SEMIGROUPS OF LINEAR OPERATORS

NOTES BY YOUSUF SOLIMAN

1. Introduction

Our goal is to define exponentials of linear operators. We will try to construct e^{tA} as a linear operator, where $A : \mathscr{D}(A) \to X$ is a general linear operator, not necessarily bounded. Notationally, it seems like we are looking for a solution to $\dot{\mu}(t) = A\mu(t)$, $\mu(0) = \mu_0$, and we would like to write $\mu(t) = e^{tA}\mu_0$. It turns out that this will hold once we make sense of the terms.

How can we construct e^{tA} when A is a finite matrix? The most obvious way is to write down the power series: $\sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n$. This series is absolutely convergent for every A and $t \in \mathbf{R}$. In fact, this method works for $A \in \mathscr{L}(X;X)$, even if X is infinite dimensional.

A second method is to consider the connection with the *explicit Euler scheme*. Consider the system of ordinary differential equations:

$$\begin{cases} \dot{\mu}(t) = A\mu(t), \\ \mu(0) = \mu_0. \end{cases}$$

Partition $[0,t]$ into n parts and write

$$\dot{\mu}\left(\frac{kt}{n}\right) = \frac{n}{t}\left(\mu\left(\frac{(k+1)t}{n}\right) - \mu\left(\frac{kt}{n}\right)\right),$$

the *forward difference quotient* approximation. From the ODE, we get

$$\begin{aligned} A\mu\left(\frac{kt}{n}\right) &= \frac{n}{t}\left(\mu\left(\frac{(k+1)t}{n}\right) - \mu\left(\frac{kt}{n}\right)\right), \\ \mu\left(\frac{(k+1)t}{n}\right) &= \left(\mathbb{1} + \frac{t}{n}A\right)\mu\left(\frac{kt}{n}\right), \\ \mu(t) = \mu\left(\frac{nt}{n}\right) &\approx \left(\mathbb{1} + \frac{t}{n}A\right)^n \mu_0. \end{aligned}$$

Thus $\mu(t) = \lim_{n\to\infty}\left(\mathbb{1} + \frac{t}{n}A\right)^n \mu_0$ and we write $e^{tA} = \lim_{n\to\infty}\left(\mathbb{1} + \frac{t}{n}A\right)^n$.

Both of these methods are doomed to fail if A is not bounded. When the explicit method fails, one would normally try the implicit method. The third method we consider is the connection with the *implicit Euler scheme*. Partition $[0,t]$ into n parts and write

$$\dot{\mu}\left(\frac{(k+1)t}{n}\right) = \frac{n}{t}\left(\mu\left(\frac{(k+1)t}{n}\right) - \mu\left(\frac{kt}{n}\right)\right),$$

the *backward difference quotient* approximation. From the ODE, we get

$$\begin{aligned} A\mu\left(\frac{(k+1)t}{n}\right) &= \frac{n}{t}\left(\mu\left(\frac{(k+1)t}{n}\right) - \mu\left(\frac{kt}{n}\right)\right), \\ \mu\left(\frac{(k+1)t}{n}\right) &= \left(\mathbb{1} - \frac{t}{n}A\right)^{-1}\mu\left(\frac{kt}{n}\right), \\ \mu(t) = \mu\left(\frac{nt}{n}\right) &\approx \left(\mathbb{1} - \frac{t}{n}A\right)^{-n} \mu_0. \end{aligned}$$

Thus $\mu(t) = \lim_{n\to\infty}\left(\mathbb{1} - \frac{t}{n}A\right)^{-n} \mu_0$ and we write $e^{tA} = \lim_{n\to\infty}\left(\mathbb{1} - \frac{t}{n}A\right)^{-n}$. This works for some unbounded A as well. The key point will be the behavior of $\|R(\lambda;A)^n\|$ for large n.

E-mail address: ysoliman@cmu.edu.
Date: July 21, 2017.

An engineer might consider the Laplace transform. If $f(t) = e^{tA}$ then it can be shown that $\widehat{f}(\lambda) = (\lambda \mathbb{1} - A)^{-1} = R(\lambda; A)$. There is an inversion formula, namely

$$e^{tA} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} R(\lambda; A) \, d\lambda,$$

where γ is chosen such that the spectrum of A lies to the left of the line over which we are integrating. This formula can be interpreted and works for many important unbounded iperators.

A fifth method works for self-adjoint matrices. Let $\{e_k\}_{k=1}^N$ be an orthonormal basis of X of eigenvectors of A. For any $v \in X$, $v = \sum_{k=1}^N (v, e_k) e_k$ and $Av = \sum_{k=1}^N \lambda_k (v, e_k) e_k$. We take

$$e^{tA} v = \sum_{k=1}^N e^{\lambda_k t} (v, e_k) e_k.$$

In general, if X is a Hilbert space and $A : \mathscr{D}(A) \to X$ is self-adjoint then

$$A = \int_{-\infty}^{\infty} \lambda \, dP(\lambda),$$

where $\{P(\lambda) \,:\, \lambda \in \mathbf{R}\}$ is the *spectral family* associated with A. We know that $\sigma(A) \subseteq \mathbf{R}$, so if $\sigma(A)$ is bounded above then we could define

$$e^{tA} = \int_{-\infty}^{\infty} e^{\lambda t} \, dP(\lambda).$$

Note that the matrix A can be recovered from its exponential via the formula

$$A = \lim_{t \downarrow 0} \frac{1}{t} \left(e^{tA} - \mathbb{1} \right).$$

2. Linear C_0-semigroups

Let X be a Banach space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$.

Definition 2.1. A *linear C_0-semigroup* (or a *strongly continuous semigroup*) is a mapping $T : [0, \infty) \to \mathscr{L}(X; X)$ such that

(i) $T(0) = \mathbb{1}$,
(ii) $T(t + s) = T(t)T(s)$ for all $s, t \in [0, \infty)$, and
(iii) for all $x \in X$, $\lim_{t \downarrow 0} T(t)x = x$.

⋄

Remark 2.2.

(i) By the second condition $T(t)T(s) = T(s)T(t)$ for all s, t.
(ii) Sometimes we will use the notation $\{T(t)\}_{t \geq 0}$.
(iii) If we have a mapping $T : [0, \infty) \to \mathscr{L}(X; X)$ satisfying conditions (i) and (ii), (called a semigroup of bounded linear operators) then if the following condition holds so does (iii).
(iii') $\lim_{t \downarrow 0} \langle x^*, T(t)x \rangle = \langle x^*, x \rangle$ for all $x^* \in X^*$ and $x \in X$.
(iv) The condition $\lim_{t \downarrow 0} \|T(t) - \mathbb{1}\| = 0$ implies that $T(t) = \sum_{n=0}^{\infty} \frac{1}{n!} (tA)^n$ for all t, for some $A \in \mathscr{L}(X; X)$. This condition is too strong for practical purposes.
(v) The "C_0" in the name may come form "continuous at zero" or it may refer to the fact that these semigroups are (merely) continuous, as opposed to differentiable, etc.

