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Ultraproducts and possible worlds semantics in institutions

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Abstract

We develop possible worlds (Kripke) semantics at the categorical abstract model theoretic level provided by the so-called 'institutions'. Our general abstract modal logic framework provides a method for systematic Kripke semantics extensions of logical systems from computing science and logic. We also extend the institution-independent method of ultraproducts of [R. Diaconescu, Institution-independent ultraproducts, Fundamenta Informaticæ55 (3–4) (2003) 321–348] to possible worlds semantics and prove a fundamental preservation result for abstract modal satisfaction. As a consequence we develop a generic compactness result for possible worlds semantics.

Keywords: Institutions; Possible worlds; Kripke models; Modal logic; Ultraproducts

1. Introduction

The theory of "institutions" [2] is a categorical abstract model theory which formalizes the intuitive notion of logical system, including syntax, semantics, and the satisfaction between them. It provides the most complete form of abstract model theory, the only one including signature morphisms, model reducts, and even mappings (morphisms) between logics as primary concepts. Institutions are more general than Barwise's 'abstract model theory' [3] which still keeps a strong commitment to classical logic. They are also more general than categorical approaches to model theory represented by works on sketches [4–6] or on satisfaction as cone injectivity [7–12] since in institution theory the satisfaction relation is axiomatized rather than being defined. Institutions have been recently also extended towards proof theory [13,14] in the spirit of categorical logic [15].

The concept of institution arose within computing science (algebraic specification) in response to the population explosion among logics in use there, with the ambition of doing as much as possible at a level of abstraction independent of commitment to any particular logic [2,16,17]. Besides its extensive use in specification theory (it has become the most fundamental mathematical structure in algebraic specification theory), there have been several

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substantial developments towards an "institution-independent" (abstract) model theory [18,19,1,20–23]. A textbook dedicated to this topic is under preparation [24,25] is a recent survey.

The significance of institution-independent model theory is manifold. First, it provides model theoretic results and analysis for various logics in a generic way. Apart from reformulation of standard concepts and results in a very general setting, thus applicable to many logical systems, institution-independent model theory has already produced a series of new significant results in classical model theory [20,22,26].

Then, institution-independent model theory provides a new *top-down* way of doing model theory, making explicit the generality and power of concepts by placing them at the right level of abstraction and thus extracting the essence of the results independently of the largely irrelevant details of the particular logic in use. This leads to a deeper conceptual understanding guided by a structurally clean causality. Concepts come naturally as presumptive features that "a logic" might exhibit or not, hypotheses are kept as general as possible and introduced on a by-need basis, results and proofs are modular and easy to track down despite their sometimes very deep content.

Possible worlds semantics (also called *Kripke semantics* [27]) is a development in the area of non-classical logics. Apart from its great influence in philosophy, logic, and linguistics, possible worlds semantics have been repeatedly applied to computing and in particular in the dynamic logic of programs [28–30], process algebra [31,32] and the temporal logic's approach to concurrency [33–35].

In this paper we introduce possible worlds semantics at an institution-independent level. This means we can develop modal satisfaction (and consequently treat the satisfaction of modalities) on top of an abstract satisfaction relation. To be a bit more specific, given a base institution with amalgamation property of models, we can define a concept of Kripke model employing the models of the base institution. The sharing constraint for the Kripke models (i.e. how much of the structure is shared between the models of the Kripke model) is managed abstractly by an institution morphism from the base institution to a simpler 'domain' institution. This provides a flexible method for tuning the level of rigidness of the Kripke models.

This internalization of possible worlds semantics allows an extension of the satisfaction relation of the base institution to modal satisfaction for sentences extending the base sentences with modalities, Boolean connectives, quantifiers. We prove that this yields a 'modal' institution on top of the base institution.

Our institution-independent study of modal satisfaction extends the institution-independent method of ultraproducts [1] to possible worlds semantics in a rather smooth way thanks to the categorical concept of filtered products which applies uniformly both for the base models and for the Kripke models. The main result here is that modal satisfaction is preserved by ultraproducts of Kripke models. By employing general results of [1], an immediate consequence of this is model compactness of possible worlds semantics.

Because our institutional approach to modal logic is a model theoretic one, it differs from what is generally known under the name of 'categorical modal logic' which is proof theoretic inspired (a good reference of the latter is [36]).

For reasons of simplicity of presentation in this paper we consider only standard modalities possibility $\Diamond$ and necessity $\Box$. However our method can be extended to other modalities and more refined types of possible worlds semantics.

The main relevance of our work to computing science is that it provides an uniform smooth method for the combination of modalities and possible worlds semantics with any other logical formalism. This constitutes a good foundation for a sound combination between computing paradigms based upon some form of modal logic and other computing paradigms.

The audience of this work consists primarily of people with some background in formal specification theory (including the institutional approaches) and only secondarily of people from the modal logic community.

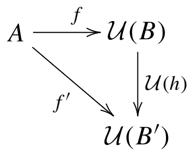
2. Institutional preliminaries

2.1. Categories

We assume the reader is familiar with some basic categorical notions, such as category, functor, (co)limits, and with the standard notations from category theory; e.g., see [37] for an introduction to this subject. Here we recall very briefly some of the notations and of the concepts used by our work.

By way of notation, $|\mathbb{C}|$ denotes the class of objects of a category $\mathbb{C}$, $\mathbb{C}(A, B)$ the set of arrows with domain A and codomain B, and composition is denoted by ";" and in diagrammatic order. The category of sets (as objects) and functions (as arrows) is denoted by $\mathbb{S}et$, and $\mathbb{CAT}$ is the category of all categories (as objects)[1] and functors (as arrows). The opposite of a category $\mathbb{C}$ (obtained by reversing the arrows of $\mathbb{C}$) is denoted $\mathbb{C}^{op}$.

Given a functor $\mathcal{U}\colon \mathbb{C}' \to \mathbb{C}$, for any object $A \in |\mathbb{C}|$, the *comma category* $A/\mathcal{U}$ has arrows $f\colon A \to \mathcal{U}(B)$ as objects (sometimes denoted as (f, B)) and $h \in \mathbb{C}'(B, B')$ with $f;\mathcal{U}(h) = f'$ as arrows $(f, B) \to (f', B')$.



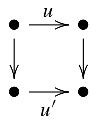
When $\mathbb{C} = \mathbb{C}'$ and $\mathcal{U}$ is the identity functor the category $A/\mathcal{U}$ is denoted by $A/\mathbb{C}$.

A *J-(co)limit* in a category $\mathbb{C}$ is a (co)limit of a functor $J \to \mathbb{C}$. When J is a directed partial order the J-colimits are called *directed* colimits.

A functor $L\colon J' \to J$ is called *final* if for each object $j \in |J|$ the comma category j/L is non-empty and connected. Consequently, a subcategory $J' \subseteq J$ is final when the corresponding inclusion functor is final. Let us recall the following important result.

Theorem 1 ([37]). *For each final functor $L\colon J' \to J$ and each functor $D\colon J \to \mathbb{C}$, there exists a colimit $\mu\colon D \Rightarrow Colim(D)$ and a canonical isomorphism $h\colon Colim(L; D) \to Colim(D)$ when a colimit $\mu'\colon L; D \Rightarrow Colim(L; D)$ exists.*

A class of arrows $\mathcal{S} \subseteq \mathbb{C}$ in a category $\mathbb{C}$ is *stable under pushouts* if for any pushout square in $\mathbb{C}$



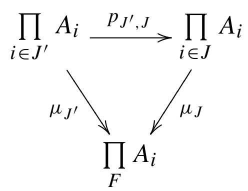
$u' \in \mathcal{S}$ whenever $u \in \mathcal{S}$.

2.2. Categorical filtered products

The categorical concept of filtered product is necessary for the ultraproducts part of our work.

Let $\mathbb{C}$ be a category with small products (denoted $\prod_{i\in I} A_i$). Consider a family of objects $\{A_i\}_{i\in I}$. Each filter F over the set of indices I determines a functor $A_F\colon F \to \mathbb{C}$ such that $A_F(J \subset J') = \prod_{i\in J'} A_i \xrightarrow{p_{J',J}} \prod_{i\in J} A_i$ for each $J, J' \in F$ with $J \subset J'$, and with $p_{J',J}$ being the canonical projection.

Then a *filtered product of $\{A_i\}_{i\in I}$ modulo F* is a colimit $\mu\colon A_F \Rightarrow \prod_F A_i$ of the functor A_F.



Obviously, filtered products, when they exist, are unique up to isomorphisms. If F is ultrafilter then the filtered product modulo F is called an *ultraproduct*.

The filtered product construction from classical model theory (see Chapter 4 of [38]) has been probably defined categorically for the first time in [10] and has been used in some abstract model theoretic works, such as [7]. The equivalence between the category theoretic and the set theoretic definitions of the filtered products is shown in [39].[2]

[1] Strictly speaking, this is only a quasi-category living in a higher set-theoretic universe.

[2] However this relies upon an appropriate concept of model homomorphism avoiding the usual classical model theoretic restrictions to 'embeddings' (i.e. closed inclusive model homomorphisms) or even to 'elementary embeddings'. In fact it is easy to see that the categorical filtered products makes essential use of projections, which are rather far from any concept of model 'embedding'.

Let F be a filter over I and $I' \subseteq I$. The *reduction of F to I'* is denoted $F|_{I'}$ and defined as $\{I' \cap X \mid X \in F\}$. We can easily check that the reduction of a filter is still a filter. We say that a class $\mathcal{F}$ of filters is *closed under reductions* if and only if $F|_J \in \mathcal{F}$ for each $F \in \mathcal{F}$ and $J \in F$. Examples of classes of filters closed under reductions include the class of all filters, the class of all ultrafilters, the class $\{\{I\} \mid I \text{ set}\}$, etc.

By noticing that the filter inclusion $(F|_J, \supseteq) \subseteq (F, \supseteq)$ is a final functor, we get the following immediate corollary of Theorem 1.

