Instrumentation Laboratory
Massachusetts Institute of Technology
Cambridge, Massachusetts

Space Guidance Analysis Memo #12

To: SGA Distribution
From: James E. Potter
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Subject: Storing Position of Moon

I have been examining the problem of how to efficiently store, in the Apollo Guidance Computer, ephemeris data giving the position of the moon as a function of time. The concrete results obtained so far indicate that is is possible to give three, eighth degree polynomials in time whose values agree within about a mile with the rectangular coordinates of the moon over a two week period.

A program has been written for the Honeywell 800 Computer which calculates the coefficients of the polynomial of given degree which approximates the input data to the program with the least maximum error. This program was used to calculate polynomials fitting the x, y and z coordinates of the moon in the earth centered ecliptic coorcinate system for the period from January 1, 1951 to January 15, 1951. The maximum errors are:

x coordinate	0.97 mi.
y coordinate	0.84 mi.
z coordinate	0.11 mi.

Since the maximum errors in the three coordinates occur simultaneously on January 1 and January 15, the maximum vector error is

$$1.3 \text{ mi} = \sqrt{(0.97)^2 + (0.84)^2 + (0.11)^2}\,.$$

Because the z axis in this coordinate system is perpendicular to the ecliptic plane and the moon's motion is nearly in the ecliptic plane, the variation of the moon's z coordinate is only about one tenth as large as the variations of its x and y coordinates as it revolves about the earth. This fact probably accounts for the relatively small z coordinate error.

More computer runs are planned to determine how rapidly the maximum errors decrease as the degrees of the approximating polynomials are increased and to determine the effect of the length of the time interval over which the ephemeris data is approximated on the error. Lunar position data from other two week periods will also be fitted to find out if the error depends appreciably on the date.

The remainder of this note is a brief discussion of the lunar position approximation problem. Two approaches to this problem are (a) to solve the equations of motion of the moon and (b) to fit the rectangular coordinates of the moon with ordinary polynomials or truncated Fourier series. Approach (a) is an approximation if, to facilitate computation, either some of the forces acting on the moon are left out of the equations of motion or the differential equations are not solved exactly. The tables of the position of the moon given in the American Ephermeris were obtained by evaluating 1600 terms of a series solution of the complete set of lunar equations of motion. This series solution was worked out by the American astronomer Brown.

Mr. R. Hutchinson of the Instrumentation Laboratory has investigated the feasibility of using Brown's series for on-board computation of the moon's position. If all of the 1600 terms worked out by Brown are retained, the series is supposed to give the position of the center of the moon within a mile for a century or more. (Reference 1, page 99. The Naval Observatory workers probably feel that their tabulated positions of the moon are more accurate than this. In the American Ephemeris the Longitude and Latitude of the moon are given to the nearest 0.01 second corresponding to 0.012 mile and the horizontal parallax of the moon is tabulated to

0.001 second corresponding to 0.06 mile. The question of the accuracy of ephemeris tables needs further clarification.) It was hoped that if high accuracy is only required for a period of a few weeks almost all of the terms of Brown's series could be ignored. However, Mr. Hutchinson's calculations indicate that, for accuracies on the order of a mile, a prohibitively large number of terms must be retained even to represent the moon's motion over short periods of time. Mr. Hutchinson's conclusions are consistent with Dr. Battin's experience in writing the moon-planet subroutine used with the Honeywell 800 Computer for preliminary space trajectory calculations. This subroutine calculates the moon's position using the thirty largest terms from Brown's series, and the errors in the lunar positions calculated using the subroutine are on the order of 200 miles, although the approximation is this accurate over a period of years. (Reference 2)

The differential equation approach to calculating lunar position can be made feasible for on-board calculation by leaving some of the forces acting on the moon out of the equations of motion. If all forces except the earth's attraction are neglected the problem becomes particularly simple since the two-body problem can be solved analytically. Reference 1 states that the perturbative force of the sun on the moon is only 89 times smaller than the earth's attraction. The author of reference 1 calculates from this that, after three days, the moon's orbit will deviate from an originally oscullating two-body trajectory by about 600 miles. Therefore, the two-body approximation is not adequate for primary lunar position calculations although it might be useful for interpolation. After the attraction of the sun the next largest perturbing force, the force due to the oblateness of the earth, is a million times smaller than the primary earth attracting force according to reference 1. A perturbing force of this magnitude would cause a deviation of the order of a mile over a two week period. Thus the three body, moon-earth-sun, approximation is capable of one mile or better accuracy while the differential equations can be handled by numerical methods without difficulty. My opinion is that this approach to lunar position calculation is less "rough and ready" than fitting the lunar position data with polynomials and that it is more susceptible to human error but that it is a practical method.

