SHEAVES OVER TOPOLOGICAL SPACES

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ABSTRACT. The category of sheaves over a topological space has a rich internal logic, but they also have a very rich geometric interpretation. In these notes I review the basic notions of sheaves over topological spaces, with an emphasis on the geometry of sheaves. I prove that sheaves are toposes, present the proof of the equivalence between the category of sheaves over X and the set of local homeomorphisms into X. Finally, I define geometric morphisms and show that there is a correspondence between continuous functions and geometric morphisms. This material can mostly be found in Chapter II of *Sheaves in Geometry and Logic* by Mac Lane and Moerdijk [MM12].

1. PRELIMINARIES

Definition 1.1. A **topological space** is a pair $(X, \mathscr{O}(X))$, where X is a set and $\mathscr{O}(X) \subseteq \mathscr{P}(X)$ satisfies

(i) $\emptyset, X \in \mathscr{O}(X)$,
(ii) $\mathscr{O}(X)$ is closed under arbitrary unions,
(iii) $\mathscr{O}(X)$ is closed under finite intersections.

We call $\mathscr{O}(X)$ the **topology**, and the elements of $\mathscr{O}(X)$ are called the **open sets**.

We will view $\mathscr{O}(X)$ as a category where the objects are the open sets and the morphisms are determined by set inclusion. The main building block in sheaf theory is a *presheaf*.

Definition 1.2. Let $\mathbf{C}$ be a category. The **presheaf category** is the usual functor category $\widehat{\mathbf{C}} := [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$.

In the special case when we are working over topological spaces we say that presheaves over $(X, \mathscr{O}(X))$ are presheaves over the $\mathscr{O}(X)$, viewed as a category.

Definition 1.3. Let $\mathscr{F}$ be a presheaf (of sets) over a topological space $(X, \mathscr{O}(X))$. An element $s \in \mathscr{F}(U)$ is called a **section of $\mathscr{F}$ over** U, and an element of $\mathscr{F}(X)$ is called a **global section**.

Notation 1.4. Consider a presheaf $\mathscr{F}$ over $(X, \mathscr{O}(X))$, and $U, V \in \mathscr{O}(X)$ with $V \subset U$. Then we have a map $\mathscr{F}(V \subset U) : \mathscr{F}(U) \to \mathscr{F}(V)$. We call this map the **restriction map** and write the action of this map for every $s \in \mathscr{F}(U)$ as $s \mapsto s|_V$, as if it is were actual restriction of functions.

The notion of sheaves allows one to patch locally defined objects together to produce an object defined on a union of open sets. In some sense, we obtain a framework for dealing with *local to global* analysis.

Definition 1.5. Let $(X, \mathscr{O}(X))$ be a topological space, and let $\mathscr{F}$ be a presheaf over X. We say that $\mathscr{F}$ is a **sheaf** if both of the following conditions hold:

(i) if $U \in \mathscr{O}(X)$ and $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of U, that is if $U = \bigcup_{\alpha \in \Lambda} U_\alpha$ and $U_\alpha \in \mathscr{O}(X)$, and if $s, t \in \mathscr{F}(U)$ satisfy $s|_{U_\alpha} = t|_{U_\alpha}$ for all $\alpha \in \Lambda$ then $s = t$. *If just (i) holds then we call $\mathscr{F}$ **separated**.*
(ii) if $U \in \mathscr{O}(X)$ and $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of U, and if for every $\alpha \in \Lambda$ there exists $s_\alpha \in \mathscr{F}(U_\alpha)$ with the family $(s_\alpha)_{\alpha \in \Lambda}$ satisfying
$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$$
for all $\alpha, \beta \in \Lambda$, then there exists some $s \in \mathscr{F}(U)$ such that $s_\alpha = s|_{U_\alpha}$ for all $\alpha \in \Lambda$.

The following categorical characterization of a sheaf is equivalent; it is just saying the same thing slightly differently.

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Definition 1.6. Let $(X, \mathscr{O}(X))$ be a topological space, and let $\mathscr{F}$ be a presheaf over X. We say that $\mathscr{F}$ is a **sheaf** if for every $U \in \mathscr{O}(X)$ and any open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of U we have that

$$\mathscr{F}(U) \dashrightarrow^{e} \prod_{\alpha\in\Lambda} \mathscr{F}(U_\alpha) \underset{\pi_\beta}{\overset{\pi_\alpha}{\rightrightarrows}} \prod_{\alpha,\beta\in\Lambda} \mathscr{F}(U_\alpha \cap U_\beta)$$

is an equalizer diagram in **Set**, where

$$e : \mathscr{F}(U) \to \prod_{\alpha\in\Lambda} \mathscr{F}(U_\alpha)$$
$$s \mapsto (s|_{U_\alpha})_{\alpha\in\Lambda}$$

$$\pi_\alpha : \prod_{\alpha\in\Lambda} \mathscr{F}(U_\alpha) \to \prod_{\alpha,\beta\in\Lambda} \mathscr{F}(U_\alpha \cap U_\beta) \qquad \pi_\beta : \prod_{\alpha\in\Lambda} \mathscr{F}(U_\alpha) \to \prod_{\alpha,\beta\in\Lambda} \mathscr{F}(U_\alpha \cap U_\beta)$$
$$(s_\alpha)_{\alpha\in\Lambda} \mapsto (s_\alpha|_{U_\alpha\cap U_\beta})_{\alpha,\beta\in\Lambda} \qquad (s_\alpha)_{\alpha\in\Lambda} \mapsto (s_\beta|_{U_\alpha\cap U_\beta})_{\alpha,\beta\in\Lambda}$$

A morphism $\mathscr{F} \to \mathscr{G}$ of sheaves is a natural transformation of functors. The *category of all sheaves* $\mathscr{F}$ over $(X, \mathscr{O}(X))$ with morphisms as described will be denoted as $\mathbf{Sh}(X; \mathscr{O}(X))$. We will simply write $\mathbf{Sh}(X)$ when the topology is understood from the context.
Note that by definition we have that $\mathbf{Sh}(X)$ is a full subcategory of the presheaf category $\widehat{\mathscr{O}(X)}$.

Example 1.7. Now we give a collection of geometric examples of sheaves. Let $(X, \mathscr{O}(X))$ be a topological space.

1. Consider the presheaf $\mathscr{C} : \mathscr{O}(X) \to \mathbf{Set}$ where
$$\mathscr{C}(U) := \{\text{set of continuous functions on } U\},$$
and $\mathscr{C}(V \subset U)(f) = f|_V$.
2. Similarly, we can consider $\mathscr{C}^k$ where $k \in \mathbf{N} \cup \{\infty\}$ is the sheaf that returns the set of functions over U with regularity $\mathscr{C}^k$. Note that we obtain a sequence of subsheaves
$$\mathscr{C}^\infty \subset \cdots \subset \mathscr{C}^k \subset \mathscr{C}^{k-1} \subset \cdots \subset \mathscr{C}^1 \subset \mathscr{C}$$
3. Consider the *skyscraper presheaf*. Fix some point $x_0 \in X$. The skyscraper presheaf $\mathscr{S}_{x_0} \in \widehat{\mathscr{O}(X)}$ is defined as follows
$$\mathscr{S}_{x_0}(U) := \begin{cases} X & \text{if } x_0 \in U, \\ \{*\} & \text{if } x_0 \notin U, \end{cases}$$
and
$$\mathscr{S}_{x_0}(V \subset U)(x) = \begin{cases} x & \text{if } x_0 \in V \cap U, \\ \{*\} & \text{if } x_0 \notin V \cap U. \end{cases}$$
It is not hard to check that this is indeed a sheaf.

