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Space Guidance Analysis Memo #20

To: SGA Distribution

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Subject: Optimum Use of Acceleration Measurements to Determine Altitude-Rate During Re-entry

The successful operation of the re-entry guidance system depends upon having a fairly accurate indication, within the AGC, of the rate-of-change of altitude. This is especially important in the sensitive portion of the trajectory just after pull-up, while the velocity is still supercircular. Using standard inertial navigation techniques the indicated altitude rate will, in general, be in error. The major sources of this error are likely to be:

1. The initial error, at the start of the re-entry phase (a function of mid-course guidance) and
2. Misalignment of the IMU about an axis perpendicular to the plane of the re-entry trajectory. This results in a steadily increasing altitude-rate-error term which is proportional to the integrated horizontal component of aerodynamic forces.

This memo has two purposes: First, an alternate scheme for measuring altitude-rate is presented; this scheme makes use of our knowledge of the character of the atmosphere and measured values of drag-acceleration and the rate-of-change of drag acceleration. Second, solutions to a set of mathematical *optimization problems are given. These problems are* approximations to the real problem of how to make optimum use of the two methods of measuring altitude-rate.

Derivation of the Altitude-Rate Equation

Drag acceleration is given by:

$$a_D = \left[\frac{C_D A}{2\ m}\right] \rho\ V^2 \qquad (1)$$

Differentiating and recombining yields:

$$\frac{\dot{a}_D}{a_D} = 2\ \frac{\dot{V}}{V} + \frac{\dot{\rho}}{\rho} \qquad (2)$$

Now assume an exponential atmosphere:

$$\rho = \rho_o\ e^{-\beta h} \qquad (3)$$

It is only necessary to assume this locally; that is, take $\rho_o$ at the present value and choose the value for $\beta$ which best describes the change in $\rho$ for small changes in altitude.

Differentiating (3) and combining with (2) yields:

$$\boxed{\dot{h} = \frac{1}{\beta}\left[2\ \frac{\dot{V}}{V} - \frac{\dot{a}_D}{a_D}\right]} \qquad (4)$$

If constant-density surfaces slope with respect to the local horizontal, equation (4) is a measure of the rate at which these surfaces are being crossed. A difference equation form of equation (4) could be mechanized in the AGC to provide a measure of $\dot{h}$ which is largely independent of the measure provided by the standard inertial navigation procedure. The method based on equation (4) is likely to be most accurate when $\dot{h}$ is near zero, since a large deviation in $\beta$ would then cause only a small absolute error in $\dot{h}$. This is fortunate, since it is just after the altitude-rate goes to zero (pull-up) that an accurate indication is most needed.

Optimum Weighting Vectors

Summarized below are the statements of and solutions to a pair of optimization problems. Problem I is a simplified version of Problem II. The solutions are given in various equivalent forms. The derivations of these solutions are omitted; these are a matter of minimizing the mean-squared-errors of linear estimates.

The following definitions apply to both problems:

$a_n$ sequence of true values, $n = 1$ to $N$

$\tilde{a}_n$ sequence of measured values from one source -- for instance, indicated altitude-rate from an inertial navigator

$\tilde{a}_n'$ sequence of measured values from an alternate source -- for instance, indicated altitude-rate from equation (4)

$\alpha_n = \tilde{a}_n - a_n$ , error in $\tilde{a}_n$

$\beta_n = \tilde{a}_n' - a_n$ , error in $\tilde{a}_n'$

Statement of Problem I

assume:

$\alpha_1 = \alpha_2 = \alpha_n = \alpha$, a constant

$\overline{\alpha} = \overline{\beta_n} = \overline{\alpha\beta_n} = 0$

given: the constant $\overline{\alpha^2}$ and the correlation matrix,

$$B = \{\overline{\beta_i \beta_j}\}_{N \times N}$$

find: the weighting vector $\underline{\omega}_n$ giving the best linear estimate for $\alpha$

$$\boxed{\hat{\alpha} = \sum_{n=1}^{N} \omega_n (\tilde{a}_n - \tilde{a}_n')}$$

Interpreting $a_n$ as a sequence of true values of altitude-rate, we are trying to estimate a constant or bias error in the sequence of values, $\tilde{a}_n$, indicated by an inertial navigator. This corresponds to a case in which the initial condition error is the only significant one.

