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Space Guidance Analysis Memo #11-64

TO: SGA Distribution
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SUBJECT: The Transition Matrix for a Circular Orbit

The transition matrix is a means of determining the propagation of small deviations from a reference orbit. Thus,

$$\begin{pmatrix} \delta \underline{r} \\ \\ \delta \underline{v} \end{pmatrix} = \Phi \begin{pmatrix} \delta \underline{r}_0 \\ \\ \delta \underline{v}_0 \end{pmatrix}$$

where $\delta \underline{r}$ and $\delta \underline{v}$ are the position and velocity deviations, $\delta \underline{r}_0$ and $\delta \underline{v}_0$ the initial deviations, and Φ is the 6×6 transition matrix.

The Φ matrix satisfies the matrix differential equation

$$\dot{\Phi} = \begin{pmatrix} O & I \\ \\ G & O \end{pmatrix} \Phi \tag{1}$$

where I and O are the 3×3 identity and zero matrices, respectively, and G is the gravity gradient matrix. The initial value of the Φ matrix is the 6×6 identity matrix.

Let the position and velocity vectors have components in an inertial axis system as follows:

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

Let the reference trajectory be a circular orbit in a central force field with initial conditions

$$\underline{r}_0 = \begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{v}_0 = \begin{pmatrix} 0 \\ v_0 \\ 0 \end{pmatrix}$$

Then the reference orbit is given by

$$\underline{r}_{REF} = r_0 \begin{pmatrix} \cos \omega t \\ \sin \omega t \\ 0 \end{pmatrix}$$

where ω is the orbital angular rate.

The gravity vector is given by

$$\underline{g} = - \frac{\mu}{r^3} \underline{r}$$

where μ is the gravitational constant and $\underline{r}$ is the position of the spacecraft and is given by

$$\underline{r} = \underline{r}_{REF} + \delta \underline{r}$$

the G matrix is then

$$G = -\omega^2 \begin{pmatrix} 1 - 3\cos^2 \omega t & -3 \sin \omega t \cos \omega t & 0 \\ -3 \sin \omega t \cos \omega t & 1 - 3 \sin^2 \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

where

$$\omega^2 = \frac{\mu}{r_0^3}$$

Since the propagation of errors in the xy plane is independent of the errors along the z axis, the Φ matrix may be written as follows:

$$\Phi = \begin{pmatrix} \phi_{xx} & \phi_{xy} & 0 & \phi_{x\dot{x}} & \phi_{x\dot{y}} & 0 \\ \phi_{yx} & \phi_{yy} & 0 & \phi_{y\dot{x}} & \phi_{y\dot{y}} & 0 \\ 0 & 0 & \phi_{zz} & 0 & 0 & \phi_{z\dot{z}} \\ \phi_{\dot{x}x} & \phi_{\dot{x}y} & 0 & \phi_{\dot{x}\dot{x}} & \phi_{\dot{x}\dot{y}} & 0 \\ \phi_{\dot{y}x} & \phi_{\dot{y}y} & 0 & \phi_{\dot{y}\dot{x}} & \phi_{\dot{y}\dot{y}} & 0 \\ 0 & 0 & \phi_{\dot{z}z} & 0 & 0 & \phi_{\dot{z}\dot{z}} \end{pmatrix}$$

Let

$$\Phi_{xy} = \begin{pmatrix} \phi_{xx} & \phi_{xy} & \phi_{x\dot{x}} & \phi_{x\dot{y}} \\ \phi_{yx} & \phi_{yy} & \phi_{y\dot{x}} & \phi_{y\dot{y}} \\ \phi_{\dot{x}x} & \phi_{\dot{x}y} & \phi_{\dot{x}\dot{x}} & \phi_{\dot{x}\dot{y}} \\ \phi_{\dot{y}x} & \phi_{\dot{y}y} & \phi_{\dot{y}\dot{x}} & \phi_{\dot{y}\dot{y}} \end{pmatrix} \qquad (3)$$

and

$$\Phi_z = \begin{pmatrix} \phi_{zz} & \phi_{z\dot{z}} \\ \phi_{\dot{z}z} & \phi_{\dot{z}\dot{z}} \end{pmatrix} \tag{4}$$

