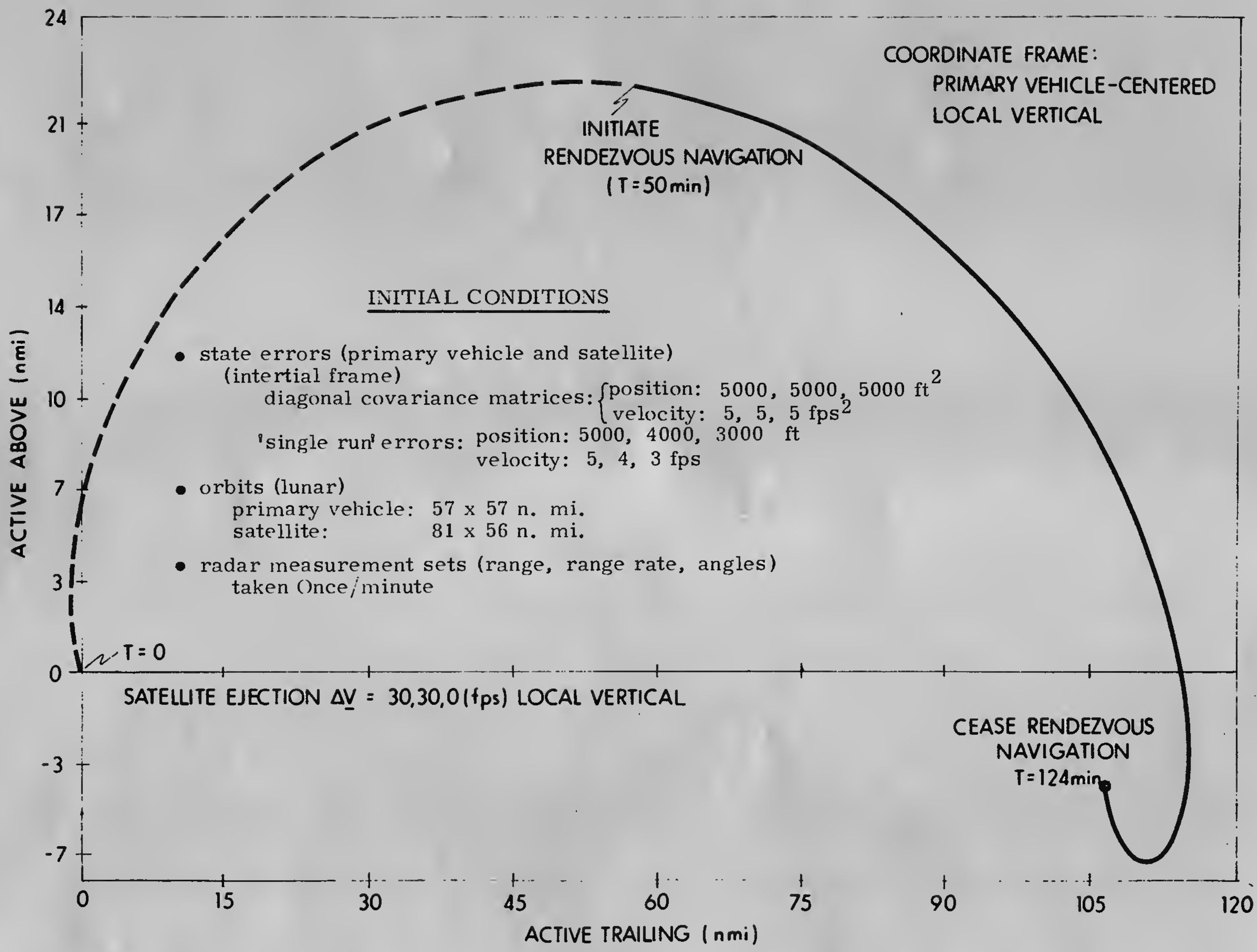


Fig. II-25 Relative Motion Plot - Orbit Navigation Simulation

Relative state estimation errors are shown in Figs. II-26 and II-27. It can be seen that the relative errors, which of course are zero before ejection, grow considerably before rendezvous navigation is initiated. This is predominantly due to the initial inertial state errors given to the primary vehicle and its satellite. (These errors were again specifically chosen since they grow inordinately large without the benefit of navigation updates.) A component of relative error also results from the uncertainty in the ejection maneuver (1σ Apollo accelerometer performance was assumed, resulting in a 0.7 fps ΔV uncertainty for the ejection maneuver.)

Of primary importance, of course, is the performance of the rendezvous navigation in resolving the inertial state estimation errors of the primary vehicle. These errors are shown in Figs. II-28 and II-29. (Errors in estimation of the satellite inertial state are also shown in Figs. II-30 and II-31.) Both statistical and "single run" results are excellent. At the start of rendezvous navigation (T = 50 min), the statistical and "single run" inertial errors were 61,000 ft, 53 fps and 116,000 ft, 96 fps respectively. After about one half orbit of navigation (T = 110 nm), these errors are reduced to 3,800 ft, 3 fps and 2,100 ft, 0.3 fps respectively. It should be noted again that these results are somewhat optimistic since no stable platform misalignment errors have been assumed, and these errors directly affect inertial errors. Of course, a similar problem exists with standard orbit navigation schemes such as landmark tracking. Should the stable platform errors prove significant, the filter performance can be improved by including the platform misalignment errors in the estimated state, and estimating them along with the vehicle inertial state.

This simulation, though not conclusive, certainty illustrates the feasibility of using this rendezvous navigation system to perform orbit navigation.