⋄

Let T be a linear C_0-semigroup. The *infintesimal generator* of T is the linear operator $A : \mathscr{D}(A) \to X$ defined as follows.

$$\mathscr{D}(A) := \left\{ x \in X \,:\, \lim_{t \downarrow 0} \frac{1}{t} \left(T(t)x - x \right) \text{ exists} \right\}$$

and for all $x \in \mathscr{D}(A)$, $Ax = \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x)$. It is not immediately obvious that $\mathscr{D}(A) \neq \{0\}$. We will show that $\mathscr{D}(A)$ is dense and that A is a closed linear operator.

Example 2.3. Let $X = BUC(\mathbf{R})$ =bounded uniformly continuous functions $\mathbf{R} \to \mathbf{K}$. Define $(T(t)f)(x) := f(t+x)$ for all $t \in [0,\infty)$ and $x \in \mathbf{R}$. Clearly T satisfies (i) and (ii) of the definition. Uniform continuity is essential to get (iii). Indeed, if f is uniformly continuous then

$$\|T(t)f - f\|_\infty = \sup\{|f(t+x) - f(x)| : x \in \mathbf{R}\} \to 0 \text{ as } t \to 0.$$

The infinitesimal generator is

$$Af = \lim_{t\downarrow 0} \frac{f(t+x) - f(x)}{t} = f'(x),$$

i.e. differentiation. Note that the solution to the PDE $\mu_t(x,t) = \mu_x(x,t)$, $\mu(x,0) = \mu_0$ is $\mu(x,t) = \mu_0(x+t) = (T(t)\mu_0)(x)$. ⋄

Lemma 2.4. *Let T be a linear C_0-semigroup. Then there are $M, \omega \in \mathbf{R}$ such that $\|T(t)\| \le Me^{\omega t}$ for all $t \in [0,\infty)$.*

Proof. We claim that there is some $\eta > 0$ such that $\sup\{\|T(t)\| : t \in [0,\eta]\}$ is finite. Indeed, assume for the sake of contradiction there is no such η. Choose $\{t_n\}_{n=1}^\infty$ such that $t_n \downarrow 0$ and $\{T(t_n)x\}_{n=1}^\infty$ is unbounded. However, for all $x \in X$, since $T(t_n)x \to x$, $\{T(t_n)x\}_{n=1}^\infty$ is a convergent sequence, so $\sup\{\|T(t_n)x\| : n \in \mathbf{N}\}$ is finite for each $x \in X$. By the Banach-Steinhaus theorem we deduce that $\sup\{\|T(t_n)\| : n \in \mathbf{N}\}$ is finite, a contradiction.

Now let $\eta > 0$ be as above. Set $M := \sup\{\|T(t)\| : t \in [0,\eta]\} \ge 1\}$. Let $t \in [0,\infty)$ be given. Choose $n \ge 0$ and $\alpha \in [0,\eta)$ such that $t = n\eta + \alpha$. Then $T(t) = T(n\eta + \alpha) = (T(\eta))^n T(\alpha)$ by the semigroup property. Hence,

$$\|T(t)\| \le \|T(\alpha)\|\|T(\eta)\|^n \le MM^n.$$

Now let $\omega = \frac{1}{\eta}\log M \ge 0$, so that $\omega t \ge n \log M$, and $\|T(t)\| Me^{\omega t}$. □

Definition 2.5. Let T be a linear C_0-semigroup. We say that T is

(i) *uniformly bounded* if there is $M \in \mathbf{R}$ such that $\|T(t)\| leqM$ for all $t \ge 0$.
(ii) *contractive* if $\|T(t)\| \le 1$ for all $t \ge 0$.
(iii) *quasi-contractive* provided there is $\omega \in \mathbf{R}$ such that $\|T(t)\| \le e^{\omega t}$ for all $t \ge 0$.

⋄

Contractive semigroups are much easier to study than general linear C_0-semigroups. If T is a linear C_0-semigroup satisfying $\|T(t)\| leqMe^{\omega t}$ then $S(t) := e^{-\omega t}T(t)$ is a uniformly bounded linear C_0-semigroup. Note that the infinitesimal generator of S is related to that of T as follows.

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{S(t)x - x}{t} &= \lim_{t\downarrow 0} \frac{e^{-\omega t}T(t)x - x}{t} \\
&= \lim_{t\downarrow 0} \frac{e^{-\omega t} - 1}{t} T(t)x + \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \\
&= -\omega x + Ax = (A - \omega \mathbb{1})x.
\end{aligned}$$

Further, there is an equivalent norm $|||\cdot|||$ on X such that S is contractive with respect to $|||\cdot|||$. In fact, we may take $|||x||| := \sup\{\|S(t)x\| : t \in [0,\infty)\}$. Indeed, for all $x \in X$,

$$|||S(t)x||| = \sup\{\|S(t+s)x\| : s \in [0,\infty)\} \le |||x|||.$$

Warning: The norm $|||\cdot|||$ need not preserve all "nice" geometric properties of $\|\cdot\|$, such as the parallelogram law.

Lemma 2.6. *Let T be a linear C_0-semigroup and let $x \in X$ be given. Then the mapping $t \mapsto T(t)x$ is continuous on $[0,\infty)$.*

Proof. For continuity from the right, let $t \ge 0$ be given and notice that

$$T(t+h)x - T(t)x = (T(h) - \mathbb{1})(T(t)x) \to 0 \text{ as } h \to 0.$$

For continuity from the left, let $t > 0$ and $h(0,t)$ be given. Choose $M \geq 1$ and $\omega \geq 0$ such that $\|T(s)\| \leq Me^{\omega s}$ for all $s \in [0,\infty)$.

$$\begin{aligned}\|T(t-h)x - T(t)x\| &= \|T(t-h)(\mathbb{1} - T(h))x\| \\ &\leq \|T(t-h)\|\|T(h)x - x\| \\ &\leq Me^{\omega(t-h)}\|T(h)x - x\| \to 0 \text{ as } h \to 0.\end{aligned}$$

□

Lemma 2.7. *Let T be a linear C_0-semigroup with infinitesimal generator A, and let $x \in X$ be given.*

(i) For all $t \geq 0$, $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x\,ds = T(t)x$ (where the limit is one sided if $t = 0$).
(ii) For all $t \geq 0$, $\int_0^t T(s)x\,ds \in \mathscr{D}(A)$ and $A\int_0^t T(s)x\,ds = T(t)x - x$.

Proof.