Now for some examples of presheaves that are not sheaves:

1. Consider the presheaf $\mathscr{C}_b : \mathscr{O}(X)^{\text{op}} \to \mathbf{Set}$ which returns the set of continuous and bounded functions over an open set. This is clearly not a sheaf since gluing together bounded functions over an unbounded domain may be unbounded.
2. Consider the constant presheaf defined as $\mathscr{F}(U) = X$. Now consider $U_1, U_2 \in \mathscr{O}(X)$ disjoint (i.e. $U_1 \cap U_2 = \emptyset$ and so the conditions for the gluing condition are vacuously true), and $s_1 \in \mathscr{F}(U_1)$ and $s_2 \in \mathscr{F}(U_2)$. Now notice that there does not exist any $s \in \mathscr{F}(U)$ such that $s|_{U_1} = s|_{U_2}$.

Note that for any sheaf $\mathscr{F}$ over X and open set $U \subset X$, we can restrict $\mathscr{F}$ to U to obtain a sheaf $\mathscr{F}|_U$. We now present a very important theorem regarding the construction of sheafs.

Theorem 1.8. *Let $(X, \mathscr{O}(X))$ be a topological space, let $(U_\alpha)_{\alpha\in\Lambda}$ be an open cover of X, and for each $\alpha \in \Lambda$ let $\mathscr{F}_\alpha$ be a sheaf over U_α. If, for $\alpha, \beta \in \Lambda$ satisfying $U_\alpha \cap U_\beta \neq \emptyset$, we have a sheaf isomorphism*

$$\phi_{\alpha,\beta} : \mathscr{F}_\alpha|_{U_\alpha\cap U_\beta} \to \mathscr{F}_\beta|_{U_\alpha\cap U_\beta}$$

satisfying the co-cycle condition: $\phi_{\beta,\gamma} \circ \phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ *on triple overlaps* $U_\alpha \cap U_\beta \cap U_\gamma$*, then there exists a sheaf* $\mathscr{F}$ *over* X*, unique up to isomorphism, and and isomorphisms for every* $\alpha \in \Lambda$ *of the form* $\psi_\alpha : \mathscr{F}|_{U_\alpha} \to \mathscr{F}_\alpha$ *such that the following diagram commutes for every* $\alpha, \beta \in \Lambda$*:*

$$\begin{tikzcd} \mathscr{F}|_{U_\alpha \cap U_\beta} \arrow[r, "\psi_\alpha"] \arrow[d, equal] & \mathscr{F}_\alpha|_{U_\alpha \cap U_\beta} \arrow[d, "\phi_{\alpha,\beta}"] \\ \mathscr{F}|_{U_\alpha \cap U_\beta} \arrow[r, "\psi_\beta"'] & \mathscr{F}_\beta|_{U_\alpha \cap U_\beta} \end{tikzcd}$$

Proof. We break the proof up into several steps.

Step 1. *Constructing a candidate sheaf.*

For every $U \in \mathscr{O}(X)$ define

$$\mathscr{F}(U) := \left\{ (s_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} \mathscr{F}_\alpha(U \cap U_\alpha) \;:\; \text{for all } \alpha, \beta \in \Lambda,\; \phi_{\alpha,\beta}(s_\alpha|_{U \cap U_\alpha \cap U_\beta}) = s_\beta|_{U \cap U_\alpha \cap U_\beta} \right\},$$

and let the restriction maps $\mathscr{F}(V \subseteq U) : \mathscr{F}(U) \to \mathscr{F}(V)$ be defined as follows:

$$(s_\alpha)_{\alpha \in \Lambda}|_V = (s_\alpha|_{V \cap U_\alpha})_{\alpha \in \Lambda}.$$

Now we verify that $\mathscr{F}$ is indeed a sheaf over X.

Consider any open set $W \in \mathscr{O}(X)$ and let $(W_i)_{i \in I}$ be an open cover of W. Let $s, t \in \mathscr{F}(W)$ be such that $s|_{W_i} = t|_{W_i}$ for all $i \in I$. Now write $s = (s_\alpha)_{\alpha \in \Lambda}$ and $t = (t_\alpha)_{\alpha \in \Lambda}$ then we see that

$$s_\alpha|_{W_i \cap U_\alpha} = t_\alpha|_{W_i \cap U_\alpha} \quad \text{for all } \alpha \in \Lambda,\; i \in I.$$

Since each $\mathscr{F}_\alpha$ is a sheaf, and hence a separated presheaf, we obtain that $s_\alpha = t_\alpha$ for all $\alpha \in \Lambda$, and hence $s = t$. So we have that $\mathscr{F}$ is *separated*.

Now let $W \in \mathscr{O}(X)$ and let $(W_i)_{i \in I}$ be an open cover of W. For each $i \in I$ let $s_i \in \mathscr{F}(W_i)$ and suppose that $s_i|_{W_i \cap W_j} = s_j|_{W_i \cap W_j}$ for all $i, j \in I$. Now write $s_i = (s_{i,\alpha})_{\alpha \in \Lambda}$ for all $i \in I$. note that

$$s_{i,\alpha}|_{W_i \cap W_j \cap U_\alpha} = s_{j,\alpha}|_{W_i \cap W_j \cap U_\alpha} \quad \text{for all } i, j \in I,\; \alpha \in \Lambda.$$

Now since each $\mathscr{F}_\alpha$ satisfies the gluing property, we have that there exists some $s_\alpha \in \mathscr{F}_\alpha(W \cap U_\alpha)$ such htat

$$s_\alpha|_{W_i \cap U_\alpha} = s_{i,\alpha} \quad \text{for all } i \in I,\; \alpha \in \Lambda.$$

Now let us define $s = (s_\alpha)_{\alpha \in \Lambda}$. Note that we have that

$$\phi_{\alpha,\beta}(s_{i,\alpha}|_{W_i \cap U_\alpha \cap U_\beta}) = s_{i,\beta}|_{W_i \cap U_\alpha \cap U_\beta} \quad \text{for all } i \in I,\; \alpha, \beta \in \Lambda.$$

It now directly follows that

$$\phi_{\alpha,\beta}(s_{i,\alpha})|_{W_i \cap U_\alpha \cap U_\beta} = s_{i,\beta}|_{W_i \cap U_\alpha \cap U_\beta} \quad \text{for all } i \in I,\; \alpha, \beta \in \Lambda.$$

Since $\mathscr{F}_\beta$ is a sheaf we conclude that $\phi_{\alpha,\beta}(s_{i,\alpha}) = s_{i,\beta}$ for every $i \in I$ and $\alpha, \beta \in \Lambda$. Hence,

$$\phi_{\alpha,\beta}(s_\alpha)|_{W_i \cap U_\beta} = s_\beta|_{W_i \cap U_\beta},$$

and so we conclude that $\phi_{\alpha,\beta}(s_\alpha) = s_\beta$ for all $\alpha, \beta \in \Lambda$. Finally, from this we conclude that

$$\phi_{\alpha,\beta}(s_\alpha|_{W \cap U_\alpha \cap U_\beta}) = s_\beta|_{W \cap U_\alpha \cap U_\beta} \quad \text{for all } \alpha, \beta \in \Lambda.$$

So s as is constructed is an element of $\mathscr{F}(W)$. Now we see that $\mathscr{F}$ is indeed a sheaf.