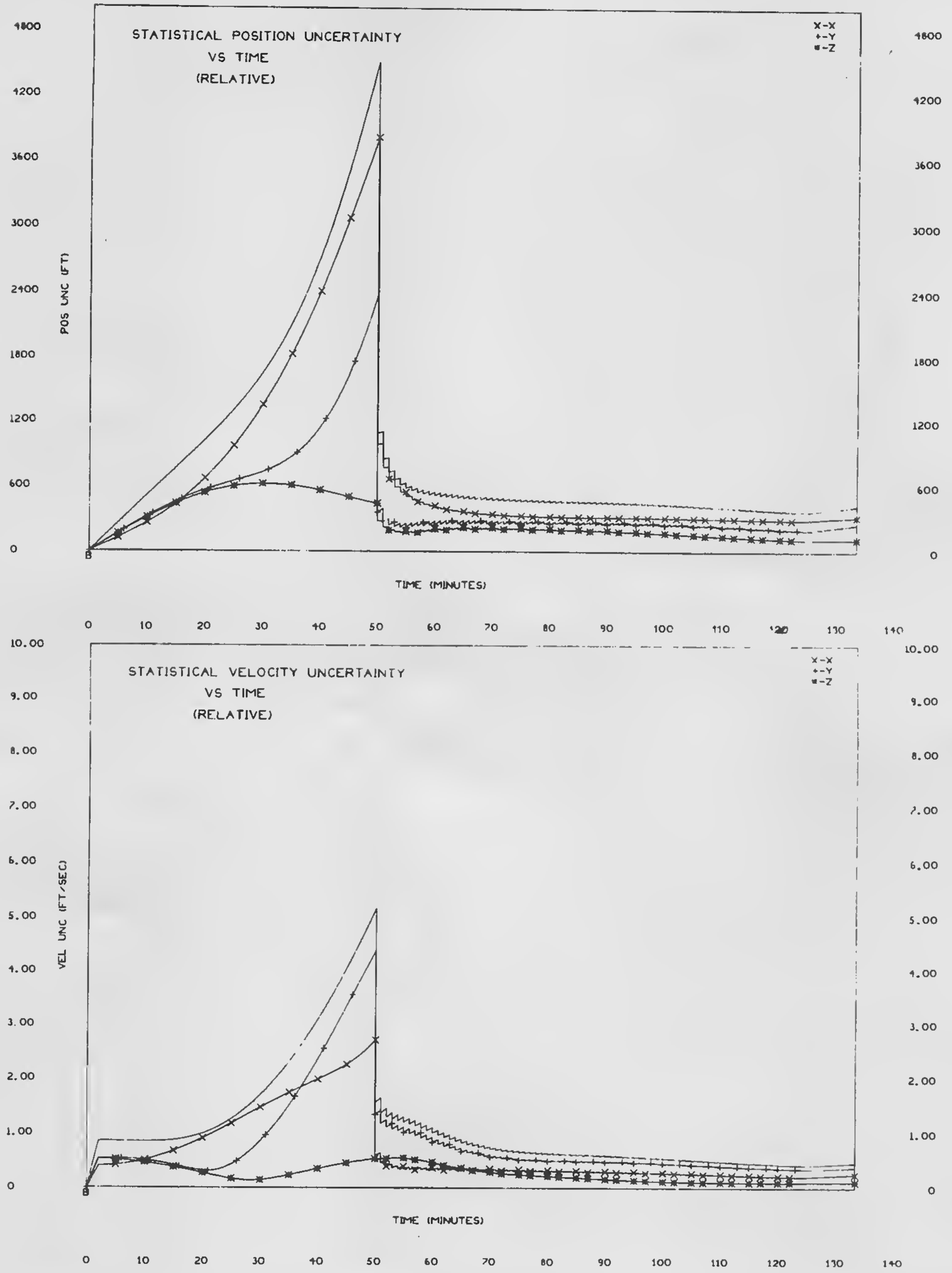


Fig. II-26 RMS Relative Position and Velocity Uncertainties (Orbit Navigation Simulation)

