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PROJECTIVE SYNCHRONIZATION OF A CLASS OF UNCERTAIN CHAOTIC SYSTEMS VIA FEEDBACK IMPULSIVE CONTROL

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Abstract. *Robust function projective synchronization (FPS) of a class of an uncertain chaotic system with parameters and external disturbances on both master and impulsive slave systems is investigated in this paper. Based on the stability theory of impulsive differential equations, some new sufficient conditions guaranteeing the slave system will robustly synchronize the drive system up to a scaling function matrix are obtained via impulsive control and nonlinear feedback control. Compared with existing works, the considered parameters and external disturbances in master and slave systems are different. For illustration, a numerical example and the corresponding contrast test are given to show the feasibility and effectiveness of derived results.*

Keywords: Impulsive control, Robust synchronization, Function projective synchronization, Feedback control, Chaotic systems

1. **Introduction.** In recent years, control theory for chaos synchronization has attracted increasing attention for their potential applications in some engineering fields, such as image processing, chemical and biological systems, information science and secure communication [7, 9, 30]. A chaotic system has complex dynamical behaviors that possess some special features, such as excessive sensitivity to initial conditions, broad spectrums of Fourier transform, and bounded and fractal properties of the motion in the phase space. Since the pioneering works of Pecora and Carroll [31], many different types of synchronization have been investigated, such as complete synchronization [17], lag synchronization [4], generalized synchronization [14], impulsive synchronization [22], and phase synchronization [32]. To achieve chaos synchronization, a wide variety of schemes such as nonlinear feedback control [28], adaptive control [18], active control [34], sliding mode control [5], nonlinear observer [26] and impulsive control approach [13] are proposed.

In 1996, projective synchronization phenomenon was first reported and discussed by González-Miranda [11]. In 1999, Mainieri and Rehacek first proposed the concept of projective synchronization which is characterized that the drive and the response systems could be synchronized up to a scaling factor [25]. Then, Li [20, 21] considered a new synchronization method called modified projective synchronization (MPS), where the responses of the synchronized dynamical states synchronized up to a constant scaling matrix α. Especially, projective synchronization will reduce to the complete synchronization and

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anti-phase synchronization when $\alpha = I$ and $\alpha = -I$ (I is identity matrix), respectively. More recently, Chen et al. [2, 6] extended the modified projective synchronization and proposed a new projective synchronization called FPS, where the responses of the synchronized dynamical states synchronized up to a constant scaling function matrix. In the application of secure communication, the scaling function may also be a useful utility to improve the security of the secure communication scheme. So the study of function projective synchronization is well worth doing. Recently, some authors studied the problem of FPS and obtained some results. For example, [36] investigated the cluster modified FPS of a generalized linearly coupled network with asymmetric. [10] designed a reduced-order FPS scheme for synchronizing the hyperchaotic Rikitake system. [19] studied the hybrid projective synchronisation for incommensurate, integer and commensurate fractional-order financial systems with unknown disturbance and [37] investigated the bounded scaling FPS of uncertain chaotic systems using adaptive finite-time control.

In fact system parameters always fluctuate within some scopes in engineering applications because they are inevitably perturbed by external non-artificial factors such as environment temperature, voltage fluctuation, and mutual interfere among components. In addition, external disturbance is inevitable. Therefore, it is very necessary to take the fluctuation of system parameters and external disturbance into full account for the synchronization problem of two uncertain chaotic systems in hardware realizes. For uncertain systems, the problem is to devise a control that uses the dynamic equation to govern the trajectory of the system with acceptable performance. In terms of control tools, robust control is capable of compensation of both structured and unstructured uncertainties. In recent years, robust control of uncertain nonlinear systems is one of the most significant issues in control engineering sciences, and has attracted a lot of attention [8, 27, 29]. On the other hand, impulsive phenomena exist in many biological science and mechanics fields which are characterized by abrupt changes in the state of the systems at certain instants. Impulsive control has been studied and gradually become an interesting and useful synchronization approach [16, 23, 24, 39]. The importance of impulsive control lies in that, in some cases, impulse control may give an efficient method to deal with systems which cannot endure continuous disturbance. Using the impulsive control method, the response system needs to receive the information from the drive system only at some discrete instants and the responsive velocity is rapid; therefore, it has very strong advantage in practice due to reduced control cost. Moreover, it has been proved that impulsive synchronization approach is effective and robust in synchronization of chaotic system. In addition, the impulsive controllers, which are discontinuous, usually have simple structures. What is more, in an impulsive synchronization scheme, only the synchronization impulses are sent to the receiving system at the impulsive instants, which can decrease the information redundancy in the transmitted signal and increase robustness against the disturbances. In this sense, impulsive synchronization strategy is very useful in practical application, such as in digital communication systems.

To our best knowledge, there are few results about robust FPS of uncertain chaotic systems. [1, 12, 33, 35] investigated robust function projective synchronization of uncertain chaotic system with uncertain parameter, but did not consider the external disturbances on both master and slave systems. Motivated by the above mentioned comments, in the paper, we will discuss robust FPS of a class of uncertain chaotic systems with external disturbance and parameter perturbation. By using nonlinear feedback and impulsive control technique, some sufficient conditions for robust FPS are derived, which overcome some limitations of the previous work where function projective synchronization has been investigated only in chaotic systems without external disturbance and parameter perturbation.