Step 2. *Constructing the sheaf isomorphisms.*

Note that our construction of $\mathscr{F}$ is simply the following limit:

$$\mathscr{F}(V) = \varprojlim_{\alpha \in \Lambda} \mathscr{F}_\alpha(V \cap U_\alpha).$$

Now for every open subset $V \subseteq U_\alpha \subset X$ we define the map $\psi_{\alpha,V} : \mathscr{F}(V) \to \mathscr{F}_\alpha(V)$ as the natural projection map from the limit: $\mathscr{F}(V) \to \mathscr{F}_\alpha(V)$. Now we can check that $\psi_\alpha : \mathscr{F}|_{U_\alpha} \to \mathscr{F}_i$ is indeed a sheaf morphism: for any open sets $V' \subseteq V \subseteq U_\alpha \subseteq X$ we see that

$$\begin{tikzcd}
\mathscr{F}|_{U_\alpha}(V) \arrow[r, "\psi_{\alpha,V}"] \arrow[d] & \mathscr{F}_\alpha(V) \arrow[d] \\
\mathscr{F}|_{U_\alpha}(V') \arrow[r, "\psi_{\alpha,V'}"'] & \mathscr{F}_\alpha(V')
\end{tikzcd}$$

commutes since $\mathscr{F}|_{U_\alpha}(V)$ is a limit. So we see that ψ_α is indeed a sheaf morphism. Furthermore, we have that $\psi_\beta = \phi_{\alpha,\beta} \circ \psi_\alpha$ on $U_\alpha \cap U_\beta$ for exactly the same reason as is depicted in the following diagram: let $V \subseteq U_\alpha \cap U_\beta$ be an open set

$$\begin{tikzcd}
\mathscr{F}|_{U_\alpha \cap U_\beta}(V) \arrow[d, "\psi_{\alpha,V}"'] \arrow[dr, "\psi_{\beta,V}"] & \\
\mathscr{F}_\alpha(V) \arrow[r, "(\phi_{\alpha,\beta})_V"'] & \mathscr{F}_\beta(V)
\end{tikzcd}$$

Now we claim that $\psi_\alpha : \mathscr{F}|_{U_\alpha} \to \mathscr{F}_\alpha$ is indeed a sheaf isomorphism. Define $\psi^{-1}_{\alpha,V} : \mathscr{F}_\alpha(V) \to \mathscr{F}|_{U_\alpha}(V)$ as follows: let $s \in \mathscr{F}_\alpha(V)$ and

$$\psi^{-1}_{\alpha,V}(s) = \left(\phi^{-1}_{\beta,\alpha}(s)\right)_{\beta \in \Lambda} =: \widetilde{s},$$

in the above if $V \cap U_\beta = \emptyset$ then with a bit of abuse of notation we just take the "inverse" to be a single point, since $\mathscr{F}(\emptyset) = \{*\}$ and since the inverse isn't well defined. Now we need to show that $\widetilde{s} \in \mathscr{F}(V)$. First note that for all $\beta \in \Lambda$ we have that $\widetilde{s}_\beta \in \mathscr{F}_\beta(V \cap U_\beta)$. Now consider any $\beta, \gamma \in \Lambda$. We want to show that $\phi_{\beta,\gamma}(\widetilde{s}_\beta|_{V \cap U_\beta \cap U_\gamma}) = \widetilde{s}_\gamma|_{V \cap U_\beta \cap U_\gamma}$. Indeed, this follows from the co-cycle condition since we have that

$$\phi_{\beta,\gamma}(\widetilde{s}_\beta|_{V \cap U_\beta \cap U_\gamma}) = \phi_{\beta,\gamma}(\phi^{-1}_{\beta,\alpha}(s)|_{V \cap U_\beta \cap U_\gamma}) = \phi^{-1}_{\gamma,\alpha}(s)|_{V \cap U_\alpha \cap U_\beta} = \widetilde{s}|_{V \cap U_\alpha \cap U_\beta}.$$

The fact that this map ψ^{-1}_α is a sheaf morphism is immediate from the definition. Furthermore, note that ψ^{-1}_α is indeed the inverse of ψ_α. So we have a sheaf isomorphism, as desired.

Step 3. *The sheaf $\mathscr{F}$ is unique, up to isomorphism.*

Suppose that $\mathscr{F}$ and $\mathscr{G}$ are two sheaves satisfying the properties of the theorem, equipped with sheaf isomorphisms $\psi_\alpha : \mathscr{F}|_{U_\alpha} \to \mathscr{F}_\alpha$ and $\xi_\alpha : \mathscr{G}|_{U_\alpha} \to \mathscr{F}_\alpha$. Now consider the sheaf morphisms

$$\theta_\alpha = \xi^{-1}_\alpha \circ \psi_\alpha : \mathscr{F}|_{U_\alpha} \to \mathscr{G}|_{U_\alpha}.$$

For any open subset $V \subseteq X$, and for any $s \in \mathscr{F}(V)$ we claim that the family

$$\{(\theta_\alpha)_{V \cap U_\alpha}(s|_{V \cap U_\alpha})\}_{\alpha \in \Lambda}$$

is a coherent assignment (in the sense that the image coincides on overlaps) for the sheaf $\mathscr{G}$, and hence it defines a unique element $\theta(s) \in \mathscr{G}(V)$. To check that this family is coherent note that on any overlap $U_\alpha \cap U_\beta$ we have that

$$\theta_\alpha = \xi^{-1}_\alpha \circ \psi_\alpha = \xi^{-1}_\alpha \circ \phi^{-1}_{\alpha,\beta} \circ \phi_{\alpha,\beta} \circ \psi_\alpha = (\phi_{\alpha,\beta} \circ \xi_\alpha)^{-1} \circ (\phi_{\alpha,\beta} \circ \psi_\alpha) = \xi^{-1}_\beta \circ \psi_\beta = \theta_\beta$$

To show that this is indeed an isomorphism we can define the inverse

$$\rho_\alpha = \psi^{-1}_\alpha \circ \xi_\alpha : \mathscr{G}|_{U_\alpha} \to \mathscr{F}|_{U_\alpha},$$

which can be checked in an identical way to be a sheaf morphism. Now it is clear that $\rho \circ \theta = \mathrm{id}_{\mathscr{F}}$ and $\theta \circ \rho = \mathrm{id}_{\mathscr{G}}$, and so we have that $\mathscr{F}$ is unique up to isomorphism.

□

We call sheaf constructions of this form *collations*. In particular, we see that we can glue together sheaves that agree on the overlaps of an open cover.

Corollary 1.9. *Let $(X, \mathscr{O}(X))$ be a topological space, let $(U_\alpha)_{\alpha\in\Lambda}$ be an open cover of X, and for each $\alpha \in \Lambda$ let $\mathscr{F}_\alpha$ be a sheaf over U_α. If for all $\alpha, \beta \in \Lambda$ satisfying $U_\alpha \cap U_\beta \neq \emptyset$ we have that*

$$\mathscr{F}_\alpha|_{U_\alpha\cap U_\beta} = \mathscr{F}_\beta|_{U_\alpha\cap U_\beta},$$

then there exists a sheaf $\mathscr{F}$ over X such that $\mathscr{F}|_{U_\alpha} = \mathscr{F}_\alpha$ for all $\alpha \in \Lambda$.

2. The relation between pre/sheaves and bundles/local homeomorphisms

Definition 2.1. Let $(X, \mathscr{O}(X))$ and $(Y, \mathscr{O}(Y))$ be topological spaces. A continuous map $f : Y \to X$ is called a **space over** X or a **bundle over** X.

Remark 2.2. It is helpful to think about such a map as a family of sets $(E_x)_{x\in X}$, where each $E_x := f^{-1}(\{x\}) \subseteq Y$ is called the **fiber** of Y over x.

Definition 2.3. A **cross-section** of a bundle $f : Y \to X$ is a continuous map $s : X \to Y$ such that $ps = \mathrm{id}$.

