

Massachusetts Institute of Technology
Instrumentation Laboratory
Cambridge, Massachusetts

Space Guidance Analysis Memo #44

TO: SGA Distribution
FROM: Robert J. Fitzgerald
DATE: May 27, 1963
SUBJECT: Optimization of Midcourse Correction Times

Section 4 of reference [1] discusses the possibility of varying the times of midcourse velocity corrections to minimize a function ϕ , which involves the terminal deviation variances and a statistical measure of the total velocity change used for control.

It is suggested that $\partial\phi/\partial t_j$ be calculated by changing t_j to $t_j + \Delta t_j$ and recalculating the $X(t)$ and $E(t)$ histories. In such an approach Δt_j would have to be kept small to make the assumed linearization valid. But when Δt is small, the computation of the derivative inevitably involves taking differences of nearly-equal numbers, a procedure to be avoided if reasonable accuracy is to be attained. The following procedure allows direct computation of the derivative.

A general performance criterion of this type is

$$\phi = \sum_{i=1}^n \text{tr } f_i (X(t_i^-), E(t_i^-)) = \min.$$

where $n-1$ corrections are made and t_n is the final time.

The basic problem is that of finding the partial derivatives

$$\frac{\partial X(t_i^-)}{\partial t_k} \quad \text{and} \quad \frac{\partial E(t_i^-)}{\partial t_k} \quad (k \leq i \leq n)$$

We first derive the relations describing the propagation of perturbations in the variance matrices, $\delta X(t)$ and $\delta E(t)$. From (3-1)* and (4-7), with $\Lambda = Q = 0$ and $K = EH^T R^{-1}$:

*Hyphenated equation numbers refer to reference [1]

$$\dot{X} = XF^T + FX \quad (1)$$

$$\dot{E} = EF^T + FE - EH^T R^{-1} HE \quad (2)$$

Equation (1) is linear and the corresponding perturbation equation, from (4-29), has the solution

$$\delta X_{i+1-} = \Phi_{i+1, i} \delta X_{i+} \Phi_{i+1, i}^T \quad (i > k) \quad (3)$$

Equation (2) is of the Riccati type, and its perturbed solution (see Appendix A) satisfies the equation

$$\delta E_{i+1-} = \left[E_{i+1-} \Phi_{i+1, i}^{-T} E_{i+}^{-1} \right] \delta E_{i+} \left[E_{i+}^{-1} \Phi_{i+1, i}^{-1} E_{i+1-} \right] \quad (i > k) \quad (4)$$

From Eq. (4-33),

$$E_{i+} = E_{i-} + G_i Q_i G_i^T \quad (4-33)$$

where Q is the control implementation error covariance matrix, we see that

$$\delta E_{i+} = \delta E_{i-} + G_i \delta Q_i G_i^T$$

where δQ_i can be expressed in terms of δX_{i-} and δE_{i-} , or may be zero. Combining this with Eq. (4) we can write δE_{i+1-} in the form

$$\delta E_{i+1-} = \sum_j N_{xij} \delta X_{i-} N'_{xij} + \sum_j N_{eij} \delta E_{i-} N'_{eij} \quad (5)$$

The propagation of δX across a correction is derived by perturbing Eq. (4-28)

$$X_{i+} = X_{i-} + G_i \left[B(X-E)B^T + Q \right]_{i-} G_i^T + \left[(X-E)B^T G^T + GB(X-E) \right]_{i-} \quad (4-28)$$

where B (derived by Battin) determines the correction $B \delta \hat{x}$.

$$\begin{aligned}
\delta X_{i+} &= \delta X_{i-} + G_i B_i \delta X_{i-} B_i^T G_i^T \\
&+ \delta X_{i-} B_i^T G_i^T + G_i B_i \delta X_{i-} \\
&- G_i B_i \delta E_{i-} B_i^T G_i^T - \delta E_{i-} B_i^T G_i^T - G_i B_i \delta E_{i-} + G_i \delta Q_i G_i^T \quad (6)
\end{aligned}$$

Combination of Eq. (3) with Eq. (6) gives an expression for δX_{i+1-} of the form

$$\delta X_{i+1-} = \sum_j M_{xij} \delta X_{i-} - M'_{xij} - \sum_j M_{ejj} \delta E_{i-} - M'_{ejj} \quad (7)$$

