Space Guidance Analysis Memo #32-65

TO: SGA Distribution

FROM: Donald C. Fraser

DATE: 31 December 1965

SUBJECT: Updating the Determinant of the Covariance Matrix

   The formulation of the recursive filter equations includes the method for updating the covariance matrix when a measurement is incorporated. A method for updating the determinant of this matrix is derived here.*

   A sequence of coordinate transformations in both state and measurement space is made on the covariance matrix update equation until its format is amenable to evaluation of the desired determinants. The result is an expression for the determinant of the covariance matrix after incorporation of the measurement which involves only pre-incorporation parameters. The result in the vocabulary described below is:

$$\frac{\det E}{\det E'} = \frac{\det R}{\det (R + H^T E' H)}$$

The departure point is the covariance matrix update equation:

$$E = E' - E'H(R + H^T E'H)^{-1}H^T E' \tag{1}$$

where: E = $n \times n$ = covariance matrix at time t after incorporation of the measurement

     E' = $n \times n$ = covariance matrix at time t before incorporation of the measurement (non-singular, positive definite)

     R = $r \times r$ = covariance of the measurement noise (assumed non-singular, positive definite)

     H = $n \times r$ = geometry matrix (of rank r) where $\underline{m} = H^T \underline{x}$

     $\underline{x}$ =   state vector (dimension n)

     $\underline{m}$ =   measurement vector (dimension r)

   First make a rotation in state space to principal axes such that E'

---

* This derivation is due originally to Dr. James E. Potter.

1

is diagonal, then scale the result such that this diagonal matrix is the identity matrix. The result is a new E' matrix:

$$E'' = I = CE'C^T$$

This rotation is not, however, unique since we can make an orthogonal transformation:

$$PE''P^T = PE''P^{-1} = PIP^{-1} = I$$

which leaves the E'' matrix unchanged. Use this degree of freedom by choosing P such that the second term of equation (1) is diagonalized. Thus we have:

$$E'' = PCE'C^TP^T = QE'Q^T = I \qquad (2)$$

By noting that the second term on the right side of equation (1) is of rank r and by ordering the elements of the rotated term we may write:

$$QE'H(R + H^TE'H)^{-1}H^TE'Q^T = \text{diagonal } [\lambda_1, \lambda_2 \ldots \lambda_r, 0, 0 \ldots 0] \qquad (3)$$

Next a similar pair of transformations is made in measurement space. Choose a matrix K such that:

$$K(R + H^TE'H)K^T = I \qquad (4)$$

and:

$$KH^TE'HK^T = \text{diagonal } [\mu_1, \mu_2 \ldots \mu_r] \qquad (5)$$

Noting that Q is non-singular (because of the way in which it was defined - see equation (2)) we may rewrite equation (3):

$$QE'Q^TQ^{T^{-1}}HK^TK^{T^{-1}}(R + H^TE'H)^{-1}K^{-1}KH^TQ^{-1}QE'Q^T =$$

$$Q^{T^{-1}}HK^T[K(R + H^TE'H)K^T]^{-1}KH^TQ^{-1} =$$

$$Q^{-1^T}HK^TKH^TQ^{-1} = \text{diagonal } (\lambda_1, \ldots \lambda_r, 0, \ldots 0)$$

defining:

$$H' = Q^{-1^T}HK^T$$

which is the geometry matrix in terms of the new coordinate systems in both state and measurement space, there results:

$$H'H'^T = \text{diagonal}(\lambda_1, \ldots \lambda_r, 0, \ldots 0) \tag{6}$$

Rewriting equation (5):

$$KH^TQ^{-1}QE'Q^TQ^{-1^T}HK^T = KH^TQ^{-1}Q^{-1^T}HK^T =$$

$$H'^TH' = \text{diagonal}(\mu_1, \mu_2, \ldots \mu_r) = D_\mu \tag{7}$$

In order to show that $D_\mu = D_\lambda$ (D_λ is defined below) partition H' as:

$$H'^T = [A \quad B]$$

where:

$$A = r \times r$$
$$B = r \times (n - r)$$

then:

$$H'H'^T = \begin{bmatrix} A^TA & A^TB \\ B^TA & B^TB \end{bmatrix} = \begin{bmatrix} D_\lambda & 0 \\ 0 & 0 \end{bmatrix}$$

But:

$$\text{tr}(B^TB) = 0 = \sum_{i=1}^{n} \sum_{j=1}^{n-r} B_{ij}^2$$

Since all the B_{ij} are real $B = 0$ and:

$$A^TA = D_\lambda$$

Similarly:

$$H'^TH' = [A \quad 0] \begin{bmatrix} A^T \\ 0 \end{bmatrix} = D_\mu$$

Then:

$$A^{-1}D_\mu A = A^{-1}AA^TA = A^TA = D_\lambda$$

or:

$$A^{-1}D_\mu A = D_\lambda$$

Since the characteristic polynomials of D_μ and D_λ are the same

$$D_\mu = D_\lambda \tag{8}$$

assuming the elements of each have been ordered (e.g. in increasing magnitude along the diagonal).

From (8) we may conclude:

$$\mu_i = \lambda_i \qquad i = 1 \text{ to } r \tag{9}$$

From equation (6):

$$\det (QEQ^T) = \det (I - H'H'^T) = \prod_{i=1}^{r} (1 - \lambda_i)$$

$$\det E = \frac{1}{(\det Q)^2} \prod_{i=1}^{r} (1 - \lambda_i) \tag{10}$$

From equation (2):

$$\det (QE'Q^T) = \det I$$

$$\det E' = \frac{1}{(\det Q)^2} \tag{11}$$

dividing equation (10) by equation (11):

$$\frac{\det E}{\det E'} = \prod_{i=1}^{r} (1 - \lambda_i) \tag{12}$$

Rewriting equation (4):

$$KRK^T = I - KH^T E'HK^T$$
$$= I - D_\mu$$
$$\det R (\det K)^2 = \prod_{i=1}^{r} (1 - \mu_i)$$

Applying equation (9) and rearranging terms:

$$\prod_{i=1}^{r} (1 - \mu_i) = \prod_{i=1}^{r} (1 - \lambda_i) = \det R (\det K)^2 \tag{13}$$

From equation (4):

$$\det K(R + H^T E'H)K^T = 1$$

$$(\det K)^2 = \frac{1}{\det (R + H^T E'H)} \tag{14}$$

Combining equations *(12)*, *(13)*, and *(14)* gives the final result:

$$\frac{\det E}{\det E'} = \frac{\det R}{\det (R + H^T E H)} \tag{15}$$

Summary

An expression has been derived (equation (15)) which determines how to update the determinant of the covariance matrix when a measurement is incorporated. It yields possible computational advantages because the resulting equation involves the evaluation of determinants which are usually of lower order than that of the covariance matrix itself. More saving is possible for the special case of R = constant or r = 1.