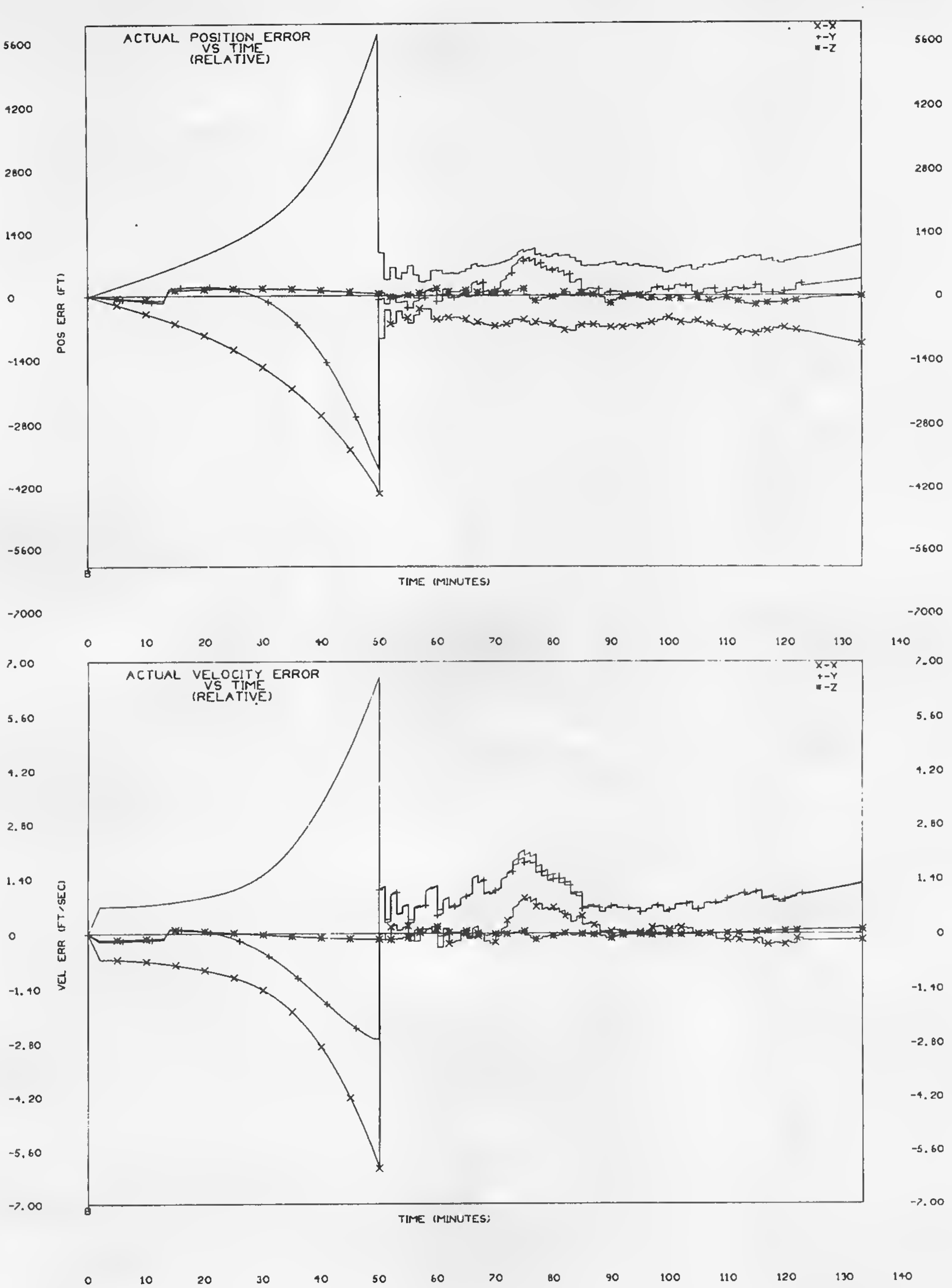


Fig. II-27 "Single Run" Relative Position and Velocity Errors (Orbit Navigation Simulation)

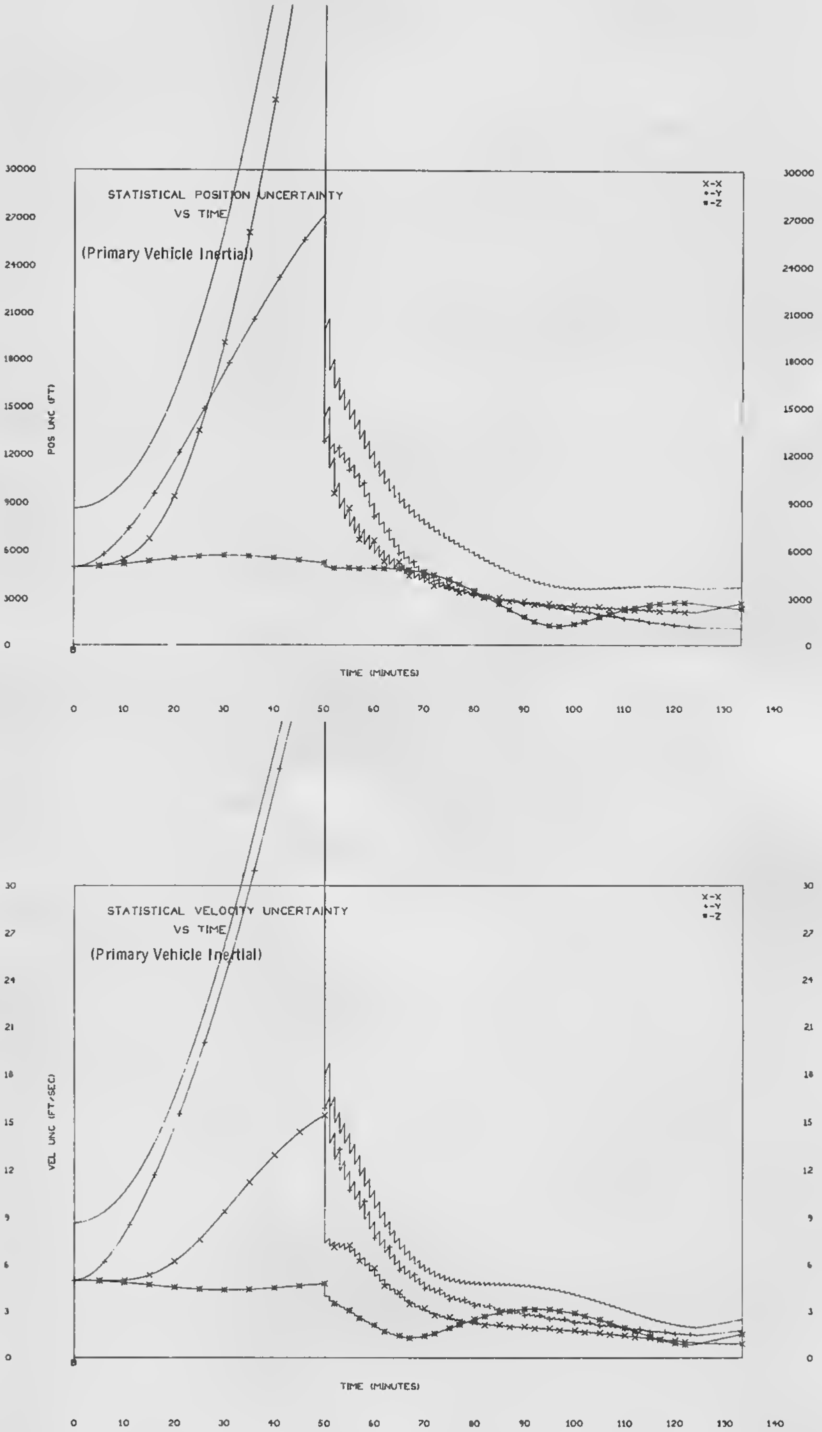


Fig. II-28 RMS Primary Vehicle Position and Velocity Uncertainties (Orbit Navigation Simulation)