Proposition 1. *Let F be a filter over I and $\{A_i\}_{i\in I}$ a family of objects in a category $\mathbb{C}$ with small products and directed colimits. For each $J \in F$, the filtered products $\prod_{F|_J} A_i$ and $\prod_F A_i$ are isomorphic.*

Definition 1 (*Preservation of Categorical Filtered Products [1]*). Consider a functor $G: \mathbb{C}' \to \mathbb{C}$ and F a filter over a set I. Then G *preserves the filtered product* $\mu': B_F \Rightarrow \prod_F B_i$ (for $\{B_i\}_{i\in I}$ a family of objects in $\mathbb{C}'$), if $\mu' G: B_F; G \Rightarrow \prod_F G(B_i)$ is also a filtered product in $\mathbb{C}$ of $\{G(B_i)\}_{i\in I}$. For any class $\mathcal{F}$ of filters, we say a functor *preserves $\mathcal{F}$-filtered products* if it preserves all filtered products modulo F for each filter $F \in \mathcal{F}$.

Definition 2 (*Lifting of Categorical Filtered Products [1]*). Let $\mathcal{F}$ be a class of filters closed under reductions. A functor $G: \mathbb{C}' \to \mathbb{C}$ *lifts $\mathcal{F}$-filtered products* when for each $F \in \mathcal{F}$, and each filtered product $\mu: A_F \Rightarrow \prod_F A_i$ (for $\{A_i\}_{i\in I}$ a family of objects in $\mathbb{C}$), for each object B in $\mathbb{C}'$ such that $G(B) = \prod_F A_i$, there exists $J \in F$ and $\{B_i\}_{i\in J}$ a family of objects in $\mathbb{C}'$ such that $G(B_i) = A_i$ for each $i \in J$ and such that there exists a filtered product $\mu': B_{F|_J} \Rightarrow B$ such that $G(\mu'_{J'} = \mu_{J'}$ for each $J' \in F|_J$. When $J = I$ we say that G *lifts completely* the respective filtered product.

2.3. Institutions

Definition 3 (*Institutions*). [2]

An *institution* $\mathcal{I} = (\mathbb{S}ig^{\mathcal{I}}, \mathsf{Sen}^{\mathcal{I}}, \mathsf{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$ consists of

(i) a category $\mathbb{S}ig^{\mathcal{I}}$, whose objects are called *signatures*,
(ii) a functor $\mathsf{Sen}^{\mathcal{I}}: \mathbb{S}ig^{\mathcal{I}} \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
(iii) a functor $\mathsf{Mod}^{\mathcal{I}}: (\mathbb{S}ig^{\mathcal{I}})^{\mathrm{op}} \to \mathbb{CAT}$ giving for each signature Σ a category whose objects are called *Σ-models*, and whose arrows are called Σ-(*model*) *morphisms*, and
(iv) a relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathsf{Mod}^{\mathcal{I}}(\Sigma)| \times \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in |\mathbb{S}ig^{\mathcal{I}}|$, called *$\Sigma$-satisfaction*,

such that for each morphism $\varphi: \Sigma \to \Sigma'$ in $\mathbb{S}ig^{\mathcal{I}}$, the *satisfaction condition*

$$M' \models^{\mathcal{I}}_{\Sigma'} \mathsf{Sen}^{\mathcal{I}}(\varphi)(\rho) \text{ iff } \mathsf{Mod}^{\mathcal{I}}(\varphi)(M') \models^{\mathcal{I}}_{\Sigma} \rho$$

holds for each $M' \in |\mathsf{Mod}^{\mathcal{I}}(\Sigma')|$ and $\rho \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$. We denote the *reduct* functor $\mathsf{Mod}^{\mathcal{I}}(\sigma)$ by $\_\restriction_{\varphi}$ and the sentence translation $\mathsf{Sen}^{\mathcal{I}}(\varphi)$ by $\varphi(\_)$. When $M = M' \restriction_{\varphi}$ we say that M is a *φ-reduct of M'*, and that M' is a *φ-expansion of M*. When there is no danger of ambiguity, we may skip the superscripts from the notations of the entities of the institution; for example $\mathbb{S}ig^{\mathcal{I}}$ may be simply denoted $\mathbb{S}ig$.

General assumption: We assume that all our institutions are such that satisfaction is invariant under model isomorphism, i.e. if Σ-models M, M' are isomorphic, denoted $M \cong M'$, then $M \models_{\Sigma} \rho$ iff $M' \models_{\Sigma} \rho$ for all Σ-sentences ρ.

The following is the most classical example of institution.

Example 1 (*Classical Logic*). Let **FOL** be the institution of *many sorted first order logic with equality*.

Its *signatures* (S, F, P) consist of

– a set of sort symbols S,

– a family $F = \{F_{w \to s} \mid w \in S^*, s \in S\}$ of sets of function symbols indexed by arities (for the arguments) and sorts (for the results), and
– a family $P = \{P_w \mid w \in S^*\}$ of sets of relation (predicate) symbols indexed by arities.

Signature morphisms map the three components in a compatible way. This means that a signature morphism $\varphi\colon (S, F, P) \to (S', F', P')$ consist of

– a function $\varphi^{\mathrm{st}}\colon S \to S'$,
– a family of functions $\varphi^{\mathrm{op}} = \{\varphi^{\mathrm{op}}_{w \to s}\colon F_{w \to s} \to F'_{\varphi^{\mathrm{st}}(w) \to \varphi^{\mathrm{st}}(s)} \mid w \in S^*, s \in S\}$, and
– a family of functions $\varphi^{\mathrm{rl}} = \{\varphi^{\mathrm{rl}}_{w \to s}\colon P_w \to P'_{\varphi^{\mathrm{st}}(w)} \mid w \in S^*, s \in S\}$.

Models M are first order structures interpreting each sort symbol s as a set M_s, each function symbol σ as a function M_σ from the product of the interpretations of the argument sorts to the interpretation of the result sort, and each relation symbol π as a subset M_π of the product of the interpretations of the argument sorts. In order to avoid the existence of empty interpretations of the sorts, which may complicate unnecessarily our presentation, we assume that each signature has at least one *constant* (i.e. operation symbol with empty arity) for each sort. A model homomorphism $h\colon M \to M'$ is an indexed family of functions $\{h_s\colon M_s \to M'_s\}_{s \in S}$ such that

– h is a F-algebra homomorphism $M \to M'$, i.e., $h_s(M_\sigma(m)) = M'_\sigma(h_w(m))$ for each $\sigma \in F_{w \to s}$ and each $m \in M_w$, and
– $h_w(m) \in M'_\pi$ if $m \in M_\pi$ (i.e. $h_w(M_\pi) \subseteq M'_\pi$) for each relation $\pi \in P_w$ and each $m \in M_w$.

where $h_w\colon M_w \to M'_w$ is the canonical component-wise extension of h, i.e. $h_w(m_1 \ldots m_n) = h_{s_1}(m_1) \ldots h_{s_n}(m_n)$ for $w = s_1 \ldots s_n$ and $m_i \in M_{s_i}$.

For each signature morphism φ, the *reduct* $M' \upharpoonright_\varphi$ of a model M' is defined by $(M' \upharpoonright_\varphi)_x = M'_{\varphi(x)}$ for each x sort, function, or relation symbol from the domain signature of φ.

Sentences are the usual first order sentences built from equational and relational atoms by iterative application of Boolean connectives and quantifiers. Sentence translations along signature morphisms just rename the sorts, function, and relation symbols according to the respective signature morphisms. They can be formally defined by induction on the structure of the sentences. While the induction step is straightforward for the case of the Boolean connectives it needs a bit of attention for the case of the quantifiers. For any signature morphism $\varphi\colon (S, F, P) \to (S', F', P')$,

$$\mathsf{Sen}^{\mathbf{FOL}}(\varphi)((\forall X)\rho) = (\forall X^\varphi)\mathsf{Sen}^{\mathbf{FOL}}(\varphi')(\rho)$$

for each finite set of variables X, each $(S, F \uplus X, P)$-sentence ρ, and where $X^\varphi = \{(x\colon \varphi^{\mathrm{st}}(s)) \mid (x\colon s) \in X\}$, and $\varphi'\colon (S, F \uplus X, P) \to (S', F' \uplus X^\varphi, P')$ extends φ canonically (here by $(x\colon \varphi^{\mathrm{st}}(s))$ we denote the fact that x is a variable of sort s).

$$\begin{array}{ccc} (S, F, P) & \longrightarrow & (S, F \uplus X, P) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ (S', F', P') & \longrightarrow & (S', F' \uplus X^\varphi, P') \end{array}$$

The satisfaction of sentences by models is the usual Tarskian satisfaction defined inductively on the structure of the sentences.

Propositional logic, denoted **PL**, is the sub-institution of **FOL** determined by the signatures with empty sets of sort symbols (and therefore empty sets of operation symbols).

Examples of institutions abound, conventional or less conventional. Besides the three examples below, a very short list of institutions in use in computing science include rewriting [40], higher-order [41], polymorphic [42], various modal logics such as temporal [43], process [43], behavioral [44], coalgebraic [45], object-oriented [46], and multi-algebraic (non-determinism) [47] logics.

Example 2 (*Partial Algebra*). The institution **PA** of partial algebra [48,49] is defined as follows.

A *partial* algebraic signature is a tuple (S, TF, PF), where TF is the set of *total* operations and PF is the set of *partial* operations. Signature morphisms map the three components in a compatible way.

A partial algebra is just like an ordinary algebra (i.e. a **FOL** model without relations) but interpreting the operations of PF as partial rather than total functions. A *partial algebra homomorphism* $h\colon A \to B$ is a family of (total) functions $\{h_s\colon A_s \to B_s\}_{s\in S}$ indexed by the set of sorts S of the signature such that $h_w(A_\sigma(a)) = B_\sigma(h_s(a))$ for each operation $\sigma \in (TF \cup PF)_{w\to s}$ and each string of arguments $a \in A_w$ for which $A_\sigma(a)$ is defined.

The sentences have three kinds of atoms: *definedness* $\mathsf{def}(t)$, *strong* equality $t \stackrel{s}{=} t'$, and *existence* equality $t \stackrel{e}{=} t'$. The definedness $\mathsf{def}(t)$ of a term t holds in a partial algebra A when the interpretation A_t of t is defined. The strong equality $t \stackrel{s}{=} t'$ holds when both terms are undefined or both of them are defined and are equal. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and are equal.[3] The sentences are formed from these atoms by means of Boolean connectives and quantifications over total first order variables.

