A LADDER THERMOELECTRIC PARALLELEPIPED GENERATOR

by

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Submitted to ICT2002 on the 5th of May 2002.

ABSTRACT:

The frequency behaviour of a thermoelectric generator becomes very important when a high frequency switching regulator is used. The operating frequency of switching regulators has steadily increased over the past few years and >1MHz is now practical. A thermoelectric generator is constructed from a large number of series connected parallelepipeds of some thermoelectric crystalline material. The hot and/or cold reservoir is made of some electrically conductive metal, and the fluid is to some extent conductive to the ground. This topology generates a number of small capacitors formed by each two parallelepiped crystal faces and the usually grounded thermal reservoir. We analyse the frequency behaviour of such a thermoelectric generator, which typically contains thousands of parallelepipeds, each generating few milliwatts.

INTRODUCTION:

Electronic network theory of linear circuit elements has a strong connection to the mathematics and the algebra of Polynomials in the Complex Domain. This is due to the fact that a dynamical differential equation can be Laplace transformed from the time domain into the frequency domain, and in the process, is turned into a polynomial in the complex variable "$s = \beta + i\omega$", where (ω) is the angular frequency "$f = \omega/2\pi$" and (β) is the dissipative (or generative) time-constant. A thermoelectric generator consisting of a large number of small crystals connected serially together can be considered as a naturally occurring network of lumped elements, and is ideally suited to network analyses in the frequency domain.

CONTENTS:

1. Thermoelectric parallelepipeds connected in series:

Let a thermoelectric body be a rectangular parallelepiped with two opposite metal faces, each of area (A). The thermal and electrical paths are both along the length (l_m) of the body, and it's volume is "$V_m = A\ l_m$". Now consider a device consisting of a number of such rectangular parallelepipeds, all connected in series by a metallic conductor. The thermoelectric solid body possess a dielectric constant (ε_m) between its two metallic faces. This results in a small capacitor of capacitance "$C = \varepsilon_m A / l_m$". A small resistor of resistance "$R = l_m / \sigma A$", where (σ) is the electrical conductivity of the thermoelectric material, is also present. Further, due to the inertia of the charge carriers and magnetic fields, we also have an inductive behaviour as represented in the inductance "$L = \mu_m\ l_m$" where (μ_m) is the permeability or specific inductance of the material in question. To prevent excessive build-up of accumulated inductance of the complete device, with hundreds of series connected blocks, the blocks are arranged in a non-inductive arrangement such that no loops are formed. The negative mutual inductance so formed, will speed up the transmission of electrical disturbances along the segmented path through all the thermoelectric blocks and metal joints.

Now consider four series connected blocks of thermoelectric parallelepipeds as symbolized in the following figure. Observe new ground capacitor that is different from the body capacitor of the parallelepipeds above. Its area is larger, the dielectric length and the dielectric constant is different. We express this capacitor as "$C = \varepsilon_i A_i / l_i$", where the i-subscript refers to the insulating topology, the length (l_i) also being the thermal length of the insulating layer from the metal face to the thermal reservoir:

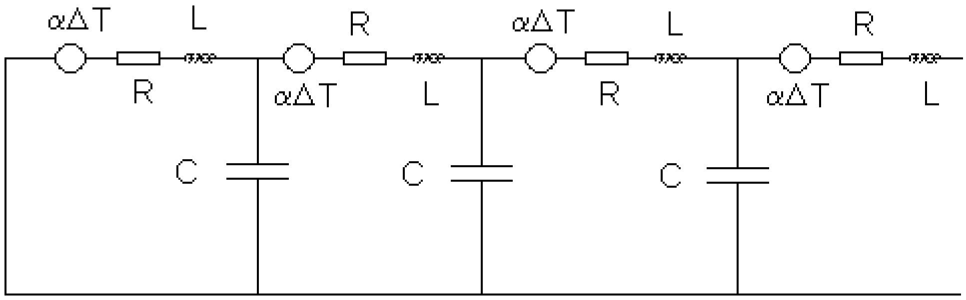


FIG 1, A GROUNDED LADDER GENERATOR

With four parallelepipeds we can calculate the impedance, looking into the generator from the right, with the left side firmly grounded to both hot and cold reservoirs:

$$Z_4 = R + L \cdot s + \cfrac{1}{C \cdot s + \cfrac{1}{R + L \cdot s + \cfrac{1}{C \cdot s + \cfrac{1}{R + L \cdot s + \cfrac{1}{C \cdot s + \cfrac{1}{R + L \cdot s}}}}}}$$

This continued fraction can be extended to any number of parallelepipeds or generating elements. At the highest frequency where "$R + Ls >> 1/C\ s$", only one element is connected and gives "$Z_4 = R + Ls$", and at the lowest frequency, the impedance is "$Z_4 = 4\ (R + Ls)$", displaying the number of parallelepipeds or thermoelectric elements. By expanding the continued fraction and using the complex variable "$a = R\ C\ s + LCs^2$" as a shorthand, we get successively up to six generating elements:

$$Z_1 = R + L \cdot s, \qquad Z_2 = Z_1 \cdot \left(\frac{2+a}{1+a}\right), \qquad Z_3 = Z_1 \cdot \left(\frac{3 + 4 \cdot a + a^2}{1 + 3 \cdot a + a^2}\right), \qquad Z_4 = Z_1 \cdot \left(\frac{4 + 10 \cdot a + 6 \cdot a^2 + a^3}{1 + 6 \cdot a + 5 \cdot a^2 + a^3}\right)$$

By looking at the continued fraction we can deduce the recursive relation: "$Z_k = Z_1 + Z_{k-1} / (1+CsZ_{k-1})$", and with it, easily calculate for any number of generating elements. For up to six elements we get accordingly:

$$Z_5 = Z_1 \cdot \left(\frac{5 + 20 \cdot a + 21 \cdot a^2 + 8 \cdot a^3 + a^4}{1 + 10 \cdot a + 15 \cdot a^2 + 7 \cdot a^3 + a^4}\right), \qquad Z_6 = Z_1 \cdot \left(\frac{6 + 35 \cdot a + 56 \cdot a^2 + 36 \cdot a^3 + 10 \cdot a^4 + a^5}{1 + 15 \cdot a + 35 \cdot a^2 + 28 \cdot a^3 + 9 \cdot a^4 + a^5}\right)$$

2. Analytical evaluation for large number of generating elements:

The coefficients of the numerator and denominator polynomials can be arranged in a Pascal's-like triangle. This will aid in the seek for a simplified expression when using tens to hundreds of thermoelectric parallelepipeds:

Denominator:

$$\begin{matrix} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&3&&1&&& \\ &&1&&6&&5&&1&& \\ &1&&10&&15&&7&&1& \\ 1&&15&&35&&28&&9&&1 \end{matrix}$$

Numerator:

$$\begin{matrix} &&&&&1&&&&& \\ &&&&2&&1&&&& \\ &&&3&&4&&1&&& \\ &&4&&10&&6&&1&& \\ &5&&20&&21&&8&&1& \\ 6&&35&&56&&36&&10&&1 \end{matrix}$$

Let D(n,m) and N(n,m) symbolize the denominator and numerator coefficients respectively, where (n) is the number of elements and the row counter, and (m) is the coefficient counter, also the column index. Starting with the denominator, which is simpler of the two, we see first that "*D(n,1)= D(n,n)=1"*, and that the 2nd coefficient is just the sum of integers: (1), (1+2), (1+2+3), (1+2+3+4), etc. This gives "*D(n,2)=n(n-1)/2*". Finally, the next to last coefficient is "*D(n,n-1)=2n-3*" or the odd integers 1, 3, 5, · · ·.