The remainder of this paper is organized as follows. In Section 2, some necessary definitions and preliminary results are presented. In Section 3, some new criteria are obtained to ensure the robust FPS of a class of uncertain chaotic systems via nonlinear feedback impulsive control. In Section 4, a simulation is given to illustrate the effectiveness of the main results. Finally, conclusions are drawn in Section 5.

Notation: Throughout this paper, R^n denotes the n-dimensional Euclidean Space. $R^+ = [0, +\infty)$. The norm of $x = [x_1, x_2, \ldots, x_n]^T \in R^n$, $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. For any matrix $A = [a_{ij}]_{n\times n}$, A^T and A^{-1} denote the transpose and the inverse of A, respectively. $\|A\| = \sqrt{\rho(A^T A)}$ denotes the spectral norm of matrix A, where $\rho(M)$ denotes the spectral of matrix M.

2. **Problem Description and Preliminaries.** Consider the following uncertain chaotic drive systems with external disturbance and parameter perturbation:

$$\dot{x}(t) = (A + \Delta A_1(t))\, x(t) + f(x(t)) + h_1(t), \tag{1}$$

where $x \in R^n$ is the state vector, $A \in R^{n\times n}$ is a constant matrix, $\Delta A_1 \in R^{n\times n}$ is a perturbation matrix norm-bounded by $\|\Delta A_1\| \le l_1$, which represents the system parametric uncertainties, $h_1(t)$ is external disturbance, and $f(x(t)) \in R^n$ stands for the nonlinear part of the system, satisfying Lipschitz condition, namely

$$\|f(y(t)) - f(x(t))\| \le L\|y(t) - x(t)\|. \tag{2}$$

The response system for (1) is constructed as follows:

$$\begin{cases} \dot{y}(t) = (A + \Delta A_2(t))y(t) + f(y(t)) + h_2(t) + U, \ t \ne t_k, \\ \Delta y = y(t_k^+) - y(t_k^-) = B_k(y - \alpha(t)x(t)), \ t = t_k, \ k = 1, 2, \ldots, \\ y(t_0^+) = y_0, \end{cases} \tag{3}$$

where U is a continuous robust controller, f is the same function as defined before, $\Delta A_2 \in R^{n\times n}$ is a perturbation matrix norm-bounded by $\|\Delta A_2\| \le l_2$, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{t\to t_k^+} y(t)$, $y(t_k^-) = \lim_{t\to t_k^-} y(t)$. In general, for simplicity, it is assumed that $y(t_k) = y(t_k^-)$, which means that $y(t)$ is left continuous. The impulsive instant sequence $\{t_k\}_{k=1}^{+\infty}$ satisfies $0 \le t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots \to \infty (k \to \infty)$, $B_k \in R^{n\times n}$ denotes the impulsive control gains. $\alpha(t)$ is a continuously differentiable scaling function factor with norm-bounded by $\|\alpha(t)\| \le l_3$. In the same way we assume that there must exit positive constant l_4, l_5 such that $\|h_2(t) - h_1(t)\| \le l_4$ and $\|\Delta A_2(t) - \Delta A_1(t)\| \le l_5$.

Remark 2.1. *Many chaotic systems with external disturbance and parameter perturbation can be described by mode (1). Some examples are listed in Table 1.*

Definition 2.1. *Systems (1) and (3) are robust FPS, if there exists a scaling function matrix $\alpha(t)$ such that*

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t) - \alpha(t)x(t)\| = 0. \tag{4}$$

It follows from systems (1) and (3) that the error dynamical system is governed as follows:

$$\begin{cases} \dot{e}(t) = \dot{y}(t) - \alpha(t)\dot{x}(t) - \dot{\alpha}(t)x(t) \\ \qquad = Ae(t) + f(y(t)) - \alpha(t)f(x(t)) + h_2(t) - \alpha(t)h_1(t) + \Delta A_2(t)y(t) \\ \qquad - \alpha(t)\Delta A_1(t)x(t) - \dot{\alpha}(t)x(t) + U, \ t \ne t_k, \\ \Delta e(t_k) = [y(t_k^+) - \alpha(t_k^+)x(t_k^+)] - \left[y(t_k^-) - \alpha(t_k^-)x(t_k^-)\right] = B_k e(t_k), \ t = t_k. \end{cases} \tag{5}$$

Table 1. Systems described using model (1)

System	Matrix A	Nonlinear parts $f(x)$
Van der Pol oscillator	$\begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}$	$\begin{pmatrix} 0 \\ -\varepsilon x_2^2 x_1 \end{pmatrix}$
Lorzenz system	$\begin{pmatrix} -10 & 10 & 0 \\ P & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}$
Chen system	$\begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}$
Lü system	$\begin{pmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}$

It is obvious that FPS between the system (1) and the system (3) will be achieved under feedback impulsive control if and only if the error system (5) is robustly asymptotically stable at zero point. So, the main objective of this paper is to design a suitable nonlinear feedbck control inputs U, impulsive control gains B_k and impulsive distances $\tau_k = t_k - t_{k-1}$ such that error system (5) is robustly asymptotically stable at zero point.

In order to obtain our results, we present some definitions and lemmas.