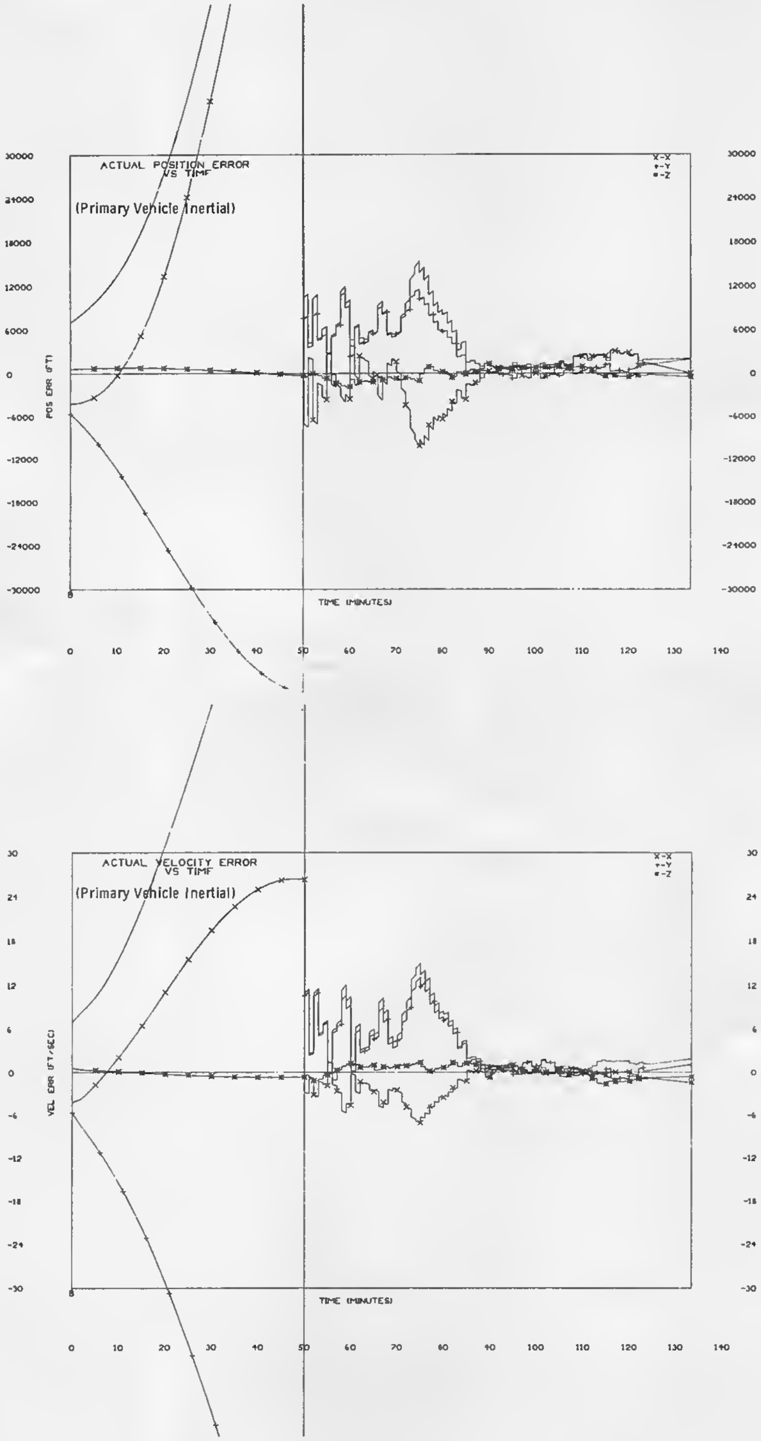


Fig. II-29 "Single Run" Primary Vehicle Position and Velocity Errors (Orbit Navigation Simulation)