Then, using Eqs. (2), (3), and (4), Eq. (1) may be written as the following two equations:

$$\dot{\Phi}_{xy} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega^2(1 - 3\cos^2 \omega t) & 3\omega^2 \sin \omega t \cos \omega t & 0 & 0 \\ 3\omega^2 \sin \omega t \cos \omega t & -\omega^2(1 - 3\sin^2 \omega t) & 0 & 0 \end{pmatrix} \Phi_{xy} \tag{5}$$

$$\dot{\Phi}_z = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \Phi_z \tag{6}$$

The initial values of Φ_{xy} and Φ_z are the identity matrices of the appropriate dimensions.

Differentiating Eq. (6) yields

$$\ddot{\Phi}_z = -\omega^2 \Phi_z$$

which has solution

$$\Phi_z = \begin{pmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{pmatrix}$$

To solve Eq. (5), let a column of the matrix Φ_{xy} be

$$\underline{\phi} = \begin{pmatrix} p \\ q \\ s \\ w \end{pmatrix}$$

Then, each column satisfies the same differential equation as the Φ_{xy} matrix. Thus,

$$\dot{\underline{\phi}} = \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{s} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} s \\ w \\ -\omega^2 \left[(1 - 3 \cos^2 \omega t)\, p - (3 \sin \omega t \cos \omega t)\, q\right] \\ -\omega^2 \left[-(3 \sin \omega t \cos \omega t)\, p + (1 - 3 \sin^2 \omega t)\, q\right] \end{pmatrix} \tag{7}$$

Eliminating the variables s and w from Eq. (7) yields

$$\ddot{p} = -\omega^2 \left[(1 - 3 \cos^2 \omega t)\, p - (3 \sin \omega t \cos \omega t)\, q\right] \tag{8}$$

$$\ddot{q} = -\omega^2 \left[-(3 \sin \omega t \cos \omega t)\, p + (1 - 3 \sin^2 \omega t)\, q\right] \tag{9}$$

Let

$$\theta = \omega t$$

and denote differentiation with respect to θ by a prime. Then, Eqs. (8) and (9) may be written as follows:

$$p'' - \frac{1}{2} (1 + 3 \cos 2 \theta)\, p - (\frac{3}{2} \sin 2\theta)\, q = 0 \tag{10}$$

$$q'' - (\frac{3}{2} \sin 2\theta)\, p - \frac{1}{2} (1 - 3 \cos 2\theta)\, q = 0 \tag{11}$$

Equations (10) and (11) may be written in terms of the complex variables

$$u = p + iq \qquad v = p - iq$$

Multiplying Eq.(11) by i, and forming the sum and difference of the result with Eq. (10) gives

$$u'' - \frac{1}{2} u - \frac{3}{2} e^{2i\theta} v = 0 \tag{12}$$

$$v'' - \frac{1}{2} v - \frac{3}{2} e^{-2i\theta} u = 0 \tag{13}$$

After some algebraic manipulation of Eqs. (12), (13) and the first two derivatives of Eq. (12), the following equation for u alone is obtained

$$u'''' - 4i\, u''' - 5u'' + 2i\, u' = 0 \tag{14}$$

If a solution

$$u = e^{ik\theta}$$

is assumed for u and substituted into Eq. (14), the following fourth degree polynomial for k results:

$$k^4 - 4k^3 + 5k^2 - 2k = 0$$

or

$$k(k - 1)^2 (k - 2) = 0$$

The general solution of Eq. (14) is then given by

$$u = (A_0 + i\,B_0) + (A_1 + i\,B_1)\,e^{i\theta} + (A_2 + i\,B_2)\,\theta\,e^{i\theta} + (A_3 + i\,B_3)\,e^{2i\theta} \tag{15}$$

where the A's and B's are real constants.