Consider the impulsive functional differential equation

$$\begin{cases} \dot{x}(t) = f(t, x(t)),\ t \neq t_k \\ x(t_k^+) = B_k(x(t_k^-)),\ t = t_k,\ k = 1, 2, \dots \end{cases} \tag{6}$$

where $f : R^+ \times PC \to R^n$ is a continuous function and $B_k(x) : S(\rho) \to R^n$ for each $k \in N$, and PC= $\{\varphi$ is continuous every-where except at the finite number of point $\bar{t}$ at which $\varphi(\bar{t}^+)$ and $\varphi(\bar{t}^-)$ exist and $\varphi(\bar{t}^+) = \varphi(\bar{t}^-)\}$, $S(\rho) = \{x \in R^n : |x| < \rho\}$.

Definition 2.2. [38] *Let $V : R_+ \times S(\rho) \to R^n$, and then V is said to belong to class ν_0 if*
1) V is continuous in each of the sets $[t_{k-1,t_k}) \times S(\rho)$, and for each $x \in S(\rho)$

$$\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$$

exist for $k = 1, 2, \dots$;
2) V is locally Lipschitzian in $x \in S(\rho)$, $V(t, 0) = 0$.

Definition 2.3. [38] *For $(t, x) \in [t_{k-1}, t_k) \times R^n$, we define*

$$D^+V(t, x) = \lim_{h\to 0^+} \frac{1}{h}\{V(t + h, x + hf(t, x)) - V(t, x)\}.$$

Lemma 2.1. [38] *The zero solution of system (6) is uniformly stable if there exist $V \in V_0$, $\omega_1, \omega_2 \in K$, $\psi \in K^*$ and $H \in \Omega$ such that*
1) $\omega_1(\|x\|) \le V(t, x) \le \omega_2(\|x\|)$ for $(t, x) \in R^+ \times S(\rho)$;
2) for all $x \in S(\rho_1)$, $0 < \rho_1 \le \rho$, $V(t_k, B_k(x)) \le \varphi(V(t_k^-, x))$ for all k;
3) for any solution $x(t)$ of Equation (6),

$$V(t, x(t)) \le \varphi^{-1}(V(t, x(t)))$$

implies that

$$D^+V(t,x(t)) \le g(t)H(V(t,x(t))),$$

where $g : R^+ \to R^+$ *is locally integrable, and* φ^{-1} *is the inverse function of* φ*;*

4) H *is nondecreasing and there exist constants* $\lambda_2 \ge \lambda_1 > 0$ *and* $A > 0$ *such that for any* $\mu > 0$,

$$\lambda_1 \le t_k - t_{k-1} \le \lambda_2$$

and

$$\int_{\varphi(\mu)}^{\mu} \frac{du}{H(u)} - \int_{t_{k-1}}^{t_k} g(s)ds \ge A,$$

where $K = \{\omega \in C(R^+, R^+), \omega(s)$ *is strictly increasing and* $\omega(0) = 0\}$, $K^* = \{\psi \in K, \varphi(s) < s, \forall s > 0\}$, $\Omega = \{H \in C(R^+, R^+), H(0) = 0, H(s) > 0, \forall s > 0\}$.

Lemma 2.2. [15] *For any vectors* $x, y \in R^n$ *and the positive definite matrix* $\Omega \in R^{n\times n}$*, the following inequality is satisfied:*

$$2x^Ty \le x^T\Omega x + y^T\Omega^{-1}y. \tag{7}$$

Lemma 2.3. [3] *Let* x*,* y *be real vectors of appropriate dimension,* A*,* $B(t)$ *are real matrices of appropriate dimensions,* $\|B(t)\| \le r$*, we have*

$$2x^TA^TB(t)y \le x^TA^TAx + r^2y^Ty. \tag{8}$$

In the following, by employing above lemmas, we will present controller design method such that error system (5) is asymptotically stable.

3. Main Results.

Theorem 3.1. *Let* β_k *be the largest eigenvalue of* $(I + B_k)^T(I + B_k)$*, if there exist a positive-definite symmetric matrix* P*, positive diagonal matrices* R*,* S*,* T*,* W *and constant* $\eta > 0$ *such that the following conditions hold:*

1) the continuous robust controller is defined as:

$$\begin{aligned} U = {} & \alpha(t)h_1(t) - h_1(t) - f(\alpha(t)x(t)) + \alpha(t)f(x(t)) + \dot{\alpha}(t)x(t) \\ & -\frac{R^{-1}l_4^2 + T^{-1}l_3^2l_5^2x(t)^Tx(t)}{2\|e(t)\|^2}P^{-1}e(t); \end{aligned} \tag{9}$$

2) $PA + A^TP + (R + S + T + W)P^2 + (W^{-1}l_2^2 + S^{-1}L_2^2) \le \eta P$*;*

3) $\ln(\beta_k) + \eta \sup(t_k - t_{k-1}) < 0$*,*

then the error dynamical system (5) is asymptotically stable, which means that the system (3) is robustly function projective synchronized with system (1).