Solution $I_1$:

$$\boxed{\underline{\omega}_n = (B + A)^{-1} \underline{a}}$$

where

$$\underline{a} = \begin{pmatrix} \overline{\alpha^2} \\ \overline{\alpha^2} \\ \overline{\alpha^2} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}_N$$

$$A = \begin{bmatrix} \overline{\alpha^2} & \overline{\alpha^2} & \cdots \\ \overline{\alpha^2} & \overline{\alpha^2} & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \end{bmatrix}_{N \times N}$$

Equivalent Solution $I_2$:

Define

$$D = \left\{ \frac{\overline{\beta_i \beta_j}}{\overline{\alpha^2}} + 1 \right\}_{N \times N}$$

$$\boxed{\underline{\omega}_n \triangleq D^{-1} \underline{1}}$$

where,

$$\underline{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_{N}$$

that is,

$$\boxed{\omega_n = \sum n^{th} \text{ row of } D^{-1}}$$

Equivalent Solution $I_3$:

Define

$$S_n = \text{sum of } n^{th} \text{ row of } B^{-1}$$

$$b_1 = \sum_{n=1}^{N} S_n$$

$$b_2 = \sum_{n=1}^{N} (\overline{\beta_i \beta_n} \quad S_n) \text{ for any } i$$

then

$$\underline{\omega}_n = \begin{pmatrix} S_1 K \\ S_2 K \\ \cdot \\ \cdot \\ \cdot \\ S_N K \end{pmatrix}$$

where

$$K = \frac{\overline{\alpha^2}}{b_1 \overline{\alpha^2} + b_2}$$

$$\therefore \quad \boxed{\hat{\alpha} = K \sum_{n=1}^{N} S_n (\tilde{a}_n - \tilde{a}_n')}$$

This form would be useful where the B matrix is known in advance, but the value of $\overline{\alpha^2}$ is not known until the midcourse navigation phase of the mission is completed.

Statement of Problem II

Assume:

$$\alpha_n = \alpha_0 + \theta f_n$$

$$f_n = \sum_{i=1}^{n} \Delta f_i$$

Here we interpret $\theta$ as a platform misalignment and $f_n$ as the integrated horizontal acceleration due to aerodynamic forces.

given: the constants $\overline{\alpha_0^2}$ and $\overline{\theta^2}$ and the correlation matrices,

$$B = \{ \overline{\beta_i \beta_j} \}_{N \times N}$$

and

$$F = \left\{ \frac{\overline{\Delta \beta_i \Delta \beta_j}}{\Delta f_i \; \Delta f_j} \right\}_{(N-1) \times (N-1)}$$

where,

$$\overline{\Delta\beta_i \Delta\beta_j} = \overline{(\beta_i - \beta_{i-1})(\beta_j - \beta_{j-1})} = \overline{\beta_i \beta_j} - \overline{\beta_{i-1} \beta_j} - \overline{\beta_i \beta_{j-1}} + \overline{\beta_{i-1} \beta_{j-1}}$$

find: the weighting vectors $\underline{v}_n$ and $\underline{\omega}_n$ giving the best linear estimates for $\theta$ and $\alpha_0$:

$$\hat{\theta} = \sum_{n=2}^{N} v_n \frac{\Delta(\tilde{a}_n - \tilde{a}_n')}{\Delta f_n}$$

$$\hat{\alpha}_0 = \sum_{n=1}^{N} \omega_n \left[(\tilde{a}_n - \tilde{a}_n') - \hat{\theta} f_n\right]$$

Solution $II_1$:

$$\underline{v}_n = (H + F)^{-1} \underline{h}$$

where,

$$h = \begin{pmatrix} \overline{\theta^2} \\ \overline{\theta^2} \\ \overline{\theta^2} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} (N - 1)$$

$$H = \begin{pmatrix} \overline{\theta^2} & \overline{\theta^2} & \overline{\theta^2} & \cdot & \cdot & \cdot \\ \overline{\theta^2} & \overline{\theta^2} & & & & \\ \overline{\theta^2} & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \end{pmatrix}_{(N-1) \times (N-1)}$$

$$\boxed{\underline{\omega}_n = (A + B + E)^{-1} \; \underline{a}}$$

where,

$$E = \overline{\epsilon_1^2} \; \{ f_i f_j \}_{N \times N}$$

$$\epsilon_1 = \hat{\theta} - \theta$$

Equivalent Solution $II_2$:

$$\boxed{\underline{v}_n = G^{-1} \underline{1}}$$

where,

$$G = \left\{ \frac{\overline{\Delta \beta_i \, \Delta \beta_j}}{\Delta f_i \, \Delta f_j} + 1 \right\}_{(N-1) \times (N-1)}$$

$$\boxed{\underline{\omega}_n = J^{-1} \underline{1}}$$

$$J = \left\{ \frac{\overline{\beta_i \, \beta_j}}{\overline{\alpha_0^2}} + \frac{\overline{\epsilon_1^{\,2}}}{\overline{\alpha_0^2}} \quad f_i \, f_j + 1 \right\}_{N \times N}$$