Since there are only four independent constants, the relationships between the eight A's and B's are obtained by substituting Eq. (15) and its conjugate into Eq. (12). Equation (15) then becomes

$$u = (A_0 + i\,B_0) + (A_1 + i\,B_1)\,e^{i\theta} - \frac{3}{2}\,A_1\,\theta\,e^{i\theta} - \frac{1}{3}\,(A_0 - i\,B_0)\,e^{2i\theta} \tag{16}$$

Expressions for p(t) and q(t) are obtained by taking real and imaginary parts of Eq. (16). Differentiation of p(t) and q(t) yields s(t) and w(t). The general solution of Eq. (7) is then

$$\underline{\phi} = \begin{pmatrix} A_0(1 - \frac{1}{3}\cos 2\theta) + A_1(\cos\theta + \frac{3}{2}\theta\sin\theta) - \frac{1}{3}B_0\sin 2\theta - B_1\sin\theta \\ -\frac{1}{3}A_0\sin 2\theta + A_1(\sin\theta - \frac{3}{2}\theta\cos\theta) + B_0(1 + \frac{1}{3}\cos 2\theta) + B_1\cos\theta \\ \omega\left[\frac{2}{3}A_0\sin 2\theta + \frac{1}{2}A_1(\sin\theta + 3\theta\cos\theta) - \frac{2}{3}B_0\cos 2\theta - B_1\cos\theta\right] \\ \omega\left[-\frac{2}{3}A_0\cos 2\theta - \frac{1}{2}A_1(\cos\theta - 3\theta\sin\theta) - \frac{2}{3}B_0\sin 2\theta) - B_1\sin\theta\right] \end{pmatrix} \tag{17}$$

The initial value of $\underline{\phi}$ is obtained by setting t equal to zero in Eq. (17). Thus,

$$\underline{\phi}_0 = \begin{pmatrix} \frac{2}{3}A_0 + A_1 \\ \frac{4}{3}B_0 + B_1 \\ -\omega(\frac{2}{3}B_0 + B_1) \\ -\omega(\frac{2}{3}A_0 + \frac{1}{2}A_1) \end{pmatrix}$$

Each ϕ_0 has one component equal to unity and three zero components, allowing the four constants to be evaluated. The resulting solution of Eq. (5) as follows:

$$\Phi_{xy} = \left(\begin{array}{c:c:c:c} 1+2C(1-C)+3S(\theta-S) & S(1-C) & \frac{1}{\omega}[S(2-C)] & \frac{1}{\omega}[-1+C(2-C)+3S(\theta-S)] \\ 2S(1-C)-3C(\theta-S) & 1-C(1-C) & \frac{1}{\omega}[1-C(2-C)] & \frac{1}{\omega}[S(2-C)-3C(\theta-S)] \\ \omega[S(1+C)+3C(\theta-S)] & \omega[1+C(1-2C)] & 1+2C(1-C) & S(1-C)+3C(\theta-S) \\ \omega[2-C(1+C)+3S(\theta-S)] & \omega[S(1-2C)] & 2S(1-C) & 1-C(1-C)+3S(\theta-S) \end{array}\right) \quad (18)$$

where

$$S = \sin\theta \qquad C = \cos\theta$$

It is of interest to express Eq. (18) in nominal local vertical coordinates, i.e., in an axis system rotating about the z axis at a rate equal to ω. To accomplish this, each of the 2×2 submatrices of Φ_{xy} must be premultiplied by the following matrix:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Then, the transition matrix in the local vertical system Φ'_{xy} is given by

$$\Phi'_{xy} = \begin{pmatrix} 2-\cos\omega t & \sin\omega t & \frac{1}{\omega}\sin\omega t & \frac{2}{\omega}(1-\cos\omega t) \\ 2\sin\omega t - 3\omega t & -(1-2\cos\omega t) & -\frac{2}{\omega}(1-\cos\omega t) & \frac{1}{\omega}(4\sin\omega t - 3\omega t) \\ -\omega(\sin\omega t - 3\omega t) & \omega(1-\cos\omega t) & 2-\cos\omega t & -(2\sin\omega t - 3\omega t) \\ -\omega(1-\cos\omega t) & -\omega\sin\omega t & -\sin\omega t & -(1-2\cos\omega t) \end{pmatrix}$$