Example 3 (*Preordered Algebras*). Preordered algebras are used for formal specification and verifications of algorithms [50], for automatic generation of case analysis [50], and in general about reasoning about transitions between states of systems. They constitute an unlabeled form of rewriting logic of [40]. Let **POA** denote the institution of preordered algebras.

The signatures are just the algebraic signatures, i.e. **FOL** signatures without any relation symbols. The **POA** models are *preorder algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, i.e. a monotonic function. A *preorder algebra homomorphism* is just a family of preorder functors (monotonic functions) which is an algebra homomorphism.

The sentences have two kinds of atoms: (ordinary) equations and preorder atoms. An preorder atoms $t \le t'$ is satisfied by a preorder model M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \le M_{t'}$. The sentences are formed from these atoms by means of Boolean connectives and quantifications over first order variables.

Example 4 (*First Order Modal Logic*). The *signatures* are tuples (S, S_0, F, F_0, P, P_0) where

- (S, F, P) is a **FOL** signature, and
- (S_0, F_0, P_0) is a sub-signature of (S, F, P) of *rigid* symbols.

Signature morphisms $(S, S_0, F, F_0, P, P_0) \to (S', S'_0, F', F'_0, P', P'_0)$ are just **FOL** signature morphisms $(S, F, P) \to (S', F', P')$ which preserve the rigid symbols.

A **MFOL** *model* (W, R) for a signature (S, S_0, F, F_0, P, P_0), called *Kripke model*, consists of

- a family $W = \{W^i\}_{i\in I_W}$ of 'possible worlds', which are (S, F, P)-models in **FOL**, indexed by a set I_W, and such that for all rigid symbols x, $W^i_x = W^j_x$ for all $i, j \in I_W$, and
- an 'accessibility' binary relation $R \subseteq I_W \times I_W$ between the possible worlds.

A Kripke model (W, R) is T when R is reflexive, $S4$ when it is T and R is transitive, and is $S5$ when it is $S4$ and R is symmetric.

Our definition of Kripke model corresponds to first order modal logic with 'constant domains' from the modal logic literature.

Homomorphisms between Kripke models preserve their mathematical structure. Thus a *Kripke model homomorphism* $h\colon (W, R) \to (W', R')$ consists of

- a function $h\colon I_W \to I_{W'}$ which preserves the accessibility relation, i.e. $\langle i, j\rangle \in R$ implies $\langle h(i), h(j)\rangle \in R'$, and
- for each $i \in I_W$ an S-sorted function $\{h^i_s\colon W^i_s \to {W'}^{h(i)}_s\}_{s\in S}$, which is an (S, F, P)-model homomorphism $W^i \to {W'}^{h(i)}$, and such that for each rigid sort s_0 we have that $h^i_{s_0} = h^j_{s_0}$ for any $i, j \in I_W$. (Notice the overloading of 'h' in this definition!.)

The (S, S_0, F, F_0, P, P_0)-sentences are expressions formed from **FOL** (S, F, P)-atoms by closing under usual Boolean connectives, universal and existential first order quantifications by rigid variables (i.e. quantifications by rigid new constants), and unary modal connectives $\Box$ (necessity) and $\Diamond$ (possibility). The satisfaction of **MFOL** sentences

3 Notice that $\mathsf{def}(t)$ is equivalent to $t \stackrel{e}{=} t$ and that $t \stackrel{s}{=} t'$ is equivalent to $(t \stackrel{e}{=} t') \vee (\neg def(t) \wedge \neg def(t'))$.

by the Kripke models, $(W, R) \models \rho$ is defined by $(W, R) \models^i \rho$ for each $i \in I_W$, where $\models^i$ is defined by induction on the structure of the sentences as follows:

- $(W, R) \models^i \rho$ iff $W^i \models^{\mathbf{FOL}} \rho$ for each atom ρ and each $i \in I_W$,
- $(W, R) \models^i \rho_1 \wedge \rho_2$ iff $(W, R) \models^i \rho_1$ and $(W, R) \models^i \rho_2$; and similarly for the other Boolean connectives,
- $(W, R) \models^i \Box\rho$ iff $(W, R) \models^j \rho$ for each j such that $\langle i, j\rangle \in R$,
- $\Diamond\rho$ is the same as $\neg\Box\neg\rho$,
- $(W, R) \models^i (\forall X)\rho$ when $(W', R) \models^i \rho$ for each expansion (W', R) of (W, R) to a Kripke $(S, F \uplus X, P)$-model and $(W, R) \models^i (\exists X)\rho$ if and only if $(W, R) \models^i \neg(\forall X)\neg\rho$.

Modal propositional logic (denoted **MPL**) is the sub-institution of **MFOL** determined by the signatures with empty set of sort symbols (and therefore empty sets of operation symbols) and empty sets of rigid relation symbols. Most of the classical modal logic studies are concerned with this institution.

Notation 1. For any signature Σ in an institution $\mathcal{I}$:

- For each set E of Σ-sentences, let $E^* = \{M \in \mathsf{Mod}(\Sigma) \mid M \models_\Sigma e \text{ for each } e \in E\}$, and
- For each class $\mathbb{M}$ of Σ-models, let $\mathbb{M}^* = \{e \in \mathsf{Sen}(\Sigma) \mid M \models_\Sigma e \text{ for each } M \in \mathbb{M}\}$.

If E and E' are sets of sentences of the same signature, then $E' \subseteq E^{**}$ is denoted by $E \models E'$. Two (sets of) sentences (of the same signature) are *semantically equivalent* when they are satisfied by the same class of models.

2.4. Institution morphisms

In the literature there are several concepts of structure preserving mappings between institutions. For the purpose of our work we need the original concept introduced by [2] and which corresponds to forgetting structure from a more complex to a simpler institution.

Definition 4 (*Institution Morphisms*). [2] An *institution morphism* $(\Phi, \alpha, \beta)\colon \mathcal{I}' \to \mathcal{I}$ consists of

(i) a functor $\Phi\colon \mathbb{S}ig' \to \mathbb{S}ig$,
(ii) a natural transformation $\alpha\colon \Phi; \mathsf{Sen} \Rightarrow \mathsf{Sen}'$, and
(iii) a natural transformation $\beta\colon \mathsf{Mod}' \Rightarrow \Phi^{\mathrm{op}}; \mathsf{Mod}$

such that the following *satisfaction condition* holds

$$M' \models'_{\Sigma'} \alpha_{\Sigma'}(e) \text{ if and only if } \beta_{\Sigma'}(M') \models_{\Phi(\Sigma')} e$$

for any signature $\Sigma' \in |\mathbb{S}ig'|$, any Σ'-model M' and any $\Phi(\Sigma')$-sentence e.

Examples 1. The following table provides three rather simple examples of institution morphisms.

$\mathcal{I}'$	$\mathcal{I}$	Φ	α	β
PA	**FOL**	$\Phi(S, TF, PF) = (S, TF, \emptyset)$	canonical inclusion	forgets interpretations of PF
POA	**FOL**	$\Phi(S, F) = (S, F, \emptyset)$	canonical inclusion	forgets the preorder relations
FOL	**MFOL**	$\Phi(S, F, P) = (S, S, F, F, P, P)$	erases the modalities $\Box$ and $\Diamond$	$\beta_\Sigma(M) = (W, R)$ with $I_W = \{*\}$, $W^* = M$, $R = \{\langle *, *\rangle\}$

2.5. Model amalgamation

Amalgamation properties for institutions formalize the possibility of amalgamating models of different signatures when they are consistent on some kind of 'intersection' of the signatures.

Definition 5 (*Model Amalgamation*). The commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is an *amalgamation square* if and only if for each Σ_1-model M_1 and a Σ_2-model M_2 such that $M_1 \restriction_{\varphi_1} = M_2 \restriction_{\varphi_2}$, there exists an unique Σ'-model M', denoted $M_1 \otimes_{\varphi_1,\varphi_2} M_2$, or $M_1 \otimes M_2$ for short when there is no danger of ambiguity, such that $M' \restriction_{\theta_1} = M_1$ and $M' \restriction_{\theta_2} = M_2$.

In most of the institutions formalizing conventional or non-conventional logics pushout squares of signature morphisms are model amalgamation squares [17,24].

Example 5. **FOL**, **PA**, **POA** have model amalgamation for all pushout squares of signature morphisms.

Moreover, in practice often the ‘weak’ version of model amalgamation, i.e. without the uniqueness condition for the amalgamation, suffices [51,52,49]. The model amalgamation property is a necessary condition in many institution-independent model theoretic results, thus being one of the most desirable properties for an institution. Model amalgamation can be considered even as more fundamental than the satisfaction condition since in institutions with quantifications it is used in its weak form in the proof of the satisfaction condition at the induction step corresponding to quantifiers.

2.6. Internal logic

Much of our institution-independent development of model theory relies on the possibility of defining concepts such as Boolean connectives, quantification, and atomic sentences internally to any institution. The main implication of this fact is that the abstract satisfaction relation between models and sentences can be decomposed at the level of arbitrary institutions into several concrete layers of satisfaction defined categorically in terms of (a simple form of) injectivity and reduction (see [1]). Essentially speaking, this is what gives depth to the institution-independent approach to model theory. Internal Boolean connectives and quantifiers have been introduced first time by [53] while the internal approach to the atomic sentences has been introduced in [1] under the concept of *basic sentence*.

Definition 6 (*Internal Boolean Connectives [19,1]*). Given a signature Σ in an institution

– the Σ-sentence ρ' is a *(semantic) negation* of ρ when $\rho'^{*} = |\mathsf{Mod}(\Sigma)| \setminus \rho^{*}$, and
– the Σ-sentence ρ' is the *(semantic) conjunction* of the Σ-sentences ρ_1 and ρ_2 when $\rho'^{*} = \rho_1^{*} \cap \rho_2^{*}$.

An institution *has (semantic) negation* when each sentence of the institution has a negation, and *has (semantic) conjunctions* when each two sentences (of the same signature) have a conjunction. Distinguished negations are often denoted by $\neg\_$, while distinguished conjunctions by $\_ \wedge \_$.