Turning to the numerator, after a little thought, we see that the 2nd coefficient is the sum of partial sums as shown in the following sequences: (1), (1 + 1+2), (1 + 1+2 + 1+2+3), (1 + 1+2 + 1+2+3 + 1+2+3+4), etc. After a little algebra, the 2nd coefficient for the numerator is revealed as: "*N(n,2)=n(n-1)(n+1)/6*" displaying a third order growth (n^3), while the denominators 2nd coefficient grew like the second power (n^2). Collecting all information, we have: "*N(n,1)=n*" and "*N(n,n-1)=2(n-1)*" and "*N(n,n)=1*" and the impedance of (n>1) elements is formally express as:

$$Z_n = Z_1 \cdot \left(\frac{n + \frac{1}{6} \cdot n \cdot (n^2 - 1) \cdot a + \cdots + (2 \cdot n - 2) \cdot a^{n-2} + a^{n-1}}{1 + \frac{1}{2} \cdot n \cdot (n-1) \cdot a + \cdots + (2 \cdot n - 3) \cdot a^{n-2} + a^{n-1}} \right)$$

To complete this work, we need to establish a relation between the numerator and denominator coefficients. The first such observation is that the following is true for n>1: "*N(n,2) = N(n-1,2) + D(n,2)*". Let us try to get a similar expression for m=3. In fact when we try, the expression "*N(n,m) = N(n-1,m) + D(n,m)*" is also true! If we can get a similar expression for the denominator, we are home. Believe it or not, the result is "*D(n,m) = N(n-1,m-1) + D(n-1,m)*". The complete recursive relation for the numerator is therefore: "*N(n,m) = N(n-1,m) + N(n-1,m-1) + D(n-1,m)*". The last relation enables us to obtain the general expression for the 3rd coefficient of the denominator polynomial as "*D(n,3) = N(1,2) + N(2,2) + N(3,2) + · · · +N(n-1,2)*". In the same way we get "*N(n,3) = D(1,3) + D(2,3) + D(3,3) + · · · +D(n,3)*" and our task is almost done. We accumulate this information in the following expressions:

$$D(n,1) = 1 \qquad N(n,1) = n$$

$$D(n,2) = \tfrac{1}{2} \cdot n \cdot (n-1) \qquad N(n,2) = \tfrac{1}{6} \cdot n \cdot (n^2 - 1)$$

$$D(n,3) = \tfrac{1}{24} \cdot n \cdot (n^2 - 1) \cdot (n-2) \qquad N(n,3) = \tfrac{1}{120} \cdot n \cdot (n^2 - 1) \cdot (n^2 - 2^2)$$

$$D(n,m) = \sum_{k=1}^{n-1} N(k, m-1) \qquad N(n,m) = \sum_{k=1}^{n} D(k,m)$$

$$D(n,n-1) = 2 \cdot n - 3 \qquad N(n,n-1) = 2 \cdot n - 2$$

$$D(n,n) = 1 \qquad N(n,n) = 1$$

We almost have a closed form expression for both the numerator and denominator coefficients. For example, we see that "*N/D = (n+m-1)/(m+2)*" and after a little work, we arrive at the following two statements:

$$D(n,m) = \frac{n \cdot (n+1-m)}{(2 \cdot m - 2)!} \cdot \prod_{k=1}^{m-2} (n^2 - k^2) = \frac{n+1-m}{n \cdot (2 \cdot m - 2)!} \cdot \prod_{k=0}^{m-2} (n^2 - k^2)$$

$$N(n,m) = \frac{n}{(2 \cdot m - 1)!} \cdot \prod_{k=1}^{m-1} (n^2 - k^2) = \frac{1}{n \cdot (2 \cdot m - 1)!} \cdot \prod_{k=0}^{m-1} (n^2 - k^2)$$

This pursuit has certainly paid off as we now have a closed form solution for both the numerator, and the denominator coefficients, up to any order in (n) or (m)!

3. Tabulation of values for the numerator and denominator coefficients:

By inspection, both recursive and independent expressions, for either D(n,m) or N(n,m) can be obtained. We have thus in fact, solved all the difference equations encountered so far. The following relations are also memory efficient, needing only one stored variable besides the indexes, (n) and (m):

$$D(n,1) = 1, \qquad D(n,m+1) = \frac{n^2 - m^2 - n + m}{4 \cdot m \cdot \left(m - \frac{1}{2}\right)} \cdot D(n,m)$$

$$N(n,1) = n, \qquad N(n,m+1) = \frac{n^2 - m^2}{4 \cdot m \cdot \left(m + \frac{1}{2}\right)} \cdot N(n,m)$$

Constructing a recursive spreadsheet is now easy, using rows and columns to represent the index variables (n) and (m):

D(n,1)	D(n,2)	D(n,3)	D(n,4)	D(n,5)	D(n,6)	D(n,7)	D(n,8)	D(n,9)	D(n,10)
1									
1	1								
1	3	1							
1	6	5	1						
1	10	15	7	1					
1	15	35	28	9	1				
1	21	70	84	45	11	1			
1	28	126	210	165	66	13	1		
1	36	210	462	495	286	91	15	1	
1	45	330	924	1287	1001	455	120	17	1

N(n,1)	N(n,2)	N(n,3)	N(n,4)	N(n,5)	N(n,6)	N(n,7)	N(n,8)	N(n,9)	N(n,10)
1									
2	1								
3	4	1							
4	10	6	1						
5	20	21	8	1					
6	35	56	36	10	1				
7	56	126	120	55	12	1			
8	84	252	330	220	78	14	1		
9	120	462	792	715	364	105	16	1	
10	165	792	1716	2002	1365	560	136	18	1

We are now in a position to write a simple computer programs to simulate any number of thermoelectric parallelepipeds connected in series.

4. Convergence of the ladder polynomials for low frequencies:

To investigate the speed of convergence when using a large number of elements at a low frequency in relation to the single crystal frequency "$f_0 = 1/2\pi(LC)^{1/2}$", let us write the ladder polynomials as a power series in (a) using (m) as a term index:

$$P_n(a) = n + \tfrac{1}{6} \cdot a \cdot n \cdot \left(n^2 - 1\right) + \tfrac{1}{120} \cdot a^2 \cdot n \cdot \left(n^2 - 1\right) \cdot \left(n^2 - 2^2\right) + \cdots + \frac{a^{m-1}}{n \cdot (2 \cdot m - 1)!} \cdot \prod_{k=0}^{m-1} \left(n^2 - k^2\right) + \cdots$$

$$Q_n(a) = 1 + \tfrac{1}{2} \cdot a \cdot n \cdot (n - 1) + \tfrac{1}{24} \cdot a^2 \cdot n \cdot \left(n^2 - 1\right) \cdot (n - 2) + \cdots + \frac{(n + 1 - m) \cdot a^{m-1}}{n \cdot (2 \cdot m - 2)!} \cdot \prod_{k=0}^{m-2} \left(n^2 - k^2\right) + \cdots$$