(i) Follows from Lemma 2.6 and basic calculus.
(ii) If $t = 0$ there is nothing to prove. Let $t > 0$ be given. For $h > 0$,

$$\begin{aligned}\frac{T(h)-\mathbb{1}}{h}\int_0^t T(s)x\,ds &= \frac{1}{h}\int_0^t (T(s+h) - T(s))x\,ds \\ &= \frac{1}{h}\int_0^t T(s+h)x\,ds - \frac{1}{h}\int_0^t T(s)x\,ds \\ &= \frac{1}{h}\int_h^t T(u)x\,du + \frac{1}{h}\int_t^{t+h} T(u)x\,du - \frac{1}{h}T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds \\ &= \frac{1}{h}\int_t^{t+h} T(u)x\,du - \frac{1}{h}\int_0^h T(s)x\,ds \\ &\to T(t)x - x \text{ as } h \to 0\end{aligned}$$

by part (a). The conclusion immediately follows.

□

Lemma 2.8. *Let T be a linear C_0-semigroup with infinitesimal generator A, and $x \in \mathscr{D}(A)$ be given. Put $\mu(t) = T(t)x$ for all $t \geq 0$. Then $\mu(t) \in \mathscr{D}(A)$ fo rlal $t \geq 0$, μ is differentiable on $[0,\infty)$, and for each $t \geq 0$,*

$$\dot{\mu}(t) = T(t)Ax = AT(t)x = A\mu(t).$$

Proof. Let $t \geq 0$ be given. For $h > 0$,

$$\frac{T(t+h)x - T(t)x}{h} = \left(\frac{T(h)-\mathbb{1}}{h}\right)T(t)x = T(t)\left(\frac{T(h)-\mathbb{1}}{h}\right)x \to T(t)Ax$$

as $h \downarrow 0$. In particular, $T(t)x \in \mathscr{D}(A)$ and $AT(t)x = T(t)Ax$. Furthermore, $D^+\mu(t) = x = T(t)Ax$. Let $t > 0$ be given. For $h \in (0,t)$,

$$\frac{T(t-h)x - T(t)x}{h} = T(t-h)\left(\frac{x - T(h)x}{h}\right) \to -T(t)Ax \text{ as } h \to 0.$$

So we deduce that $D^-\mu(t)x = T(t)Ax$. Since the left and right derivatives both exist and are equal, μ is differentiable and $\dot{\mu}(t) = A\mu(t)$. □

Lemma 2.9. *Let T be a linear C_0-semigroup with infinitesimal generator A, and let $x \in \mathscr{D}(A)$ be given. Then for all $s,t \in [0,\infty)$,*

$$T(t)x - T(s)x = \int_s^t AT(u)x\,du = \int_s^t T(u)Ax\,du.$$

Proof. This follows from Lemma 2.8 and the fundamental theorem of calculus. □

Theorem 2.10. *Let T be a linear C_0-semigroup with infinitesimal generator A. Then $\mathscr{D}(A)$ is dense in X and A is closed.*

Proof. Let $x \in X$. By Lemma 2.7 we see that $x = \lim_{h\downarrow 0} \int_0^h T(s)x\ ds$, and $\int_0^h T(s)x\ ds \in \mathscr{D}(A)$ for all $h \geq 0$, so $\mathscr{D}(A)$ is dense in X.

Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathscr{D}(A)$ converging to $x \in X$ and suppose that $Ax_n \to y \in X$ as $n \to \infty$. We must show that $x \in \mathscr{D}(A)$ and that $Ax = y$. For $h > 0$, by Lemma 2.9,

$$T(h)x_n - x_n = \int_0^h T(s)Ax_n\ ds,$$

so by Lemma 2.7,

$$Ax = \lim_{h\downarrow 0} \frac{T(h)x - x}{h} = \lim_{h\downarrow 0} \frac{1}{h} \int_0^h T(s)y\ ds = y.$$

It follows that $x \in \mathscr{D}(A)$ and $Ax = y$ □

Lemma 2.11. *Let S, T be linear C_0-semigroups having the same infinitesimal generator A. Then $S(t) = T(t)$ for all $t \geq 0$.*

Proof. Let $x \in \mathscr{D}(A)$ and $t > 0$ be given. Define the function $\mu : [0,t] \to X$ by $\mu(s) = T(t-s)S(s)x$ for all $x \in [0,t]$. We will show that μ is constant as follows. We claim that μ is differentiable on $[0,t]$ and

$$\dot{\mu}(s) = T(t-s)AS(s)x - T(t-s)AS(s)x = 0$$

for all $s \in [0,t]$. This will imply that μ is constant on $[0,t]$, so

$$T(t)x = \mu(0) = \mu(1) = S(t)x.$$

Since $\mathscr{D}(A)$ is dense in X, it will follow that $T(t) = S(t)$ on X for all $t \geq 0$. To prove the claim we apply Lemma 2.8.

$$\begin{aligned}
\frac{\mu(s+h) - \mu(s)}{h} &= \frac{1}{h}(T(t-s-h)S(s+h)x - T(t-s)S(s)x) \\
&= \frac{1}{h}T(t-s-h)(S(s+h) - S(s))x + \frac{1}{h}(T(t-s-h) - T(t-s))S(s)x \\
&= T(t-s-h)\left(\frac{S(s+h) - S(s)}{h}\right)x + \left(\frac{T(t-s-h) - T(t-s)}{h}\right)S(s)x \\
&\to T(t-s)AS(s)x - T(t-s)AS(s)x = 0 \text{ as } h \to 0.
\end{aligned}$$

The mean value theorem holds for calculus in Banach spaces, and so μ is constant. □

3. Infinitesimal Generators

Given a closed densely defined A, how do we tell if A generates a linear C_0-semigroup? Let $a \in \mathbf{R}$ and $n \in \mathbf{N}$ and put $f(t) = t^{n-1}e^{at}$ for all $t \geq 0$. Recall that the Laplace transform of f is

$$\widehat{f}(\lambda) = \frac{(n-1)!}{(\lambda - a)^n}.$$

Let A be an $N \times N$ matrix and put $F(t) = e^{tA}$.

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t}e^{tA}\ dt = \int_0^\infty e^{t(A-\lambda\mathbb{1})}\ dt = (A - \lambda\mathbb{1})^{-1}e^{t(A-\lambda\mathbb{1})}\Big|_0^\infty = -(A-\lambda\mathbb{1})^{-1} = R(\lambda; A).$$

Recall that $e^{tA} = \lim_{n\to\infty}\left(\mathbb{1} - \frac{t}{n}\right)^{-n} = \lim_{n\to\infty}\left(\frac{n}{t}\right)^n R\left(\frac{t}{n}; A\right)^n$. To apply this to unbounded operators, the behavior of $R(\lambda; A)^n$ for large n will be key. We conjecture that

$$R(\lambda; A)^n = \frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}e^{tA}\ dt.$$

Lemma 3.1. *Let $M, \omega \in \mathbf{R}$ and $\lambda \in \mathbf{K}$ with $\Re(\lambda) > \omega$ be given. Let T be a linear C_0-semigroup such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and let A be the infinitesimal generator of T. Then $\lambda \in \rho(A)$ and, for all $x \in X$,*

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt.$$

Proof. Put $I_1(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$ for all $x \in X$. We need to show that $\lambda \in \rho(A)$ and $R(\lambda; A) = I_1(\lambda)$. Let $x \in \mathscr{D}(A)$ be given.