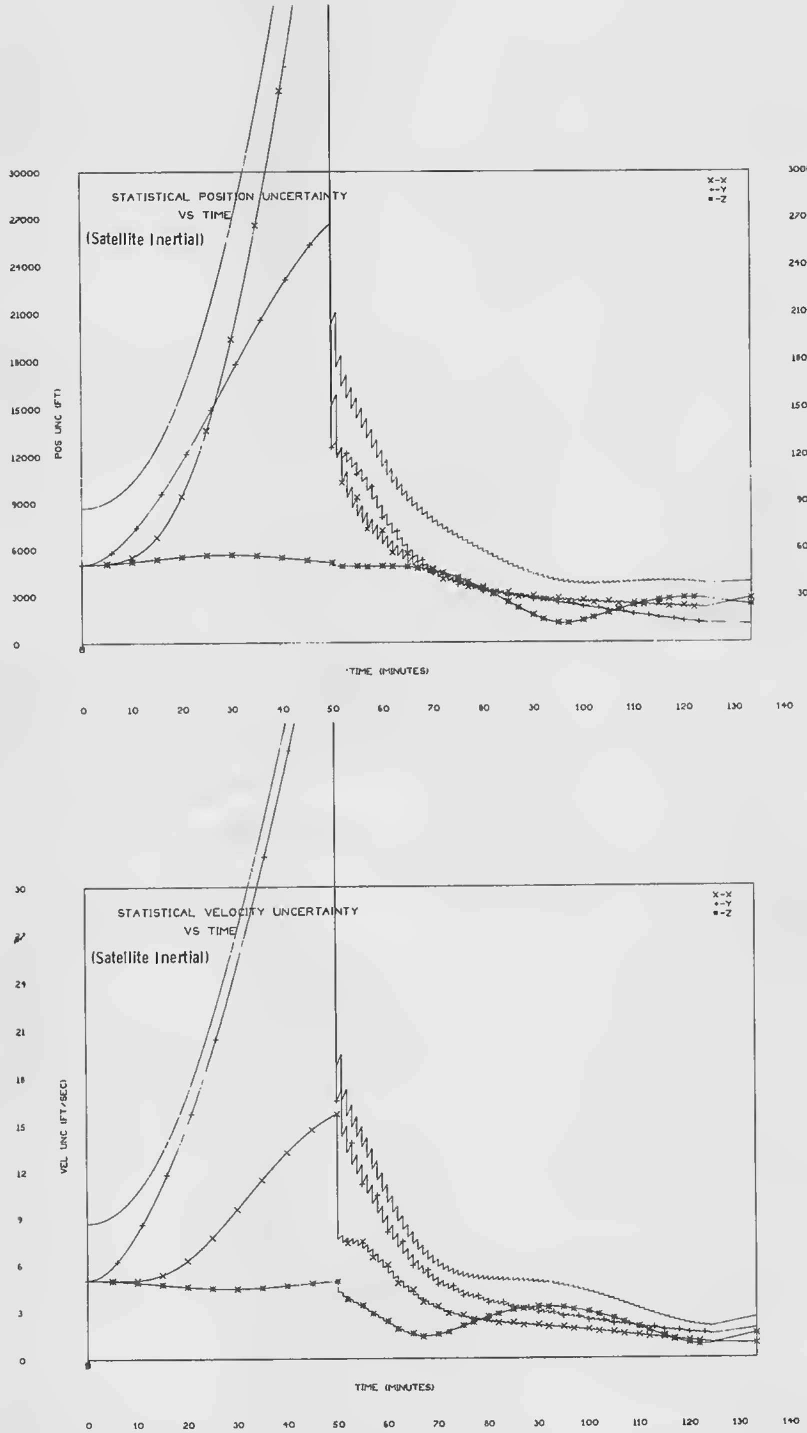


Fig. II-30 RMS Satellite Position and Velocity Uncertainties (Orbit Navigation Simulation)

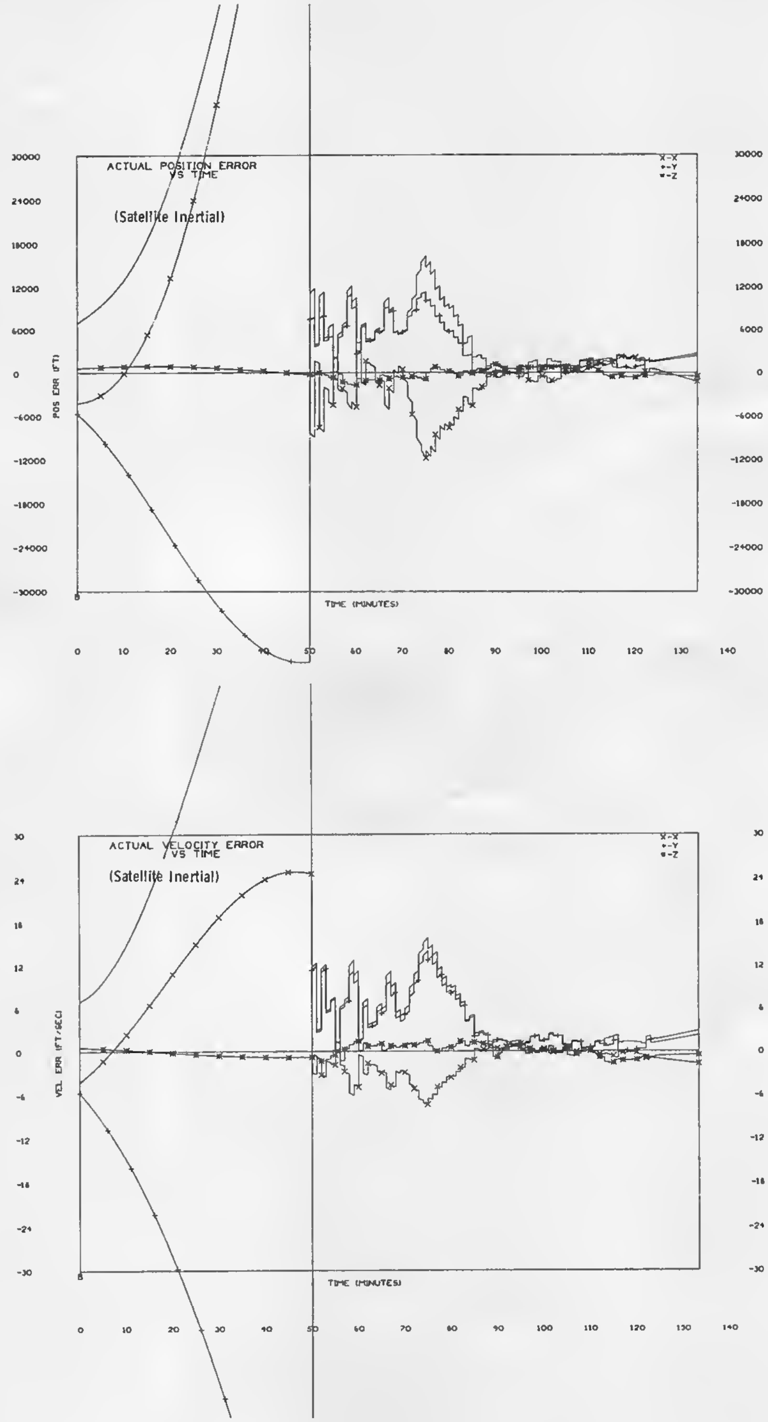


Fig. II-31 "Single Run" Satellite Position and Velocity Errors (Orbit Navigation Simulation)