The above recurrence formulas Eq. (5) and (7) require, as initial conditions, the values of $\delta E(t_{k+1-})$ and $\delta X(t_{k+1-})$, where t_k is the correction time to be varied. By differentiating Eq. (4-28) with respect to time at t_k we can express $\partial X(t_k+)/\partial t_k$ in terms of the time derivatives, at (t_k-) , of the components of the right hand side. $\dot{X}(t_k-)$ is obtained from Eq. (1), \dot{G} from the original differential equations, \dot{E} from Eq. (2), \dot{B} from Battin's derivation of B, and \dot{Q} from the (time-varying) statistical properties of the control implementation error v . If Q is also a function of X and E , its derivative must be calculated accordingly. We can thus write

$$\delta X(t_k+) = M_{k+} \delta t_k \quad (8)$$

Now Eq. (3) must be altered, for $i = k$, because $\Phi_{k+1, k}$ changes as we vary t_k . Hence

$$\delta X_{k+1-} = \Phi_{k+1, k} \delta X_{k+} \Phi_{k+1, k}^T + 2 \Phi_{k+1, k} X_{k+} \delta \Phi_{k+1, k}^T \quad (9)$$

Now the transition matrix may be written

$$\Phi_{k+1, k} = \Gamma(t_{k+1}) \Gamma^{-1}(t_k) \quad (10)$$

where

$$\dot{\Gamma}(t) = F \Gamma(t)$$

Hence

$$\begin{aligned} \frac{\partial}{\partial t_k} \Phi_{k+1, k} &= \Gamma(t_{k+1}) \frac{d}{dt_k} \Gamma^{-1}(t_k) = - \Gamma_{k+1} \Gamma_k^{-1} \dot{\Gamma}_k \Gamma_k^{-1} \\ &= - \Phi_{k+1, k} F(t_k) \end{aligned} \quad (11)$$

Hence, combining Eqs. (8), (9) and (11) we may write

$$\delta X(t_{k+1}^-) = M_{k+1} \delta t_k \quad (12)$$

In similar fashion, by differentiating (4-33) we can express $\delta E(t_k^+)$ as

$$\delta E(t_k^+) = N_{k+} \delta t_k \quad (13)$$

The relation between $\delta E(t_k^+)$ and $\delta E(t_{k+1}^-)$ is derived in Appendix B, and allows us to write

$$\delta E(t_{k+1}^-) = N_{k+1} \delta t_k \quad (14)$$

With Eqs. (12) and (14) and the recurrence formulas Eqs(5) and (7) we can now evaluate the matrices M_i and N_i ($i > k+1$) which enable us to write

$$\delta X(t_i^-) = M_i \delta t_k \quad (i > k) \quad (15)$$

$$\delta E(t_i^-) = N_i \delta t_k \quad (i > k) \quad (16)$$

The matrices M_k and N_k are just the time derivatives of X and E at t_k , as determined from Eqs. (1) and (2).

Appendices

The Perturbed Riccati Equation

A. Fixed Initial and Final Times.

The solution of equation (2) is given by Kalman [2] as

$$E = \begin{bmatrix} \theta_{21} + \theta_{22} E_0 \end{bmatrix} \begin{bmatrix} \theta_{11} + \theta_{12} E_0 \end{bmatrix}^{-1} \quad (A1)$$

E_0 and E here will represent values at the beginning and end of an interval between corrections, t_{i+} and t_{i+1-} . The matrix

$$\theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \quad (A2)$$

is the transition matrix of

$$\dot{\underline{x}} = -F^T \underline{x} + H^T R^{-1} H \underline{w} \quad (A3)$$

$$\dot{\underline{w}} = F \underline{w} \quad (A4)$$

Obviously, $\theta_{21} = 0$ in this case and $\theta_{22} = \Phi_{i+1,i}$. θ_{11} is the transition matrix of the equation (A3), ~~and~~ ^{or} joint to (A4), and can be shown to be

$$\theta_{11} = \Phi_{i+1,i}^{-T} \quad (A5)$$

From Eq. (A1) we can deduce that, with $\theta_{21} = 0$,

$$\theta_{12} = E^{-1} \theta_{22} - \theta_{11} E_0^{-1} \quad (A6)$$

By perturbing Eq. (A1) we find

$$\begin{aligned}
\delta E &= \begin{bmatrix} \theta_{22} \delta E_0 \end{bmatrix} \begin{bmatrix} \theta_{11} + \theta_{12} E_0 \end{bmatrix}^{-1} \\
&\quad - \begin{bmatrix} \theta_{21} + \theta_{22} E_0 \end{bmatrix} \begin{bmatrix} \theta_{11} + \theta_{12} E_0 \end{bmatrix}^{-1} \theta_{12} \delta E_0 \begin{bmatrix} \theta_{11} + \theta_{12} E_0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \theta_{22} - E \theta_{12} \end{bmatrix} \delta E_0 \begin{bmatrix} \theta_{11} + \theta_{12} E_0 \end{bmatrix}^{-1}
\end{aligned} \tag{A7}$$