$$\begin{aligned} I_1(\lambda)Ax &= \int_0^\infty e^{-\lambda t} T(t)Ax \, dt \\ &= \int_0^\infty e^{-\lambda t} \frac{\mathrm{d}}{\mathrm{d}t}(T(t)x) \, dt && \text{Lemma 2.8} \\ &= -x + \lambda \int_0^\infty e^{-\lambda t} T(t)x \, dt && \text{integration by parts} \\ &= \lambda I_1(\lambda)x - x. \end{aligned}$$

Now let $x \in X$ be given. We will show that $I_1(\lambda)x\mathscr{D}(A)$ and

$$AI_1(\lambda)x = \lambda I_1(\lambda)x - x.$$

Fix $h > 0$ and compute the difference quotient:

$$\begin{aligned} \left(\frac{T(h) - \mathbb{1}}{h}\right) I_1(\lambda)x &= \frac{1}{h} \int_0^\infty e^{-\lambda t}(T(t+h)x - T(t)x) \, dt \\ &= \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t+h)x \, dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t)x \, dt \\ &= \frac{1}{h} \int_h^\infty e^{-\lambda(s-h)} T(s)x \, ds - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t)x \, dt \\ &= \frac{1}{h} \int_0^\infty e^{-\lambda(t-h)} T(t)x \, dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t)x \, dt - \frac{1}{h} \int_0^h e^{-\lambda(t-h)} T(t)x \, dt \\ &= \int_0^\infty \frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h} T(t)x \, dt - e^{\lambda h} \frac{1}{h} \int_0^h e^{-\lambda(t-h)} T(t)x \, dt \\ &\to \lambda I_1(\lambda)x - x \text{ as } h \to 0. \end{aligned}$$

This proves the lemma. □

Lemma 3.2. *Let $M, \omega \in \mathbf{R}$ and $\lambda \in \mathbf{K}$ with $\Re(\lambda) > \omega$ be given. Let T be a linear C_0-semigroup such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and let A be the infinitesimal generator of T. Then $\lambda \in \rho(A)$ and, for all $n \in \mathbf{N}$ and all $x \in X$,*

$$R(\lambda; A)^n x = \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x \, dt.$$

Proof. We already know that $\rho(A) \supseteq \{\mu \in \mathbf{K} : \Re(\mu) > \omega\}$. We also know that $\mu \mapsto R(\mu; A)$ is analytic. In particular, we have

$$R(\mu; A) = \sum_{n=0}^\infty (\lambda - \mu)^n R(\lambda; A)^{n+1} = \sum_{n=0}^\infty R(\lambda; A)^{n+1} (\mu - \lambda)^n$$

for $|\mu - \lambda$ sufficiently small. Let $R^{(k)}(\lambda; A)$ denote the k^{th} derivative of $R(\mu; A)$ evaluated at $\mu = \lambda$. From the power series, for all $n \in \mathbf{N}$,

$$\frac{R^{(n-1)}(\lambda; A)}{(n-1)!} = (-1)^{n-1} R(\lambda; A)^n.$$

By Lemma 3.1, $R(\lambda;A)x=\int_0^\infty e^{-\lambda t}T(t)x\ dt$ for all $x\in X$. From this,

$$R^{(n-1)}(\lambda;A)x=(-1)^{n-1}\int_0^\infty e^{-\lambda t}t^{n-1}T(t)x\ dt.$$

This proves the result. □

Theorem 3.3 (Hille-Yosida, 1948). *Let $M,\omega\in\mathbf{R}$ be given. Suppose that $A:\mathscr{D}(A)\to X$ is a linear operator with $\mathscr{D}(A)\subseteq Z$. Then A is the infinitesimal generator of a linear C_0-semigroup T satisfying $\|T(t)\|\le Me^{\omega t}$ for all $t\ge 0$ if and only if the following hold.*

(i) A is closed and $\mathscr{D}(A)$ is dense in X; and
(ii) $\rho(A)\supseteq\{\lambda\in\mathbf{R}:\lambda>\omega\}$ and $\|R(\lambda;A)^n\|\le\frac{M}{(\lambda-\omega)^n}$ for all $\lambda\in\mathbf{R}$ with $\lambda>\omega$ and all $n\in\mathbf{N}$.

Remark 3.4. Note that the condition that $\|R(\lambda;A)^n\|\le\frac{M}{(\lambda-\omega)^n}$ may be difficult to verify in practice. Notice that $\|R(\lambda;A)\|\le\frac{M}{(\lambda-\omega)}$ implies that $\|R(\lambda;A)^n\|\le\frac{M^n}{(\lambda-\omega)^n}$, so if $M=1$, i.e. if the semigroup is quasi-contractive, then it is enough to verify the inequality for $n=1$ only. ⋄

Proof.

Step 1. Necessity.

We have already seen that (i) holds, by Theorem 2.10, and that $\rho(A)$ contains $\{\lambda\in\mathbf{R}:\lambda>\omega\}$, by Lemma 3.1. By Lemma 3.2,

$$\begin{aligned}R(\lambda;A)^n&=\frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}T(t)x\ dt\\ \|R(\lambda;A)^n x\|&\le\frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}\|T(t)x\|\ dt\\ &\le\frac{M}{(n-1)!}\|x\|\int_0^\infty e^{-\lambda t}t^{n-1}e^{\omega t}\ dt\\ &=\frac{M}{(n-1)!}\frac{(n-1)!}{(\lambda-\omega)^n}\|x\|\\ &=\frac{M}{(\lambda-\omega)^n\|x\|.}\end{aligned}$$

This concludes the proof of necessity.

Step 2. Sufficiency.

Should we try using the inverse Laplace transform? If we could write

$$T(t)=\frac{1}{2\pi i}\int_{\gamma-\infty}^{\gamma+\infty}e^{\lambda t}R(\lambda;A)\ d\lambda$$

then T would have higher order regularity in general. This method would work for so called “analytic” semigroups, but not for general C_0-semigroups.

How about the limit obtained from considering the implicit scheme? In general $T(t)=\lim_{n\to\infty}\left(\mathbb{1}-\frac{t}{n}A\right)^{-n}$, and this method can be used, but we will not use it here. What we will do is approximate A with bounded operators $\{A_\lambda\}_{\lambda>\omega}$ and put $T_\lambda(t)=\sum_{n=0}^\infty\frac{1}{n!}(tA_\lambda)^n$. Then in theory $T_\lambda(t)\to T(t)$ as $\lambda\to\infty$.