We now show how to construct a sheaf from an arbitrary bundle.

Construction 2.4. Given a continuous map $f : Y \to X$, for each $U \in \mathscr{O}(X)$ define

$$\Gamma_f(U) = \{s \,:\, s : U \to Y \text{ and } ps = i : U \subseteq X\},$$

where $i : U \to X$ is the inclusion map. For every pair of open sets $V \subseteq U$, we obtain a restriction $\Gamma_f U \to \Gamma_f V$. So we see that $\Gamma_f(-)$ is a presheaf over X. It is straightforward to check that Γ_f is a sheaf. This sheaf is called **the sheaf of cross-sections of the bundle** f.

Remark 2.5. Note that we obtain a functor $\Gamma_{(-)} : \mathbf{Top}/X \to \mathbf{Sh}(X)$.

Definition 2.6. Let $x \in X$ be an arbitrary point. Let $\mathscr{O}_x \subseteq \mathscr{O}(X)$ denote the set of open sets containing x. For $U_1, U_2 \in \mathscr{O}_x$, we say that $s_1 \in \mathscr{F}(U_1)$ and $s_2 \in \mathscr{F}(U_2)$ are **equivalent** if there exists $V \in \mathscr{O}_x$ such that $V \subseteq U_1 \cap U_2$ satisfying $s_1|_V = s_2|V$. This equivalence class is denoted as $\mathrm{germ}_x s$, and this is called the **germ** of s at x.

Example 2.7. When we are considering the presheaf $\mathscr{C}^\infty$ of smooth functions, then we see that two functions have the same germ at a point x if they agree on some neighborhood of x.

Definition 2.8. Let $(X, \mathscr{O}(X))$ be a topological space, and let $\mathscr{F}$ be a presheaf over X. For $x \in X$, the **stalk** of $\mathscr{F}$ at x is the set of equivalence classes under the equivalence relation above:

$$\mathscr{F}_x := \{(U,s) \,:\, U \in \mathscr{O}_x,\ s \in \mathscr{F}(U)\}/\!\sim\ = \{\mathrm{germ}_x s \,:\, U \in \mathscr{O}_x,\ s \in \mathscr{F}(U)\},$$

where in the above $(U,s) \sim (V,t)$ if there is some open set $W \subseteq U \cap V$ such that $s|_W = t|_W$.

Proposition 2.9. $\mathscr{F}_x = \underset{U\in\mathscr{O}_x}{\underrightarrow{\mathrm{colim}}}\, \mathscr{F}(U)$.

Proof. Note that $\mathscr{O}_x$ is a filtered category since given any two such open sets, there is a third such set contained in both. This now follows immediately from the explicit construction of filtered colimits in **Set**. □

Definition 2.10. Let $(X, \mathscr{O}(X))$ be a topological space and let $\mathscr{F}$ be a presheaf over X. The **étalé space of a presheaf** $\mathscr{F}$ is the disjoint union of the stalks of $\mathscr{F}$:

$$\mathrm{Et}(\mathscr{F}) := \coprod_{x\in X} \mathscr{F}_x.$$

We endow $\mathrm{Et}(\mathscr{F})$ with the **étalé topology**, that is the topology with basis elements subsets of the form

$$\mathscr{B}(U,s) = \{\mathrm{germ}_x(s) \,:\, x \in U\}, \qquad U \in \mathscr{O}(X),\ s \in \mathscr{F}(U).$$

Now let $\pi_{\mathscr{F}} : \mathrm{Et}(\mathscr{F}) \to X$ denote the canonical projection:

$$\pi_{\mathscr{F}}(\mathrm{germ}_x s) = x.$$

Proposition 2.11. *Let $(X, \mathscr{O}(X))$ be a topological space and let $\mathscr{F}$ be a presheaf over X. The following hold for* $\mathrm{Et}(\mathscr{F})$*:*

- *(i) The family $\mathscr{B}(U,s)$ for $U \in \mathscr{O}(X)$ and $s \in \mathscr{F}(U)$ form a basis for a topology.*
- *(ii) The projection $\pi_{\mathscr{F}}$ is a local homeomorphism.*

Proof.

(i) To show that this is a base for a topology is suffices to show that for every $\mathscr{B}(U_1, s_1)$, $\mathscr{B}(U_2, s_2)$, and $\mathrm{germ}_x\, s \in \mathscr{B}(U_1, s_1) \cap \mathscr{B}(U_2, s_2)$ that there exists a neighborhood $\mathscr{B}(V, t) \subseteq \mathscr{B}(U_1, s_1) \cap \mathscr{B}(U_2, s_2)$ and $\mathrm{germ}_x\, s \in \mathscr{B}(V, t)$. It is now easy to see that this is always possible. Let $V \subseteq U_1 \cap U_2$ be any neighborhood of x such that
$$s(y) = s_1(y) = s_2(y), \quad y \in V.$$
Note that this is possible since $\mathrm{germ}_x\, s \in \mathscr{B}(U_1, s_1) \cap \mathscr{B}(U_2, s_2)$. It is now clear that $\mathscr{B}(V, s)$ satisfies the desired properties.

(ii) By definition of $(\mathrm{Et}(F), \mathscr{O}(\mathrm{Et}(\mathscr{F})))$ we see that $\pi_F|_{b(U,s)}$ is a homeomorphism onto U (the inverse map is just s). Hence $\pi_{\mathscr{F}}$ is a local homeomorphism.

□

Example 2.12. Let E be a set endowed with the discrete topology. Let $\mathscr{F}_E$ be the constant sheaf (sheafification of constant presheaf) with values in E over X. The corresponding etale space is $(X \times E, \pi_1)$. This follows since for an open set $U \subseteq X$ we have that the sections of π_1 over U are just the maps $x \mapsto (x, s(x))$ where $s : U \to E$ is locally constant.

Theorem 2.13. *Let $(X, \mathscr{O}(X))$ be a topological space. There is a pair of adjoint functors*
$$\begin{tikzcd} \textbf{\textit{Top}}/X \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\vdash"] & \textbf{\textit{Psh}}(X) \arrow[l, bend left, "\Lambda"] \end{tikzcd}$$
where $\textbf{Psh}(X) = \widehat{\mathscr{O}(X)}$ is the category of presheaves over X, Γ is the functor which assigns to each bundle $f : Y \to X$ the sheaf of cross sections as in Definition 2.4, and where the left adjoint Λ assigns to each presheaf P the bundle of germs of P. There are natural transformations
$$\eta_{\mathscr{F}} : \mathscr{F} \to \Gamma\Lambda\mathscr{F}, \qquad \varepsilon_f : \Lambda\Gamma f \to f,$$
for $\mathscr{F}$ a presheaf and $f : Y \to X$ a bundle which are unit and counit making Λ a left adjoint for Γ.

Proof. Let $\Gamma : \mathbf{Top}/X \to \mathbf{Psh}(X)$ be defined as
$$\Gamma(f : Y \to X) := \Gamma_f,$$
as in Construction 2.4. Now let $\Lambda : \mathbf{Psh}(X) \to \mathbf{Top}/X$ be defined as
$$\Lambda(\mathscr{F}) := \pi_{\mathscr{F}} : \mathrm{Et}(\mathscr{F}) \to X.$$

Step 1. *The natural transformation $\eta_{\mathscr{F}}$.*

Let $\mathscr{F}$ be a presheaf, and consider the sheaf of cross-sections $\Gamma\Lambda(\mathscr{F})$. Note that each section $s \in \mathscr{F}(U)$ determines a cross-section (of s)
$$\dot{s} : U \to \mathrm{Et}(\mathscr{F}), \qquad x \mapsto \mathrm{germ}_x\, s, \ x \in U.$$
For each open set $U \in \mathscr{O}(X)$ we have a function
$$\eta_U : \mathscr{F}(U) \to \Gamma\Lambda(\mathscr{F})(U), \qquad s \mapsto \dot{s}.$$
Note that the restriction operator commutes with η, and so we have that η is a natural transformation of between the functors $(\mathscr{F} \implies \Gamma\Lambda\mathscr{F})$.