Other Boolean connectives, such as disjunction ($\vee$), implication ($\Rightarrow$), equivalence ($\Leftrightarrow$), etc., can be derived as usually from negations and (finite) conjunctions.

Example 6. **FOL**,**PA** and **POA** have all semantic Boolean connectives while **MFOL** and **MPL** have only semantic conjunction.

Fact 1. The semantic Boolean connectives are unique modulo semantical equivalence.

Definition 7 (*Internal Quantifiers [19,1]*). For any signature morphism $\chi : \Sigma \to \Sigma'$ in an arbitrary institution,

– a Σ-sentence ρ is a *(semantic) existential χ-quantification* of a χ-sentence ρ' when $\rho^{*} = (\rho'^{*}) \restriction_{\chi}$; in this case we may write ρ as $(\exists\chi)\rho'$,
– a Σ-sentence ρ is a *(semantic) universal χ-quantification* of a χ-sentence ρ' when $\rho^{*} = |\mathsf{Mod}(\Sigma) \setminus (|\mathsf{Mod}(\Sigma')| \setminus \rho'^{*}) \restriction_{\chi}$; in this case we may write ρ as $(\forall\chi)\rho'$.

For a class $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms, we say that the institution has universal/existential *$\mathcal{D}$-quantification* when for each $\chi\colon \Sigma \to \Sigma'$ in $\mathcal{D}$, each Σ'-sentence has a universal/existential χ-quantification.

Examples 2. Notice that the concept of 'internal quantification' of Definition 7 captures ordinary quantification of the actual institutions, for example **FOL** has $\mathcal{D}$-quantification for $\mathcal{D}$ the class of signature extensions with a finite number of constants, while in the case of second order logic $\mathcal{D}$ is the class of signature (finite) extensions with any relation and any operation symbols.

PA has $\mathcal{D}$-quantification for $\mathcal{D}$ the class of signature extensions with a finite number of *total* constants, while **POA** has quantification similar to **FOL**.

Note that **MFOL** has only semantic universal $\mathcal{D}$-quantification for $\mathcal{D}$ the class of signature extensions with a finite number of rigid constants, it does *not* have the existential one.

3. Internal modal logic

In this section we define Kripke models and modal satisfaction on top of an arbitrary 'base institution' by using its semantic and its syntactic structures, and the satisfaction relation between them. A typical rather classical example of the relationship between the 'modal institution' thus obtained and the base institution is given by the relationship between **MFOL** (as modal institution) and **FOL**$'$ (as base institution), where **FOL**$'$ is the variation of **FOL** having **MFOL** signatures (S, S_0, F, F_0, P, P_0) rather than **FOL** signatures (S, F, P).

3.1. Models

Definition 8 (*Internal Kripke Models*). Given an institution morphism $(\Phi^{\Delta}, \alpha^{\Delta}, \beta^{\Delta})\colon (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$ (from a 'base' institution to a 'domain' institution), for any signature Σ in $\mathbb{S}ig$, a *Kripke Σ-model* (W, R) consists of

– a family of Σ-models $W\colon I_W \to |\mathsf{Mod}(\Sigma)|$ such that the sharing condition

$$\beta^{\Delta}_{\Sigma}(W^i) = \beta^{\Delta}_{\Sigma}(W^{i'})$$

holds for each $i, i' \in I_W$, and

– an binary "accessibility" relation R on the index set I_W.

A Kripke model (W, R) is T when R is reflexive, $S4$ when it is T and R is transitive, and is $S5$ when it is $S4$ and R is symmetric.

A *Kripke Σ-model homomorphism* $(h^W, h^I)\colon (W, R) \to (W', R')$ consists of

– a function $h^I\colon I_W \to I_{W'}$ between the index sets which is a relation homomorphism, i.e. $\langle i, j\rangle \in R$ implies $\langle h^I(i), h^I(j)\rangle \in R'$; note this means that h^I is a **FOL**-model homomorphism $(I_W, R) \to (I_{W'}, R')$, and
– a natural transformation $h^W\colon W \Rightarrow h^I; W'$ such that $\beta^{\Delta}_{\Sigma}((h^W)^i) = \beta^{\Delta}_{\Sigma}((h^W)^{i'})$ for each $i, i' \in I_W$.

Notice that h^W being natural transformation means just that h^W consists of a family of Σ-model homomorphisms $\{(h^W)^i\colon W^i \to W'^{h^I(i)}\}_{i\in I_W}$. When the context is clear we may omit the superscripts W and I from the notation of h^W, respectively h^I, and simply use h instead.

Fact 2. The Kripke Σ-models and their homomorphisms form a category denoted $\mathsf{K\text{-}Mod}(\Sigma)$.

Remark 1. Given a signature morphism $\varphi\colon \Sigma \to \Sigma'$, each Kripke Σ'-model (W', R') can be reduced to the Kripke Σ-model $(W'; \mathsf{Mod}(\varphi), R')$. Note that by the naturality of β^{Δ} and by the sharing constraint for (W', R), we obtain the sharing constraint for the reduced Kripke model. Similarly, each Kripke model homomorphism (h^W, h^I) can be reduced to $(h^W\mathsf{Mod}(\varphi), h^I)$. This defines a Kripke model functor $\mathsf{K\text{-}Mod}\colon \mathbb{S}ig^{\mathrm{op}} \to \mathbb{CAT}$.

Example 7. The table below provides a list of examples showing various Kripke model concepts (in all entries Sen^{Δ} is empty).

	base inst.	Φ^{Δ}	$\mathsf{Mod}^{\Delta}(\Sigma)$	*sharing constraint*
1.	$\mathbf{FOL}'$	$\Phi^{\Delta}(S, S_0, F, F_0, P, P_0) = (S_0, F_0, P_0)$	$\mathsf{Mod}^{\mathbf{FOL}}(S_0, F_0, P_0)$	the interpretations of the rigid symbols
2.	**POA**	forgets non-constant operation symbols	$\mathsf{Mod}^{\mathbf{FOL}}(S, C, \emptyset)$	underlying carrier sets and interpretations of constants
3.	**POA**	identity	$\mathsf{Mod}^{\mathbf{FOL}}(S, F, \emptyset)$	underlying algebras
4.	**PA**	forgets non-constant total operation and all partial operation symbols	$\mathsf{Mod}^{\mathbf{FOL}}(S, C, \emptyset)$	underlying carrier sets and interpretations of total constants
5.	**PL**	$\mathbb{S}ig^{\Delta}$ is the terminal category	terminal category	without

In the setup 1. which defines the Kripke models of **MFOL**, the base institution $\mathbf{FOL}'$ is like **FOL** but with signatures with marked rigid symbols, i.e. signatures of the form (S, S_0, F, F_0, P, P_0). Then $\mathsf{Mod}^{\mathbf{FOL}'}(S, S_0, F, F_0, P, P_0) = \mathsf{Mod}^{\mathbf{FOL}}(S, F, P)$ and $\mathsf{Sen}^{\mathbf{FOL}'}(S, S_0, F, F_0, P, P_0) = \mathsf{Sen}^{\mathbf{FOL}}(S, F, P)$ (although at this stage we do not need to care about sentences and satisfaction). The setup 5. gives the Kripke models of **MPL**.

Note also that the setup 3. supports uniform valuations of second order variables. The most interesting aspect of this example is that although all symbols of the signatures are 'rigid', it does not collapse possible worlds semantics to 'single world' semantics because the preorder relations are not shared.

The result below shows that the model amalgamation properties of the base institution carries to the Kripke model functor.

Proposition 2. *Given an institution morphism* $(\Phi^{\Delta}, \alpha^{\Delta}, \beta^{\Delta})\colon (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$ *(from a 'base' institution to a 'domain' institution) any commuting square of signature morphisms in* $\mathbb{S}ig$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

such that

(i) *it is a model amalgamation square in the base institution, and*
(ii) Φ^{Δ} *maps it to a model amalgamation square in the domain institution*

it is a model amalgamation square with respect to the Kripke model functor $\mathsf{K\text{-}Mod}$.

Proof. Let (W_1, R_1) be a Kripke Σ_1-model and (W_2, R_2) be a Kripke Σ_2-model such that $(W_1, R_1) \restriction_{\varphi_1} = (W_2, R_2) \restriction_{\varphi_2}$. This means that $R_1 = R_2$ and $I_{W_1} = I_{W_2}$, and for each $i \in I_{W_1} = I_{W_2}$, $(W_1)^i \restriction_{\varphi_1} = (W_2)^i \restriction_{\varphi_2}$.

We define the Kripke Σ'-model (W', R') such that $R' = R_1 = R_2$, $I_{W'} = I_{W_1} = I_{W_2}$, and for each index $i \in I_{W'}$, W'^i is the amalgamation of $(W_1)^i$ and $(W_2)^i$. We can easily notice that $(W', R') \restriction_{\theta_1} = (W', R') \restriction_{\theta_2}$ and that (W', R') is the *unique* common expansion of (W_1, R_1) and (W_2, R_2). We still need to show the sharing condition for (W', R'), that for each $i, j \in I_{W'}$ we have that $\beta^{\Delta}_{\Sigma'}(W'^i) = \beta^{\Delta}_{\Sigma'}(W'^j)$.

Because

$$\begin{array}{ccc} \Phi^{\Delta}(\Sigma) & \xrightarrow{\Phi^{\Delta}(\varphi_1)} & \Phi^{\Delta}(\Sigma_1) \\ {\scriptstyle \Phi^{\Delta}(\varphi_2)}\downarrow & & \downarrow{\scriptstyle \Phi^{\Delta}(\theta_1)} \\ \Phi^{\Delta}(\Sigma_2) & \xrightarrow[\Phi^{\Delta}(\theta_2)]{} & \Phi^{\Delta}(\Sigma') \end{array}$$

is an amalgamation square in the domain institution Δ it is enough to show that $\beta^{\Delta}_{\Sigma'}(W'^{i}) \upharpoonright_{\Phi^{\Delta}(\theta_k)} = \beta^{\Delta}_{\Sigma'}(W'^{j}) \upharpoonright_{\Phi^{\Delta}(\theta_k)}$ for $k \in \{1, 2\}$. By the naturality of β^{Δ} this is equivalent to $\beta^{\Delta}_{\Sigma_k}(W'^{i} \upharpoonright_{\theta_k}) = \beta^{\Delta}_{\Sigma_k}(W'^{j} \upharpoonright_{\theta_k})$ which means $\beta^{\Delta}_{\Sigma_k}(W^{i}_{k}) = \beta^{\Delta}_{\Sigma_k}(W^{j}_{k})$. This holds by the sharing condition for $(W_k, R' = R_k)$. ■

An instance of Proposition 2 corresponding to the first entry in the table of Example 7 gives the following model amalgamation property for **MFOL**.