Lemma 3.5. *Let $A:\mathscr{D}(A)\to X$ be a linear operator with $\mathscr{D}(A)\subseteq X$. Assume that (i) and (ii) of the Hille-Yosida theorem hold. Then, for all $x\in X$, $\lim_{\lambda\to\infty}\lambda R(\lambda;A)x=x$.*

Proof. Let $x \in \mathscr{D}(A)$ be given. For any $\lambda > \omega$,

$$\begin{aligned}(\lambda \mathbb{1} - A)R(\lambda;A)x &= x, \\ \lambda R(\lambda;A)x - x &= AR(\lambda;A)x = R(\lambda;A)Ax, \\ \|\lambda R(\lambda;A)x - x\| &= \|R(\lambda;A)Ax\| \\ &\le \frac{M}{\lambda-\omega}\|Ax\| \\ &\to 0 \text{ as } \lambda \to \infty.\end{aligned}$$

Since $\mathscr{D}(A)$ is dense in X, the result follows. □

Now we define the *Yosida approximation* A_λ of λ for $\lambda > \omega$. It is defined as

$$A_\lambda x := \lambda AR(\lambda;A)x = (\lambda^2 R(\lambda;A) - \lambda\mathbb{1})x.$$

By Lemma 3.5, $A_\lambda x \to Ax$ as $\lambda \to \infty$ for all $x \in \mathscr{D}(A)$.

Lemma 3.6. *Let $B \in \mathscr{L}(X;X)$ and define $e^{tB} = \sum_{n=0}^{\infty} \frac{1}{n!}(tB)^n$ for all $t \in \mathbf{R}$.*

(i) *$\{e^{tB}\}_{t\ge 0}$ is a linear C_0-semigroup with infinitesimal generator B.*
(ii) $\lim_{t\to 0} \|e^{tB} - \mathbb{1}\| = 0$.
(iii) *For all $\lambda \in \mathbf{K}$, $e^{t(B-\lambda\mathbb{1})} = e^{-\lambda t}e^{tB}$.*

Proof.

(i) Since $B \in \mathscr{L}(X;X)$ we have $\|B\| < +\infty$, and so for any $x \in X$ and $t \ge 0$,

$$\left\|\sum_{i=n}^{m} \frac{1}{i!}(tB)^i x\right\| \le \sum_{i=n}^{m} \frac{(t\|B\|)^i}{i!}\|x\| \le \|x\|e^{t\|B\|},$$

hence the sequence of partial sums $\{\sum_{n=0}^{m} \frac{1}{n!}(tB)^n\}_{n\in\mathbf{N}}$ is Cauchy in X. Hence the series converges, e^{tB} is well defined, and $e^{tB} \in \mathscr{L}(X;X)$.
Since for $a, b \in \mathbf{R}$

$$\left(\sum_{n=0}^{\infty}\frac{1}{n!}a^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}b^n\right) = \sum_{n=0}^{\infty}\frac{1}{n!}(a+b)^n,$$

we deduce that the semigroup property holds by the same argument which shows that e^{tB} is well defined. Clearly, $\lim_{t\downarrow 0} e^{tB} = e^{0B} = \mathbb{1}$. Finally, to show that $\{e^{tB}\}_{t\ge 0}$ is a linear C_0-semigroup note that

$$\|e^{tB}x - x\| = \left\|\sum_{n=1}^{\infty}\frac{1}{n!}(tB)^n x\right\| \le \sum_{n=1}^{\infty}\frac{1}{n!}t^n\|B\|^n\|x\| \le (e^{t\|B\|}-1)\|x\| \to 0 \text{ as } t \downarrow 0.$$

We claim that $e^{t\|B\|}$ is differentiable with derivative Be^{tB}. To see this note that

$$B\int_0^t e^{sB}\,ds = B\int_0^t \sum_{n=0}^{\infty}\frac{1}{n!}(sB)^n\,ds = \sum_{n=0}^{\infty}B^{n+1}\int_0^t s^n\,ds = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(tB)^{n+1} = e^{tB} - \mathbb{1}.$$

Now by differentiating both sides we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tB} = \lim_{h\downarrow 0}\frac{e^{(t+h)B} - e^{tB}}{t} = Be^{tB}.$$

Now since the infinitesimal generator is derivative at $t = 0$ we deduce that the infinitesimal generator of the linear C_0-semigroup $\{e^{tB}\}_{t\ge 0}$ is simply B.

(ii) Since

$$\|e^{tB} - \mathbb{1}\| = \left\|\sum_{n=1}^{\infty}\frac{1}{n!}(tB)^n\right\| \le \sum_{n=1}^{\infty}\frac{1}{n!}t^n\|B\|^n = e^{t\|B\|} - 1,$$

and $e^{t\|B\|} - 1 \to 0$ as $t \to 0$, we deduce that $\lim_{t\to 0}\|e^{tB} - \mathbb{1}\| = 0$.

(iii) We have

$$e^{t(B-\lambda\mathbb{1})} = \sum_{n=0}^{\infty} \frac{1}{n!}(tB - t\lambda\mathbb{1})^n = \left(\sum_{n=0}^{\infty} \frac{1}{n!}(tB)^n\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!}(-t\lambda\mathbb{1})^n\right) = e^{tB}e^{-\lambda t\mathbb{1}}.$$

Now for any $x \in X$ we see that

$$e^{t(B-\lambda\mathbb{1})}x = e^{tB}e^{-\lambda t\mathbb{1}}x = e^{tB}e^{-\lambda t}x = e^{-\lambda t}e^{tB}x.$$

□

In fact, it can be shown that if T is a linear C_0-semigroup with the property that $\lim_{h\downarrow 0}\|T(h)-\mathbb{1}\| = 0$ then $T(t) = e^{tB}$ for some $B \in \mathscr{L}(X;X)$.

Now assume that conditions (i) and (ii) of the Hille-Yosida theorem hold, and let A_λ be the Yosida approximation of A. Notice that for any $\lambda > \omega$,

$$\begin{aligned} e^{tA_\lambda} &= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^{2n}t^nR(\lambda;A)^n}{n!} \\ \|e^{tA_\lambda}\| &\le Me^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^{2n}t^n}{(\lambda-\omega)^n n!} && \text{by (ii)} \\ &= Me^{-\lambda t}\exp\left(\frac{\lambda^2}{\lambda-\omega}t\right) && \lambda > \omega \\ &= M\exp\left(\frac{\lambda}{\lambda-\omega}t\right). \end{aligned}$$

It follows that $\|e^{tA_\lambda}\| \le Me^{\omega_1 t}$ for any fixed $\omega_1 > \omega$, for all λ sufficiently large when compared to ω.