Step 2. *The natural transformation ε_f.*

Given a bundle $f : Y \to X$ each point of $\Lambda\Gamma(f)$ is of the form $\dot{s}x$ for some point $x \in X$ and some cross-section $s : U \to Y$. Now we define
$$\varepsilon_f(\dot{s}x) = sx \in Y, \qquad x \in U, \ s \in \Gamma_f U.$$
It is easy to check that this definition is independent of the representative of $\mathrm{germ}_x\, s$. Furthermore, it is immediate to see that $\varepsilon_f : \Gamma\Lambda(f) \to f$ is continuous, and hence a map of bundles.

Step 3. *The adjunction.*

One readily verifies that

$$\Gamma \xrightarrow{\eta\Gamma} \Gamma\Lambda\Gamma \xrightarrow{\Gamma\varepsilon} \Gamma\ , \qquad \Lambda \xrightarrow{\Lambda\eta} \Lambda\Gamma\Lambda \xrightarrow{\varepsilon\Lambda} \Lambda$$

are both identities. Indeed, consider any bundle $f : Y \to X$ and a cross section $s \in \Gamma_f(U)$. Then we see that the first composition above sends $s \to \dot{s} \in \Gamma\Lambda\Gamma_f(U) \to s$. Similarly, the second composition sends $\operatorname{germ}_x s \to \operatorname{germ}_x \dot{s} \to \dot{s}x = \operatorname{germ}_x s$. It now directly follows from these two triangle identities that Γ is left adjoint to Λ.

□

Proposition 2.14. *If $\mathscr{F}$ is a sheaf then $\eta_{\mathscr{F}}$ is an isomorphism.*

Proof. Consider $s, t \in \mathscr{F}(U)$, such that $\dot{s} = \dot{t}$. Then we see that $\operatorname{germ}_x s = \operatorname{germ}_x t$ for all $x \in U$. So for every $x \in U$ we have an open set $V_x \subseteq U$ such that $s|_{V_x} = t|_{V_x}$. These open sets form an open cover of U, and so by the first condition of $\mathscr{F}$ being a sheaf we have that $s = t$, which implies that η is injective.

Consider any $\sigma : U \to \operatorname{Et}(\mathscr{F})$ be any cross section of $\pi_{\mathscr{F}} : \operatorname{Et}(\mathscr{F}) \to X$ over some open set U. Now for every $x \in U$ we have an open set U_x and some element $s_x \in \mathscr{F}(U_x)$ such that

$$\sigma(x) = \operatorname{germ}_x s_x, \qquad x \in U_x,\ s_x \in \mathscr{F}(U_x).$$

Since sections of $\operatorname{Et}(\mathscr{F})$ are local inverses for the local homeomorphisms $\pi_{\mathscr{F}}$, we have that sections of $\operatorname{Et}(\mathscr{F})$ agreeing at x must agree in a neighborhood of x. In particular, there exists a neighborhood of x, $V_x \subseteq U_x$ such that $\sigma(y) = \operatorname{germ}_y s_x$ for all $y \in V_x$. Hence, since we have that

$$s_{x_1}|_{V_{x_1} \cap V_{x_2}} = s_{x_2}|_{V_{x_1} \cap V_{x_2}}$$

for all $x_1, x_2 \in U$ it follows that since $\mathscr{F}$ is a sheaf that there exists some $s_\sigma \in \mathscr{F}(U)$ such that $\sigma(x) = \operatorname{germ}_x s_x = \operatorname{germ}_x s_\sigma$ for all $x \in U$, as desired. Hence η is surjective, and therefore a bijection. □

Proposition 2.15. *If $f : Y \to X$ is a local homeomorphism then ε_f is an isomorphism.*

Proof. We construct and inverse $\theta_f : f \to \Lambda\Gamma f$. For every point $y \in Y$ with $f(y) = x$ we have an open neighborhood U of x in X and a cross-section $s : U \to Y$ such that $s(x) = y$. We now define $\theta_f(y) = \dot{s}x \in \Lambda\Gamma f$. This is well-defined (i.e. independent of choice of cross-section), and θ_f is trivially continuous. Furthermore, by definition it is the (two-sided) inverse of ε_f. □

Proposition 2.16. *Consider an adjunction*

$$\begin{tikzcd} \mathbf{A} \arrow[r, bend left, "\Gamma"] & \mathbf{B} \arrow[l, bend left, "\Lambda"] \end{tikzcd}$$

such that for $A \in \mathbf{A}$ and $B \in \mathbf{B}$

$$\eta_{\Gamma A} : \Gamma A \to \Gamma\Lambda\Gamma A,$$

$$\varepsilon_{\Lambda B} : \Lambda\Gamma\Lambda B \to \Lambda B$$

are isomorphisms. Then let $\widetilde{\mathbf{A}}$ be the full subcategory of $\mathbf{A}$ with objects A that are isomorphic to some ΓB. Similarly, let $\widetilde{\mathbf{B}}$ is the full subcategory of $\mathbf{B}$ with object B that are isomorphic to some ΛA. Then Λ and Γ restrict to an equivalence of these subcategories.

Proof. Found on page 90 of [MM12]. □

Corollary 2.17. *The functors Γ and Λ from Theorem 2.6 restrict to an equivalence of categories*

$$\begin{tikzcd} \boldsymbol{Sh}(X) \arrow[r, bend left, "\Gamma"] & \boldsymbol{LH}(X) \arrow[l, bend left, "\Lambda"] \end{tikzcd}$$

where $\boldsymbol{LH}(X)$ is the space of local homeomorphisms into X.

3. Categorical Properties of $\mathbf{Sh}(X)$

Theorem 3.1. *Let $(X, \mathscr{O}(X))$ be a topological space. Then $\mathbf{Sh}(X)$ is a topos.*

To prove this theorem we will first prove a series of propositions.

Definition 3.2. A full subcategory $\mathbf{C}$ of a category $\mathbf{D}$ is **reflective** if the inclusion function $\mathbf{C} \hookrightarrow \mathbf{D}$ has a left adjoint. The left adjoint is called the **reflector**.

Proposition 3.3. *Let $(X, \mathscr{O}(X))$ be a topological space. The category $\mathbf{Sh}(X)$ is reflective in the category $\widehat{\mathscr{O}(X)}$.*

Proof. The composition $\Gamma \circ \Lambda$ (as in the Theorem 2.12) is left adjoint to the inclusion $\mathbf{Sh}(X) \hookrightarrow \widehat{\mathscr{O}(X)}$.

