Riemannian Geometry and Geometric Analysis by Jürgen Jost: *Solutions*

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Exercise 1. Give five more examples of differentiable manifolds besides those discussed in the text.

1. *The real projective space* $\mathbf{RP}^n$. $\mathbf{RP}^n$ is defined to be the set of 1-dimensional linear subspaces of $\mathbf{R}^{n+1}$. Formally, we say that $v, w \in \mathbf{R}^{n+1} \setminus \{0\}$ are related, $v \sim w$, if and only if $v = tw$ for some $t \in \mathbf{R} \setminus \{0\}$. Clearly $\sim$ is an equivalence relation, and each equivalence class represents a line in $\mathbf{R}^{n+1}$. Now we define

$$\mathbf{RP}^n := (\mathbf{R}^{n+1} \setminus \{0\})/\!\sim .$$

We endow $\mathbf{RP}^n$ with the quotient topology. Let $\pi : \mathbf{R}^{n+1} \setminus \{0\} \to \mathbf{RP}^n$ denote the projection map. Now write $[x^0 : x^1 : \cdots : x^n] = [(x^0, x^1, \ldots, x^n)]_\sim$ for an element in $\mathbf{RP}^n$. Now we define the coordinate charts. Let $V_i := \{\boldsymbol{x} \in \mathbf{R}^{n+1} \setminus \{0\} \,:\, x^i \neq 0\}$ and $\psi : V_i \to \mathbf{R}^n$ be given by

$$\psi((x^0, \ldots, x^n)) = \left(\frac{x^0}{x^i}, \frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^n}{x^i} \right).$$

Note that for $\boldsymbol{v} \sim \boldsymbol{w}$ we have that $\psi_i(\boldsymbol{v}) = \psi_i(\boldsymbol{w})$. Now let $U_i := \pi(V_i)$ $\varphi_i : U_i \to \mathbf{R}^n$ as

$$\varphi_i(\boldsymbol{v}) = \psi_i \circ \pi^{-1}(\boldsymbol{v}),$$

where by an abuse of notation we write $\pi^{-1}(\boldsymbol{v})$ to indicate an arbitrary element in the preimage. φ_i is well defined since ψ does not depend on the choice of representative in the equivalence class. We claim that U_i is open in $\mathbf{RP}^n$. Then $\pi^{-1}(U_i) = \pi^{-1}(\pi(V_i)) = V_i$, which is clearly open in $\mathbf{R}^{n+1}$. Hence U_i is open in the quotient topology in $\mathbf{RP}^n$.

Now we show that φ_i is surjective. Fix $(x^1, \ldots, x^n) \in \mathbf{R}^n$, and note that

$$\varphi_i([x^1 : \cdots : x^{i-1} : 1 : x^i : \cdots : x^n]) = (x^1, \ldots, x^n).$$

Hence $\varphi_i(U_i) = \mathbf{R}^n$.

We claim that φ_i is a homeomorphism. Fix an open set $V \subseteq \mathbf{R}^n$. Since ψ_i is continuously differentiable we deduce that $\psi^{-1}(V)$ is open in $\mathbf{R}^{n+1} \setminus \{0\}$. Now $\pi^{-1} \circ \varphi_i^{-1}(V) = \psi_i^{-1}(V)$, and so $\varphi_i^{-1}(V)$ is open in U_i. This establishes that φ_i is continuous. To see that φ_i is injective consider $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{R}^{n+1}$ satisfying $\varphi_i([\boldsymbol{v}]) = \varphi_i([\boldsymbol{w}])$. Then we see that for all $j = 1, \ldots, i-1, i+1, \ldots, n$ that

$$\frac{v^j}{v^i} = \frac{w^j}{w^i}.$$

Hence $\boldsymbol{w} = \frac{w^i}{v^i}\boldsymbol{v}$, and so $\boldsymbol{w} \sim \boldsymbol{v}$. So φ_i is injective, and we now deduce that φ_i is in fact bijective. In particular, we have that $\varphi_i^{-1} : \mathbf{R}^n \to U_i$ is well defined. In light of this we have

$$\varphi_i^{-1}(x^1, \ldots, x^n) = [x^1 : \cdots , x^{i-1} : 1 : x^i : \cdots : x^n].$$

Now define $f_i : \mathbf{R}^n \to \mathbf{R}^{n+1}$ by

$$f_i(\boldsymbol{x}) = (x^1, \ldots, x^{i-1}, 1, x^i, \ldots, x^n).$$

Note that f_i is $\mathscr{C}^1(\mathbf{R}^n, \mathbf{R}^{n+1})$ and since $\varphi_i^{-1} = \pi \circ f_i$ we see that $\varphi_i^{-1} : \mathbf{R}^n \to \mathbf{RP}^n$ is continuous. We have shown that (U_i, φ_i) is a chart of $\mathbf{RP}^n$.

We claim that $\{(U_i, \varphi_i) \,:\, i = 1, \ldots, n+1\}$ is an atlas for $\mathbf{RP}^n$. First we notice that

$$\mathbf{R}^{n+1} \setminus \{0\} = \bigcup_{i=1}^{n+1} V_i, \text{ and so } \mathbf{RP}^n = \bigcup_{i=1}^{n+1} U_i.$$

Fix $1 \le i < j \le n+1$ and consider $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$. We see that

$$\begin{aligned}\varphi_j(\varphi_i^{-1}(x^1, \ldots, x^n)) &= \varphi_j([x^1 : \cdots : x^{i-1} : 1 : x^{i+1} : \cdots x^n]) \\ &= \left(\frac{x^1}{x^j}, \ldots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \ldots, \frac{x^{i-1}}{x^j}, \frac{1}{x^j}, \frac{x^{i+1}}{x^j}, \ldots, \frac{x^n}{x^j} \right)\end{aligned}$$

Note that all of the components of $\varphi_j \circ \varphi_i^{-1}$ are rational functions with $\mathbf{0} \notin \varphi_i(U_i \cap U_j)$, and so $\varphi_j \circ \varphi_i^{-1}$ is smooth. By an identical argument we see that $\varphi_j \circ \varphi_i^{-1}$ is smooth when $j > i$. So we conclude that $\{(U_i, \varphi_i) : i = 1, \ldots, n+1\}$ is an atlas for $\mathbf{RP}^n$.