Proof: Let

$$\triangle = -\frac{R^{-1}l_4^2 + T^{-1}l_3^2l_5^2x(t)^Tx(t)}{2\|e(t)\|^2}P^{-1}e(t), \tag{10}$$

and construct the Lyapunov function

$$V(t,e(t)) = e(t)^TPe(t).$$

Taking the Dini derivative of V along the trajectories of Equation (5) and substituting (3) into (5), when $t \ne t_k$, we obtain

$$\begin{aligned} D^+V(t,e(t)) &= \dot{e}(t)^TPe(t) + e(t)^TP\dot{e}(t) \\ &= \big\{Ae(t) + f(y(t)) - f(\alpha(t)x(t)) + \triangle + h_2(t) - h_1(t) + \Delta A_2(t)y(t) \\ &\quad - \alpha(t)\Delta A_1(t)x(t)\big\}^TPe(t) + e(t)^TP\big\{Ae(t) + f(y(t)) - f(\alpha(t)x(t)) \end{aligned}$$

$$
\begin{aligned}
&+\triangle+h_2(t)-h_1(t)+\Delta A_2(t)y(t)-\alpha(t)\Delta A_1(t)x(t)\big\} \\
=\;& e(t)^T\left\{PA+A^TP+\Delta A_2(t)^TP+P\Delta A_2(t)\right\}e(t) \\
&+2(f(y(t))-f(\alpha(t)x(t)))^TPe(t)+2e(t)^TP(h_2(t)-h_1(t)) \\
&+2e(t)^TP(\Delta A_2(t)-\Delta A_1(t))\alpha(t)x(t)+2e(t)^TP\triangle. \qquad (11)
\end{aligned}
$$

It follows from Lemmas 2.2 and 2.3 that

$$2e(t)^TP(h_2(t)-h_1(t))e(t)\le e(t)^TRP^2e(t)+R^{-1}l_4^2, \qquad (12)$$

$$2(f(y(t))-f(\alpha(t)x(t)))^TPe(t)\le e(t)^TSP^2e(t)+S^{-1}L^2e(t)^Te(t), \qquad (13)$$

$$2e(t)^TP\Delta A_2(t)e(t)\le e(t)^TWP^2e(t)+W^{-1}l_2e(t)^Te(t), \qquad (14)$$

$$2e(t)^TP(\Delta A_2(t)-\Delta A_1(t))\alpha(t)x(t)\le e(t)^TTP^2e(t)+T^{-1}l_3^2l_5^2x(t)^Tx(t). \qquad (15)$$

By virtue of inequalities (12)-(15) and combining (10) with (11), one has

$$
\begin{aligned}
D^+V(t,e(t))\le\;& e(t)^T\left\{PA+A^TP+(R+S+T+W)P^2+\left(W^{-1}l_2^2+S^{-1}L_2^2\right)\right\}e(t) \\
&+R^{-1}l_4^2+2e(t)^TP\triangle+T^{-1}l_3^2l_5^2x(t)^Tx(t)\le\eta e(t)^TPe(t). \qquad (16)
\end{aligned}
$$

On the other hand, when $t=t_k$

$$
\begin{aligned}
V(t_k,e(t_k))&=e(t_k)^TPe(t_k)=\left[(I+B_k)e(t_k)\right]^TP\left[(I+B_k)e(t_k)\right] \\
&\le\lambda_{\max}\left[(I+B_k)^T(I+B_k)\right]e(t_k)^TPe(t_k)=\beta_kV(t_k,e(t_k)). \qquad (17)
\end{aligned}
$$

Let $\varphi(s)=\beta_ks$, namely, $\varphi\in K^*$, we chose $g(t)=1$, $H(s)=\eta s$ and $A=-\ln(\beta_k)-\eta\sup(t_k-t_{k-1})>0$ in Lemma 2.1, then for any $\mu>0$ and all k,

$$
\begin{aligned}
\int_{\varphi(\mu)}^{\mu}\frac{du}{H(u)}-\int_{t_{k-1}}^{t_k}g(s)ds&=\frac{-\ln(\beta_k)}{\eta}-(t_k-t_{k-1}) \\
&\ge-\ln(\beta_k)-\eta\sup(t_k-t_{k-1})=A>0. \qquad (18)
\end{aligned}
$$

Thus, it follows from Lemma 2.1 that the origin of error dynamical system (5) is asymptotically stable. This completes the proof.

Remark 3.1. *The aim of introduction of positive diagonal matrices R, S, T, W is to increase the flexibility of the design. The robust controller U can be uniquely determined only if the matrices R, S, T, W satisfy the condition 2) in Theorem 3.1.*

In practice, for convenience, the gain matrix B_k is always selected as a constant matrix and the impulsive intervals $\tau_k=t_k-t_{k-1}$ $(k=1,2,\ldots)$ are set to be a positive constant. Thus, we have the following corollary.

Corollary 3.1. *Assume $\tau_k=\tau$ and matrix $B_k=B$ $(k=1,2,\ldots)$. Let β be the largest eigenvalue of $(I+B)^T(I+B)$, if there exist a positive-definite symmetric matrix P, positive diagonal matrices R, S, T, W and constant scalar $\eta>0$ such that the following conditions hold:*

1) the continuous robust controller is defined as:

$$
\begin{aligned}
U=\;&\alpha(t)h_1(t)-h_1(t)-f(\alpha(t)x(t))+\alpha(t)f(x(t))+\dot\alpha(t)x(t) \\
&-\frac{R^{-1}l_4^2+T^{-1}l_3^2l_5^2x(t)^Tx(t)}{2\|e(t)\|^2}P^{-1}e(t); \qquad (19)
\end{aligned}
$$

2) $\left(PA+A^TP+(R+S+T+W)P^2+\left(W^{-1}l_2^2+S^{-1}L_2^2\right)\right)\le\eta P$;

3) $\ln(\beta)+\eta\tau<0$,

then the error dynamical system (5) is asymptotically stable, which means that system (3) is robustly function projective synchronized with system (1).