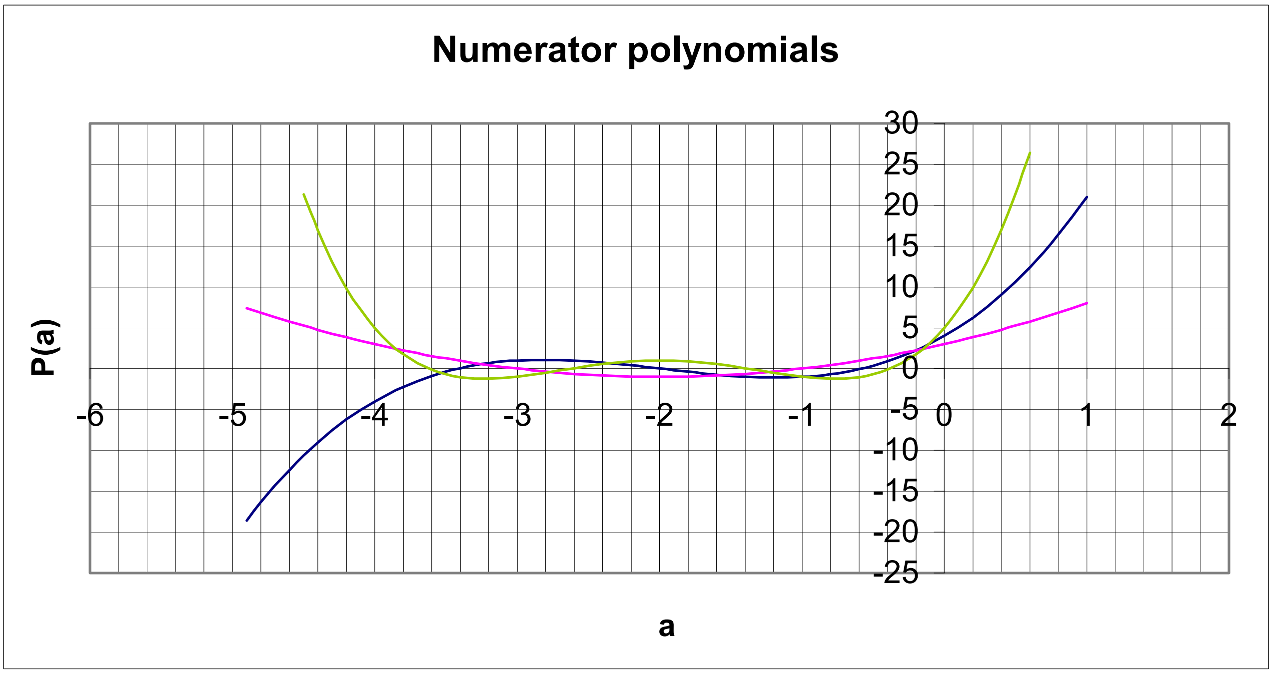
By comparing terms it is apparent that if "$|a| < 1/n^2$", convergence is guarantied and only few terms in (m) are needed in cases where (n) is large. If we now transform from (a) back to the frequency (f), convergence is secured if "$f < f_0/n$". By performing the ratio-test on the numerator series, a larger interval of convergence is obtained as "$|a| < 4m^2/n^2$" and corresponding "$f < 2f_0 m/n$". This allows us to determine how many terms (m) are needed when (n) is fixed.

5. Zeros and factors of the numerator polynomials:

It is now stated without proof that the zeros of the numerator polynomials which we label "$P_n(a)$" lie on the negative "a" axis where "$a = RCs + LCs^2$". The reader is reminded of the fact that "$P_n(0)=n$". With elementary algebra we can factor the first few numerator polynomials as:

$$P_1 = 1$$
$$P_2 = (a+2)$$
$$P_3 = (a+2-1)\cdot(a+2+1)$$
$$P_4 = \left(a+2-\sqrt{2}\right)\cdot(a+2)\cdot\left(a+2+\sqrt{2}\right)$$
$$P_5 = \left(a+2-\tfrac{1+\sqrt{5}}{2}\right)\cdot\left(a+2-\tfrac{2}{1+\sqrt{5}}\right)\cdot\left(a+2+\tfrac{2}{1+\sqrt{5}}\right)\cdot\left(a+2+\tfrac{1+\sqrt{5}}{2}\right), \qquad \tfrac{1+\sqrt{5}}{2} = \text{Golden Section}$$
$$P_6 = \left(a+2-\sqrt{3}\right)\cdot(a+2-1)\cdot(a+2)\cdot(a+2+1)\cdot\left(a+2+\sqrt{3}\right)$$

Observe that the zeros are centred about "a=-2" which is also a zero for all the even order polynomials. Also note a new numerical sequence 0, 1, $2^{1/2}$, ϕ, $3^{1/2}$, … where (ϕ) is the golden section. The limit of this sequence can only be 2 if our initial statement is proved: The negative definiteness of the roots of the numerator polynomials $P_n(a)$.
Below is a graph of $P_3(a)$, $P_4(a)$ and $P_5(a)$ that display the confinement of the roots to the interval ($-4 < a < 0$) .



An important frequency of concern is the smallest frequency where P(a) =0. If (ω) is the angular frequency in [rad/sec], we can write "$a = RCs + LCs^2 = -\omega^2 LC + \underline{i}\,\omega RC = -\omega^2 LC\,(1 - \underline{i}\,R/\omega L) = -(f/f_0)^2\,(1 - \underline{i}\,/Q_f)$", where "$f_0 = 1/2\pi(LC)^{1/2}$" is the free resonance frequency in [Hz] for a single crystal and "$Q_f = 2\pi f\, L/R$" is the frequency dependant inductive quality-factor. At very low frequencies, (a) is almost a pure imaginary number, whereas at the critical frequency (a) is equally real and imaginary and finally, at a very high frequency, (a) is almost a pure negative real number!

We now present a formula to calculate the smallest root when we go from the origin to the left. This formula is based on empirical experimental mathematics and will be shown to be exact! We label this first zero "a_{Z1}" where the Z1-subscript refers to the numerator 1^{st} zero.

$$a_{Z1} = -4\cdot\sin^2\left(\frac{\pi}{2\cdot n}\right) \qquad \Rightarrow f_{Z1} = 2\cdot f_0\cdot\sin\left(\frac{\pi}{2\cdot n}\right)$$

For large (n), an asymptotic formula is easily derived by expanding the sinus function around zero:

$$a_{Z1} = -\frac{\pi^2}{n^2} + \frac{\pi^4}{24\cdot n^4} - \cdots \qquad \Rightarrow f_{Z1} = f_0\cdot\left(\frac{\pi}{n} - \frac{\pi^3}{24\cdot n^3} + \cdots\right) \approx \frac{\pi\cdot f_0}{n}$$

6. Zeros and factors of the denominator polynomials:

To factor the denominator polynomials, we apply the same algebraic methods as we did with the numerator polynomials. Label the denominator polynomial for n elements $Q_n(a)$ in the hope of not to confuse the reader with the frequent discussion of the "Quality factor" also labelled Q. Starting with the first few, we immediately see generally more irrational roots and almost no integer roots, a different situation from the numerator polynomials. To our pleasure, the distribution of the roots turn out to be very similar with a centre at "a = -2" and confinement, just as in the numerator case.