Finally, it remains to show that $\mathbf{RP}^n$ is Hausdorff and second countable. To see that $\mathbf{RP}^n$ is second countable consider the maps $\alpha_t : \mathbf{R}^{n+1} \setminus \{0\} \to \mathbf{R}^{n+1} \setminus \{0\}$ given by $\alpha_t(\boldsymbol{x}) = t\boldsymbol{x}$. Now we see that $\alpha_t^{-1} = \alpha_{1/t}$ and α_t are homeomorphisms. Now consider any open set $U \in \mathbf{R}^{n+1} \setminus \{0\}$ and note that $\alpha_t(U)$ is open in $\mathbf{R}^{n+1} \setminus \{0\}$. Then $\pi^{-1}([U]) = \bigcup_{t \in \mathbf{R}} \alpha_t(U)$ is open in $\mathbf{R}^{n+1} \setminus \{0\}$. So $[U]$ is open in $\mathbf{RP}^n$. Now we see that π is an open map. Since $\mathbf{R}^{n+1} \setminus \{0\}$ is second countable and π is an open map we deduce that $\pi(\mathbf{R}^{n+1} \setminus \{0\}) = \mathbf{RP}^n$ is second countable.

To see that $\mathbf{RP}^n$ is Hausdorff it suffices to show that $\{(\boldsymbol{v}, \boldsymbol{w}) : \boldsymbol{v} \sim \boldsymbol{w}\}$ is closed in $\mathbf{R}^{n+1} \setminus \{0\} \times \mathbf{R}^{n+1} \setminus \{0\}$. Consider the map $f : \mathbf{R}^{n+1} \setminus \{0\} \times \mathbf{R}^{n+1} \setminus \{0\} \to \mathbf{R}$ given by

$$f(v^0, \ldots, v^n, w^0, \ldots, w^n) = \sum_{i \neq j} (v^i w^j - w^i v^j)^2.$$

Since f is a polynomial, it is continuous. Suppose that $\boldsymbol{v} = t\boldsymbol{w}$ for some $t \in \mathbf{R} \setminus \{0\}$. Then a direct computation shows that $f(\boldsymbol{v}, \boldsymbol{w}) = 0$. Furthermore, if $f(\boldsymbol{v}, \boldsymbol{w}) = 0$ then we see that

$$v^i w^j - w^i v^j = 0$$

for all $i \neq j$. Since $\boldsymbol{v} \neq 0$ there exists some i_0 such that $v^{i_0} \neq 0$ and so $w^j = \frac{w^{i_0}}{v^{i_0}} v^j$ and so $\boldsymbol{v} \sim \boldsymbol{w}$. So we see that

$$\{(\boldsymbol{v}, \boldsymbol{w}) : \boldsymbol{v} \sim \boldsymbol{w}\} = f^{-1}(\{0\})$$

is closed since $\{0\}$ is closed in $\mathbf{R}$. So we deduce that $\mathbf{RP}^n$ is Hausdorff. Concluding, we see that $\mathbf{RP}^n$ is a smooth manifold of dimension n.

2. *The graph of a smooth function:* Consider a smooth function $f : \mathbf{R}^n \to \mathbf{R}^m$. Consider the graph $\mathrm{graph}(f) := \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m : y = f(x)\}$. Now define a map $F : \mathbf{R}^{n+m} \to \mathbf{R}^m$ given by $F(x, y) = y - f(x)$. Now we see that $\mathrm{graph}(f) = F^{-1}(\{0\})$. A direct computation of the Jacobian of F shows that

   $$DF = \left[\begin{array}{c|c} -Df & \mathbf{1}_{m \times m} \end{array} \right],$$

   To use the implicit function theorem we need to show that 0 is a regular value of F. Note that $\operatorname{rank} DF$ is always maximal since $\operatorname{rank} \mathbf{1}_{m \times m} = m$. Now by the implicit function theorem we deduce that $\mathrm{graph}(f)$ is a smooth parameterizable n-dimensional manifold in a neighborhood of each of it's points.

3. *The general linear group:* Let $\mathbf{GL}(n, \mathbf{R}) := \{A \in \mathbf{R}^{n \times n} : \det(A) \neq 0\}$. Consider the map $\det : \mathbf{GL}(n, \mathbf{R}) \to \mathbf{R}$. Recall that the determinant is a polynomial in the entries of the matrices. So we deduce that $\mathbf{GL}(n, \mathbf{R}) = \det^{-1}(\{0\})$ is an open subset of $\mathbf{R}^{n \times n} \cong \mathbf{R}^{n^2}$. Hence $\mathbf{GL}(n, \mathbf{R})$ inherits the smooth structure of $\mathbf{R}^{n^2}$, and is therefore a smooth submanifold of $\mathbf{R}^{n^2}$.

4. *The connected sum of two manifolds:* Let M and N be smooth manifolds of the same dimension n. Now fix $p \in M$ and $q \in N$ and find open neighborhoods, U and V, of p and q, respectively. By shrinking U and V if necessary we can assume that we have charts $\varphi : U \to B(0, 2)$ and $\psi : V \to B(0, 2)$, where $B(0, 2) \subseteq \mathbf{R}^n$. Now let $\widetilde{U} \subseteq U$ and $\widetilde{V} \subseteq V$ be given by

   $$\widetilde{U} := \varphi^{-1}(\{1/2 < |x| < 2\}) \qquad \widetilde{V} := \psi^{-1}(\{1/2 < |x| < 2\}).$$

For simplicity let $A := \{1/2 < |x| < 2\} \subseteq B(0,2)$. Now consider the map $\alpha : A \to A$ be given by

$$\alpha(x) = \frac{x}{|x|^2}.$$

Note that α simply switches the two boundary components of the annulus A, and reverses the orientation of the radial directions. Now we *glue* $\widetilde{U}$ to $\widetilde{V}$ using map given by $\psi^{-1} \circ \alpha \circ \varphi : \widetilde{U} \to \widetilde{V}$. In this way, we have a new topological space with a natural smooth structure induced from M and N. Up to a diffeomorphism, this new manifold is independent of the choices of local coordinates and is called the connected sum of M and N. We denote this new manifold by $M \# N$.