We also present a sketch of a direct proof here. Let $\mathscr{F}$ be a sheaf, $\mathscr{G}$ be a presheaf, and $\theta : \mathscr{G} \to \mathscr{F}$ be a morphism of presheaves. Since $\eta : \mathscr{F} \to \Gamma\Lambda\mathscr{F}$ is an isomorphism, we can find a unique morphism of sheaves $\gamma : \Gamma\Lambda\mathscr{G} \to \mathscr{F}$ such that $\gamma = \eta^{-1}\Gamma\theta$. So in the following diagram we have that

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\eta"] \arrow[d, "\theta"'] & \Gamma\Lambda\mathscr{G} \arrow[d, "\Gamma\Lambda\theta"] \arrow[dl, "\gamma"'] \\ \mathscr{F} \arrow[r, "\eta"'] & \Gamma\Lambda\mathscr{F} \end{tikzcd}$$

the bottom triangle commutes. Since η is a natural transformation, the square commutes, and so $\gamma \circ \eta = \theta$ as desired. It now isn't very difficult to show that γ is unique. $\square$

The left adjoint in the above proposition is the **sheafification functor**

Remark 3.4 (Intuition regarding sheafification). A presheaf can fail to be a sheaf in one of two ways:

(1) the presheaf has local sections that should glue together to give a global section, but don't
(2) it has global non-zero sections which are locally zero.

The sheafification functor adds the sections necessary to fix the issues of (1) and remove the sections necessary to fix the issues of (2).

Here are a few concrete examples:

(a) Consider the presheaf $\mathscr{C}_b(X)$ the continuous and bounded functions over a topological space X. This is a violation of type (1), and the sheafification of $\mathscr{C}_b(X)$ is simply $\mathscr{C}(X)$, the space of all continuous functions on X. This is obtained by adding all of the sections necessary to allow us to glue all bounded functions.
(b) Consider the presheaf $\mathscr{C}_{p=q}(\mathbb{S}^1)$ over the topological space $\mathbb{S}^1 \subseteq \mathbf{R}^2$ of all continuous functions which have the same value at two distinct points $p, q \in \mathbb{S}^1$.

Note that locally, if we consider neighborhoods of p and q that separate these two points, this condition disappears and we locally have the continuous function sheaf. Now consider any function $f \in \mathscr{C}(\mathbb{S}^1)$ such that $f(p) \neq f(q)$. Then by restricting to a sufficiently fine open cover $\{U_\alpha\}$ we have that each local section is in $\mathscr{C}_{p=q}(U_\alpha)$; however, the function $f \notin \mathscr{C}_{p=q}(\mathbb{S}^1)$. So we see that the gluing condition doesn't hold in this case and we see that $\mathscr{C}_{p=q}(\mathbb{S}^1)$ is *not* a sheaf.

We now can sheafify, by adding in all of these sections, and we obtain (yet again) the sheaf $\mathscr{C}(\mathbb{S}^1)$. Another way to intuitively see that this is the correct sheafification is that any two sheaves that coincide locally must be the same.
(c) Consider the sheaf of smooth functions $\mathscr{C}^\infty(\mathbb{S}^1)$. Consider the morphism of sheaves $\partial_\theta : \mathscr{C}^\infty(\mathbb{S}^1) \to \mathscr{C}^\infty(\mathbb{S}^1)$, which is the map which differentiates the function with respect to the angle θ. Now consider any $U \subseteq \mathbb{S}^1$ and we see that cokernel presheaf $\mathscr{C}^\infty(\mathbb{S}^1; U)/\partial_\theta\mathscr{C}^\infty(\mathbb{S}^1; U)$ is
   - ⋆ zero if $U \neq \mathbb{S}^1$ since every smooth function on U is the derivative of it's indefinite integral (which one can easily see by pulling back into local coordinates)
   - ⋆ $\mathbf{R}$ if $U = \mathbb{S}^1$. Note that for any $f \in \mathscr{C}^\infty(\mathbb{S}^1)$ we have that

$$\int_{\mathbb{S}^1} \partial_\theta f \, \mathrm{d}\theta = 0.$$

So we see that constant functions are not derivatives of any smooth function over $\mathscr{C}^\infty(\mathbb{S}^1)$.

In this case we see that (2) is violated – there are non-zero global sections which are locally zero. The sheafification of the cokernel presheaf $\mathscr{C}^\infty(\mathbb{S}^1;-)/\partial_\theta\mathscr{C}^\infty(\mathbb{S}^1;-)$ is the zero sheaf, since we just need to throw away all constant functions.

Lemma 3.5. *Let $(X,\mathscr{O}(X))$ be a topological space. Then $\mathbf{Sh}(X)$ has a subobject classifier.*

Proof. Define the presheaf $\Omega : \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ by

$$\Omega(U) = \{V \in \mathscr{O}(X) \,:\, V \subseteq U\}.$$

On morphisms $V \subseteq U$ we just intersect the sets in $\Omega(U)$ with V.

Step 1. *Ω is a sheaf.*

Consider any open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of some open set $U \in \mathscr{O}(X)$. Now given a family of open sets $\{V_\alpha\}_{\alpha\in\Lambda}$ satisfying $V_\alpha \subseteq U_\alpha$ for all $\alpha \in \Lambda$, and $V_\alpha \cap U_\beta = V_\beta \cap U_\alpha \subseteq U_\alpha \cap U_\beta$ for all $\alpha, \beta \in \Lambda$, we have that there exists a unique open set

$$V = \bigcup_{\alpha\in\Lambda} V_\alpha,$$

such that

$$V \cap U_\alpha = V_\alpha$$

for all $\alpha \in \Lambda$. So we have the gluing property of the sheaf. Alternatively, we see that the separability of Ω is trivial, and so the equalizer characterization of a sheaf is satisfied by V as above. So we have shown that Ω is a sheaf.

Step 2. *$\top : 1 \to \Omega$ is a subobject classifier.*

Define the sheaf morphism $\top : 1 \to \Omega$ componentwise as follows:

$$(\top)_U(*) = U.$$

Now consider any subobject $\mathscr{G} \rightarrowtail \mathscr{F}$ in $\mathbf{Sh}(X)$. Without loss of generality, we may assume that $\mathscr{G}$ is a subsheaf of $\mathscr{F}$, that is to say $\mathscr{G}(U) \subseteq \mathscr{F}(U)$ for all $U \in \mathscr{O}(X)$. Now for every $s \in \mathscr{F}(U)$ we have that there is a unique largest $W_U^s \subseteq U$ such that $s|_{W_U^s} \in \mathscr{G}(W_U^s)$. Namely, we have that

$$W_U = \bigcup \{V \in \mathscr{O}(X) \,:\, V \subseteq U \text{ and } s|_V \in \mathscr{G}(V)\}.$$

Now we define the characteristic function of $\mathscr{G}$ componentwise as follows:

$$\begin{aligned}(\chi_{\mathscr{G}})_U : \mathscr{F}(U) &\to \Omega(U)\\ s &\mapsto W_U^s,\end{aligned}$$

using the notation above. Now consider the pullback of $\top$ along $\chi_{\mathscr{G}}$

$$\begin{tikzcd} \mathscr{P} \arrow[r, dashed] \arrow[d, dashed] & 1 \arrow[d, "\top"] \\ \mathscr{F} \arrow[r, "\chi_{\mathscr{G}}"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} \mathscr{P}(U) \arrow[r, dashed] \arrow[d, dashed] & 1 \arrow[d, "\top"] \\ \mathscr{F}(U) \arrow[r, "(\chi_{\mathscr{G}})(U)"'] & \Omega(U) \end{tikzcd}$$

where we draw the diagram on the right to emphasize that pullbacks are computed componentwise in $\mathbf{Sh}(X)$. We know that the componentwise pullback in $\mathbf{Set}$ is

$$\mathscr{P}(U) = \{s \in \mathscr{F}(U) \,:\, (\chi_{\mathscr{G}})_U(s) = U\} = \mathscr{G}(U),$$

one can easily show that $\chi_{\mathscr{G}}$ is unique, and so we see that $\top : 1 \to \Omega$ is indeed the subobject classifier of $\mathbf{Sh}(X)$.