Remark 3.2. *It is very simple to calculate the largest eigenvalue of* $\big(PA+A^TP+(R+S+T+W)P^2+\big(W^{-1}l_2^2+S^{-1}L_2^2\big)\big)=\delta$*. According to condition 3) in Corollary 3.1, the interval* τ *of impulses for robust FPS scheme can be estimated as follows:*

$$\tau < \frac{-\ln\beta}{\delta}. \tag{20}$$

Moreover, let $P=I$ (I is identity matrix) and $\lambda_{\max}\big(PA+A^TP+(R+S+T+W)P^2+\big(W^{-1}l_2^2+S^{-1}L_2^2\big)\big)=\delta$. One obtains the following corollary.

Corollary 3.2. *Assume* $\tau_k=\tau$ *and matrix* $B_k=B$ $(k=1,2,\ldots)$*. Let* β *be the largest eigenvalue of* $(I+B)^T(I+B)$*, if there exist positive diagonal matrices* R*,* S*,* T*,* W *such that the following conditions hold:*
1) the continuous robust controller is defined as:

$$\begin{aligned} U &= \alpha(t)h_1(t)-h_1(t)-f(\alpha(t)x(t))+\alpha(t)f(x(t))+\dot{\alpha}(t)x(t)\\ &\quad -\frac{W^{-1}l_4^2+S^{-1}l_3^2l_5^2x(t)^Tx(t)}{2\|e(t)\|^2}e(t); \end{aligned} \tag{21}$$

2) $\ln\beta+\delta\tau<0$*, where* $\lambda_{\max}\big(A+A^T+(R+S+T+W)+\big(W^{-1}l_2^2+S^{-1}L_2^2\big)\big)=\delta$*,*
then the error dynamical system (5) is asymptotically stable, which means that system (3) is robustly function projective synchronized with system (1).

Proof: Let $V(t,e(t))=e(t)^Te(t)$, and we can easily obtain the corollary on the basis of the proof procedure of Theorem 3.1.

Specially, by letting $R=S=T=W=I$ in Corollary 3.2, we get the following corollary.

Corollary 3.3. *Assume* $\tau_k=\tau$ *and matrix* $B_k=B$ $(k=1,2,\ldots)$*. Let* β *be the largest eigenvalue of* $(I+B)^T(I+B)$*, if the following conditions hold:*
1) the continuous robust controller is defined as:

$$\begin{aligned} U &= \alpha(t)h_1(t)-h_1(t)-f(\alpha(t)x(t))+\alpha(t)f(x(t))+\dot{\alpha}(t)x(t)\\ &\quad -\frac{l_4^2+l_3^2l_5^2x(t)^Tx(t)}{2\|e(t)\|^2}e(t); \end{aligned} \tag{22}$$

2) $\ln\beta+\delta\tau<0$*, where* $\lambda_{\max}\big(A+A^T+\big(4+l_2^2+L_2^2\big)I\big)=\delta$*,*
then the error dynamical system (5) is asymptotically stable, which means that system (3) is robustly function projective synchronized with system (1).

From Equations (12)-(15), we find that there exist the external disturbances but no parameter perturbation if $h_1(t)=h_2(t)$ and $\Delta A_2(t)=\Delta A_1(t)=0$. Then system (1) is an external disturbance system. So we have the following criteria to ensure the FPS.

Corollary 3.4. *Let* β_k *be the largest eigenvalue of* $(I+B_k)^T(I+B_k)$*, if there exist a positive-definite symmetric matrix* P*, a positive diagonal matrix* S *and constant* $\eta>0$ *such that the following conditions hold:*
1) the continuous robust controller is defined as:

$$U=\alpha(t)h_1(t)-h_1(t)-f(\alpha(t)x(t))+\alpha(t)f(x(t))+\dot{\alpha}(t)x(t); \tag{23}$$

2) $\big(PA+A^TP+SP^2+S^{-1}L_2^2\big)\le\eta P$*;*
3) $\ln(\beta_k)+\eta\sup(t_k-t_{k-1})<0$*,*

then the drive and response systems with external disturbances are robust FPS.

Remark 3.3. *In this paper, we aim to investigate robust FPS of a class of uncertain chaotic system with external disturbances and parameter perturbation via feedback and impulsive control. Results about the model are scarcely explored in the literature. In fact, from the process of the proof in Theorem 3.1, it is very easy to deduce that robust function projective synchronization of a class of uncertain chaotic system with external disturbances and parameter perturbation could also be achieved if conditions 1) and 2) in Theorem 3.1 are satisfied. However, there may exist the problem of optimization that response time is long. In order to solve this question, we adopt the way of impulsive control to realize synchronization, whose advantage lies in reducing the information redundancy in the transmitted signal and increasing robustness against the disturbances, especially, the responsive velocity is high and reducing the control cost.*

Remark 3.4. *From a practical point of view, the impulses cannot be too large, because of the practical impossibility of applying impulses of very large amplitude. Note that the impulsive gain matrix and the impulsive interval have important influence on achieving synchronization. According to the actual demand, we can choose an adequate matrix B_k such that the impulses are small based on condition 3) in Theorem 3.1.*

Remark 3.5. [1, 12, 33, 35] *considered robust function projective synchronization of uncertain chaotic system with uncertain parameter, but did not address the external disturbances on both master and slave systems. Therefore, the considered model (1) is more general and the obtained results have greater potential application.*

4. **Numerical Example.** In this section, a numerical example will be given to verify theoretical results obtained in the previous section. The simulation is performed using Matlab software, and the simulation step size is selected as 0.0001.