Corollary 1. *Any commuting square of* **MFOL** *signature morphisms*

$$\begin{array}{ccc}
(S, S_0, F, F_0, P, P_0) & \xrightarrow{\varphi_1} & (S^1, S^1_0, F^1, F^1_0, P^1, P^1_0) \\
\Big\downarrow{\scriptstyle \varphi_2} & & \Big\downarrow{\scriptstyle \theta_1} \\
(S^2, S^2_0, F^2, F^2_0, P^2, P^2_0) & \xrightarrow[\theta_2]{} & (S', S'_0, F', F'_0, P', P'_0)
\end{array}$$

is an amalgamation square whenever both

$$\begin{array}{ccc}
(S, F, P) & \longrightarrow & (S^1, F^1, P^1) \\
\Big\downarrow & & \Big\downarrow \\
(S^2, F^2, P^2) & \longrightarrow & (S', F', P')
\end{array}
\qquad\qquad
\begin{array}{ccc}
(S_0, F_0, P_0) & \longrightarrow & (S^1_0, F^1_0, P^1_0) \\
\Big\downarrow & & \Big\downarrow \\
(S^2_0, F^2_0, P^2_0) & \longrightarrow & (S'_0, F'_0, P'_0)
\end{array}$$

are pushout squares of **FOL** *signature morphisms.*

Proof. Follows directly from Proposition 2 by using the well known model amalgamation property of **FOL** (see Example 5) that each pushout of signature morphisms is an amalgamation square. ■

3.2. Modal satisfaction

Definition 9. For any fixed signature Σ, for each Kripke Σ-model (W, R) and each Σ-sentence ρ we define the *satisfaction of* ρ *in* (W, R) *at the possible world* $i \in I_W$, denoted $(W, R) \models^{i} \rho$. Then $(W, R) \models \rho$ if and only if $(W, R) \models^{i} \rho$ at each possible world $i \in I_W$.

The modal satisfaction of the Boolean connectives and of the quantifiers is defined by their standard internal logic semantics (cf. Section 2) but applied to $\models^{i}$ rather than to $\models$.

The satisfaction of modalities ‘necessity’ and ‘possibility’ is defined by

$(W, R) \models^{i} \Box\rho$ if and only if $(W, R) \models^{j} \rho$ for each $\langle i, j\rangle \in R$

$(W, R) \models^{i} \Diamond\rho$ if and only if there exists $\langle i, j\rangle \in R$ such that $(W, R) \models^{j} \rho$.

Remark 2. Note that the modal negation is not semantic in the sense that $(W, R) \models \neg\rho$ is not the same with $(W, R) \not\models \rho$ (while $(W, R) \models^{i} \neg\rho$ is *defined* as $(W, R) \not\models^{i} \rho$). The same situation holds for most of the Boolean connectives or quantifiers, however there are some notable exceptions: conjunctions and universal quantifiers are semantic with respect to the modal satisfaction, i.e.

– for each Kripke Σ-model (W, R) and any Σ-sentences ρ_1 and ρ_2, $(W, R) \models \rho_1 \wedge \rho_2$ if and only if $(W, R) \models \rho_1$ and $(W, R) \models \rho_2$, and
– for each signature morphism $\chi : \Sigma \to \Sigma'$, each Kripke Σ-model (W, R) and each Σ-sentence ρ, $(W, R) \models (\forall\chi)\rho$ if and only if $(W', R) \models \rho$ for each χ-expansion (W', R) of (W, R).

In standard modal logic terminology this situation is explained by the difference between ‘local’ and ‘global’ satisfaction.

In order to complete the definition of a ‘modal institution’ on top of a ‘base institution’ we need to define a ‘modal sentence’ functor.

Definition 10. Let $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta)\colon (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$ be an institution morphism (from a 'base' institution to a 'domain' institution). We extend Sen to a 'modal' sentence functor $\mathsf{M\text{-}Sen}\colon \mathbb{S}ig \to \mathbb{S}et$ such that each $\mathsf{M\text{-}Sen}$ sentence is syntactically accessible from the sentences of the base institution by

– Boolean connectives,
– modalities ($\Box$ and $\Diamond$), and
– $\mathcal{D}$-quantifiers, for a class $\mathcal{D}$ of signature morphisms stable under pushouts and such that any pushout between any morphism from $\mathcal{D}$ and any other signature morphism
(EB) is an amalgamation square in the base institution, and
(ED) gets mapped by Φ^Δ to an amalgamation square in the domain institution.

Then we define a satisfaction relation between Kripke models and $\mathsf{M\text{-}Sen}$ sentences inductively on the structure of the sentences according to the internal modal satisfaction described above and by defining

$$(W, R) \models^i \rho \text{ if and only if } W^i \models \rho \text{ when } \rho \in \mathsf{Sen}(\Sigma).$$

Example 8. The **MFOL** sentences and their satisfaction by the **MFOL** Kripke models is an instance of the general process defined above as follows:

– We replace the base institution **FOL**$'$ used for defining the Kripke models with its 'atomic' sub-institution **AFOL**$'$ which has only the atoms as sentences. This is necessary because some of the Boolean connectives and of the quantifications obtained by internal modal logic will not be semantic (in the sense of Definition 6), and thus semantically different from the Boolean connectives and the quantifiers of **FOL**$'$.
– We consider all sentences constructed from the atoms by iteratively applying Boolean and modal connectives and $\mathcal{D}$-quantifications for the signature extensions $(S, S_0, F, F_0, P, P_0) \hookrightarrow (S, S_0, F \uplus X, F_0 \uplus X, P, P_0)$ with a finite set of rigid constants X.

The conditions (EB) and (ED) hold easily because both squares involved in these conditions represent pushout squares of **FOL** signature morphisms and cf. Example 5 all pushout squares of signatures morphisms are amalgamation squares.

Theorem 2. *For any institution morphism* $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta)\colon (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$ *(from a 'base' institution to a 'domain' institution), for any modal sentence functor constructed by a process described by Definition* 10*,* $(\mathbb{S}ig, \mathsf{M\text{-}Sen}, \mathsf{K\text{-}Mod}, \models)$ *is an institution.*

Proof. The satisfaction condition for $(\mathbb{S}ig, \mathsf{M\text{-}Sen}, \mathsf{K\text{-}Mod}, \models)$ follows from the fact that

$$(W', R') \models^i \varphi(\rho) \text{ if and only if } (W', R') \upharpoonright_\varphi \models^i \rho$$

for each signature morphism $\varphi\colon \Sigma \to \Sigma'$, each $\rho \in \mathsf{M\text{-}Sen}(\Sigma)$, for each Kripke Σ'-model (W', R'), and for each $i \in I_{W'}$. This can be shown easily by induction on the structure of the sentence ρ. Note that when $\rho \in \mathsf{Sen}(\Sigma)$, this relation follows from the satisfaction condition of the base institution. The induction step can be checked easily for the Boolean connectives, and for the modalities. The quantifiers are less straightforward and will be treated here in detail.

Consider a signature morphism $(\chi : \Sigma \to \Sigma_1) \in \mathcal{D}$. For any Σ_1-sentence ρ, $\varphi((\forall\chi)\rho)$ is semantically equivalent to $(\forall\chi')\varphi_1(\rho)$ for a pushout square as below

$$\begin{array}{ccc} \Sigma & \xrightarrow{\chi} & \Sigma_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ \Sigma' & \xrightarrow[\chi']{} & \Sigma'_1 \end{array}$$

Let (W', R') be a Kripke Σ'-model. Then $(W', R') \models^i_{\Sigma'} \varphi((\forall\chi)\rho)$ if and only if $(W', R') \models^i_{\Sigma'} (\forall\chi')\varphi_1(\rho)$ if and only if $(W'_1, R') \models^i_{\Sigma'_1} \varphi_1(\rho)$ for each χ'-expansion (W'_1, R') of (W', R').

By the induction hypothesis $(W'_1, R') \models^i \varphi_1(\rho)$ is equivalent to $(W'_1, R') \upharpoonright_{\varphi_1} \models^i_{\Sigma_1} \rho$. By the model amalgamation property (cf. Proposition 2) the following are equivalent:

(i) $(W_1', R')\restriction_{\varphi_1} \models^i_{\Sigma_1} \rho$ for each χ'-expansion (W_1', R') of (W', R')
(ii) $(W_1, R') \models^i \rho$ for each χ-expansion (W_1, R') of $(W', R')\restriction_\varphi$.

The latter item just means $(W', R')\restriction_\varphi \models (\forall\chi)\rho$. ■

By specializing Theorem 2 above to the process by which **MFOL** is obtained as internal modal logic (see Example 8) we obtain the following result which now fully explains Example 4.

Corollary 2. **MFOL** *is an institution.*

4. Ultraproducts of Kripke models

The aim of this section is to develop an extension of the institution-independent method of ultraproducts of [1] to possible worlds semantics and to modal satisfaction. The first step is to show that categorical filtered products can be lifted from the categories of the base models to the categories of Kripke models. In the second part of this section we will develop an ultraproduct fundamental theorem for the modal satisfaction.

4.1. Filtered products of Kripke models

Let us assume

- a class $\mathcal{F}$ of filters, and
- an institution morphism from a base institution to a domain institution

$$(\Phi^\Delta, \alpha^\Delta, \beta^\Delta) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$$

such that the following two properties hold:

(FP) for each signature Σ the category of Σ-models $\mathsf{Mod}(\Sigma)$ has products and has $\mathcal{F}$-filtered products which are preserved by β^Δ_Σ, and

(LI) for any signature Σ, β^Δ_Σ *lifts isomorphisms*, i.e. if $\beta^\Delta_\Sigma(M)$ is isomorphic to N' there exists N isomorphic to M such that $N' = \beta^\Delta_\Sigma(N)$.

$$\begin{array}{ccc} \beta^\Delta_\Sigma(M) & \overset{\cong}{\longrightarrow} & N' = \beta^\Delta_\Sigma(N) \\ M & \underset{\cong}{\longrightarrow} & (\exists)N \end{array}$$

Remark 3. The assumption (FP) is expected and constitutes the basis for the existence of filtered products of Kripke models. The assumption (LI) is rather technical and is very easily satisfied in the applications. For example, it is obvious for all entries of the table of Example 7.