5. *The stable manifold theorem:* I state the theorem without proof, and compute an explicit example. This example is not to provided to show the technicalities of complicated manifolds, but rather to show an example of manifold theory being used outside of Riemannian geometry.

Theorem 1. *Let $\Omega \subseteq \mathbf{R}^n$ be an open set containing the origin. Let $f \in \mathscr{C}^1(\Omega; \mathbf{R}^n)$, and Φ_t be the flow of the nonlinear system*

$$\dot{x} = f(x).$$

Suppose that $f(0) = 0$ and $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional manifold S tangent to the stable subspace E^s of the linear system

$$\dot{x} = Df(0)x,$$

such that for all $t \geq 0$, $\Phi_t(S) \subseteq S$ and for all $x_0 \in S$

$$\lim_{t\to\infty} \Phi_t(x_0) = 0;$$

there exists an $n-k$-dimension smooth manifold U tangent to the unstable subspace E^u of $\dot{x} = Df(0)x$ such that for all $t < 0$, $\Phi_t(U) \subseteq U$ and for all $x_0 \in U$

$$\lim_{t\to-\infty} \Phi_t(x_0) = 0.$$

Consider the system:

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= x_3 + x_1^2. \end{aligned}$$

We can rewrite this as

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ x_1^2 \\ x_1^2 \end{pmatrix}$$

The flow map is easily seen to be

$$\Phi_t(\boldsymbol{a}) = \begin{pmatrix} a_1 e^{-t} \\ a_2 e^{-t} + a_1^2(e^{-t} + e^{-2t}) \\ a_3 e^t + \frac{a_1^2}{3}(e^t - e^{-2t}) \end{pmatrix},$$

where $\boldsymbol{a} = (a_1, a_2, a_3) = x(0)$. Clearly $\lim_{t\to\infty} \Phi_t(\boldsymbol{a}) = 0$ only if $a_3 = -a_1^2/3$. So we deduce that the stable manifold is

$$S = \left\{ \boldsymbol{a} \in \mathbf{R}^3 \,:\, a_3 = -\frac{a_1^2}{3} \right\}.$$

Similarly, we have that the unstable manifold is

$$U = \left\{ \boldsymbol{a} \in \mathbf{R}^3 \,:\, a_1 = a_2 = 0 \right\}.$$

■

Exercise 2. Determine the tangent space of $\mathbf{S}^n$. (Give a concrete description of the tangent bundle of $\mathbf{S}^n$ as a submanifold of $\mathbf{S}^n \times \mathbf{R}^{n+1}$.)

If $c : I \to \mathbf{S}^n$ is a curve then we have that $\|c\|^2 = 1$ and consequently we have that

$$\dot{c}c = 0.$$

This tells us that the velocity is always perpendicular to the base vector. Hence, the tangent space can be identified as follows

$$T_p\mathbf{S}^n := \{v \in \mathbf{R}^{n+1} \ : \ p \cdot v = 0\}$$

Consequently, the tangent bundle of $\mathbf{S}^n$ is the following:

$$T\mathbf{S}^n \cong \left\{(p, v) \in \mathbf{S}^n \times \mathbf{R}^{n+1} \ : \ \|p\| = 1 \text{ and } p \cdot v = 0\right\}.$$

■

Exercise 3. Let M be a differentiable manifold, $\tau : M \to M$ an involution without fixed points, i.e. $\tau \circ \tau = \mathrm{id}$, $\tau(x) \neq x$ for all $x \in M$. We call x and y equivalent if $y = \tau(x)$. Show that the space M/τ of equivalence classes possesses a unique differentiable structure for which the projection $M \to M/\tau$ is a local diffeomorphism. Discuss the example $M = \mathbf{S}^n \subseteq \mathbf{R}^{n+1}$, $\tau(x) = -x$.

We claim that since τ has no fixed points, for every point $p \in M$ we can find an open set U containing p such that $U \cap \tau(U) = \emptyset$. Assume, for the sake of contradiction, that this is not the case. In particular, there exists some point $p \in M$ such that for all open sets U containing p we have that $U \cap \tau(U) \neq \emptyset$. Now consider a decreasing sequence of connected open sets U_n such that $\bigcap_{n\in\mathbf{N}} \overline{U_n} = \{p\}$. Since M is locally compact without loss of generality we can assume that all of the U_n are precompact. Now we see that

$$U_n \cap \tau(U_n) \neq \emptyset \implies \overline{U_n} \cap \tau(\overline{U_n}) \neq \emptyset.$$

Since compactness is characterized by the finite intersection property we see that

$$\bigcap_{n\in\mathbf{N}} (\overline{U}_n \cap \tau(\overline{U}_n)) \neq \emptyset,$$

and so in particular,

$$\bigcap_{n\in\mathbf{N}} (\overline{U}_n \cap \tau(\overline{U}_n)) = \{p\}.$$

Now we have a contradiction since

$$\tau\left(\bigcap_{n\in\mathbf{N}} (\overline{U}_n \cap \tau(\overline{U}_n))\right) = \bigcap_{n\in\mathbf{N}} (\overline{U}_n \cap \tau(\overline{U}_n)) = \tau(\{p\}) \neq \{p\}.$$

So we have shown that the group action induced by $G = (\{\mathrm{id}, \tau\}, \circ)$ is *properly discontinuous*.