Approach (b) mentioned above is to apply curve fitting methods directly to the ephemeris data, without making use of the lunar equations of motion. Since the moon's orbit is nearly circular, it was suggested that the moon's position might be closely approximated by equations of the form

$$x = A_o + A_1 \sin(wt + \theta_1)$$
$$y = B_o + B_1 \sin(wt + \phi_1)$$
$$z = C_o + C_1 \sin(wt + \psi_1) \quad . \qquad (1)$$

In order to make a rough estimate of the accuracy of such an approximation, assume that the moon's orbit is a two-body ellipse whose eccentricity e is 0.055, the mean eccentricity of the moon's true orbit, and whose semi-major axis a is 240,000 miles. Since the motion in a two-body ellipse is periodic, the x component of the vector from one body to the other can be expanded in a Fourier series,

$$\frac{x}{a} = b_0 + b_1 \sin(wt + \rho_1) + b_2 \sin(2wt + \rho_2) + \ldots \qquad (2)$$

The coefficient b_2 of the second harmonic term should be an estimate of the accuracy of equation (1). The coefficients of the series (2) are given on page 79 of reference 3. The coefficient of the second harmonic term is given by the formula

$$b_2 = \frac{1}{2} e - \frac{1}{3} e^3 + \frac{1}{16} e^5 + \ldots$$

Plugging in e = 0.055 we find that $b_2 = 0.027$, and the accuracy of the approximation given in equation (1) is

$$(0.027)\,(240{,}000) = 6{,}000 \text{ miles}$$

This error estimate may be several times too large, but at least it indicates that additional terms are necessary in the approximation formulas. Probably the most appropriate additional terms are higher harmonic trigonometric terms. Since trigonometric sines are calculated on the Apollo Guidance Computer by evaluating a ninth degree polynomial approximation, it would take less computer time if polynomial approximation were used directly.

The numerical results on polynomial fitting of ephemeris data have already been given. Theoretically, according to the Weierstrass approximation theorem, any continuous function can be approximated to any desired accuracy by a polynomial of sufficiently high degree. From a practical standpoint the degree of the polynomial may be prohibitively high. It must also be noted that polynomials cannot be evaluated without some roundoff error. Chebyshev proved that for a given continuous function there is a unique polynomial of given degree with least maximum deviation from the given continuous function. When approximating the continuous function with the best possible polynomial of degree N, the maximum deviation will occur for at least N + 2 values of the independent variable. (Under ordinary circumstances the maximum deviation occurs exactly N + 2 times). Furthermore, one has the best polynomial approximation of degree N in the sense of least maximum deviation if he has an Nth degree polynomial approximation whose error curve has N + 2 extrema with equal heights and alternating signs. Reference 4 discusses this method of polynomial fitting and gives many examples.

The velocity of the moon is needed in on-board calculations when changing from earth centered to moon centered coordinates and vice-versa. An easy way to obtain the approximate velocity of the moon would be to differentiate the approximating polynomial. There has not yet been time to test this method thoroughly by comparing the numerical derivative of the ephemeris data with the derivative of the approximating polynomials. The error in the value of a velocity component obtained by differentiating the approximating polynomial is equal to the derivative of the error curve for the approximating polynomial. Chords were drawn to the error curves from the January 1951 curve fitting run and their slopes were calculated to obtain the following values for the maximum velocity errors.

maximum x coordinate velocity error	0.43 mph
maximum y coordinate velocity error	0.35 mph
maximum z coordinate velocity error	0.04 mph
maximum vector velocity error	0.54 mph

REFERENCES

1. Y. Ryabou, An Elementary Survey of Celestial Mechanics, Dover, 1961

2. Battin, R. H., Class Notes

3. D. Brouwer and G. M. Clemence, Methods of Celestial Mechanics, Academic Press, 1961

4. B. Hastings, Approximations for Digital Computers, Princeton, 1955