Put $T_\lambda(t) := e^{tA_\lambda}$ for all $t \ge 0$ and $\lambda > \omega$. Notice that $A_\lambda A_\mu = A_\mu A_\lambda$ and $A_\lambda T_\mu(t) = T_\mu(t)A_\lambda$ for all $\lambda,\mu > \omega$. Fix $x \in \mathscr{D}(A)$.

$$\begin{aligned} T_\lambda(t)x - T_\mu(t)x &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\big(T_\mu(t-s)T_\lambda(s)x\big)\,ds \\ &= \int_0^t T_\mu(t-s)A_\lambda T_\lambda(s)x - T_\mu(t-s)A_\mu T_\lambda(s)x\ ds \\ &= \int_0^t \big(T_\mu(t-s)T_\lambda(s)\big)\big(A_\lambda x - A_\mu x\big)\,ds. \end{aligned}$$

So we deduce that

$$\|T_\lambda(t)x - T_\mu(t)x\| \le M^2e^{\omega_1 t}t\|A_\lambda x - A_\mu x\|.$$

Hence $\{T_\lambda(t)x\}_{\lambda>\omega}$ is uniformly Cauchy in t on bounded intervals. Since $\mathscr{D}(A)$ is dense in X and since we have a bound on $\|T_\lambda(t)\|$ (in λ), we have for all $x \in X$, $\lim_{\lambda\to\infty} T_\lambda(t)x$ exists.

For all $t \ge 0$ and $x \in X$, put $T(t)x = \lim_{\lambda\to\infty} T_\lambda(t)x$. Note that $\|T(t)\| \le Me^{\omega_1 t}$, $T(t)T(s) = T(t+s)$ for all $s,t \ge 0$, and $T(0) = \mathbb{1}$ – this follows since these relations all hold for each T_λ. Continuity follows since the convergence is uniform for t bounded intervals. Hence, we have shown that T is linear C_0-semigroup. Let B be the infinitesimal generator of T. Now we must show that $B = A$. First we will show that B is an extension of A, and then we will use a resolvent argument to show that $\mathscr{D}(A) = \mathscr{D}(B)$. Let $x \in \mathscr{D}(A)$ be given.

$$\begin{aligned} \|T_\lambda(t)A_\lambda x - T(t)Ax\| &\le \|T_\lambda(t)(A_\lambda x - Ax)\| + \|(T_\lambda(t) - T(t))Ax\| \\ &\le Me^{\omega_1 t}\|A_\lambda x - Ax\| + \|(T - \lambda(t) - T(t))Ax\| \\ &\to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

Since the convergence is uniform in t on bounded intervals,

$$T(t)x - x = \lim_{\lambda\to\infty} T_\lambda(t)x - x = \lim_{\lambda\to\infty}\int_0^t T_\lambda(s)A_\lambda x\ dx = \int_0^t T(s)Ax\ dx.$$

Now by the definition of B, for any $h > 0$,

$$\frac{T(h)x - x}{h} = \frac{1}{h}\int_0^h T(s)Ax\ ds \to Ax \text{ as } h \downarrow 0.$$

Hence, $x \in \mathscr{D}(B)$ and $Bx = Ax$. B is closed since it is the infinitesimal generator of a linear C_0-semigroup, and A is closed by assumption. Since $\|T(t)\| \le Me^{\omega_1 t}$ for any $\omega_1 > \omega$, by Lemma 3.1 $\rho(B) \supseteq (\omega, \infty)$, so it follows that $\rho(B) \cap \rho(A) \neq \emptyset$. Choose $\lambda \in \rho(A) \cap \rho(B)$. By standard spectral theory since A and B are closed, $(\lambda\mathbb{1} - A)[\mathscr{D}(A)] = X$ and $(\lambda\mathbb{1} - B)[\mathscr{D}(B)] = x$. Furthermore, since B extends A, $(\lambda\mathbb{1} - B)[\mathscr{D}(A)] = (\lambda\mathbb{1} - A)[\mathscr{D}(A)] = X$. To conclude the proof of the Hille-Yosida theorem, note that $\mathscr{D}(A) = R(\lambda; B)[X] = \mathscr{D}(B)$.

□

Remark 3.7. Let $A : \mathscr{D}(A) \to X$ be a linear operator with $\mathscr{D}(A) \subseteq X$. The following are equivalent.

(i) A is closed,
(ii) $(\lambda\mathbb{1} - A) : \mathscr{D}(A) \to X$ is a bijection for some $\lambda \in \rho(A)$,
(iii) $(\lambda\mathbb{1} - A) : \mathscr{D}(A) \to X$ is a bijection for all $\lambda \in \rho(A)$.

⋄

Corollary 3.8. *Assume that $A : \mathscr{D}(A) \to X$ is linear with $\mathscr{D}(A) \subseteq X$, and that $\mathscr{D}(A)$ is dense and A is closed. Then A generates a contractive linear C_0-semigroup if and only if $\rho(A) \supseteq (0, \infty)$ and $\|R(\lambda; A)\| \le \frac{1}{\lambda}$ for all $\lambda > 0$.*

4. Contractive semigroups

Let $T : [0, \infty) \to \mathscr{L}(X; X)$ be a contractive semigroup. For all $t, h \in [0, \infty)$,

$$\|T(t + h)\| = \|T(h)T(t)\| \le \|T(h)\|\|T(t)\| \le \|T(t)\|,$$

so $t \mapsto \|T(t)\|$ is a decreasing function. Assume for now that X is a Hilbert space. Let $x \in \mathscr{D}(A)$ be given, and put $\mu(t) = \|T(t)x\|^2 = (T(t)x, T(t)x)$. For all $t \ge 0$, since μ is decreasing,

$$0 \ge \dot{\mu}(t) = (T(t)x, T(t)Ax) + (T(t)Ax, T(t)x) = 2\Re(AT(t)x, x).$$

In particular, for $t = 0$, $\Re(Ax, x) \le 0$ for all $x \in \mathscr{D}(A)$.

We will prove that if X is a Hilbert space and $A : \mathscr{D}(A) \to X$ is a linear operator then A generates a contractive semigroup if and only if both of the following hold.

(i) $\Re(Ax, x) \le 0$ for all $x \in \mathscr{D}(A)$, and
(ii) there exists $\lambda_0 > 0$ such that $\lambda_0\mathbb{1} - A$ is surjective.

Definition 4.1. Let X be a Banach space over $\mathbf{K}$ with norm $\|\cdot\|$. A *semi-inner product* on X is a mapping $[\cdot, \cdot] : X \times X \to \mathbf{K}$ such that

(i) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in X$,
(ii) $[\alpha x, y] = \alpha[x, y]$ for all $x, y \in X$ and $\alpha \in \mathbf{K}$,
(iii) $[x, x] = \|x\|^2$ for all $x \in X$, and
(iv) $|[x, y]| \le \|x\|\|y\|$ for all $x, y \in X$.

⋄

Remark 4.2. The term "semi-inner product" is often used in a more general sense that is not linked to a pre-existing norm. ⋄

Now we ask: do semi-inner products exists, and can there be more than one associated with any given norm? The answer to both is yes in general. However, if X^* is strictly convex then there cannot be more than one. We will see that if $\Re[Ax, x] \le 0$ with respect to one semi-inner product then it holds with respect to any semi-inner product.