Now for each $p \in M$ choose a parameterization $x : V \to M$ such that $x(V) \subseteq U$, where $U \subseteq M$ is an open set containing p such that $U \cap \tau(U) = \emptyset$. Now we see that $\pi|_U : M \to M/\tau$ is injective, and so the map $y = \pi \circ x : V \to M/\tau$ is injective. Now we see that the family $\{(V, y)\}$ clearly covers M/τ. Now to show that such a collection forms an atlas it suffices to consider two maps $y_1 = \pi \circ x_1 : V_1 \to M/\tau$ and $y_2 = \pi \circ x_2 : V_2 \to M/\tau$ satisfying $y_1(V_1) \cap y_2(V_2) \neq \emptyset$. Let $\pi_i = \pi|_{x_i(V_i)}$ for $i = 1, 2$. Now fix $q \in y_1(V_1) \cap y_2(V_2)$ and let $\widetilde{q} = x_2^{-1} \circ \pi_2^{-1}(q)$. Now let $W \subseteq V_2$ be a neighborhood of $\widetilde{q}$ such that $(\pi_2 \circ x_2)(W) \subseteq y_1(V_1) \cap y_2(V_2)$. Then the restriction to W is given by

$$y_1^{-1} \circ y_2|_W = x_1^{-1} \circ \pi_1^{-1} \circ \pi_2 \circ x_2.$$

So it suffices to show that $\pi_1^{-1} \circ \pi_2$ is smooth at $p_2 = \pi_2^{-1}(q)$. Let $p_1 = \pi_1^{-1} \circ \pi_2(p_2)$. Now we have by definition of the projection that

$$p_1 = \tau(p_2).$$

Now we see that $\pi_1^{-1} \circ \pi_2|_{x_2(W)}$ coincides with $\tau|_{x_2(W)}$, which shows that $\pi_1^{-1} \circ \pi_2$ is smooth at p_2, as desired. Note that $M \to M/\tau$ is a local diffeomorphism by definition of the smooth structure we endowed M/τ.

Now we show the uniqueness of the smooth structure. Suppose that M/τ has two smooth structures $(M/\tau)_1$ and $(M/\tau)_2$ making $\pi : M \to M/\tau$ a local diffeomorphism. The identity map is smooth from $(M/\tau)_1 \to (M/\tau)_2$, and so is it's inverse, which shows that the smooth structures are identical.

The example of $\mathbf{S}^n$ with $\tau(x) = -x$ gives us the real projective space $\mathbf{RP}^n$. We clearly have the same interpretation since the space of 1-dimensional subspaces of $\mathbf{R}^{n+1}$ can be identified with any point on the hemisphere. The smooth structure is identical to that described in Exercise 1. So we see that $\mathbf{RP}^n \cong \mathbf{S}^n/\sim$. ∎

Exercise 4.

(a) Let N be a differentiable manifold, $f : M \to N$ a homeomorphism. Introduce a smooth structure of a differentiable manifold on M such that f becomes a diffeomorphism. Show that such a differentiable structure is unique.

(b) Can the boundary of a cube, i.e. the set $\{x \in \mathbf{R}^n : \max\{|x_i| : i = 1, \dots, n\} = 1\}$ be equipped with a structure of a differentiable manifold.

(a) First we show that such a smooth structure is unique. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are two smooth structures on M making $f : M \to N$ a diffeomorphism. Then we see that the composition

$$(M, \mathscr{A}) \xrightarrow{f} N \xrightarrow{f^{-1}} (M, \mathscr{B})$$

is smooth and the identity. Hence $\mathrm{id} : (M, \mathscr{A}) \to (M, \mathscr{B})$ is a diffeomorphism, and so $\mathscr{A} = \mathscr{B}$.

Now we show the existence of such a smooth structure. Let $\mathscr{A} := \{(U_i, \varphi_i)\}_{i \in I}$ be the smooth structure on N, where $U_i \subseteq N$ and $\varphi_i : U_i \to \mathbf{R}^n$ for some $n \in N$. Now consider the family

$$\widetilde{\mathscr{A}} := \{(f^{-1}(U_i), \varphi_i \circ f)\}_{i \in I}.$$

We claim that $\widetilde{\mathscr{A}}$ is a smooth structure on M making $f : M \to N$ a diffeomorphism. Since f is a homeomorphism we have that $f^{-1}(U_i)$ is open in M for all $i \in I$ and that $\varphi_i \circ f$ is a homeomorphism for all $i \in I$. Note that for any $i, j \in I$ we have

$$(\varphi_i \circ f) \circ (\varphi_j \circ f)^{-1} = \varphi_i \circ (f \circ f^{-1}) \circ \varphi_j^{-1} = \varphi_i \circ \varphi_j^{-1},$$

which is smooth. This shows that $\widetilde{\mathscr{A}}$ is a smooth atlas on M. Now it remains to show that f is a diffeomorphism between $(M, \widetilde{\mathscr{A}})$ and $(N, \mathscr{A})$. This clearly holds by definition since for any $i, j \in I$ we have that

$$\varphi_i \circ f \circ (\varphi_j \circ f)^{-1} = \varphi_i \circ \varphi_j^{-1}$$

is smooth (whenever the map is defined).

(b) The boundary of a cube *can* be equipped with a smooth structure. Note that the n-cube is homeomorphic to $\mathbf{S}^{n-1}$ and so the boundary of the n-cube is homeomorphic to $\partial \mathbf{S}^{n-1}$. Now by Exercise 4(a), we can pullback the smooth structure of the sphere back onto the boundary of the cube, as desired.

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Exercise 5. We equip $\mathbf{R}^{n+1}$ with the inner product

$$\langle x, y\rangle := -x^0y^0 + x^1y^1 + \cdots + x^ny^n$$

for $x = (x^0, x^1, \ldots, x^n)$, $y = (y^0, y^1, \ldots, y^n)$. We put

$$H^n := \{x \in \mathbf{R}^{n+1} \,:\, \langle x, x\rangle = -1,\ x_0 > 0\}.$$

Show that $\langle\cdot,\cdot\rangle$ induces a Riemannian metric on the tangent spaces $T_pH^n \subseteq T_p\mathbf{R}^{n+1}$ for $p \in H^n$. H^n is called hyperbolic space.

We remark that the inner product is called the *Lorentzian inner product* on $\mathbf{R}^{n+1}$, and $(\mathbf{R}^{n+1}, \langle\cdot,\cdot\rangle)$ is often denoted $\mathbf{R}^{1,n}$. This is the *hyperboloid model* of hyperbolic space. It is easy to see geometrically that $H^n \subseteq \mathbf{R}^{n+1}$ is the upper sheet of a hyperboloid.