With $\theta_{21} = 0$, this can be written, using Eq. (A1),

$$\delta E = \begin{bmatrix} \theta_{22} - E \theta_{12} \end{bmatrix} \delta E_0 \begin{bmatrix} \theta_{22} E_0 \end{bmatrix}^{-1} E \tag{A8}$$

which, together with Eq. (A6), gives

$$\delta E = \begin{bmatrix} E \Phi^{-T} E_0^{-1} \end{bmatrix} \delta E_0 \begin{bmatrix} E_0^{-1} \Phi^{-1} E \end{bmatrix} \tag{A9}$$

B. Variable Initial Time.

We now consider the effect of varying the initial time on the Eq. (A9). We consider only the special case $\theta_{21} = 0$. E_0 and E represent E_{k+} and E_{k+1} . Equation (A1) can be written

$$E = \Phi E_0 \begin{bmatrix} \Phi^{-T} + \theta_{12} E_0 \end{bmatrix}^{-1} \tag{B1}$$

and Eq. (A6) can be written

$$\theta_{12} = E^{-1} \Phi - \Phi^{-T} E_0^{-1} \tag{B2}$$

Hence, from Eq. (B1),

$$\begin{aligned}
\delta E &= \begin{bmatrix} \delta \Phi E_0 + \Phi \delta E_0 \end{bmatrix} \begin{bmatrix} \Phi^{-T} + \theta_{12} E_0 \end{bmatrix}^{-1} \\
&\quad - E \begin{bmatrix} \delta \Phi^{-T} + \delta \theta_{12} E_0 + \theta_{12} \delta E_0 \end{bmatrix} \begin{bmatrix} \Phi^{-T} + \theta_{12} E_0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \delta \Phi E_0 + \Phi \delta E_0 \end{bmatrix} E_0^{-1} \Phi^{-1} E \\
&\quad - E \begin{bmatrix} \delta \Phi^{-T} + \delta \theta_{12} E_0 + \theta_{12} \delta E_0 \end{bmatrix} E_0^{-1} \Phi^{-1} E
\end{aligned} \tag{B3}$$

where, using Eq. (11),

$$\delta \Phi = -\Phi F_k \delta t_k \quad (B4)$$

$$\delta \Phi^{-T} = -\Phi^{-T} \delta \Phi^T \Phi^{-T} = \Phi^{-T} F_k^T \delta t_k \quad (B5)$$

The quantity $\delta \theta_{12}$ is determined by noting that the solution of Eqs. (A3) and (A4) can be expressed as

$$\underline{w}(t) = \Phi(t, t_k) \underline{w}(t_k) \quad (B6)$$

$$\underline{x}(t) = \Phi^{-T}(t, t_k) \underline{x}(t_k) + \int_{t_k}^t \Phi^{-T}(t, \tau) H^T(\tau) R^{-1}(\tau) H(\tau) \underline{w}(\tau) d\tau \quad (B7)$$

Since the last term represents $\theta_{12}(t, t_k) \underline{w}(t_k)$, it follows that

$$\theta_{12}(t, t_k) = \int_{t_k}^t \Phi^{-T}(t, \tau) H^T(\tau) R^{-1}(\tau) H(\tau) \Phi(\tau, t_k) d\tau \quad (B8)$$

from which

$$\begin{aligned} \frac{\partial}{\partial t_k} \theta_{12}(t_{k+1}, t_k) &= -\Phi^{-T} H_k^T R_k^{-1} H_k + \int_{t_k}^{t_{k+1}} \Phi^{-T}(t_{k+1}, \tau) H^T(\tau) R^{-1}(\tau) H(\tau) \frac{\partial}{\partial t_k} \Phi(\tau, t_k) d\tau \\ &= -\Phi^{-T} H_k^T R_k^{-1} H_k - \theta_{12}(t_{k+1}, t_k) F_k \end{aligned} \quad (B9)$$

where we have made use of Eqs. (B4) and (B8) in evaluating the last term.

Combining Eqs. (B2) - (B5), (B9) and (13), and simplifying,

$$\delta E = E \Phi^{-T} \left[H_k^T R_k^{-1} H_k + E_0^{-1} N_{k+} E_0^{-1} - F_k^T E_0^{-1} - E_0^{-1} F_k \right] \Phi^{-1} E \delta t_k \quad (B10)$$

References

- [1] Denham, W. F., and Speyer, J. L., "Optimal Measurement and Velocity Correction Programs for Midcourse Guidance," Raytheon report BR-2386, April 24, 1963.
- [2] Kalman, R. E., and Bucy, R. S., "New Results in Linear Filtering and Prediction Theory," ASME Journal of Basic Engineering, March 1961.