Consider uncertain Lorenz systems with different parameters and external perturbations. Assume drive system (1) be an uncertain Lorenz system with the following parameters:

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & b \end{bmatrix}, \quad f(x(t)) = \begin{bmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{bmatrix},$$

$$h_1(t) = \begin{bmatrix} 0.2\sin 2t \\ 0.5\cos 6t \\ 0.6\sin 4t \end{bmatrix}, \quad \Delta A_1(t) = \begin{bmatrix} -0.13\cos(2t) & 0.13\cos 2t & 0 \\ 0.24\sin 5t & 0 & 0 \\ 0 & 0 & -0.6\cos 3t \end{bmatrix},$$

where $\sigma = 10$, $r = 28$, $b = 8/3$. Lorenz system will show chaotic behavior with parameter $\sigma = 10$, $r = 28$, $b = 8/3$. The attractors of the Lorenz model with parameters and external perturbations are as shown in Figure 1. Figure 2 displays the trajectory of drive system for the Lorenz model with parameters and external perturbation. Moreover, the parameter and the external disturbances in the response system (3) are chosen as:

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & b \end{bmatrix}, \quad f(y(t)) = \begin{bmatrix} 0 \\ -y_1(t)y_3(t) \\ y_1(t)y_2(t) \end{bmatrix},$$

$$h_2(t) = \begin{bmatrix} 0.12\cos 3t \\ 0.25\cos 5t \\ 0.36\sin 6t \end{bmatrix}, \quad \Delta A_2(t) = \begin{bmatrix} -0.2\sin 4t & 0.2\sin 2t & 0 \\ -0.12\cos 2t & 0 & 0 \\ 0 & 0 & 0.25\sin 6t \end{bmatrix}.$$

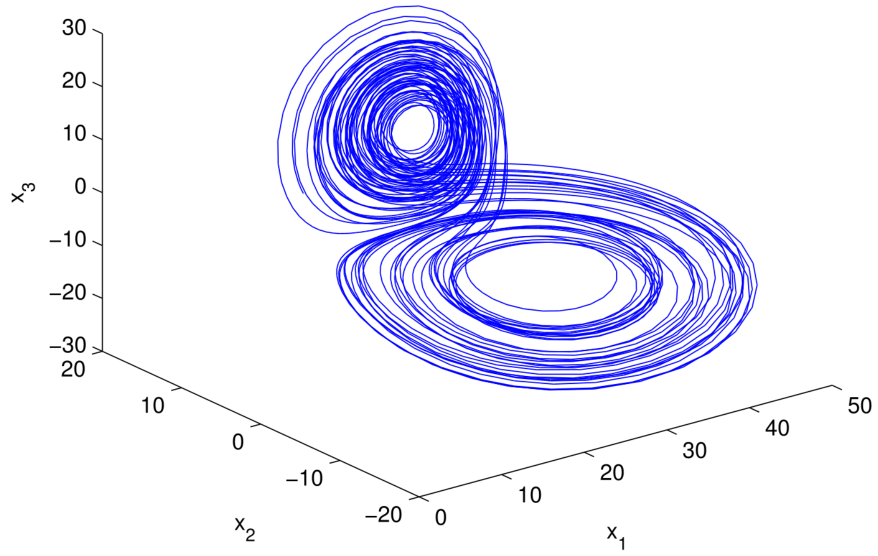

Figure 1. Attractors of the Lorenz model with parameter and external perturbations

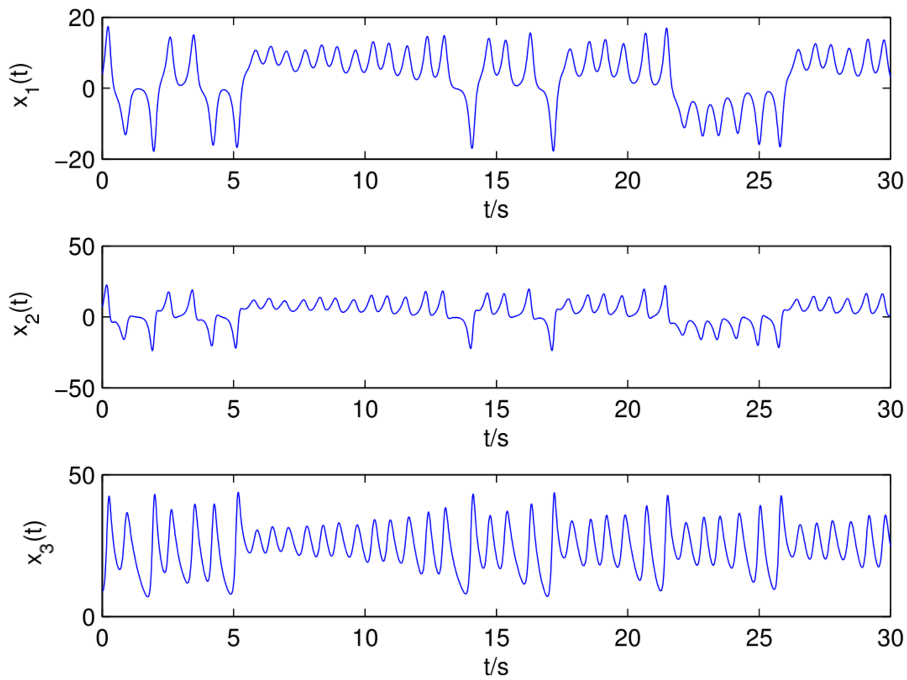

Figure 2. The trajectory of the Lorenz system with parameter and external perturbations