First we show that H^n is a smooth oriented submanifold of $\mathbf{R}^{n+1}$. Consider the map $h : \mathbf{R}^{n+1} \to \mathbf{R}$ be given by $h(x) = \langle x, x\rangle$. Then it is clear that h is everywhere differentiable and that -1 is a regular value of f. Note that since $f(x+y) = f(x) + f(y) + 2\langle x, y\rangle$ we have

$$df_x(y) = 2\langle x, y\rangle,$$

where we identify T_pH^n with $\mathbf{R}^{n+1}$. Note that $\operatorname{rank} df_x = 1$ if and only if $x \neq 0$. Now we see that

$$T_pH^n \cong \ker df_p = \{q \in \mathbf{R}^{n+1} \,:\, \langle p, q\rangle = 0\}.$$

Now we define $g_p : T_pH^n \times T_pH^n \to \mathbf{R}$ via

$$g_p(v, w) = \langle v, w\rangle.$$

We claim that $g : H^n \to T^2_0H^n$ is a Riemannian metric, i.e. a $(0,2)$-type symmetric positive definite tensor field. Fix $p \in H^n$. The symmetry of g_p is immediate:

$$g_p(v, w) = \langle v, w\rangle = -v^0w^0 + v^1w^1 + \cdots + v^nw^n = \langle w, v\rangle = g_p(w, v).$$

The bilinearity follows immediately from the bilinearity of $\langle\cdot,\cdot\rangle$ (which is elementary to show). Similarly, we see that $g_p(v, v) = 0$ if and only if $v = 0$. Finally, we show that g_p is positive-definite. Write $p = (p_0, \widehat{p}) \in H^n$ and consider $v = (v_0, \widehat{v}) \in T_pH^n$ such that $v \neq 0$. That is to say $\langle p, v\rangle = 0$. Note that if $v_0 = 0$ then $g_p(v, v) = \langle v, v\rangle = v \cdot v \geq 0$. Now consider the case when $x_0 \neq 0$. Then since

$$0 = \langle p, v\rangle = \widehat{p}\cdot\widehat{v} - p_0v_0 \qquad \text{and} \qquad -1 = \langle p, p\rangle = \widehat{p}\cdot\widehat{p} - (p_0)^2$$

we have that

$$(v_0)^2(\widehat{p}\cdot\widehat{p} + 1) = (p_0v_0)^2 = (\widehat{p}\cdot\widehat{v})^2 \leq (\widehat{v}\cdot\widehat{v})(\widehat{p}\cdot\widehat{p})$$

by the Cauchy-Schwartz inequality. So we see that

$$\langle v, v\rangle(\widehat{p}\cdot\widehat{p}) \geq v_0^2,$$

which implies that $\langle v, v\rangle > 0$. So pointwise we have that $g_p : T_pH^n \times T_pH^n \to \mathbf{R}$ is an inner product.

It remains to show that $g \in \Gamma(T^2_0H^n)$. Clearly, $\pi \circ g = \mathrm{id}$, and so we just need to show that g is smooth. This follows since $\langle\cdot,\cdot\rangle$ is smooth in each coordinate and that T_pH^n varies acording to the zero set of $\langle\cdot,\cdot\rangle$. Since this all holds, we see that $\langle\cdot,\cdot\rangle$ induces a Riemannian metric, g, on H^n. ∎

Exercise 6. In the notation of Exercise 5, let

$$s = (-1, 0, \ldots, 0) \in \mathbf{R}^{n+1}$$
$$f(x) = s - \frac{2(x-s)}{\langle x-s, x-s\rangle}.$$

Show that $f : H^n \to \{\xi \in \mathbf{R}^n : |\xi| < 1\}$ is a diffeomorphism (here, $\mathbf{R}^n = \{(0, x^1, \ldots, x^n)\} \subset \mathbf{R}^{n+1}$). Show that in this chart, the metric assumes the form

$$\frac{4}{(1-|\xi|^2)^2} d\xi^i \otimes d\xi^i.$$

First note that for $p \in H^n$ that $f(p)$ is simply the *unique* point in the ball $B := B(0,1) \cap \mathbf{R}^n$ which intersects the line connecting p and s. Write $p = (p^0, \widehat{p})$, and recall that $\langle p, p\rangle = -1$ and $p^0 > 0$. So we see that

$$f(p) = s - \frac{2(p-s)}{\langle p-s, p-s\rangle} = s - \frac{2(p-s)}{-(p^0+1)^2 + \widehat{p}\cdot\widehat{p}} = s + \frac{2(p-s)}{2(p^0+1)} = \frac{p}{p^0+1} + \left(\frac{p^0}{p^0+1}\right)s = \frac{(0,\widehat{p})}{p^0+1}.$$

Now we see that $f(p)$ is clearly smooth as a map $H^n \to \mathbf{R}^n$. We claim that the inverse map is

$$f^{-1}(x) = h(x) = \left(\frac{1+\|x\|^2}{1-\|x\|^2}, \frac{2x^1}{1-\|x\|^2}, \ldots, \frac{2x^n}{1-\|x\|^2}\right).$$

We check for $x \in B$ that

$$(f\circ h)(x) = \frac{(0, 2x^1, \ldots, 2x^n)}{(1-\|x\|^2)\left(1+\frac{1+\|x\|^2}{1-\|x\|^2}\right)} = (0, x^1, \ldots, x^n) = x.$$

Another direct computation shows that $(h\circ f)(p) = p$ for $p \in H^n$. It is clear that $h : B \to H^n$ is smooth, and so f is a diffeomorphism. To compute the metric in these local coordinates, we need to pullback the metric of H^n back onto the unit disk via h.