Proposition 3. *For each signature Σ, the category of Kripke models* $\mathsf{K\text{-}Mod}(\Sigma)$ *has filtered products.*

Proof. Let $F \in \mathcal{F}$ be any filter over a set I and let $\{(W_j, R_j) \mid j \in I\}$ be an I-indexed family of Kripke models for a fixed signature Σ. For each $J \in F$ we denote the Kripke model product $\prod_{j\in J}(W_j, R_j)$ by (W_J, R_J). This product can be obtained in the following two steps:

- (I_{W_J}, R_J) is the product $\prod_{j\in J}(I_{W_j}, R_j)$ in the category of **FOL** models for a single sorted signature with only one binary relation symbol; then if we write $k \in I_{W_J}$ as $(k_j)_{j\in J}$ with $k_j \in I_{W_j}$ for each $j \in J$, we have that

$$\langle k, k' \rangle \in R_J \text{ if and only if } \langle k_j, k_j' \rangle \in R_j \text{ for each } j \in J$$

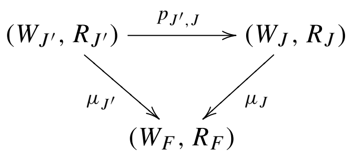
- for each $k = (k_j)_{j\in J} \in I_{W_J}$ we have $W_J^k = \prod_{j\in J} W_j^{k_j}$.

Then for each $i \in J$ the canonical projection $p_{J,i} : (W_J, R_J) \to (W_i, R_i)$ is defined by

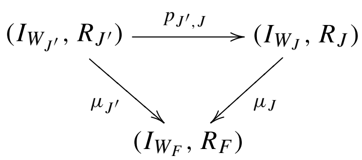
- $p_{J,i}(k) = k_i$ for each $k \in I_{W_J} = \prod_{i\in J} I_{W_i}$, and

- for each $k \in I_{W_J}$, $p^k_{J,i} \colon W^k_J \to W^{k_i}_i$ is the projection $\prod_{i \in J} W^{k_i}_i \to W^{k_i}_i$.

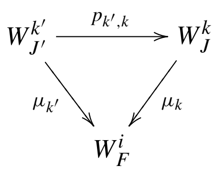
For each $J \subseteq J'$ where $J, J' \in F$, let $p_{J',J}$ denote the canonical projection $(W_{J'}, R_{J'}) \to (W_J, R_J)$. The filtered product (W_F, R_F) of $\{(W_i, R_i) \mid i \in I\}$ modulo F is the colimit of the directed diagram made of all these projections $p_{J',J}$.



This colimit is constructed in two steps. We first do the filtered product (I_{W_F}, R_F) of the family of **FOL** models $\{(I_{W_i}, R_i) \mid i \in I\}$



Recall that $\mu_I(k) = \mu_I(k')$ if and only if $\{j \mid k_j = k'_j\} \in F$. At the second step, for each $i \in I_{W_F}$ we define W^i_F as the colimit of the directed diagram constituted of the canonical projections $p_{k',k} \colon W^{k'}_{J'} \to W^k_J$ for each $J \subseteq J'$ in F, and each $k \in {\mu_J}^{-1}(i)$ and $k' \in {\mu_{J'}}^{-1}(i)$ with $p_{J',J}(k') = k$



By conditions (FP) and (LI) we can see that W^i_F can be chosen such that $\beta^{\Delta}_{\Sigma}(W^i_F) = \beta^{\Delta}_{\Sigma}(W^{i'}_F)$ for each i and i' in I_{W_F}. ■

Remark 4. Because Horn sentences are preserved by filtered products of **FOL** models we have that if the accessibility relations $\{R_j\}_{j \in I}$ satisfy some properties expressed as Horn sentences (such as T, $S4$ or $S5$) then the accessibility relation R_F of the filtered product does satisfy the same properties. This extends the existence of filtered products in subcategories of Kripke models determined by some Horn conditions on the accessibility relations. By a similar argument, in the case of ultraproducts this can be extended to subcategories of Kripke models determined by any first-order conditions.

Lemma 1. *For each $i \in I_{W_F}$, and each $(k_j)_{j \in I} \in \mu_I^{-1}(i)$, W^i_F is the filtered product modulo F of the family $\{W^{k_j}_j \mid j \in I\}$.*

Proof. For each $k \in \mu_I^{-1}(i)$ and each $J \in F$, let $k_J = p_{I,J}(k)$. Then the diagram formed by the projections $p_{k_{J'},k_J}$ for all $J \subseteq J'$ in F is a final sub-diagram of the diagram defining W^i_F. The conclusion of the lemma now follows by the general categorical result of Theorem 1 showing that final sub-diagrams of directed diagrams give isomorphic colimits. ■

Note that the ultraproducts of Kripke models in **MPL** as defined in [54] are an instance of our institution-independent ultraproducts of Kripke models of Proposition 3.

4.2. The ultraproduct fundamental theorem for modal satisfaction

Let us recall the following important preservation concepts:

Definition 11. [1] For a signature Σ in an institution, a Σ-sentence e is

– *preserved by $\mathcal{F}$-filtered factors* if $\prod_F A_i \models_\Sigma e$ implies $\{i \in I \mid A_i \models_\Sigma e\} \in F$,
– *preserved by $\mathcal{F}$-filtered products* if $\{i \in I \mid A_i \models_\Sigma e\} \in F$ implies $\prod_F A_i \models_\Sigma e$, and

for each filter $F \in \mathcal{F}$ over a set I and for each family $\{A_i\}_{i\in I}$ of Σ-models.

(Note that $\{i \in I \mid A_i \models_\Sigma e\} \in F$ is the same with 'for some $J \in F$, $A_i \models_\Sigma e$ for each $i \in J$'. We will often use the latter formulation.)

A sentence is a *Łoś sentence* when it is preserved by all ultrafactors and all ultraproducts. An institution is a *Łoś institution* when it has all filtered products of models and all its sentences are Łoś sentences.

The institution-independent method of ultraproducts has been developed in [1]. The classical Fundamental Ultraproducts Theorem shows that **FOL** is a Łoś institution, its institution-independent generalization of [1] shows that a multitude of very diverse institutions are also Łoś institutions. Examples include **PA**, **POA**, etc.

The following definition refines Definition 11 to possible worlds semantics and modal satisfaction.

Definition 12. Let $\mathcal{F}$ be a class of filters. For a signature Σ, a sentence ρ is

– *modally preserved by $\mathcal{F}$-filtered factors* when for each $i \in I_{W_F}$, $\prod_F(W_j, R_j) = (W_F, R_F) \models^i \rho$ implies "there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} \rho$ for each $j \in J$", and
– *modally preserved by $\mathcal{F}$-filtered products* when for each $i \in I_{W_F}$, "there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} \rho$ for each $j \in J$" implies $\prod_F(W_j, R_j) = (W_F, R_F) \models^i \rho$.

for each filter $F \in \mathcal{F}$ over a set I and for each family $\{(W_j, R_j)\}_{j\in I}$ of Kripke Σ-models.

Theorem 3 (*Modal Fundamental Theorem*). 1. *Each sentence of the base institution which is preserved by $\mathcal{F}$-filtered products (in the base institution) is also modally preserved by $\mathcal{F}$-filtered products (of Kripke models).*
2. *Each sentence of the base institution which is preserved by $\mathcal{F}$-filtered factors (in the base institution) is also modally preserved by $\mathcal{F}$-filtered factors (of Kripke models).*
3. *The sentences modally preserved by $\mathcal{F}$-filtered products (of Kripke models) are closed under possibility $\Diamond$.*
4. *The sentences modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) are closed under possibility $\Diamond$.*

Moreover if $\mathcal{F}$ is closed under reductions,

5. *The sentences modally preserved by $\mathcal{F}$-filtered products (of Kripke models) are closed under existential χ-quantification, when χ preserves $\mathcal{F}$-filtered products in the base institution (i.e.* $\mathsf{Mod}(\chi)$ *preserves $\mathcal{F}$-filtered products).*
6. *The sentences modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) are closed under existential χ-quantification, when χ lifts $\mathcal{F}$-filtered products of Kripke models (i.e.* $\mathsf{K\text{-}Mod}(\chi)$ *lifts $\mathcal{F}$-filtered products).*
7. *The sentences modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) and the sentences modally preserved by $\mathcal{F}$-filtered products (of Kripke models) are both closed under (finite) conjunctions.*
8. *The sentences modally preserved by $\mathcal{F}$-filtered products (of Kripke models) are closed under infinite conjunctions.*
9. *If a sentence is modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) then its negation is modally preserved by $\mathcal{F}$-filtered products (of Kripke models).*

And finally, if we further assume that $\mathcal{F}$ contains only ultrafilters,

10. *If a sentence is modally preserved by $\mathcal{F}$-filtered products (of Kripke models) then its negation is modally preserved by $\mathcal{F}$-filtered factors (of Kripke models).*
11. *The sentences modally preserved by both $\mathcal{F}$-filtered products and factors (of Kripke models) are closed under negation.*

Proof. Let F be any filter in $\mathcal{F}$ over set I, let $\{(W_j, R_j) \mid j \in I\}$ be a family of Kripke models, and let (W_F, R_F) be its filtered product modulo F. As usually, for any $k = (k_j)_{j\in J'} \in I_{W_{J'}} = \prod_{j\in J'} I_{W_j}$ and $J \subseteq J'$, by k_J we denote the tuple $(k_j)_{j\in J}$. Also, recall for any $J \in F$ its reduction to J is denoted by $F|_J$ and is defined as $\{J \cap X \mid X \in F\}$.