The initial states of the drive and response systems are taken as $x(0) = (4, 7, 9)^T$, $y(0) = (0, 1, 1)^T$ and $L = 88$, respectively. It is easy to verify that $\|h_1(t) - h_2(t)\| < 1.3$, $\|\Delta A_2(t) - \Delta A_1(t)\| < 1.1$, $\|\Delta A_2(t)\| < 0.4$. Choose $\tau_k = \tau$ and matrix $B_k = \text{diag}(-0.45, -0.57, -0.7)$ $(k = 1, 2, \ldots)$, then $\beta = \lambda_{\max}(I + B)^T(I + B) = 0.3025$, let $P = W = T = R = I$ and $S = \text{diag}(44, 44, 44)$, one has $\delta = \lambda_{\max}\big(A + A^T + W + S + R + T + \big(W^{-1}l_2^2 + S^{-1}L_2^2\big)\big) = 234.9877$. According to Equation (20), $\tau < \frac{-\ln\beta}{\delta} = 0.0051$. In the simulation, we set the scaling function as $\alpha(t) = \text{diag}(\cos 2t, \sin t, \sin 3t)$ and let $\tau = 0.005$, the controller U is given by Equation (21). According to Corollary 3.2, the equilibrium point of error dynamical system is asymptotically stable, which implies robust FPS is achieved. Simulation results are shown in Figure 3.