$$Q_1 = 1$$

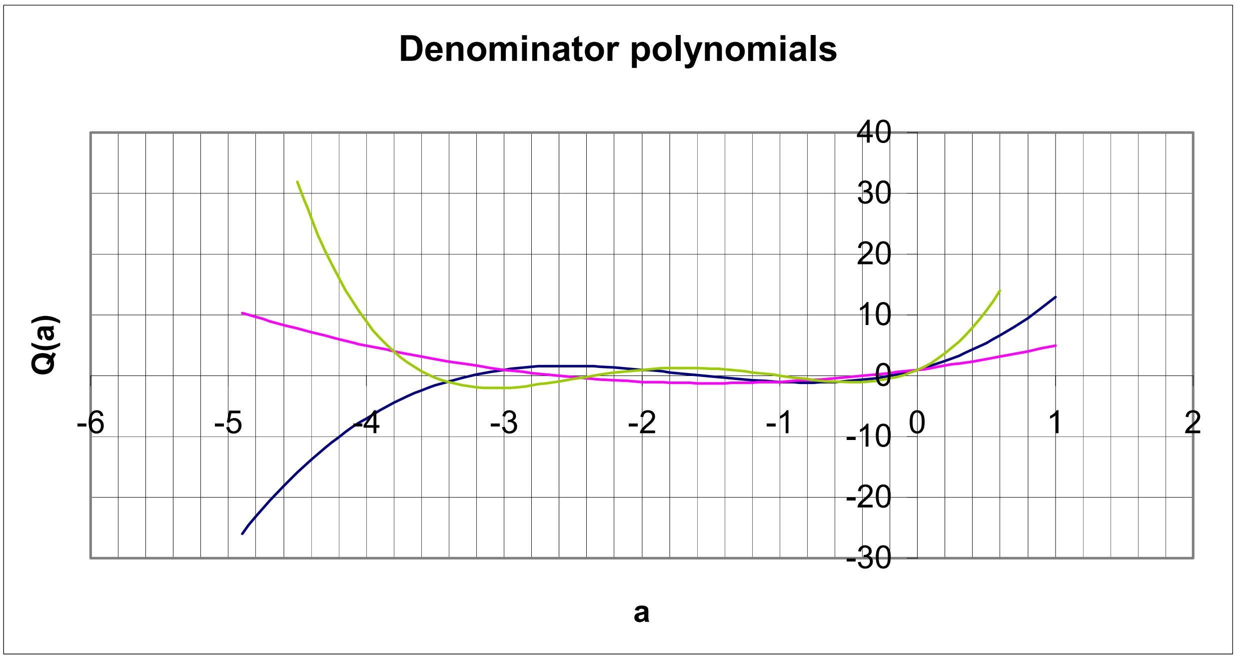
$$Q_2 = (a+1)$$

$$Q_3 = \left(a + \frac{3+\sqrt{5}\cdot\cos(0)}{2}\right)\cdot\left(a + \frac{3+\sqrt{5}\cdot\cos(\pi)}{2}\right)$$

$$Q_4 = \left(a + \frac{5+2\cdot\sqrt{7}\cdot\cos\left(\frac{\theta}{3}\right)}{3}\right)\cdot\left(a + \frac{5+2\cdot\sqrt{7}\cdot\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}{3}\right)\cdot\left(a + \frac{5+2\cdot\sqrt{7}\cdot\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}{3}\right), \qquad \theta = \arccos\left(\frac{1}{2\cdot\sqrt{7}}\right)$$

$$Q_5 = \left(a+2+2\cdot\cos\left(\tfrac{2\pi}{3}\right)\right)\cdot\left(a+2+2\cdot\cos\left(\tfrac{2\pi}{9}\right)\right)\cdot\left(a+2+2\cdot\cos\left(\tfrac{2\pi}{9}+\tfrac{2\pi}{3}\right)\right)\cdot\left(a+2+2\cdot\cos\left(\tfrac{2\pi}{9}+\tfrac{4\pi}{3}\right)\right)$$

To prepare the reader with the general expression up to any n, we have written $Q_3(a)$ using cos(0) and cos(π) as a fancy way to express 1 and –1. Also notice that the 1st factor of $Q_5(a)$ is in fact $Q_2(a)$ in a trigonometric disguise! This is to display the similarity and common attributes among the denominator polynomials. Below is a graph of $Q_3(a)$, $Q_4(a)$ and $Q_5(a)$ that displays the confinement of the roots to the negative interval $(-4 < a < 0)$.



We now present a formula to calculate the smallest root when we go from the origin to the left. This will be the first pole of the total impedance. This formula is based on empirical experimental mathematics but will be shown to be exact! We label this first pole "a_{P1}" where the P1-subscript refers to the numerator 1st root.

$$a_{P1} = -4\cdot\sin^2\left(\frac{\pi}{4\cdot n-2}\right) \qquad \Rightarrow f_{P1} = 2\cdot f_0\cdot\sin\left(\frac{\pi}{4\cdot n-2}\right)$$

For large (n), an asymptotic formula is easily derived by expanding the sinus function around zero:

$$a_{P1} = -\frac{\pi^2}{4\cdot n^2-4\cdot n+1}+\cdots \qquad \Rightarrow f_{P1} = \frac{\pi\cdot f_0}{2\cdot n-1}-\cdots \approx \frac{\pi\cdot f_0}{2\cdot n}$$

It is by now noticed, that the 1st zero and the 1st pole are related by a surprisingly simple relation:

$$a_{P1}(n) = a_{Z1}(2\cdot n-1) \qquad \Rightarrow \qquad a_{Z1}(n) \approx a_{P1}\left(\tfrac{n+1}{2}\right)$$

7. The Zeta function, Riemann Hypothesis and the Ladder Polynomials:

All recursive relations of both numerator and denominator coefficients have been derived without a single proof. We will now present an equivalent set of equations easier to prove. By inspection and simple algebraic operations, the following can be derived by some work. Notice a complete factorisation formula for both ladder polynomials!

$$Q_1(a) = 1 \qquad\qquad P_1(a) = 1$$

$$Q_n(a) = Q_{n-1}(a) + a \cdot P_{n-1}(a) \qquad\qquad P_n(a) = Q_{n-1}(a) + (1+a) \cdot P_{n-1}(a)$$

$$Q_n(a) = 1 + a \cdot \sum_{k=1}^{n-1} P_k(a) \qquad\qquad P_n(a) = 1 + P_{n-1}(a) + a \cdot \sum_{k=1}^{n-1} P_k(a)$$

$$Q_n(a) = \prod_{k=1}^{n-1} \left(a + 4 \cdot \sin^2\left(\frac{\pi \cdot (2k-1)}{2 \cdot (2n-1)} \right) \right) \qquad\qquad P_n(a) = \prod_{k=1}^{n-1} \left(a + 4 \cdot \sin^2\left(\frac{\pi \cdot k}{2 \cdot n} \right) \right)$$

To prepare for the proof of our so-far unproven statements, the following is a consequence of the factored polynomials and is deduced from the fact, that "$P_n(0) = n$" and "$Q_n(0) = 1$".