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APPENDIX A

LIST OF NOTATIONS

AB:	variables defined by uppercase letter(s) denotes $n \times m$ matrix
$\underline{a}$:	denotes $n \times 1$ vector
$\bar{a}$:	denotes expected value of a
A':	denotes value of variable after given integration time (extrapolation period)
$\hat{a}$:	estimated value of variable a
$\tilde{a}$:	measured value of variable a
$A^T, \underline{a}^T$:	denotes transpose of matrix, vector

APPENDIX B

OPTIMUM WEIGHTING VECTORS

1. Optimum Weighting Vector $\underline{\omega}_R$ for Relative State Formulation of the Rendezvous Filter

The covariance matrix of relative state estimation errors following a measurement update is as follows (Eq. 7 of text):

$$E_R \quad E'_R - \underline{\omega}_R \left[\underline{b}_R^T E'_R + \underline{b}_\beta^T \, GR\beta'^T \right] - \left[E'_R \underline{b}_R + GR\beta' \, b_\beta \right] \underline{\omega}_R^T$$

$$+ \underline{\omega}_R \left[\underline{b}_R^T E'_R \underline{b}_R + \underline{b}_\beta^T E'_\beta \underline{b}_\beta + 2 \, \underline{b}_R^T GR\beta' \underline{b}_\beta \overline{+ \alpha^2} \right] \underline{\omega}_R^T \tag{B-1}$$

The mean squared errors in the estimate of the relative position and velocity deviations are the traces of the position and velocity submatrices of E_R, defined as:

$$E_R = \left[\begin{array}{c|c} E_R \text{ (pos)} & \overline{\underline{e}_{RP} \underline{e}_{RV}^T} \\ \hline \overline{\underline{e}_{RV} \underline{e}_{RP}^T} & E_R \text{ (vel)} \end{array} \right]$$

If $\underline{\omega}_R$ is written as

$$\begin{bmatrix} \underline{\omega}_R \text{ (pos)} \\ \underline{\omega}_R \text{ (vel)} \end{bmatrix},$$

and this is substituted in Eq. B-1 and expanded it is seen that E_R (pos) is a function of only $\underline{\omega}_R$ (pos) and E_R (vel) a function of only $\underline{\omega}_R$ (vel).

Because of this fact, the optimum $\underline{\omega}_R$ (pos) and $\underline{\omega}_R$ (vel) will each be optimum for the estimates of relative position and relative velocity respectively, and the mean squared error in the state estimate, $e_R^2(\underline{\omega}_R)$, is the trace of the matrix $E_R(\underline{\omega}_R)$.

Using the technique of variational calculus to determine the optimum $\underline{\omega}_R$, let $\underline{\omega}_R = \underline{\omega}_{R\,opt} + \delta\underline{\omega}_R$ and substitute into Eq. B-1. This becomes, after collecting terms and neglecting products of $\delta\underline{\omega}_R$:

$$\begin{aligned} \overline{e_R^2(\underline{\omega}_R)} = t_r(E_R) = t_r \Big\{ & E_R' - \underline{\omega}_{R\,opt}\left[\underline{b}_R^T E_R' + \underline{b}_\beta^T GR\beta'^T\right] \\ & - \left[E_R'\underline{b}_R + GR\beta'\underline{b}_\beta\right]\underline{\omega}_{R\,opt}^T \\ & + \underline{\omega}_{R\,opt}\left[\underline{b}_R^T E_R'\underline{b}_R + \underline{b}_\beta^T E_\beta'\underline{b}_\beta + 2\underline{b}_R^T GR\beta'\underline{b}_\beta + \overline{\alpha^2}\right]\underline{\omega}_{R\,opt}^T\Big\} \\ & + t_r\Big\{\Big[-2\left(E_R'\underline{b}_R + GR\beta'\underline{b}_\beta\right) \\ & + 2\underline{\omega}_{R\,opt}\left(\underline{b}_R^T E_R'\underline{b}_R + \underline{b}_\beta^T E_\beta'\underline{b}_\beta + 2\underline{b}_R^T GR\beta'\underline{b}_\beta + \overline{\alpha^2}\right)\Big]\delta\underline{\omega}_R^T\Big\} \end{aligned} \quad \text{(B-2)}$$

The trace of the term involving $\delta\underline{\omega}_R$ has to be zero for all variations ($\delta\underline{\omega}_R$) in the optimum weighting vector. This requires that

$$\boxed{\underline{\omega}_{R\,opt} = \left(E_R'\underline{b}_r + GR\beta'\underline{b}_\beta\right)\Big/\left(\underline{b}_R^T E_R'\underline{b}_R + \underline{b}_\beta^T E_\beta'\underline{b}_\beta + 2\underline{b}_R^T GR\beta'\underline{b}_\beta + \overline{\alpha^2}\right)} \quad \text{(B-3)}$$

Hence the form of the optimum weighting vector $\underline{\omega}_R$ has been determined.