Write $p = (t, \xi)$ and we see that the metric on H^n is simply

$$g_{H^n} = -dt\otimes dt + \sum_{i=1}^n d\xi^i \otimes d\xi^i.$$

Write $h(x) = (t(x), \xi(x)) \in H^n$, where

$$t(x) = \frac{1+\|x\|^2}{1-\|x\|^2} \qquad \text{and} \qquad \xi(x) = \frac{2x}{1-\|x\|^2}.$$

Now we see (by a slight abuse of notation where we use the variables t and ξ twice) that

$$h^* g_{H^n} = -dt\otimes dt + \sum_{j=1}^n d\xi^j \otimes d\xi^j.$$

A direct computation shows that $d(1-\|x\|^2) = -2\langle x, dx\rangle$, where $\langle x, dx\rangle$ simply means $\sum x^j dx^j$, and so

$$\begin{aligned}
dt &= \frac{2\langle x, dx\rangle}{1-\|x\|^2} + \frac{2(1+\|x\|^2)\langle x, dx\rangle}{(1-\|x\|^2)^2} \\
&= \frac{2\langle x, dx\rangle - 2\|x\|^2\langle x, dx\rangle + 2\langle x, dx\rangle + 2\|x\|^2\langle x, dx\rangle}{(1-\|x\|^2)^2} \\
&= \frac{4}{(1-\|x\|^2)^2}\langle x, dx\rangle, \\
d\xi^j &= \frac{2dx^j}{1-\|x\|^2} + \frac{4x^j\langle x, dx\rangle}{(1-\|x\|^2)^2}.
\end{aligned}$$

We now compute

$$
\begin{aligned}
\sum_{j=1}^{n} d\xi^j \otimes d\xi^j &= \sum_{j=1}^{n}\left(\frac{2dx^j}{1-\|x\|^2}+\frac{4x^j\langle x,dx\rangle}{(1-\|x\|^2)^2}\right)\otimes\left(\frac{2dx^j}{1-\|x\|^2}+\frac{4x^j\langle x,dx\rangle}{(1-\|x\|^2)^2}\right)\\
&= \frac{16\|x\|^2\langle x,dx\rangle\otimes\langle x,dx\rangle}{(1-\|x\|^2)^4}+\frac{16\langle x,dx\rangle\otimes\langle x,dx\rangle}{(1-\|x\|^2)^3}+\frac{4}{(1-\|x\|^2)^2}\sum_{j=1}^{n}dx^j\otimes dx^j\\
&= \frac{16\|x\|^2\langle x,dx\rangle\otimes\langle x,dx\rangle+16\langle x,dx\rangle\otimes\langle x,dx\rangle-16\|x\|^2\langle x,dx\rangle\otimes\langle x,dx\rangle}{(1-\|x\|^2)^4}+\frac{4}{(1-\|x\|^2)^2}\sum_{j=1}^{n}dx^j\otimes dx^j\\
&= \frac{16}{(1-\|x\|^2)^4}\langle x,dx\rangle\otimes\langle x,dx\rangle+\frac{4}{(1-\|x\|^2)^2}\sum_{j=1}^{n}dx^j\otimes dx^j\\
&= dt\otimes dt+\frac{4}{(1-\|x\|^2)^2}\sum_{j=1}^{n}dx^j\otimes dx^j.
\end{aligned}
$$

Now it is clear that

$$h^*g_{H^n}=\frac{4}{(1-\|x\|^2)^2}\sum_{j=1}^{n}dx^j\otimes dx^j+dt\otimes dt-dt\otimes dt=\frac{4}{(1-\|x\|^2)^2}\sum_{j=1}^{n}dx^j\otimes dx^j.$$

By relabeling the x-variables to ξ-variables we have the desired result (in the form stated in the problem). ■

Exercise 7. Determine the geodesics of H^n in the chart given in Exercise 6. (The geodesics through 0 are the easiest ones.)

Fix $p \in H^n$ and $v \in T_pH^n$. Define

$$\gamma_p^v(t) := \begin{cases}\cosh(t\|v\|_p)p+\left(\frac{\sinh(t\|v\|_p)}{\|v\|_p}\right)v & \text{if } v\neq 0,\\ p & \text{if } v=0.\end{cases}$$

Note that $\gamma_p^v(0)=p$. Now suppose that $v\neq 0$. Since $\langle\cdot,\cdot\rangle$ is bilinear we have

$$
\begin{aligned}
\langle\gamma_p^v(t),\gamma_p^v(t)\rangle &= \cosh^2(t\|v\|_p)\langle p,p\rangle+2\cosh(t\|v\|_p)\sinh(t\|v\|_p)\langle p,v\rangle+\sinh^2(t\|v\|_p)\\
&= -\cosh^2(t\|v\|_p)+\sinh^2(t\|v\|_p)\\
&= -1.
\end{aligned}
$$

Hence, $\gamma_p^v:\mathbf{R}\to H^n$. Now we show that γ_p^v is a constant speed curve. A direct computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_p^v(t)=\|v\|_p\sinh(t\|v\|_p)p+\cosh(t\|v\|_p)v.$$

Since the Riemannian metric at every point $q\in H^n$ is simply given by $\langle\cdot,\cdot\rangle$ we have that

$$
\begin{aligned}
\|(\gamma_p^v)'(t)\|^2 &= \langle(\gamma_p^v)'(t),(\gamma_p^v)'(t)\rangle\\
&= \sinh^2(t\|v\|_p)\|v\|_p^2\langle p,p\rangle+\cosh^2(t\|v\|_p)\langle v,v\rangle\\
&= -\sinh^2(t\|v\|_p)\|v\|_p^2+\cosh^2(t\|v\|_p)\|v\|_p^2\\
&= \|v\|_p^2\left(\cosh^2(t\|v\|_p)-\sinh^2(t\|v\|_p)\right)\\
&= \|v\|_p^2.
\end{aligned}
$$

So we see that $\|(\gamma_p^v)'(t)\|_{\gamma_p^v(t)} = \|v\|_p$. Recall that the length functional is given by

$$L(\gamma) := \int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\| \, \mathrm{d}t.$$

Let $z := (1, 0, \dots, 0) \in H^n$. Now fix $t_0 > 0$ and consider any curve $\gamma = (\gamma^0, \dots, \gamma^n)$ connecting z to

$$\gamma(1) := (\cosh t_0, 0, \dots, 0, \sinh t_0).$$

Now using the mapping in Exercise 6, we have that

$$L_{H^n}(\gamma) = L_B(f \circ \gamma) = L_B\left(\frac{\gamma^1}{1+\gamma^0}, \dots, \frac{\gamma^n}{1+\gamma^0} \right).$$