1. Assume that ρ is preserved by $\mathcal{F}$-filtered products in the base institution and let us fix $i \in I_{W_F}$. Let us assume that there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $W_j^{k_j} \models \rho$. Then we can find $k' \in \mu_I^{-1}(i)$ such that $k = k'_J$. By

Lemma 1, W_F^i is the filtered product of $\{W_j^{k'_j}\}_{j\in I}$ modulo F, hence because ρ is preserved by $\mathcal{F}$-filtered products, $W_F^i \models \rho$.

2. Assume that ρ is preserved by $\mathcal{F}$-filtered factors in the base institution and let us fix $i \in I_{W_F}$. Let us assume that $W_F^i \models \rho$ and take arbitrary $k' \in \mu_I^{-1}(i)$. By Lemma 1, W_F^i is the filtered product of $\{W_j^{k'_j}\}_{j\in I}$ modulo F, hence there exists $J \in F$ such that $W_j^{k'_j} \models \rho$ for each $j \in J$. We can then take $k = k'_J$.

3. Assume ρ is modally preserved by $\mathcal{F}$-filtered products (of Kripke models) and fix $i \in I_{W_F}$. Let us assume that $(W_j, R_j) \models^{k_j} \Diamond\rho$ for each $j \in J$, for some $J \in F$ and some $k \in \mu_J^{-1}(i)$. Then, for each $j \in J$ there exists k'_j with $\langle k_j, k'_j\rangle \in R_j$ such that $(W_j, R_j) \models^{k'_j} \rho$. We define $i' = \mu_J((k'_j)_{j\in J})$ and we notice that $\langle i, i'\rangle \in R_F$. Because ρ is modally preserved by filtered products we deduce that $(W_F, R_F) \models^{i'} \rho$. Because $\langle i, i'\rangle \in R_F$ this means $(W_F, R_F) \models^i \Diamond\rho$.

4. Assume ρ is modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) and fix $i \in I_{W_F}$. Let us assume that $(W_F, R_F) \models^i \Diamond\rho$. Then there exists i' with $\langle i, i'\rangle \in R_F$ such that $(W_F, R_F) \models^{i'} \rho$. This means that there exists $J' \in F$ and $l \in \mu_{J'}^{-1}(i)$ and $l' \in \mu_{J'}^{-1}(i')$ such that $\langle l, l'\rangle \in R_{J'}$. Because ρ is modally preserved by $\mathcal{F}$-filtered factors, there exists $J \in F$ and $k' \in \mu_J^{-1}(i')$ such that $(W_j, R_j) \models^{k'_j} \rho$ for each $j \in J$. Because $\mu_{J'}(l') = \mu_J(k') = i'$ there exists $J'' \subseteq J \cap J'$ in F such that $l'_{J''} = k'_{J''}$ denoted by k''. Let $k = l_{J''}$. Note that $k \in \mu_{J''}^{-1}(i)$. We have that $(W_j, R_j) \models^{k''_j = k'_j} \rho$ for each $j \in J''$ and since $\langle k, k''\rangle \in R_{J''}$ we have that $(W_j, R_j) \models^{k_j} \Diamond\rho$ for each $j \in J''$.

5. Consider $(\exists\chi)\rho$ for signature morphism $\chi\colon \Sigma \to \Sigma'$ and a Σ'-sentence ρ modally preserved by $\mathcal{F}$-filtered products (of Kripke models). For an arbitrary fixed $i \in I_{W_F}$, we assume there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} (\exists\chi)\rho$ for each $j \in J$. We have to prove that $(W_F, R_F) \models^i (\exists\chi)\rho$.

For each $j \in J$ there exists a χ-expansion (W'_j, R_j) of (W_j, R_j) such that $(W'_j, R_j) \models^{k_j} \rho$. Because $F|_J \in \mathcal{F}$ and because ρ is preserved by $\mathcal{F}$-filtered products (of Kripke models), we have that $(W'_{F|_J}, R_{F|_J}) \models^i \rho$ where $(W'_{F|_J}, R_{F|_J})$ is the filtered product of $\{(W'_j, R_j) \mid j \in J\}$ modulo $F|_J$. Because χ preserves $\mathcal{F}$-filtered products of models in the base institution, it also preserves $\mathcal{F}$-filtered products of Kripke models, hence $(W'_{F|_J}, R_{F|_J})$ is a χ-expansion of $(W_{F|_J}, R_{F|_J})$. Therefore $(W_{F|_J}, R_{F|_J}) \models^i (\exists\chi)\rho$ and since by Proposition 1 $(W_{F|_J}, R_{F|_J}) \cong (W_F, R_F)$ we have that $(W_F, R_F) \models^i (\exists\chi)\rho$.

6. Consider $(\exists\chi)\rho$ for signature morphism $\chi\colon \Sigma \to \Sigma'$ and a Σ'-sentence ρ modally preserved by $\mathcal{F}$-filtered factors. Assume $(W_F, R_F) \models^i (\exists\chi)\rho$ for some $i \in I_{W_F}$. Then there exists a χ-expansion (W', R_F) of (W_F, R_F) such that $(W', R_F) \models^i \rho$. Because χ lifts filtered products of Kripke models, there exists $J \in F$ such that for each $j \in J$ there exists a χ-expansion (W'_j, R_j) of (W_j, R_j) such that (W', R_F) is the filtered product $\prod_{F|_J}(W'_j, R_j)$.

By hypothesis ρ is modally preserved by $\mathcal{F}$-filtered factors, hence there exists $J' \in F|_J$ and $k \in \mu_{J'}^{-1}(i)$ such that $(W'_j, R_j) \models^{k_j} \rho$ for each $j \in J'$. But this implies that $(W_j, R_j) \models^{k_j} (\exists\chi)\rho$ for each $j \in J'$.

7. The preservation by filtered products is immediate. Therefore we focus on the preservation by filtered factors.

Assume that $(W_F, R_F) \models^i \rho_1 \wedge \rho_2$. Then for each $l \in \{1, 2\}$, there exists $J^l \in F$ and $k^l \in \mu_{J^l}^{-1}(i)$ such that $(W_j, R_j) \models^{k^l_j} \rho_l$ for each $j \in J^l$. Because $\mu_{J^1}(k^1) = \mu_{J^2}(k^2)$ there exists $J \subseteq J^1 \cap J^2$ in F such that $k^1_J = k^2_J$; let us denote this by k. Note that $\mu_J(k) = i$. Then for each $j \in J$ we have that $(W_j, R_j) \models^{k_j} \rho_1 \wedge \rho_2$.

8. Immediate.

9. Let ρ be a sentence which is modally preserved by $\mathcal{F}$-filtered factors. For some $i \in I_{W_F}$ assume there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that for each $j \in J$ we have that $(W_j, R_j) \models^{k_j} \neg\rho$. We have to prove that $(W_F, R_F) \models^i \neg\rho$.

If we assume the contrary, it means that $(W_F, R_F) \models^i \rho$. Since ρ is modally preserved by $\mathcal{F}$-filtered factors, there exists $J' \in F$ and $k' \in \mu_{J'}^{-1}(i)$ such that for each $j \in J'$ we have that $(W_j, R_j) \models^{k'_j} \rho$. Because $\mu_J(k) = \mu_{J'}(k')$ we can find a non-empty $J'' \subseteq J \cap J'$ in F such that $k_{J''} = k'_{J''}$. Let us denote this by k''. For each $j \in J''$ we then have that $(W_j, R_j) \models^{k''_j} \neg\rho$ and $(W_j, R_j) \models^{k''_j} \rho$ which is a contradiction. This shows that $(W_F, R_F) \models^i \neg\rho$.

10. Let ρ be any sentence which is modally preserved by $\mathcal{F}$-filtered products and assume $(W_F, R_F) \models^i \neg\rho$. For any fixed $i \in I_{W_F}$ take an arbitrary $k \in \mu_I^{-1}(i)$. If $\{j \in I \mid (W_j, R_j) \models^{k_j} \neg\rho\} \notin F$ then its complement

$\{j \in I \mid (W_j, R_j) \models^{k_j} \rho\}$ belongs to F (because F is an ultrafilter). Because ρ is preserved by ultraproducts, this would imply $(W_F, R_F) \models^i \rho$ which contradicts $(W_F, R_F) \models^i \neg\rho$, therefore $\{j \in I \mid (W_j, R_j) \models^{k_j} \neg\rho\} \in F$.

11. From 9. and 10. ■

The following is an important consequence of the modal fundamental Theorem 3.

Corollary 3. *Each modal sentence which is accessible from the Łoś-sentences of the base institution by (modal) Boolean connectives, possibility $\Diamond$ and (modal) χ-quantifications for which χ preserves filtered products of models (in the base institution), and lifts filtered products of Kripke models*

– *is modally preserved by ultraproducts and ultrafactors, and*
– *is preserved by ultraproducts.*

Proof. In Theorem 3 we consider $\mathcal{F}$ to be the class of all ultrafilters. The first item follows immediately from the conclusions of Theorem 3.

The second item follows from the first one. To see this let us consider an ultrafilter U over a set I and let (W_U, R_U) be an ultraproduct of Kripke models $\prod_U (W_j, R_j)$ for a family $\{(W_j, R_j)\}_{j\in I}$ of Kripke models. Assume that $\{j \mid (W_j, R_j) \models \rho\} \in U$ and that $(W_U, R_U) \not\models \rho$. Then there exists $i \in I_{W_F}$ such that $(W_U, R_U) \not\models^i \rho$ which means $(W_U, R_U) \models^i \neg\rho$. Because ρ is preserved by ultrafactors, there exists $J \in U$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} \neg\rho$ for each $j \in J$. Note that $J \cap \{j \mid (W_j, R_j) \models \rho\} \in U$. Then for any of its elements j we have both that $(W_j, R_j) \models^{k_j} \neg\rho$ and that $(W_j, R_j) \models^{k_j} \rho$ which is a contradiction. Hence $(W_U, R_U) \models \rho$. ■

Remark 5. The (ordinary) preservation by ultrafactors cannot be established for the possible worlds semantics mainly because modal negation is not a semantic negation (in the sense of Definition 6). This can be easily seen if one tries to replicate the argument for the preservation of sentences by ultraproducts of Corollary 3 above to the preservation by ultrafactors. However, preservation by ultraproducts is still sufficient to derive a series of important results, most notably model compactness.