$$2 \cdot \sin\left(\tfrac{\pi}{2n}\right) \cdot 2 \cdot \sin\left(\tfrac{2\pi}{2n}\right) \cdot 2 \cdot \sin\left(\tfrac{3\pi}{2n}\right) \cdots 2 \cdot \sin\left(\tfrac{\pi \cdot (n-1)}{2n}\right) = 2^{n-1} \cdot \prod_{k=1}^{n-1} \sin\left(\frac{\pi \cdot k}{2 \cdot n} \right) = \sqrt{n}$$

$$2 \cdot \sin\left(\tfrac{\pi}{4n-2}\right) \cdot 2 \cdot \sin\left(\tfrac{3\pi}{4n-2}\right) \cdot 2 \cdot \sin\left(\tfrac{5\pi}{4n-2}\right) \cdots 2 \cdot \sin\left(\tfrac{\pi \cdot (2n-3)}{4n-2}\right) = 2^{n-1} \cdot \prod_{k=1}^{n-1} \sin\left(\frac{\pi \cdot (2k-1)}{4 \cdot n - 2} \right) = 1$$

I t is the belief of this author, that proving these statements will be sufficient in proving most of the statements put forth so-far. We herby name the problem: The Ladder Hypothesis. To get more acquainted with the roots or zeros of the ladder polynomials, it is educational to connect each zero to its angle in the sinus argument. We further choose to use the degree unit instead of the more mathematical radian, to stress the fact, that we have extended the so-called "regular angles" to an infinite set which contain familiar historic angles like: 10°, 15°, 18°, 22½°, 30°, 45°.

Denominator:

30					
18	54				
$12\frac{6}{7}$	$38\frac{4}{7}$	$64\frac{2}{7}$			
10	30	50	70		
$8\frac{2}{11}$	$24\frac{6}{11}$	$40\frac{10}{11}$	$57\frac{3}{11}$	$73\frac{7}{11}$	
$6\frac{12}{13}$	$20\frac{10}{13}$	$34\frac{8}{13}$	$48\frac{6}{13}$	$62\frac{4}{13}$	$76\frac{2}{13}$

Numerator:

45					
30	60				
$22\frac{1}{2}$	45	$67\frac{1}{2}$			
18	36	54	72		
15	30	45	60	75	
$12\frac{6}{7}$	$25\frac{5}{7}$	$38\frac{4}{7}$	$51\frac{3}{7}$	$64\frac{2}{7}$	$77\frac{1}{7}$

It is apparent that we need only one table of roots, the Numerator roots which also have a perfect central symmetry about the 45° centre column. The Denominator roots are left biased starting at 30°, but will tend to 45° in the limit, when n grows large.

8. The Ladder Polynomials and the Finite Products of Sinuses:

In the next sections we will explore ways to obtain the simplest proof.

9. Finite products of the Sinus function:

By inserting "a=0" into the factored polynomials Q_n and P_n, two statements are derivable from "$Q_n(0) = 1$" and "$P_n(0) = n$":

$$\sin\left(\tfrac{\pi}{2n}\right)\cdot\sin\left(\tfrac{2\pi}{2n}\right)\cdot\sin\left(\tfrac{3\pi}{2n}\right)\cdots\sin\left(\tfrac{\pi\cdot(n-1)}{2n}\right)=\prod_{k=1}^{n-1}\sin\left(\frac{k\cdot\pi}{2\cdot n}\right)=2^{1-n}\cdot\sqrt{n}$$

$$\sin\left(\tfrac{\pi}{4n-2}\right)\cdot\sin\left(\tfrac{3\pi}{4n-2}\right)\cdot\sin\left(\tfrac{5\pi}{4n-2}\right)\cdots\sin\left(\tfrac{\pi\cdot(2n-3)}{4n-2}\right)=\prod_{k=1}^{n-1}\sin\left(\frac{\pi\cdot(2k-1)}{2\cdot(2n-1)}\right)=2^{1-n}$$

A simpler version is known in the literature*, the Fundamental Sinus Product. It has recently gained some attention*:

$$\sin\left(\tfrac{\pi}{n}\right)\cdot\sin\left(\tfrac{2\pi}{n}\right)\cdot\sin\left(\tfrac{3\pi}{n}\right)\cdots\sin\left(\tfrac{\pi\cdot(n-1)}{n}\right)=\prod_{k=1}^{n-1}\sin\left(\frac{\pi\cdot k}{n}\right)=2^{1-n}\cdot n$$

Even integers "n = 2n" and the 90° symmetry of sinus will transform this into the half angle sinus product squared:

$$\sin\left(\tfrac{\pi}{2n}\right)\cdot\sin\left(\tfrac{2\pi}{2n}\right)\cdot\sin\left(\tfrac{3\pi}{2n}\right)\cdots\sin\left(\tfrac{\pi\cdot(n-1)}{2n}\right)\cdot 1\cdot\sin\left(\tfrac{\pi\cdot(n-1)}{2n}\right)\cdot\sin\left(\tfrac{\pi\cdot(n-2)}{2n}\right)\cdots\sin\left(\tfrac{\pi}{2n}\right)=\left(\prod_{k=1}^{n-1}\sin\left(\frac{\pi\cdot k}{2n}\right)\right)^2=4^{1-n}\cdot n$$

By the same method, we can generate a relation for the odd integers (2n-1) as seen in the next section (3).

10. The finite sums of Logarithms of Sinus functions:

Now let n>1 and take the natural logarithm of all four products of sinus to get four equally interesting statements:

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{n}\right)=n \quad\Leftrightarrow\quad \sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot k}{n}\right)=-(n-1)\cdot\ln(2)+\ln(n)$$

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)=\sqrt{n} \quad\Leftrightarrow\quad \sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot k}{2n}\right)=-(n-1)\cdot\ln(2)+\frac{1}{2}\cdot\ln(n)$$

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{2n-1}\right)=\sqrt{2n-1} \quad\Leftrightarrow\quad \sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot k}{2n-1}\right)=-(n-1)\cdot\ln(2)+\frac{1}{2}\cdot\ln(2n-1)$$

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot(2k-1)}{4n-2}\right)=1 \quad\Leftrightarrow\quad \sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot(2k-1)}{4n-2}\right)=-(n-1)\cdot\ln(2)$$

We acknowledge the fact, that the last statement has not been proved yet.

11. Positive integers expressed as products of "zeros": $n=(1-1^{-1/n})(1-1^{-2/n})\ldots$

Some peculiar products involving only the number "1" and powers of it, can generate the integers, the square root of integers and much more. The Finite Products of sin simplifies by the exponential substitution "$\sin x = (2i)^{-1}(e^{ix} - e^{-ix})$", also known as the hyperbolic sinus of an imaginary argument. As the arguments are harmonic in our case, we can easily derive:

$$\prod_{k=1}^{n-1}\left(1-1^{-k/n}\right)=n \qquad \prod_{k=1}^{n-1}\left(1-\mathrm{e}^{-\frac{i2\pi\cdot k}{n}}\right)=i^{n-1}\cdot\mathrm{e}^{-\frac{i\pi\cdot(n-1)}{2}}\cdot n$$

$$\prod_{k=1}^{n-1}\left(1-(-1)^{-k/n}\right)=\sqrt{n\cdot i^{(n-1)}} \qquad \prod_{k=1}^{n-1}\left(1-\mathrm{e}^{-\frac{i\pi\cdot k}{n}}\right)=i^{n-1}\cdot\mathrm{e}^{-\frac{i\pi\cdot(n-1)}{4}}\cdot\sqrt{n}=i^{(n-1)/2}\cdot\sqrt{n}$$

$$\prod_{k=1}^{n-1}\left(1-1^{-k/(2n-1)}\right)=i^{(n-1)^2/(2n-1)}\cdot\sqrt{2n-1} \qquad \prod_{k=1}^{n-1}\left(1-\mathrm{e}^{-\frac{i2\pi\cdot k}{2n-1}}\right)=i^{n-1}\cdot\mathrm{e}^{-\frac{i\pi\cdot n\cdot(n-1)}{2\cdot(2n-1)}}\cdot\sqrt{2n-1}$$

$$\prod_{k=1}^{n-1}\left(1-(-1)^{-(2k-1)/(2n-1)}\right)=i^{n\cdot(n-1)/(2n-1)} \qquad \prod_{k=1}^{n-1}\left(1-\mathrm{e}^{-\frac{i\pi\cdot(2k-1)}{2n-1}}\right)=i^{n-1}\cdot\mathrm{e}^{-\frac{i\pi\cdot(n-1)^2}{4n-2}}=\mathrm{e}^{\frac{i\pi\cdot n\cdot(n-1)}{4n-2}}$$