Furthermore, using the result of Exercise 6 we obtain

$$L_B\left(\frac{\gamma^1}{1+\gamma^0}, \dots, \frac{\gamma^n}{1+\gamma^0} \right) \geq L_B\left(\frac{\gamma^1}{1+\gamma^0}, \dots, 0 \right).$$

Now define the curve

$$\alpha(t) := \frac{\gamma^1(t)}{1+\gamma^0(t)}.$$

A computation shows us that

$$\begin{aligned} L_B(\alpha, 0, \dots, 0) &= \int_0^1 \frac{2\|\alpha'(t)|}{1-\alpha(t)^2} \, \mathrm{d}t \geq \int_0^1 \frac{2\alpha'(t)}{1-\alpha(t)^2} \, \mathrm{d}t \\ &= 2\int_0^{\alpha(1)} \frac{1}{1-s^2} \, \mathrm{d}s = 2\tanh^{-1}(\alpha(1)). \end{aligned}$$

Recall that

$$\cosh(2\tanh^{-1}(t)) = \frac{1+t^2}{1-t^2},$$

and

$$(\gamma^0(1))^2 - (\gamma^1(1))^2 = 1.$$

So $\cosh(2\tanh^{-1}(\alpha(1))) = \gamma^0(1) = \cosh(t_0)$, and we deduce that

$$L_{H^n}(\gamma) \geq L_B(\alpha, 0, \dots, 0) \geq t_0.$$

On the other hand, note that the curve $\gamma_z^{(t_0, 0, \dots, 0)}$ connects $\gamma(0)$ to $\gamma(1)$ and that

$$L_{H^n}(\gamma_z^{(t_0,0,\dots,0)}) = t_0.$$

So we see that $\gamma_z^{(t_0,0,\dots,0)}$ is globally length minimizing, and hence a critical point of both the length functional and the energy

$$E(\gamma) := \int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\|^2 \, \mathrm{d}t.$$

Since geodesics are critical points of the energy E, we deduce that $\gamma_z^{(t_0,0,\dots,0)}$ is a geodesic.

Note that rotations around the x^0-coordinate are isometries of H^n. To formalize this, let

$$\mathbf{SO}_{H^n} := \{A \in \mathbf{SO}(n+1, \mathbf{R}) \, : \, A^\top \mathbf{1}_{1,n} A = \mathbf{1}_{1,n}\},$$

where

$$\mathbf{1}_{1,n} := \left(\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathbf{1}_{n\times n} & \\ 0 & & & \end{array}\right).$$

Note that $\mathbf{1}_{1,n}$ is the matrix which corresponds to the quadratic form $\langle \cdot, \cdot \rangle$. Now fix $A \in \mathbf{SO}_{H^n}$ and let γ be a curve in H^n. Now let

$$\gamma_A(t) := \gamma(t)A^\top.$$

Note that $\gamma_A'(t) = \gamma'(t)A^\top$ and that

$$\langle vA^\top, wA^\top \rangle_x = vA^\top \mathbf{1}_{1,n} w^\top = v\mathbf{1}_{1,n} w^\top = \langle v, w \rangle_z,$$

for any $v, w \in T_z H^n$ and $x = zA^\top$. So we see that γ_A is a geodesic if and only if γ is a geodesic. Furthermore, we see that

$$(\gamma_z^v)A^\top = \gamma_{zA^\top}^{vA^\top}.$$

Since $\mathbf{SO}(n)$ acts transitively on $\mathbf{S}^n$ we see that

$$\{(0, t, 0, \dots, 0)A^\top, t \geq 0, A \in \mathrm{Stab}(\mathbf{SO}_{H^n})\} = T_z H^n,$$

in particular, we see that

$$\{\gamma_p^v \,:\, p \in H^n,\ v \in T_p H^n\} = \{\gamma_z^{(0,t_0,\dots,0)} A \,:\, A \in \mathbf{SO}_{H^n}\}.$$

Since $\gamma_z^{(0,t_0,0,\dots,0)}$ is a geodesic we deduce that γ_p^v is a geodesic for any $p \in H^n$ and $v \in T_p H^n$.

Now we verify that these are geodesics by using the Euler-Lagrange equations for the energy. We begin by computing the Christoffel symbols in the x-coordinates introduced in Exercise 6:

$$\begin{aligned}
\Gamma^i_{j,k} &= \frac{1}{2}\left(\frac{(1-\|x\|^2)^2}{4}\right)\delta_i^\ell \left(\frac{16}{(1-\|x\|^2)^3}\right)\left(x^k \delta^j_\ell + x^j \delta^k_\ell - x^\ell \delta^j_k\right) \\
&= \frac{2}{1-\|x\|^2}\sum_{\ell=1}^n \left(x^k \delta_i^\ell \delta_\ell^j + x^j \delta_i^\ell \delta_\ell^k - x^\ell \delta_i^\ell \delta_k^j\right) \\
&= \frac{2}{1-\|x\|^2}\left(x^k \delta_i^j + x^j \delta_i^k - (x^j + x^k)\delta_j^k\right),
\end{aligned}$$

where δ_i^j is the Kronecker delta symbol. Now consider $\widetilde{\gamma}(t) = t(0, 1, 0, \dots, 0) \in B$ for $0 \leq t < 1$. We reparameterize $\widetilde{\gamma}$ to by arc length to obtain

$$\gamma(s) = \widetilde{\gamma}(\alpha(s)) = (p(s), 0, \dots, 0)$$

for some function $\alpha : [0, S] \to [0, 1]$ such that $\|\gamma'(s)\| = 1$ for all $s \in [0, 1)$. We compute

$$\|\gamma'(s)\| = |\alpha'(s)|\langle (0, 1, 0, \dots, 0), (0, 1, 0, \dots, 0)\rangle = \frac{|\alpha'(s)|}{1 - |\alpha(s)|^2}.$$