□

Corollary 3.6. *Let $(X,\mathscr{O}(X))$ be a topological space. Then $\mathbf{Sh}(X)$ is a topos.*

Proof. From Homework #3, we see that $\mathbf{Sh}(X)$ as a reflexive subcategory of $\mathbf{Psh}(X)$ has finite limits and exponentials. From the previous proposition we see that $\mathbf{Sh}(X)$ has a subobject classifier. Hence $\mathbf{Sh}(X)$ is a topos. □

4. Geometric Morphisms and Continuous functions

Definition 4.1. Let $\mathscr{E}$ and $\mathscr{F}$ be toposes. A **geometric morphism** $f : \mathscr{F} \to \mathscr{E}$ is a pair of functors $f^* : \mathscr{E} \to \mathscr{F}$ and $f_* : \mathscr{F} \to \mathscr{E}$ such that f^* is left adjoint to f_* and f^* is left exact. We call f_* the **direct image** part of f, and f^* the **inverse image** part of f.

Given the equivalence between the Etale space of X and $\mathbf{Sh}(X)$ we will see that continuous maps $f : X \to Y$ will induce functors between the associated sheaf categories.

Definition 4.2. Let $f : X \to Y$ be a continuous map. Let $\mathscr{F}$ be a sheaf over X. The **direct image** of a functor $\mathscr{F}$ under f, denoted as $f_*\mathscr{F}$, is defined as follows:

$$(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}U),$$

and where for $V \subseteq U$ open sets, and $s \in (f_*\mathscr{F})(U)$ we have that $(f_*\mathscr{F})(V \subseteq U)(s) = \mathscr{F}(f^{-1}V \subseteq f^{-1}U)(s)$. So we see that

$$f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$$

is a functor.

Definition 4.3. Let $f : X \to Y$ be a continuous map. Let $p : Z \to Y$ be a local homeomorphism over Y. The pullback of p along f is just the usual pullback in **Top**.

Proposition 4.4. *The pullback of a local homeomorphism is a local homeomorphism.*

Proof. We have the following commutative diagram for the pullback

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Now consider $(x, z) \in X \times_Y Z$. That is $x \in X$ and $z \in Z$ such that $f(x) = p(e)$. Since p is a local homeomorphism there exists an open neighborhood U of e such that $p|_U$ is a homeomorphism. Then we see that $V := f^{-1}(p(U)) \times U$ is clearly an open neighborhood of (x, e) in the product topology on $X \times Z$. So we see that $V \cap X \times_Y Z$ is an open set in the induced topology on $X \times_Y Z$ and is mapped homeomorphically onto $f^{-1}(p(U))$ in X. So we see that $f^*(p)$ is indeed a local homeomorphism. ☐

Proposition 4.5. *Let $f : X \to Y$. There exists a functor $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$.*

Proof. This is just the functor f^* as above under the association of sheaves as local homeomorphisms:

$$\mathbf{Sh}(Y) \xrightarrow{\Lambda} \mathbf{LH}(Y) \xrightarrow{f^*} \mathbf{LH}(X) \xrightarrow{\Gamma} \mathbf{Sh}(X)$$

☐

Remark 4.6. This pullback functor is called the inverse image functor on sheaves.

Definition 4.7. Let $f : X \to Y$ be a continuous map. Let $f^{-1} : \mathbf{Psh}(Y) \to \mathbf{Sh}(X)$ be the sheafification of the functor

$$\widetilde{f^{-1}}(\mathscr{G})(U) = \varinjlim_{f(U) \subseteq V \in \mathscr{O}(X)} \mathscr{G}(V).$$

Remark 4.8. Since the forward image of a continuous function is not necessarily continuous, we can intuitively think of this colimit as the best approximation of the analogous definition of the direct image as in Definition 4.2. One can easily check that if f is an *open* continuous map and $\mathscr{G}$ a presheaf on Y then $\widetilde{f^{-1}}(\mathscr{G})(U) = \mathscr{G}(f(U))$.

Proposition 4.9. *Let $f : X \to Y$ be a continuous map. f^{-1} is isomorphic to f^*.*

Proof (Sketch). Let $\mathscr{G}$ be a sheaf over Y. Let $\mathscr{F} := \widetilde{f^{-1}}\mathscr{G}$, before the sheafification to obtain $f^{-1}\mathscr{G}$. One can show that the sheafification functor preserves stalks. Now we see that for every $x \in X$ that

$$(f^{-1}\mathscr{G})_x = \mathscr{F}_x = \varinjlim_{x \in U \in \mathscr{O}(X)} \left(\varinjlim_{f(U) \subseteq V} \mathscr{G}(V) \right) \cong \varinjlim_{f(x) \in V \in \mathscr{O}(Y)} \mathscr{G}(V) = \mathscr{G}_{f(x)}$$

Now from this equality we obtain a *Cartesian* diagram in **Top** corresponding the Etale space associated to $f^{-1}\mathscr{G}$:

$$\begin{tikzcd} f^{-1}\mathscr{G} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

which precisely means that $f^{-1}\mathscr{G} \cong f^*\mathscr{G}$. □

Proposition 4.10. *Let $f : X \to Y$ be a continuous map. The inverse image functor $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ preserves all finite limits.*

Proof. Since we have an equivalence of categories between $\mathbf{LH}(X)$ and $\mathbf{Sh}(X)$ and f^* is a pullback via this equivalence it suffices to show that $f^* : \mathbf{LH}(Y) \to \mathbf{LH}(X)$ preserves all finite limits. Note that f^* is just the restriction of the pullback in the slice categories $\mathbf{Top}/X$ and $\mathbf{Top}/Y$ as shown diagrammatically below:

$$\begin{tikzcd} \mathbf{Top}/Y \arrow[r, "f^*"] & \mathbf{Top}/X \\ \mathbf{LH}(Y) \arrow[r, "f^*"'] \arrow[u] & \mathbf{LH}(X) \arrow[u] \end{tikzcd}$$

Note that $f^* : \mathbf{Top}/Y \to \mathbf{Top}/X$ preserves all finite limits since it has a left adjoint Σ_f, where Σ_f is just the functor which composes with f. Now it remains to show that $\mathbf{LH}(X) \subseteq \mathbf{Top}/X$ is closed under finite limits.

Note that if $E \to X$ and $\widetilde{E} \to X$ are both local homeomorphisms into X, then their product $(E \to X) \times (\widetilde{E} \to X)$ in $\mathbf{LH}(X)$ is just the pullback $E \times_X \widetilde{E}$ in **Top**. Furthermore, it is easy to see that this morphism is a local homeomorphism as well. Now we show that the equalizer of two local homeomorphisms is a local homeomorphism as well.

So we see that the inclusion functor $i : \mathbf{LH}(X) \to \mathbf{Top}/X$ preserves binary products, equalizers, and terminal objects. Hence, i preserves all finite limits as desired. □

Definition 4.11. Let X and Y be topological spaces. Let $\mathscr{F} \in \mathbf{Sh}(X)$ and $\mathscr{G} \in \mathbf{Sh}(Y)$, and $f : X \to Y$ be a continuous map. A family of maps $\{\phi_{VU} : \mathscr{G}(V) \to \mathscr{F}(U) : U \in \mathscr{O}(X),\ f(U) \subseteq V \in \mathscr{O}(Y)\}$ is called **compatible** if for every open $U' \subseteq U \subset X$ and $V' \subseteq V \subseteq Y$ such that $f(U) \subseteq V$ and $f(U') \subseteq V$ then the following diagram commutes:

$$\begin{tikzcd} \mathscr{G}(V) \arrow[r, "\phi_{VU}"] \arrow[d, "\mathscr{G}(V'\subseteq V)"'] & \mathscr{F}(U) \arrow[d, "\mathscr{F}(U'\subseteq U)"] \\ \mathscr{G}(V') \arrow[r, "\phi_{V'U'}"'] & \mathscr{F}(U') \end{tikzcd}$$

Let $\hom_{YX}^{(f)}(\mathscr{G}, \mathscr{F})$ be the set of all compatible collections of maps as described above.