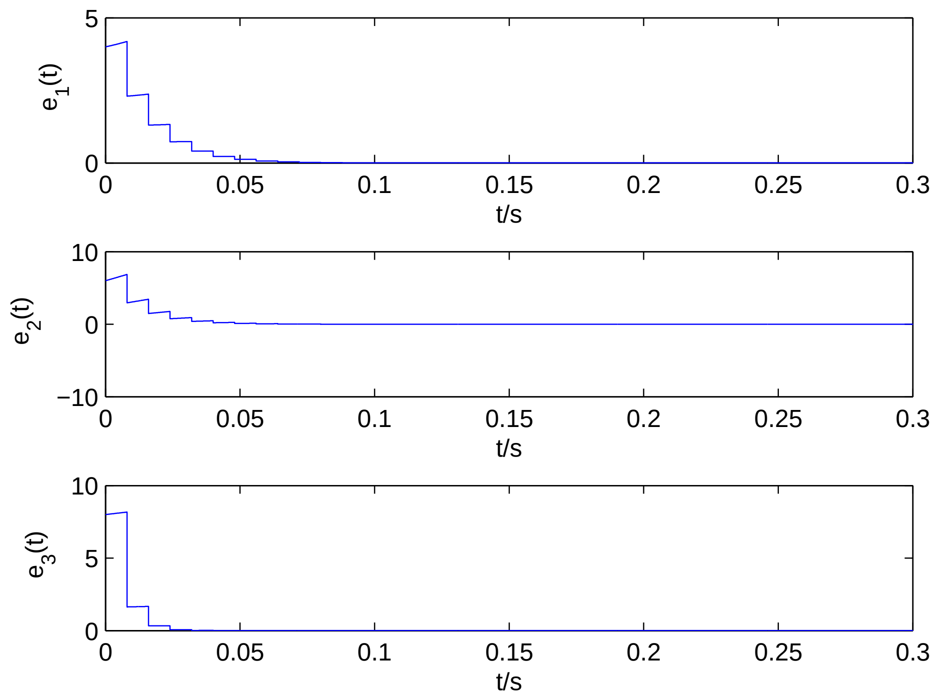

FIGURE 3. The robust FPS errors with $\tau = 0.005$

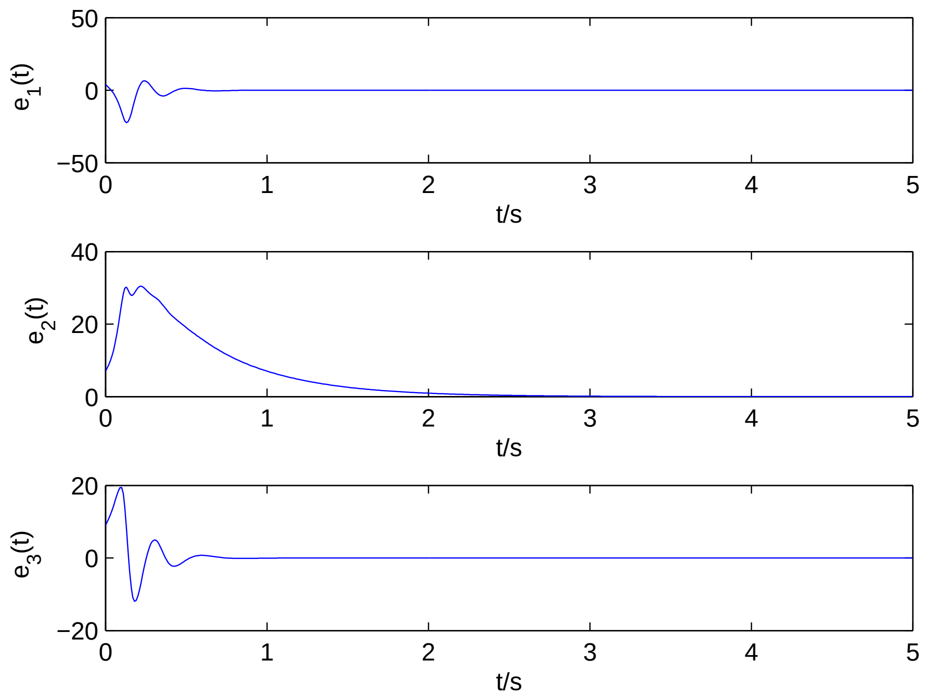

FIGURE 4. The robust FPS errors without impulsive control

Actually, if there does not exist impulsive control, robust FPS is also achieved, see Figure 4. Through the contrast numerical simulation, it is easy to find that the responsive time is more longer than the case of with impulsive control.

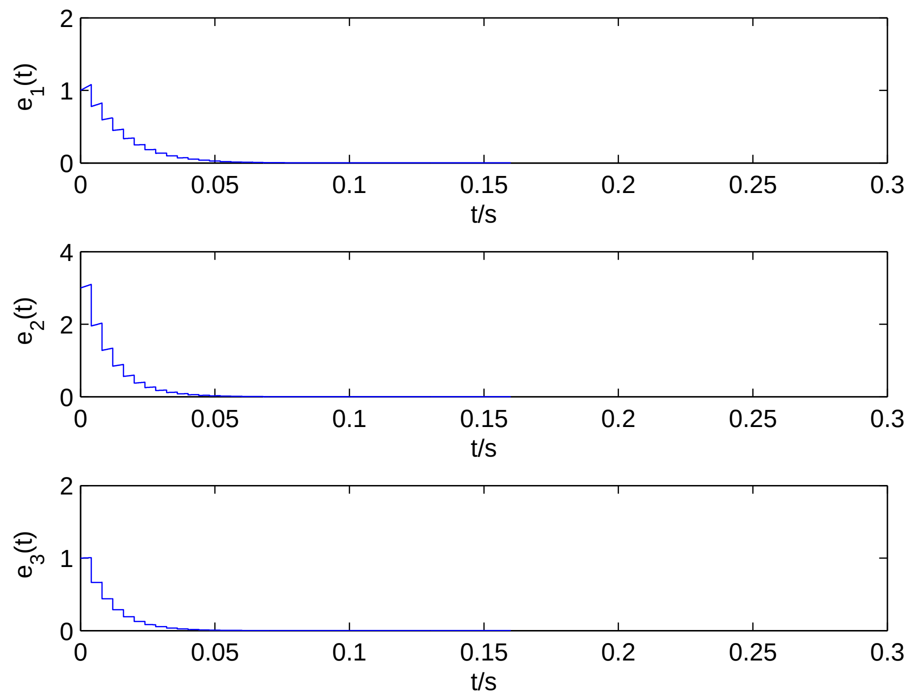

FIGURE 5. The robust FPS errors with $\tau = 0.004$

If there exist the external disturbances but no parameter perturbation, then the above uncertain Lorenz systems become the following system:

$$\dot{x}(t) = Ax(t) + f(x(t)) + h_1(t), \tag{24}$$

where A, $f(x(t))$, $h_1(t)$ are the same defined as before. We can realize the FPS of system (24) on the basis of Corollary 3.4. The initial states of the drive system and the response system are chosen as $x(0) = [3\ 4\ 5]^T$ and $y(0) = [2\ 1\ 4]^T$, respectively. Let $\tau_k = \tau$ and matrices $B_k = \text{diag}(-0.28, -0.37, -0.34)$ $(k = 1, 2, \ldots)$, then $\beta = \lambda_{\max}(I + B)^T(I + B)=$ 0.5184. Let $P = I$, $S = \text{diag}(88, 88, 88)$, $L = 88$, $\eta = \lambda_{\max}\left(A + A^T + S + S^{-1}L^2\right) =$ 160.0512, the impulsive interval can be estimated. $\tau < \frac{-\ln\beta}{\eta} = 0.0041$. In the simulation, we chose the scaling function as $\alpha(t) = \text{diag}(\cos 2t, \sin t, \sin 3t)$ and let $\tau = 0.004$. Figure 5 shows the results about robust FPS of the drive and response systems for the Lorenz mode (24) under the controller U given by Equation (24).

5. **Conclusions.** In this paper, based on stability results for impulsive FDE, some criteria are provided to ensure the function projective synchronization of chaotic systems with different parameters perturbation and different external disturbances on both master system and slave system via feedback impulsive control. There are few papers about the robust FPS of uncertain systems, which often exist in real systems. The feasibility and effectiveness of the method have been validated by computer simulation of the Lorenz model with parameter and external perturbations. Note that we only consider the system with parameter perturbation which is the coefficient of state variable, but do not discuss the case which parameter perturbation is added to the nonlinear items of the state equation. It is also an important topic that will be considered in another papers.

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