On the left we have a minimally expressed statement, on the right we have the essence of it's proof.

12. The finite product of Cosinus functions:

We would now like to make contact with the finite product of the Cosinus Function. This can be accomplished by using the identity sin 2x = 2 sinx cosx.

$$n=\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{n}\right)=\prod_{k=1}^{n-1}4\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)\cdot\cos\left(\frac{\pi\cdot k}{2n}\right)=\left(\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)\right)\cdot\left(\prod_{k=1}^{n-1}2\cdot\cos\left(\frac{\pi\cdot k}{2n}\right)\right)$$

This rather surprising result can also be deduced from the identity sin (x) = cos (π/2 – x) and the Π sin(πk/2n) product:

$$\sqrt{n}=\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)=\prod_{k=1}^{n-1}2\cdot\cos\left(\frac{\pi\cdot(n-k)}{2n}\right)=\prod_{\ell=n-1}^{1}2\cdot\cos\left(\frac{\pi\cdot\ell}{2n}\right)=\prod_{k=1}^{n-1}2\cdot\cos\left(\frac{\pi\cdot k}{2n}\right)$$

We can now add both cos(x) and tan(x) to our arsenal of products of trigonometric functions:

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)=\prod_{k=1}^{n-1}2\cdot\cos\left(\frac{\pi\cdot k}{2n}\right)=\sqrt{n}\ ,\qquad\Rightarrow\qquad\prod_{k=1}^{n-1}\tan\left(\frac{\pi\cdot k}{2n}\right)=1$$

From properties of the cos(x) function, we find that Π cos(πk/n) = {-1, 0, +1} depending on evenness and oddity of n. The zero comes from even n = 2,4,6,… the minus one from n = 3,7,11,… and plus one from n = 5,9,13,…

13. The Cotangent function and an infinite product expansion for Sinus:

The cot(x) function is rather special being the derivative of lnsin(x). The Riemann Zeta function ζ(s) appears here:

$$\cot x=\sum_{-\infty}^{+\infty}\frac{1}{\pi\cdot\ell+x}=x\cdot\sum_{-\infty}^{+\infty}\frac{1}{x^2-\pi^2\cdot\ell^2}=\frac{1}{x}-2x\cdot\sum_{\ell=1}^{+\infty}\frac{1}{\pi^2\cdot\ell^2-x^2}=\frac{1}{x}-\frac{2}{x}\cdot\sum_{k=1}^{\infty}\zeta(2k)\cdot\left(\frac{x}{\pi}\right)^{2k}$$

$$\ln\sin x=\ln x-\sum_{k=1}^{\infty}\frac{2^{2k}\cdot B_k\cdot x^{2k}}{2\cdot k\cdot(2k)!}=\ln x-\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\cdot\left(\frac{x}{\pi}\right)^{2k}=\ln x-\sum_{k,\ell=1}^{\infty}\frac{1}{k}\cdot\left(\frac{x}{\pi\cdot\ell}\right)^{2k}$$

We have used the infinite sum definition for the Riemann Zeta function to arrive at the final infinite double-sum. This result can also be obtained directly from the infinite product formula for the sinus function:

$$\sin x=x\cdot\left(1-\frac{x^2}{\pi^2}\right)\cdot\left(1-\frac{x^2}{4\pi^2}\right)\cdot\left(1-\frac{x^2}{9\pi^2}\right)\cdots=x\cdot\prod_{\ell=1}^{\infty}\left(1-\frac{x^2}{\ell^2\cdot\pi^2}\right)$$

Notice the odd x in the sin function, which shows sin(x)/x as a simpler object than sin(x). Now take the natural logarithm of the infinite product for the sin function and get:

$$\ln\sin x=\ln x+\sum_{\ell=1}^{\infty}\ln\left(1-\left(\frac{x}{\pi\cdot\ell}\right)^2\right)=\ln x-\sum_{\ell,k=1}^{\infty}\frac{1}{k}\cdot\left(\frac{x}{\pi\cdot\ell}\right)^{2k}$$

The interval of convergence is unconditional, at least on the interval [-π < x < π], which is in fact the largest interval to occur. For clarity let us now summarise this result in a formal way:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\cdot\left(\frac{x}{\pi}\right)^{2k}=\ln\left(\frac{x}{\sin x}\right)=-\ln\left(\frac{\sin x}{x}\right)\qquad,\ \zeta(2k)=\sum_{\ell=1}^{\infty}\ell^{-2k}=\frac{(2\pi)^{2k}}{(2k)!}\cdot\frac{B_k}{2}$$

The last equality support the recent attempts** to redefine the Bernoulli numbers to be even indexed, as we have related them to very fundamental functions, the natural logarithm and the sinus. The first seven (old) Bernoulli numbers are:

$$B_1=1/6,\quad B_2=1/30,\quad B_3=1/42,\quad B_4=1/30,\quad B_5=5/66,\quad B_6=691/2730,\quad B_7=7/6$$

14. Some power series with Riemann Zeta coefficients:

To get a broader view on power series with Bernoulli numbers and/or the Riemann Zeta functions, let us reproduce some known results from the power series for tan(x) and cot(x):

$$\tan x = x + \frac{x^3}{3} + \frac{2\cdot x^5}{15} + \cdots + \frac{2^{2k}\cdot\left(2^{2k}-1\right)\cdot B_k\cdot x^{2k-1}}{(2k)!} + \cdots = \frac{2}{x}\cdot\sum_{k=1}^{\infty}\left(2^{2k}-1\right)\cdot\zeta(2k)\cdot\left(\frac{x}{\pi}\right)^{2k}$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \cdots - \frac{2^{2k}\cdot B_k\cdot x^{2k-1}}{(2k)!} + \cdots = \frac{1}{x}\cdot\left(1 - 2\cdot\sum_{k=1}^{\infty}\zeta(2k)\cdot\left(\frac{x}{\pi}\right)^{2k}\right)$$

At this moment, let us pause to express the first few even Riemann Zeta values and the alternating sign series also:

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945}$$

$$\zeta(8) = \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \cdots = \frac{\pi^8}{9450}$$

$$\zeta(10) = \frac{1}{1^{10}} + \frac{1}{2^{10}} + \frac{1}{3^8} + \cdots = \frac{\pi^{10}}{93555}$$

$$\zeta(12) = \frac{1}{1^{12}} + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \cdots = \frac{691\cdot\pi^{12}}{638512875}$$

$$\zeta(14) = \frac{1}{1^{14}} + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \cdots = \frac{2\cdot\pi^{14}}{18243225}$$

$$\zeta^{\pm}(2) = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

$$\zeta^{\pm}(4) = \frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \cdots = \frac{7\cdot\pi^4}{720}$$

$$\zeta^{\pm}(6) = \frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \cdots = \frac{31\cdot\pi^6}{30240}$$

$$\zeta^{\pm}(8) = \frac{1}{1^8} - \frac{1}{2^8} + \frac{1}{3^8} - \cdots = \frac{127\cdot\pi^8}{1\,209\,600}$$

$$\zeta^{\pm}(10) = \frac{1}{1^{10}} - \frac{1}{2^{10}} + \frac{1}{3^{10}} - \cdots = \frac{511\cdot\pi^{10}}{47\,900\,160}$$

$$\zeta^{\pm}(12) = \frac{1}{1^{12}} - \frac{1}{2^{12}} + \frac{1}{3^{12}} - \cdots = \frac{1\,414\,477\cdot\pi^{12}}{1\,307\,674\,368\,000}$$

$$\zeta^{\pm}(14) = \frac{1}{1^{14}} - \frac{1}{2^{14}} + \frac{1}{3^{14}} - \cdots = \frac{8191\cdot\pi^{14}}{74\,724\,249\,600}$$

Although ζ(1) diverges to positive and negative infinity, ζ(0) is well behaved and is known to be ζ(0) = -1/2 and the corresponding Bernoulli number is B_0 = -1. We can now define a rational function "$\kappa_{2k} = 2\ \zeta(2k)\ /\ \pi^{2k}$" with "$\kappa_0 = -1$" which will simplify our series as:

$$\tan x = x + \frac{x^3}{3} + \frac{2\cdot x^5}{15} + \cdots + \left(2^{2k}-1\right)\cdot\kappa_{2k}\cdot x^{2k-1} + \cdots = \frac{1}{x}\cdot\sum_{k=0}^{\infty}\left(2^{2k}-1\right)\cdot\kappa_{2k}\cdot x^{2k}$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2\cdot x^5}{945} - \cdots - \kappa_{2k}\cdot x^{2k-1} + \cdots = \frac{1}{x}\cdot\left(1 - \sum_{k=1}^{\infty}\kappa_{2k}\cdot x^{2k}\right) = -\sum_{k=0}^{\infty}\kappa_{2k}\cdot x^{2k-1}$$

$$\ln\sin x = \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \frac{x^8}{37800} - \cdots - \frac{\kappa_{2k}\cdot x^{2k}}{2k} - \cdots = \ln x - \sum_{k=1}^{\infty}\frac{\kappa_{2k}\cdot x^{2k}}{2k}$$

The reader can verify that we can differentiate the last equation to obtain the cot(x) equation. A closer look at the tan(x) power series reveals the identity "tan(x) = cot(x) – 2 cot(2x)" a rather impressive fact! We further conclude, that the cot(x) power series converges much faster than the tax(x) power series and is simpler in expression. By integrating the tan(x) function we can obtain the ln cos(x) function as a power series:

$$\ln\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots - \frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k} - \cdots = -\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k}$$

The expression "$a_{2k}\ x^{2k}$" inside the sum can be defined for k=0 rendering the value "a_0 = -ln 2". The final result is:

$$\ln\cos x = -\ln 2 + \ln 2 - \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k} = \ln\tfrac{1}{2} - \sum_{k=0}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k}$$

15. Sinus and Gamma functions from series with Riemann Zeta coefficients:

We will now discover a relation linking the Gamma Function and the Sinus Function. In section 6 we explored the lnsin(x) power series with Riemann Zeta connection. By a change of variable "x = π z" and dividing by 2 it becomes:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k}\cdot z^{2k}=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{2k}\cdot\left(\frac{z}{\ell}\right)^{2k}=-\tfrac{1}{2}\cdot\sum_{\ell=1}^{\infty}\ln\left(1-\frac{z^2}{\ell^2}\right)=-\tfrac{1}{2}\cdot\ln\prod_{\ell=1}^{\infty}\left(1-\frac{z^2}{\ell^2}\right)=-\ln\sqrt{\frac{\sin\pi\cdot z}{\pi\cdot z}}$$

This infinite series is of even order with index 2k=2,4,6,… and can be considered as the even part of a more general series with index values k=2,3,4,5,… The odd series will accordingly have index 2k+1=3,5,7,… and it is:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{2k+1}\cdot z^{2k+1}=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{2k+1}\cdot\left(\frac{z}{\ell}\right)^{2k+1}=\sum_{\ell=1}^{\infty}\left[\tanh^{-1}\frac{z}{\ell}-\frac{z}{\ell}\right]=\ln\prod_{\ell=1}^{\infty}\left(\frac{\ell+z}{\ell-z}\right)^{1/2}\cdot e^{-z/\ell}$$

An interchange of summation order in the double sum revealed the Taylor series for $\tanh^{-1}(x)$. Now subtract the odd series from the even series to get a series alternating in sign:

$$\sum_{k=2}^{\infty}\frac{(-1)^k\cdot\zeta(k)}{k}\cdot z^k=\sum_{k=2}^{\infty}\sum_{\ell=1}^{\infty}\frac{(-1)^k}{k}\cdot\left(\frac{z}{\ell}\right)^k=\sum_{\ell=1}^{\infty}\left[\frac{z}{\ell}-\ln\left(1+\frac{z}{\ell}\right)\right]=-\ln\prod_{\ell=1}^{\infty}\left(1+\frac{z}{\ell}\right)\cdot e^{-z/\ell}$$

To complete this, we use Euler's Constant: $\gamma=\lim_{m\to\infty}\left[1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{m}-\ln m\right]$ and the Gamma Function: n!=Γ(n+1).

$$\prod_{\ell=1}^{\infty}\left(1+\frac{z}{\ell}\right)\cdot e^{-z/\ell}=\lim_{m\to\infty}\left[\prod_{k=1}^{m}e^{-z/k}\cdot\prod_{\ell=1}^{m}\left(1+\frac{z}{\ell}\right)\right]=e^{-\gamma\cdot z}\cdot\lim_{m\to\infty}\left[\frac{m^{-z}}{m!}\cdot\prod_{\ell=1}^{m}(z+\ell)\right]=\frac{e^{-\gamma\cdot z}}{\Gamma(z+1)}$$

We have thus completed the task of evaluating both the even, and the odd power series we started with, and the result is:

$$\exp\left\{\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k}\cdot z^{2k}\right\}=\prod_{\ell=1}^{\infty}\left(1-\frac{z^2}{\ell^2}\right)^{-1/2}=\sqrt{\Gamma(1+z)\cdot\Gamma(1-z)}=\sqrt{\frac{\pi\cdot z}{\sin\pi\cdot z}}$$

$$\exp\left\{\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{2k+1}\cdot z^{2k+1}\right\}=\prod_{\ell=1}^{\infty}e^{-z/\ell}\cdot\left(\frac{\ell+z}{\ell-z}\right)^{1/2}=e^{-\gamma\cdot z}\cdot\sqrt{\frac{\Gamma(1-z)}{\Gamma(1+z)}}$$

The reflective property of the Gamma Function "Γ(z) Γ(1-z) = π / sin zπ" appears here, and the odd case generates an infinite product formula to complement the even case.