2. Optimum Weighting Vector $\underline{\omega}_R$ for the Relative State Formulation of the Rendezvous Filter with Both Vehicle States Updated

From Eq. 15, the general form of the relative state covariance matrix is given by the following expression:

$$E_R = E_R' - \underline{\omega}_R \left[-\underline{b}_A^T E_A' - \underline{b}_T^T G'^T - \underline{b}_\beta^T GA\beta'^T + \underline{b}_A^T G' + \underline{b}_T^T E_T' \right.$$

$$\left. + \underline{b}_\beta^T GT\beta'^T \right]$$

$$- \left[G'^T \underline{b}_A + E_T' \underline{b}_T + GT\beta' \underline{b}_\beta - E_A' \underline{b}_A - G' \underline{b}_T - GA\beta' \underline{b}_\beta \right] \underline{\omega}_R^T \quad \text{(B-4)}$$

$$+ \underline{\omega}_R \left[\underline{b}_A^T E_A' \underline{b}_A + 2\underline{b}_A^T G' \underline{b}_T + \underline{b}_T^T E_T' \underline{b}_T \right.$$

$$\left. + 2\underline{b}_A^T GA\beta' \underline{b}_\beta + 2\underline{b}_T^T GT\beta' \underline{b}_\beta + \underline{b}_\beta^T E_\beta' \underline{b}_\beta + \overline{\alpha^2} \right] \underline{\omega}_R^T$$

It can again be shown that the position and velocity submatrices of E_R from Eq. B-4 are functions of only $\underline{\omega}_R$ (pos) and $\underline{\omega}_R$ (vel) respectively where:

$$\underline{\omega}_R = \begin{pmatrix} \underline{\omega}_R \text{ (pos)} \\ \underline{\omega}_R \text{ (vel)} \end{pmatrix}$$

Thus the mean squared relative state error estimate is the trace of $E_R(\underline{\omega}_R)$.

Using variational calculus and proceeding as before, we let $\underline{\omega}_R = \underline{\omega}_{R\,opt} + \delta \underline{\omega}_R$ in Eq. B-4. Collecting terms involving only $\delta \underline{\omega}_R$ we have the following

$$\delta \overline{e_R^2 (\underline{\omega}_R)} = 2 t_r \left\{ - \left[- E_A' \underline{b}_A + E_T' \underline{b}_T - G' \underline{b}_T + G'^T \underline{b}_A \right. \right.$$

$$\left. + GT\beta' \underline{b}_\beta - GA\beta' \underline{b}_\beta \right] \delta \underline{\omega}_R^T \quad \text{(B-5)}$$

$$+ \underline{\omega}_{R\,opt} \left[\underline{b}_A^T E_A' \underline{b}_A + \underline{b}_T^T E_T' \underline{b}_T + \underline{b}_\beta^T E_\beta' \underline{b}_\beta + 2\underline{b}_A^T G' \underline{b}_T \right.$$

$$\left. \left. + 2\underline{b}_A^T GA\beta' \underline{b}_\beta + 2\underline{b}_T^T GT\beta' \underline{b}_\beta + \overline{\alpha^2} \right] \delta \underline{\omega}_R^T \right\}$$

Since $\delta \overline{e_R^2 (\underline{\omega}_R)}$ must vanish for all deviations $\delta \underline{\omega}_R$ in the optimum filter, the coefficient of $\delta \underline{\omega}_R$ must be zero. Thus, the optimum $\underline{\omega}_R$ has the following form by setting the coefficient to zero in Eq. B-5.

$$\underline{\omega}_R = - \left[E_A' \underline{b}_a - E_T' b_T - G'^T \underline{b}_A + G' \underline{b}_T + GA\beta' \underline{b}_\beta - GT\beta' \underline{b}_\beta \right] / A$$

$$A = \underline{b}_A^T E_A' \underline{b}_A + \underline{b}^T E_T' \underline{b}_T + \underline{b}_\beta^T E_\beta' \underline{b}_\beta + 2\underline{b}_A^T G' \underline{b}_T + 2\underline{b}_A^T GA\beta' \underline{b}_\beta \quad \text{(B-6)}$$

$$+ 2\underline{b}_T^T GT\beta' \underline{b}_\beta + \overline{\alpha^2}$$

3. Optimum Weighting Vectors ($\underline{\omega}_A$, $\underline{\omega}_T$) for Inertial Updates on Both the Active and Target Vehicles

From Eqs. 13a and 13b for the active and target vehicle inertial state estimation error after measurement update the corresponding covariance matrices of these errors are respectively:

$$E_A = \left(I - \underline{\omega}_A \underline{b}_A^T\right) E_A' \left(I - \underline{b}_A \underline{\omega}_A^T\right) + \underline{\omega}_A \left(b^T E_T' b_T + \underline{b}_\beta^T E_\beta' \underline{b}_\beta\right.$$

$$\left. + \underline{b}_T^T GT\beta' \underline{b}_\beta + \underline{b}_\beta^T GT\beta'^T \underline{b}_T + \overline{\alpha^2}\right) \underline{\omega}_A^T$$

$$- \left(I - \underline{\omega}_A \underline{b}_A^T\right) \left(G' \underline{b}_T \underline{\omega}_A^T + GA\beta' \underline{b}_\beta \underline{\omega}_A^T\right) \quad \text{(B-7)}$$

$$- \left(\underline{\omega}_A b_T^T G'^T + \underline{\omega}_A \underline{b}_\beta^T GA\beta'^T\right) \left(I - \underline{b}_A \underline{\omega}_A^T\right)$$

$$E_T = \left(I - \underline{\omega}_T \underline{b}_T^T\right) E_T' \left(I - \underline{b}_T \underline{\omega}_T^T\right) + \underline{\omega}_T \left(\underline{b}_A^T E_A' b_A + \underline{b}_\beta^T E_\beta^T \underline{b}_\beta\right.$$

$$\left. + \underline{b}_A^T GA\beta' \underline{b}_\beta + \underline{b}_\beta^T GA\beta'^T b_A + \overline{\alpha^2}\right) \omega_T^T \qquad \text{(B-8)}$$

$$- \left(I - \underline{\omega}_T \underline{b}_T^T\right) \left(G'^T \underline{b}_A^T \underline{\omega}_T^T + GT\beta' b_\beta \underline{\omega}_T^T\right)$$

$$- \left(\underline{\omega}_T \underline{b}_A^T G' + \underline{\omega}_T \underline{b}_\beta^T GT\beta'^T\right) \left(I - \underline{b}_T \underline{\omega}_T^T\right)$$