Proposition 4.3. *There is at least one semi-inner product on a Banach space.*

Proof. Let X be a Banach space. For every $x \in X$ let

$$\mathscr{F}(x) := \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

By the Hahn-Banach theorem $\mathscr{F}(x)$ is nonempty for every $x \in X$. For every $x \in X$, choose $F(x) \in \mathscr{F}(x)$. Define $[\cdot,\cdot] : X \times X \to \mathbf{K}$ by $[x, y] = \langle F(y), x\rangle$ for all $x, y \in X$. □

If X^* is strictly convex then there is exactly one semi-inner product, essentially because the set $\mathscr{F}(x)$ contains only a single element.

Definition 4.4. Assume that $A : \mathscr{D}(A) \to X$ is linear with $\mathscr{D}(A) \subseteq X$. We say that A is *dissipative* if there is a semi-inner product on X such that $\Re[Ax, x] \le 0$ for all $x \in \mathscr{D}(A)$. ⋄

The notion of dissipativity depends on the particular norm used, but it will turn out that it does not depend on the particular semi-inner product used.

Remark 4.5. Consider $\mu_{tt}(x, t) = \Delta\mu(x, t) - \alpha(x)\mu_t(x, t)$ with $\mu|_{\partial\Omega} = 0$, where α is non-negative, smooth, with compact support, and $\int_\Omega \alpha > 0$. Then solutions μ tend to zero with t! ⋄

Lemma 4.6. *Assume that $A : \mathscr{D}(A) \to X$ is linear with $\mathscr{D}(A) \subseteq X$. Then A is dissipative if and only if $\|(\lambda\mathbb{1} - A)x\| \ge \lambda\|x\|$ for all $x \in \mathscr{D}(A)$ and $\lambda > 0$.*

Proof. Assume that A is dissipative. Choose a semi-inner product such that $\Re[Ax, x] \le 0$ for all $x \in \mathscr{D}(A)$. Then for all $x \in \mathscr{D}(A)$ and $\lambda > 0$, we have

$$\Re[(A - \lambda\mathbb{1})x, x] = \lambda\|x\|^2 - \Re[Ax, x] \ge \lambda\|x\|^2.$$

Combining this with the fact that

$$\Re[(\lambda\mathbb{1} - A)x, x] \le |[(\lambda\mathbb{1} - A)x, x]| \le \|(\lambda\mathbb{1} - A)x\|\|x\|$$

yields the result.

Assume now that $\|(\lambda\mathbb{1} - A)x\| \ge \lambda\|x\|$ for all $x \in \mathscr{D}(A)$ and $\lambda > 0$. As before, put

$$\mathscr{F}(x) := \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

We identify three cases: $x = 0$, $x \in \mathscr{D}(A) \setminus \{0\}$ and $x \notin \mathscr{D}(A)$.

Fix $x \in \mathscr{D}(A) \setminus \{0\}$. For all $\lambda > 0$ choose $y_\lambda^* \in \mathscr{F}(\lambda x - Ax)$ and put $z_\lambda^* = \frac{1}{\|y_\lambda^*\|} y_\lambda^*$.

$$\begin{aligned}
\lambda\|x\| &\le \|\lambda x - Ax\| && \text{by assumption}\\
&= \frac{1}{\|y_\lambda^*\|}\langle y_\lambda^*, nx - Ax\rangle && \text{since } y_\lambda^* \in \mathscr{F}(\lambda x - Ax)\\
&= \langle z_\lambda^*, \lambda x - Ax\rangle && \text{(this is a real number)}\\
&= \lambda\Re\langle z_\lambda^*, x\rangle - \Re\langle z_\lambda^*, Ax\rangle.
\end{aligned}$$

Since $\|z_\lambda^*\| = 1$ by construction,

$$\lambda\|x\| \le \lambda\Re\langle z_\lambda^*, x\rangle - \Re\langle z_\lambda^*, Ax\rangle \le \lambda\|x\| - \Re\langle z_\lambda^*, Ax\rangle.$$

Therefore, $\Re\langle z_\lambda^*, Az\rangle \le 0$ and similarly $\Re\langle z_\lambda^*, x\rangle \ge \|x\| - \frac{1}{\lambda}\|Ax\|$. Since the unit ball in X^* is weak-⋆ compact the net $\{z_\lambda^*\}_{\lambda\to\infty}$ has a weak-⋆ cluster point $z^* \in X^*$. Then $\|z^*\| \le 1$, $\Re\langle z^*, Ax\rangle \le 0$, and $\Re\langle z^*, x\rangle \ge \|x\|$. It follows that $\langle z^*, x\rangle = \|x\|$. Define a semi-inner product as before, but with

$$F(x) = \begin{cases} 0 & x = 0\\ \langle z^*, x\rangle & x \in \mathscr{D}(A) \setminus \{0\}\\ \text{anything in } \mathscr{F}(x) & x \in X \setminus \mathscr{D}(A). \end{cases}$$

□

Lemma 4.7. *Assume that $A : \mathscr{D}(A) \to X$ is linear with $\mathscr{D}(A) \subseteq X$ and that A is dissipative. Let $\lambda_0 \in (0, \infty)$ be given and assume that $\lambda_0\mathbb{1} - A$ is surjective. Then A is closed, $\rho(A) \supseteq (0, \infty)$, and $\|R(\lambda; A)\| \le \frac{1}{\lambda}$ for all $\lambda > 0$.*

Proof. Notice that, by Lemma 4.6, $\|(\lambda\mathbb{1}-A)x\| \geq \lambda\|x\|$ for all $x \in \mathscr{D}(A)$ and $\lambda > 0$. So we immediately deduce that $\|R(\lambda;A)\| \leq \frac{1}{\lambda}$, provided the resolvent exists. The key points are to show that A is closed and that $\lambda\mathbb{1}-A$ is surjective for all $\lambda > 0$.

Notice that $\lambda_0\mathbb{1}-A$ is bijective since it is surjective and bounded below, and furthermore, $\|(\lambda_0\mathbb{1}-A)^{-1}x\| \leq \frac{1}{\lambda_0}\|x\|$. So $(\lambda_0\mathbb{1}-A)^{-1} \in \mathscr{L}(X;X)$, hence it is closed, so A is closed as well.