16. The Factorial Operator and the Gamma Function:

Now we will give a rather strange expressions concerning two integers (m,n) with m>>n where (n) is fixed, but (m) will be increasing and tending towards infinity.

$$\lim_{m\to\infty}\left[m^n\cdot\frac{m!}{(m+n)!}\right]=\lim_{m\to\infty}\prod_{k=1}^{n}\frac{m}{m+k}=\lim_{m\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{m}\right)^{-1}=1$$

$$\lim_{m\to\infty}\left[m^n\cdot\frac{m!\cdot n!}{(m+n)!}\right]=\lim_{m\to\infty}\left[m^n\cdot\prod_{k=1}^{m}\frac{k}{k+n}\right]=\lim_{m\to\infty}\prod_{k=1}^{n}\frac{m\cdot k}{m+k}=\lim_{m\to\infty}\prod_{k=1}^{n}k\cdot\left(1+\frac{k}{m}\right)^{-1}=n!\equiv\Gamma(n+1)$$

The relation "$(m+n)!\geq m^n\, m!$" is an equality when n=1. To make precise the largeness of m, we do a calculus analyses and get the formula "$m > n^2/2\varepsilon$", where ε is the relative error. For example, if we want <1% for n=5, we need m>1250, a rather slow convergence! The product occurring "m" times can be used to define the general factorial function of a real or complex variable z. The result is the following definition for the Gamma Function used in section 15:

$$\Gamma(z+1)=\lim_{m\to\infty}\left[m^z\cdot\prod_{k=1}^{m}\frac{k}{k+z}\right]\quad\Rightarrow\quad\Gamma(z)=z^{-1}\cdot\lim_{m\to\infty}\left[m^z\cdot\prod_{k=1}^{m}\left(1+\frac{z}{k}\right)^{-1}\right]$$

17. The finite Product of sin(πk/n) and Γ(k/n):

We will now use two relationships among the Gamma Function Γ(1+z) to prove the Finite Product of Sinus for the fundamental argument (πk/n). The reflective property of the Gamma Function in section 8, will allow us to express the Sinus Function in terms of Gamma Functions. This allows the use of a known formula for Finite Products of the Gamma Function, which can be found in most standard mathematical handbooks:

$$\prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right)=(2\pi)^{(n-1)/2}\cdot n^{(1-2n\cdot x)/2}\cdot\Gamma(n\cdot x)$$

Now set "x=1" and use the recursive property of the Gamma Function "Γ(1+n) = n Γ(n)" and remember that "Γ(1)=1" to obtain:

$$\prod_{k=1}^{n-1}\Gamma\left(1+\frac{k}{n}\right)=\prod_{k=1}^{n-1}\left(\frac{k}{n}\right)\cdot\Gamma\left(\frac{k}{n}\right)=\frac{(n-1)!}{n^{n-1}}\cdot\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)=(2\pi)^{(n-1)/2}\cdot n^{(1-2n)/2}\cdot(n-1)!$$

$$\Rightarrow$$

$$\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)=(2\pi)^{(n-1)/2}\cdot n^{-1/2}\qquad\Rightarrow\qquad\prod_{k=1}^{n-1}\Gamma^2\left(\frac{k}{n}\right)=\frac{(2\pi)^{(n-1)}}{n}$$

Armed with the Finite Product of Squared Gamma Functions of the argument (k/n), we can now turn to the final proof:

$$\prod_{k=1}^{n-1}\sin\left(\frac{\pi\cdot k}{n}\right)=\prod_{k=1}^{n-1}\frac{\pi}{\Gamma\left(\frac{k}{n}\right)\cdot\Gamma\left(1-\frac{k}{n}\right)}=\pi^{n-1}\cdot\prod_{k=1}^{n-1}\frac{1}{\Gamma\left(\frac{k}{n}\right)\cdot\Gamma\left(\frac{n-k}{n}\right)}=\pi^{n-1}\cdot\prod_{k=1}^{n-1}\frac{1}{\Gamma^2\left(\frac{k}{n}\right)}=\frac{n}{2^{n-1}}$$

This was not so hard! The key to this result is recognising that "Π f(k)f(n-k) = Π f^2(k)" when "k=1,2…(n-1)". For example, if n=5 we get 1·4·2·3·3·2·1·4=1·1·2·2·3·3·4·4=$1^2\cdot2^2\cdot3^2\cdot4^2$ which should convince the most sceptics!

18. Stirling's formula and lnsin(πk/n) sums:

By solving together the infinite product expansion of the sinus function and our newly obtained finite sums for lnsin(x), we can generate some powerful statements about infinite sums. Starting with the prototype argument (πk/n) we can get:

$$\sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot k}{n}\right)=\sum_{k=1}^{n-1}\ln\left(\frac{\pi\cdot k}{n}\right)-\sum_{k=1}^{n-1}\sum_{\ell=1}^{\infty}\left(\frac{\zeta(2\ell)}{\ell}\right)\cdot\left(\frac{k}{n}\right)^{2\ell}$$

$$=\ln\left(\frac{(n-1)!\cdot\pi^{n-1}}{n^{n-1}}\right)-\sum_{k=1}^{n-1}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{k}{n\cdot m}\right)^{2\ell}\cdot\ell^{-1}=\ln\left(\frac{n}{2^{n-1}}\right)$$

$$\Rightarrow$$

$$\sum_{k=1}^{n-1}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{k}{n\cdot m}\right)^{2\ell}\cdot\ell^{-1}=\ln\left(\frac{(2\pi)^{n-1}\cdot(n-1)!}{n^n}\right)=-\tfrac{1}{2}+\left(n-\tfrac{1}{2}\right)\cdot\ln\left(\frac{2\pi}{e}\right)-\tfrac{1}{2}\cdot\ln(n)+\sigma(n)$$

Here we have used the Stirling's formula for n! = Γ(n+1) to eliminate "n!/n^n"! We have further taken the liberty to define a function "***$\sigma(n)$ = ln(1+1/12n+1/288n^2-…)***" which obviously tends fast to zero, as n grows larger.

19. The Factorial Triangle & Newton's Polynomials:

The core of the Gamma Function is the factor (z+k) with k=1,2,3,…n and (z) can be integer, real or complex. By performing the multiplication, a polynomial in (z) is formed. The coefficients of this polynomial can be arranged in a Pascal's-like triangle:

1

1 1

1 3 2

1 6 11 6

1 10 35 50 24

1 15 85 225 274 120

1 21 175 735 1624 1764 720

1 28 322 1960 6769 13132 13068 5040

…A WORK STILL IN PROGRESS…

27. August, 2002

Guðlaugur Kristinn Óttarsson

CREDITS

This work is partially supported by the Icelandic Science Research Fund (RANNIS), the Icelandic Industrial and Technological Foundation (ITI), the Icelandic Ministry of Industry and the Agricultural Productivity Fund of Iceland. Pro%Nil Systems[(1)] and Genergy Varmaraf[(2)] have provided additional funding and support. Warm thanks to Reykjavik Energy for providing research facilities and hot and cold water. Cool thanks to Mr. Guðbrandur Guðmundsson[2] and Mr. Sigurður Gunnarsson[1] for proofreading the initial manuscript.