Theorem 4.12. *Let $(X, \mathscr{O}(X))$ and $(Y, \mathscr{O}(Y))$ be topological spaces. If $f : X \to Y$ is a continuous map, then there is an associated geometric morphism (f^*, f_*) between $\mathbf{Sh}(X)$ and $\mathbf{Sh}(Y)$.*

Proof. Let $\mathscr{F} \in \mathbf{Sh}(X)$ and $\mathscr{G} \in \mathbf{Sh}(Y)$. We show that

$$\hom_{\mathbf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \hom_{YX}^{(f)}(\mathscr{G}, \mathscr{F}) \cong \hom_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}),$$

where the isomorphisms are natural in $\mathscr{F}$ and $\mathscr{G}$. It suffices to show this for the presheafified $f^{-1}\mathscr{G}$, since the universal property of the sheafification functor is that for any morphism of presheaves $\alpha : \mathscr{A} \to \mathscr{B}$ between a presheaf $\mathscr{A}$ and sheaf $\mathscr{B}$ we have a unique map $\widehat{\alpha}$ between $\mathscr{A}^{\text{sh}}$ and $\mathscr{B}$.

Step 1. *Showing the first natural isomorphism.*

First suppose that we have a sheaf morphism $\phi : \widetilde{f^{-1}\mathscr{G}} \to \mathscr{F}$. Fix an open set $U \subseteq X$. Now for any pair of open sets $U \in \mathscr{O}(X)$ and $V \in \mathscr{O}(Y)$ such that $f(U) \subseteq V$ we simply set $\phi_{VU} = \phi_U \circ i_V$, where i_V is the inclusion into the colimit $\mathscr{G}(V) \to (\widetilde{f^{-1}\mathscr{G}})(U)$. So we have the following commutative diagram:

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[d, "i_V"'] \arrow[dr, "\phi_{VU}"] & \\
(\widetilde{f^{-1}\mathscr{G}})(U) \arrow[r, "\phi_U"'] & \mathscr{F}(U)
\end{tikzcd}$$

Now it immediateley follows that this family of ϕ_{VU} is compatible / commutes with the restriction maps since each of the ϕ_U commute with the restriction maps. Indeed, consider the diagram:

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[rr, "\phi_{VU}"] \arrow[dr, "i_V"'] \arrow[ddd] & & \mathscr{F}(U) \arrow[ddd] \\
& (\widetilde{f^{-1}\mathscr{G}})(U) \arrow[ur, "\phi_U"'] \arrow[d] & \\
& (\widetilde{f^{-1}\mathscr{G}})(U') \arrow[dr, "\phi_{U'}"] & \\
\mathscr{G}(V') \arrow[ur, "i_{V'}"] \arrow[rr, "\phi_{V'U'}"'] & & \mathscr{F}(U')
\end{tikzcd}$$

The upper and lower triangles commute by definition, the right trapezoid commutes since $\phi : f^{-1}\mathscr{G} \to \mathscr{F}$ is a sheaf morphism, and the left trapezoid commutes since $(\widetilde{f^{-1}\mathscr{G}})(U')$ is a colimit.

Conversely, suppose that we have a compatible family of maps $\{\phi_{VU}\}$. Fix an open set $U \subseteq X$. Then consider the directed system $\{\mathscr{G}(V) \,:\, f(U) \subseteq V\}$. In particular, we have the following commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) & \\
\mathscr{G}(V) \arrow[u, "\phi_{VU}"] \arrow[r, "\mathscr{G}(V' \subseteq V)"'] & \mathscr{G}(V') \arrow[ul, "\phi_{V'U}"']
\end{tikzcd}$$

We now compute the directed colimit $(\widetilde{f^{-1}\mathscr{G}})(U)$ and obtain maps ϕ_U as in the following commutative diagram:

$$\begin{tikzcd}
(\widetilde{f^{-1}\mathscr{G}})(U) \arrow[r, dashed, "\phi_U"] & \mathscr{F}(U) \\
\mathscr{G}(V) \arrow[u, "i_V"] \arrow[ur, "\phi_{VU}"' near end] \arrow[r, "\mathscr{G}(V' \subseteq V)"'] & \mathscr{G}(V') \arrow[ul, "i_{V'}" near end] \arrow[u, "\phi_{V'U}"']
\end{tikzcd}$$

It is again easy to see that ϕ_U commute with the restriction maps since the family of maps ϕ_{VU} is compatible. It is clear that these maps are inverses of each other, and so we have established the first bijection. It also immediately follows from the above constructions that this bijection is natural in $\mathscr{F}$ and $\mathscr{G}$ – it is a little bit tedious, but the diagrams all work out.

Step 2. *Showing the second natural isomorphism.*

First suppose that $\phi : \mathscr{G} \to f_*\mathscr{F}$ is a morphism of sheaves in $\mathbf{Sh}(Y)$. Consider $U \in \mathscr{O}(X)$ and $V \in \mathscr{O}(Y)$

such that $f(U) \subseteq V$. Now define $\phi_{VU} = \mathscr{F}(U \subseteq f^{-1}(V)) \circ \phi_V$ as in the following diagram:

$$\left(\begin{tikzcd} \mathscr{G}(V) \arrow[r, "\phi_V"] \arrow[dr, "\phi_{VU}"'] & (f_*\mathscr{F})(V) \arrow[d] \\ & \mathscr{F}(U) \end{tikzcd}\right) = \left(\begin{tikzcd} \mathscr{G}(V) \arrow[r, "\phi_V"] \arrow[dr, "\phi_{VU}"'] & \mathscr{F}(f^{-1}(V)) \arrow[d] \\ & \mathscr{F}(U) \end{tikzcd}\right)$$

It is very similar as in the previous part to show that this defines a compatible family.

Now suppose we are given a compatible family $\{\phi_{VU}\}$ then define the sheaf morphism componentwise as $\phi_V = \phi_{f^{-1}(V)V}$. It is easy to see that this is indeed a natural transformation.

It is also not very difficult to show that this bijection is natural in $\mathscr{F}$ and $\mathscr{G}$.

Since we have both of these natural bijections we see that

$$\hom_{\mathbf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \hom_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}),$$

where this bijection is also natural. So we see that f^{-1} is left adjoint to f_*. It remains to show that f^{-1} is left exact, i.e. preserves all finite limits. However, from Proposition 4.9, we see that $f^{-1} \cong f^*$ and by Proposition 4.10 we see that f^* is left exact, and so therefore so is f^{-1}. So we have constructed a geometric morphism corresponding to $f : X \to Y$. □

It is now natural to ask whether every geoemetric morphism between sheaf categories corresponds to a continuous function. In general, this is not the case. However, if we require that our space be a **sober** topological space this requirement is satisfied.

Definition 4.13. A topological space X is **sober** if all of the points are exactly determined by the topology. One can easily check that that category **Sob** of sober spaces and morphisms of continuous maps is indeed a category.

Remark 4.14. Every Hausdorff space is sober. Furthermore, if X is a sober space, then the topology determines the space X up to isomorphism.

Proposition 4.15. *If X is a topological space and Y is a sober space then geometric morphisms $\boldsymbol{Sh}(X) \to \boldsymbol{Sh}(Y)$ are in bijection with local homeomorphisms $X \to Y$.*

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