4.3. Getting more concrete

Similarly to the correspondent result for Łoś-sentences (see [1]), the only conditions of Corollary 3 that in the applications narrow the set of sentences which are preserved by ultraproducts refer to the quantifiers. Except lifting of filtered products of Kripke models, the other conditions refer to the level of the base institution.

Remark 6. The preservation of filtered products of models (in the base institution) by the model reduct functors is an immediate consequence of the preservation of the direct products and of the directed colimits by the model reduct functors. While the preservation of direct products holds whenever the model reduct functors have left adjoints (which is a common property of many institutions), the model reducts *create* directed colimits of models whenever the symbols of the signatures are finitary. In many actual institutions the above arguments are valid for all signature morphisms, however for restricted classes of signature morphisms which are usually used for quantifications, we will be able to deal with this issue at a general institution-independent level.

Therefore, the key condition to be studied remains the lifting of filtered products of Kripke models. The result of Proposition 4 below reduces it to lifting of filtered products of models in the base institution.

Definition 13 (*Exact Signature Morphisms*). A signature morphism $\chi : \Sigma \to \Sigma'$ is *$(\Phi^\Delta, \beta^\Delta)$-exact* when the square of the naturality of β^Δ for χ is pullback:

$$\begin{array}{ccccc}
\Sigma & & \mathrm{Mod}^\Delta(\Phi^\Delta(\Sigma)) & \xleftarrow{\beta^\Delta_\Sigma} & \mathrm{Mod}(\Sigma) \\
\Big\downarrow \chi & & \Big\uparrow \mathrm{Mod}(\Phi^\Delta(\chi)) & & \Big\uparrow \mathrm{Mod}(\chi) \\
\Sigma' & & \mathrm{Mod}^\Delta(\Phi^\Delta(\Sigma')) & \xleftarrow[\beta^\Delta_{\Sigma'}]{} & \mathrm{Mod}(\Sigma')
\end{array}$$

Examples 3. The table below gives classes of $(\Phi^{\Delta}, \beta^{\Delta})$-exact signature extensions associated to the institution morphisms of Example 7.

Entry in table of Example 7	χ
1.	extensions with rigid constants
2.	extensions with constants
3.	extensions with constants
4.	extensions with sorts and total constants
5.	all signature morphisms

Proposition 4. *A signature morphism χ lifts filtered products of Kripke models if it is $(\Phi^{\Delta}, \beta^{\Delta})$-exact and lifts completely and preserves filtered products of models (in the base institution).*

Proof. Let (W_F, R_F) be the filtered product of a family of Σ-Kripke models $\{(W_j, R_j)\}_{j\in I}$ modulo a filter F over the set I and let (W', R_F) be a χ-expansion of (W_F, R_F).

Let $i \in I_{W_F}$ and $k \in \mu_I^{-1}(i)$. By Lemma 1, W_F^i is the filtered product modulo F of the family $\{W_j^{k_j} \mid j \in I\}$.

Because χ lifts completely filtered products of models (in the base institution), for each $j \in I$ let $W'^{k_j}_j$ be a χ-expansion of $W_j^{k_j}$ such that W'^i is the filtered product of $\{W'^{k_j}_j \mid j \in I\}$.

Because χ is $(\Phi^{\Delta}, \beta^{\Delta})$-exact, for each $j \in I$ and each $l \in I_{W_j}$ let W'^l_j be the unique Σ'-model such that $\beta^{\Delta}_{\Sigma'}(W'^l_j) = \beta^{\Delta}_{\Sigma'}(W'^{k_j}_j)$ and $W'^l_j \restriction_\chi = W^l_j$.

Now we prove that (W', R_F) is the filtered product of $\{(W'_j, R_j)\}_{j\in J}$ modulo F. Consider an arbitrary $k' \in I_{W_I} = \prod_{j\in I} I_{W_J}$ and let $i' = \mu_I(k')$. By Lemma 1, it is enough to show that $W'^{i'}$ is the filtered product of $\{W'^{k'_j}_j \mid j \in I\}$ modulo F. This follows by the $(\Phi^{\Delta}, \beta^{\Delta})$-exactness property of χ because

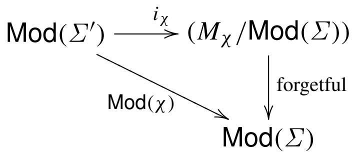
$$\begin{aligned}
\beta^{\Delta}_{\Sigma'}\left(\prod_F W'^{k'_j}_j\right) &= \prod_F \beta^{\Delta}_{\Sigma'}(W'^{k'_j}_j) && \text{(by } (FP))\\
&= \prod_F \beta^{\Delta}_{\Sigma'}(W'^{k_j}_j) && \text{(by the sharing condition)}\\
&= \beta^{\Delta}_{\Sigma'}\left(\prod_F W'^{k_j}_j\right) && \text{(by } (FP) \text{ and } (LI))\\
&= \beta^{\Delta}_{\Sigma'}(W'^i)\\
&= \beta^{\Delta}_{\Sigma'}(W'^{i'}) && \text{(by the sharing condition)}
\end{aligned}$$

and because

$$\begin{aligned}
\left(\prod_F W'^{k'_j}_j\right)\restriction_\chi &= \prod_F (W'^{k'_j}_j \restriction_\chi) && \text{(because } \chi \text{ preserves filtered products)}\\
&= \prod_F W^{k'_j}_j && \text{(by the definition of } W^{k'_j}_j)\\
&= \prod_F W^{k'_j}_j && \text{(by Lemma 1)}\\
&= W'^{i'} \restriction_\chi && \text{(by the hypothesis that } (W_F, R_F)\restriction_\chi = (W', R_F)). \quad \blacksquare
\end{aligned}$$

Many applications involve only first order quantifications. In such cases it is possible to make the conditions of Proposition 4 more concrete based on the following institution-independent generalization for the concept of 'first order' variable.

Definition 14 (*Representable Signature Morphisms [1]*). A signature morphism $\chi : \Sigma \to \Sigma'$ is *representable* if and only if there exists a Σ-model M_χ (called the *representation of* χ) and an isomorphism i_χ of categories such that the following diagram commutes:



A signature morphism $\chi : \Sigma \to \Sigma'$ is *projectively representable* when M_χ is projective.

Examples 4. In **FOL** and **POA** all signature extensions with constants are projectively representable. In **PA** all signature extensions with total constants are projectively representable.

Note that signature extensions with rigid constants are *not* representable in **MFOL**.

Corollary 4. *Assume that in the base institution all projections of model products are epis. Then a signature morphism lifts filtered products of Kripke models if it is* $(\Phi^\Delta, \beta^\Delta)$*-exact and projectively representable (in the base institution).*

Proof. Let $\chi : \Sigma \to \Sigma'$ be a representable signature morphism. The preservation of filtered products of models by $\mathsf{Mod}(\chi)$ holds because $\mathsf{Mod}(\chi)$ creates (and thus preserves) direct products and directed colimits, which can be established on the basis that the forgetful functor $M_\chi/\mathsf{Mod}(\Sigma) \to \mathsf{Mod}(\Sigma)$ does it.

Moreover, we may note that $\mathsf{Mod}(\chi)$ lifts completely filtered products by translating the problem to the forgetful $M_\chi/\mathsf{Mod}(\Sigma) \to \mathsf{Mod}(\Sigma)$ and by noticing that for any filtered product $\prod_F A_i$ the projections $p_{I,J}$ are epis, we get that $\mu_I : \prod_I A_i \to \prod_F A_i$ is epi. ■

We are now able to formulate a corollary which is immediately applicable to actual institutions.

Corollary 5. *Assume that in the base institution all projections of model products are epis. Then the modal sentences preserved by ultraproducts*

– *contain all Łoś sentences of the base institution,*
– *are closed under (modal) Boolean connectives,*
– *are closed under modalities* $\Box$ *and* $\Diamond$*, and*
– *are closed under any quantification which is* $(\Phi^\Delta, \beta^\Delta)$*-exact and projectively representable (in the base institution).*

A typical concrete instance of Corollary 5 is given by **MFOL** as a modal institution (according to Examples 7 and 8).

Corollary 6. *Each sentence of* **MFOL** *is preserved by ultraproducts.*

4.4. Compactness

In the rest of this section we develop a compactness result as a consequence of the preservation of sentences by ultraproducts.

Definition 15 (*Compactness [1]*). An institution

– is *model compact* when for each signature Σ, each set of Σ-sentences has a model whenever each of its finite subsets has a model, and
– is *compact* when for each signature Σ, any set of Σ-sentences E and each Σ-sentence e, if $E \models_\Sigma e$ then there exists a finite set E_0 of Σ-sentences such that $E_0 \models_\Sigma e$.

From [1] we know that when all sentences of an institution are preserved by ultraproducts then the institution is model compact. Hence:

Corollary 7. *If each sentence of a 'modal' institution is accessible by the operations listed in Corollary 5, then the institution is model compact.*

Corollary 8. **MFOL** *is model compact.*

Remark 7. Note that compactness of **MFOL** cannot be established from the model compactness by the general result given in [1] relating compactness to model compactness because **MFOL** has only modal negation, which is not a semantic negation as required by the above mentioned result of [1].

5. Conclusions

We have defined possible worlds semantics and modal satisfaction on top of arbitrary institutions by employing a sharing constraint formalized as an institution morphism to a 'domain' institution. We have developed an 'internal modal logic' which yields a 'modal' institution with the models being the Kripke models defined from the models of the base institution, with the sentences extending the sentences of the base institution with the usual modal operators, and with a modal satisfaction between Kripke models and sentences extending the given satisfaction relation of the base institution.

Based on the institution-independent method of ultraproducts of [1] we have developed a preservation result for the internal modal satisfaction. As an immediate application we have developed a generic compactness result for internal possible worlds semantics.

Our work can be applied as a method for systematically extending various institutions in use in computing science and logic with possible worlds semantics and modal satisfaction. This also shows that possible worlds semantics for modalities is in a certain way orthogonal to the other features of the logical systems.

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This work is dedicated to the memory of our teacher Joseph Goguen.

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