Letting

$$\underline{\omega}_A = \begin{bmatrix} \underline{\omega}_A \text{ (pos)} \\ \underline{\omega}_A \text{ (vel)} \end{bmatrix}$$

and

$$\underline{\omega}_T = \begin{bmatrix} \underline{\omega}_T \text{ (pos)} \\ \underline{\omega}_T \text{ (vel)} \end{bmatrix}$$

it can be shown that for both E_A and E_T, the position and velocity submatrices are functions of only $\underline{\omega}$ (pos) and $\underline{\omega}$ (vel) respectively; i.e.,

$$E_A \text{ (pos)} = f\left(\underline{\omega}_A \text{ (pos)}\right), \qquad E_T \text{ (pos)} = f\left(\underline{\omega}_T \text{ (pos)}\right)$$

$$E_A \text{ (vel)} = f\left(\underline{\omega}_A \text{ (vel)}\right), \qquad E_T \text{ (vel)} = f\left(\underline{\omega}_T \text{ (vel)}\right)$$

where

$$E_A = \left[\begin{array}{c|c} E_A \text{ (pos)} & \underline{e}_{AP} \underline{e}_{AV}^T \\ \hline \underline{e}_{AV} \underline{e}_{AP}^T & E_A \text{ (vel)} \end{array}\right] \text{ and similarly for } E_T \text{ submatrices.}$$

Letting $\underline{\omega}_A$ and $\underline{\omega}_T$ take on variations $\delta \underline{\omega}_A$ and $\delta \underline{\omega}_T$ respectively, substituting in Eqs. B-7 and B-8 and collecting terms involving $\delta \underline{\omega}_A$ and $\delta \underline{\omega}_T$ (again neglecting second order terms) we have:

$$\delta \overline{e_A^2 (\underline{\omega}_A)} = 2\, t_r \Big\{ - \left(E_A' \underline{b}_A + G' \underline{b}_T + GA\beta' \underline{b}_\beta \right) \delta \underline{\omega}_A^T$$

$$+ \underline{\omega}_{A\,opt} \Big(\underline{b}_A^T E_A' \underline{b}_A + \underline{b}_T^T E_T' \underline{b}_\beta + \underline{b}_\beta^T E_\beta' \underline{b}_\beta \qquad \text{(B-9)}$$

$$+ 2\underline{b}_A^T G' \underline{b}_T + 2\underline{b}_A^T GA\beta' \underline{b}_\beta + 2\underline{b}_T^T GT\beta' \underline{b}_\beta + \overline{\alpha^2} \Big) \delta \underline{\omega}_A^T \Big\}$$

$$\delta \overline{e_T^2 (\underline{\omega}_T)} = 2\, t_r \Big\{ - \left(E_T' \underline{b}_T + G' \underline{b}_A + GT\beta' \underline{b}_\beta \right) \delta \underline{\omega}_A^T$$

$$+ \underline{\omega}_{T\,opt} \Big(\underline{b}_A^T E_A' \underline{b}_A + \underline{b}_T^T E_T' \underline{b}_T + \underline{b}_\beta^T E_\beta' \underline{b}_\beta \qquad \text{(B-10)}$$

$$+ 2\underline{b}_A^T G' \underline{b}_T + 2\underline{b}_A^T GA\beta' \underline{b}_\beta + 2\underline{b}_T^T GT\beta' \underline{b}_\beta + \overline{\alpha^2} \Big) \delta \underline{\omega}_T^T \Big\}$$

$\delta \overline{e_A^2 (\underline{\omega}_A)}$ and $\delta \overline{e_T^2 (\underline{\omega}_T)}$ must vanish for all variations in $\underline{\omega}_{A\,opt}$ and $\underline{\omega}_{T\,opt}$ respectively. Hence, the coefficient of $\delta \underline{\omega}_A^T$ and $\delta \underline{\omega}_T^T$ must be equal to zero. This yields the following expressions for $\underline{\omega}_{A\,opt}$ and $\underline{\omega}_{T\,opt}$ for the optimum inertial updates of the active and target vehicles.

$$\underline{\omega}_{A\,opt} = \left(E_A' b_A + G' \underline{b}_T + GA\beta' \underline{b}_\beta \right) / A$$

$$\underline{\omega}_{T\,opt} = \left(E_T' \underline{b}_T + G'^T \underline{b}_A + GT\beta' \underline{b}_\beta \right) / A \qquad \text{(B-11)}$$

$$A = \underline{b}_A^T E_A' \underline{b}_A + \underline{b}_T^T E_T \underline{b}_T + \underline{b}_\beta^T E_\beta \underline{b}_\beta + 2\underline{b}_A^T G' \underline{b}_T$$

$$+ 2\underline{b}_A^T GA\beta' \underline{b}_\beta + 2\underline{b}_T^T GT\beta' \underline{b}_\beta + \overline{\alpha^2}$$

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