To show that $\rho(A) \supseteq (0,\infty)$ it suffices to show that $(\lambda\mathbb{1}-A)^{-1}$ is surjective for all $\lambda > 0$. Let $\Lambda = \{\lambda \in (0,\infty) : \lambda \in \rho(A)\}$, which is open (in the relative topology of $(0,\infty)$) and non-empty. We will show that Λ is closed and conclude that $\Lambda = (0,\infty)$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in Λ converging to $\lambda^* \in (0,\infty)$. We will show that $\lambda^* \in \Lambda$ by showing that $\lambda^*\mathbb{1}-A$ is surjective. Let $y \in X$ be given. For every $n \in \mathbf{N}$ let $x_n = R(\lambda_n;A)y$. Note that $\sup\left\{\frac{1}{n} : n \in \mathbf{N}\right\} < \infty$.

$$\begin{aligned}\|x_n - x_m\| &= \|(R(\lambda_n;A)-R(\lambda_m;A))y\| \\ &= |\lambda_m-\lambda_n|\|R(\lambda_n;A)R(\lambda_m;A)y\| \\ &\leq |\lambda_m-\lambda_n|\frac{\|y\|}{\lambda_n\lambda_m} \\ &\to 0 \text{ as } n,m\to\infty.\end{aligned}$$

So $x_n \to x$ for some $x \in X$. Finally, $\{x_n\}_{n=1}^\infty \subseteq \mathscr{D}(A)$, $x_n \to x$, and $Ax_n \to \lambda^* x - y$. Since A is closed, $(\lambda^*\mathbb{1}-A)x = y$. □

Theorem 4.8 (Lumer-Phillips, 1961)**.** *Assume $A : \mathscr{D}(A) \to X$ is linear with $\mathscr{D}(A)$ dense in X.*

(i) If A is dissipative and there is $\lambda_0 > 0$ such that $\lambda_0\mathbb{1}-A$ is surjective then A generates a contractive linear C_0-semigroup.
(ii) If A generates a contractive linear C_0-semigroup then $\lambda\mathbb{1}-A$ is surjective for all $\lambda > 0$ and $\mathfrak{R}[Ax,x] \leq 0$ for all $x \in \mathscr{D}(A)$ and every semi-inner product on X (in particular, A is dissipative).

Proof. The first part follows from Lemma 4.7 and the Hille-Yosida theorem, since $\|R(\lambda;A)\| \leq \frac{1}{\lambda}$ implies $\|R(\lambda;A)^n\| \leq \frac{1}{\lambda^n}$.

For the second part, the surjectivity conclusion follows from the Hille-Yosida theorem. Let $[\cdot,\cdot]$ be a semi-inner product on X. We need to show that $\mathfrak{R}[Ax,x] \leq 0$ for all $x \in \mathscr{D}(A)$. For all $h > 0$ and $x \in \mathscr{D}(A)$,

$$\begin{aligned}\mathfrak{R}[T(h)x - x, x] &= \mathfrak{R}[T(h)x,x] - \|x\|^2 \\ &\leq \|T(h)x\|\|x\| - \|x\|^2 \\ &\leq \|x\|^2 - \|x\|^2 \\ &= 0.\end{aligned}$$

Dividing by h and letting $h \downarrow 0$ yields $\mathfrak{R}[Ax,x] \leq 0$. □

Corollary 4.9. *Assume $B : \mathscr{D}(B) \to X$ is linear with $\mathscr{D}(B)$ dense in X. Let $\omega, \lambda_0 \in \mathbf{R}$ with $\lambda_0 > \omega$ be given. If $\lambda_0\mathbb{1}-B$ is surjective and there exists a semi-inner product on X such that $\mathfrak{R}[Bx,x] \leq \omega\|x\|^2$ for all $x \in \mathscr{D}(B)$, then B generates a linear C_0-semigroup T such that $\|T(t)\| \leq e^{\omega t}$.*

Proof. Let $A = B - \omega\mathbb{1}$ and apply the Lumer-Phillips theorem to A. □

Lemma 4.10. *Assume that X is reflexive and that $A : \mathscr{D}(A) \to X$ is linear with $\mathscr{D}(A) \subseteq X$. Let $\lambda_0 > 0$ be given and assume that A is dissipative and that $\lambda_0\mathbb{1}-A$ is surjective. Then $\mathscr{D}(A)$ is dense in X.*

Remark 4.11. Let M be a linear submanifold in a Banach space X (not necessarily reflexive). Then M is dense in X if and only if for all $y \in X$ there is a sequence $\{x_n\}_{n=1}^\infty \subseteq M$ such that $x_n \rightharpoonup y$ as $n \to \infty$. Indeed, one direction is trivial. For the other, if y is not in the closure of M then $\operatorname{dist}(M,y) > 0$. By the Hahn-Banach theorem there is $y^* \in X^*$ such that $\langle y^*, x\rangle = 0$ for all $x \in M$ and $\langle y^*, y\rangle \neq 0$. ⋄

Proof. Let $y \in X$ be given. It suffices to shwo that there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathscr{D}(A)$ such that $x_n \rightharpoonup y$ as $n \to \infty$. Put $x_n = \left(\mathbb{1} - \frac{1}{n}A\right)^{-1} y = nR(n;A)y \in \mathscr{D}(A)$ for all $n \in \mathbf{N}$. Then

$$\|x_n\| \le n\|R(n;A)\|\|y\| \le n\frac{1}{n}\|y\| = \|y\|.$$

Choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and $x \in X$ such that $x_{n_k} \rightharpoonup x$ as $k \to \infty$. We are done if we show that $x = y$. We have

$$A\left(\frac{x_{n_k}}{n_k}\right) = x_{n_k} - y \rightharpoonup x - y,$$

and $x_{n_k} \rightharpoonup 0$ (in fact, $x_{n_k} \to 0$). Now since $\mathbf{Gr}(A)$ is closed and convex it is weakly closed. Since $(0, x-y) \in \mathbf{Gr}(A)$, we deduce that $x = y$. □

This lemma shows that if X is reflexive then we do not need to assume that $\mathscr{D}(A)$ is dense in the Lumer-Phillips theorem. This is less helpful than it seems because in many applications it is trivial to check that the domain is dense.

Theorem 4.12 (Lumer-Phillips for Hilbert spaces)**.** *Let X be a Hilbert space and assume that $B : \mathscr{D}(B) \to X$ is linear with $\mathscr{D}(B) \subseteq X$. Let $\lambda_0, \omega \in \mathbf{R}$ and $\lambda_0 > \omega$ be given. Assume that $\Re(Bx, x) \le \omega\|x\|^2$ for all $x \in \mathscr{D}(B)$ and that $\lambda_0\mathbb{1} - B$ is surjective. Then B generates a linear C_0-semigroup T such that $\|T(t)\| \le e^{\omega t}$ for all $t \ge 0$.*

Example 4.13. Let

$$\mathscr{D}(A) := \{u \in AC[0,1] \,:\, u' \in AC[0,1],\ u'' \in L^2[0,1],\ u(0) = u(1) = 0\} \subseteq L^2[0,1],$$

and $Au := u''$. We have seen that A is closed and A is densely defined (in fact it is self-adjoint). For any $u \in \mathscr{D}(A)$,

$$(Au, u) = \int_0^1 u''u\, dx = -\int_0^1 (u')^2\, dx \le 0.$$

If we can solve the ODE $u - u'' = f$, $u(0) = u(1) = 0$ for any $f \in L^2(0,1)$, then A generates a contraction semigroup T by the Lumer-Phillips theorem. Thus the solutions to the heat equation

$$\begin{cases} u_t - u_x x = 0 & \text{on } (0,1) \\ u(t,0) = u(t,1) = 0 & \text{for all } t \ge 0 \\ u(0,x) = g(x) & \text{for all } x \in (0,1) \end{cases}$$

can be written as $u(x,t) = (T(t)g)(x)$. ⋄