Hence, if $\|\gamma'(s)\| = 1$ we have that

$$\alpha'(s) = \frac{1 - |\alpha(s)|^2}{2}$$

with $\alpha(0) = 0$. Clearly, we have that the solution is given by

$$\alpha(s) = \frac{e^s - 1}{e^s + 1}.$$

Furthermore, by differentiating the above differential equation once more we find that $\alpha'' = -\alpha\alpha'$. Now we are in a position to compute

$$\begin{aligned}
\nabla_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\left(\dot\alpha\partial_{x_1}\right) \\
&= \dot\gamma(\dot\alpha)\partial_{x_1} + (\dot\alpha)^2\sum_{k=1}^{n}\Gamma_{1,1}^{k}\partial_{x_k} \\
&= \ddot\alpha\partial_{x_1} + 2(\dot\alpha)^2\frac{\alpha}{1-\alpha^2}\partial_{x_1} \\
&= \left(-\alpha\dot\alpha + 2(\dot\alpha)^2\alpha(1-\alpha^2)^{-1}\right)\partial_{x_1} \\
&= \alpha\dot\alpha(-1+2\dot\alpha(1-\alpha^2)^{-1})\partial_{x_1} \\
&= 0.
\end{aligned}$$

Note that since the parameter s goes from 0 to ∞ we see that $L(\gamma)$ with respect to g is infinite. Nevertheless, γ is a geodesic. Note that

$$f(\gamma_z^{(0,1,0,\dots,0)}(t)) = \gamma(t) \in B,$$

and so we have indeed verified that these are geodesics of H^n in these coordinates.

This solution still needs to be cleaned up a little bit! ■

Lemma 2. *Let (M,g) be a Riemannian manifold, $p \in M$, and $f : M \to M$ an isometry such that $df_x(v) = v$ for some $v \in T_pM$. Then for the geodesic $\gamma : [a,b] \to M$ with $\gamma(0) = x$ and $\dot\gamma(0) = v$, we have that $f \circ \gamma = \gamma$.*

Proof. Note that since f is an isometry, we see that $f \circ \gamma$ is also a geodesic. By the conditions on f we see that $(f\circ\gamma)(0) = f(x) = x$ and that $(f\circ\gamma)'(0) = v$. Hence, by the uniqueness of geodesics we have that $f\circ\gamma = \gamma$. □

Exercise 8. Determine the exponential map of the sphere $\mathbf{S}^n$, for example at the north pole p. Write down normal coordinates. Compute the supremum of the radii of balls in $T_p\mathbf{S}^n$ on which $\exp_p$ is injective. Where does $\exp_p$ have maximal rank?

Recall the description of the tangent bundle:

$$T\mathbf{S}^n := \{(p,v) \in \mathbf{S}^n \times \mathbf{R}^{n+1} \; : \; p\cdot v = 0\}.$$

Fix $(p,v) \in T\mathbf{S}^n$. Consider the reflection operator $R_p^v : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ which fixes pointwise the plane spanned by $\{p,v\}$, and reverses all vectors perpendicular to p and v. Note that R_p^v is clearly an isometry from $\mathbf{R}^{n+1}$ to $\mathbf{R}^{n+1}$. In particular, since the standard metric on $\mathbf{S}^n$ is that which is induced from $\mathbf{R}^{n+1}$ we deduce that $R_p^v|_{\mathbf{S}^n} : \mathbf{S}^n \to \mathbf{S}^n$ is an isometry. Furthermore, we see $R_p^v(p) = p$ and that $(dR_p^v)_p(v) = v$.

Now let $\gamma : [a,b] \to \mathbf{S}^n \subseteq \mathbf{R}^{n+1}$ be the geodesic satisfying $\gamma(0) = p$ and $\dot\gamma(0) = v$. By Lemma 2 we see that $R\circ\gamma = \gamma$. In particular, we see that $\gamma([a,b]) \in \mathbf{S}^n \cap \operatorname{span}\{p,v\}$. So we see that the geodesic takes the form

$$\gamma(t) = c(t)p + s(t)\frac{v}{||v||}$$

for some smooth functions $c : [a,b] \to \mathbf{R}$ and $s : [a,b] \to \mathbf{R}$. Now we determine c and s. Note that since $p\cdot v = 0$ and since γ is lies on the sphere we have that

$$||\gamma(t)||^2 = |c(t)|^2 + |s(t)|^2 = 1.$$

So there exists some $r \in \mathbf{R}$ such that for all $t \in [a,b]$ that $c(t) = \cos(rt)$ and $s(t) = \sin(rt)$. Since $\|\dot{\gamma}(t)\| = \|v\|$ we find that $r = \|v\|$. So we see that the geodesic starting from p in direction v is given by

$$\gamma(t) = \cos(t\|v\|)p + \left(\frac{\sin(t\|v\|)}{\|v\|}\right)v.$$

Now we can immediately read off what the exponential map is. We have for *all* nonzero $v \in T_p\mathbf{S}^n$,

$$\exp_p(v) = \gamma_v(1) = \cos(\|v\|)p + \sin(\|v\|)\frac{v}{\|v\|},$$

and $\exp_p(0) = p$.

Let

$$V_p := \{v \in T_pM \,:\, \|v\| < \pi\},$$

and $U_p := \exp_p(V_p) = \mathbf{S}^2 \setminus \{-p\}$. So we can write normal polar coordinates of the sphere as

$$(r, v) \mapsto \exp_p(rv).$$

Now we simply compute in these coordinates

$$g = dr^2 + \sin^2(r)g_{\mathbf{S}^{n-1}}$$

Note that the *injectivity* radius is π. This is clear from the expression of $\exp_p(v)$ since $\exp_p : V_p \to U_p$ is a diffeomorphism, but $\exp_p(\partial \mathbf{S}^n_{T_p\mathbf{S}^n}(0,\pi)) = \{-p\}$. Now we compute the rank of the linear map $(d\exp_p)_v : T_v(T_p\mathbf{S}^n) \to T_{\exp_p(v)}\mathbf{S}^n$ for $v \in T_p\mathbf{S}^n$. Suppose that $v \neq 0$ since we already know that $(d\exp_p)_0 \cong \mathrm{id}$. Identify $w \in T_v(T_p\mathbf{S}^n) \cong T_p\mathbf{S}^n$ with the curve $\alpha(t) = v + tw$; then,

$$\begin{aligned}
(d\exp_p)_v(w) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_p(v+tw) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\cos(\|v+tw\|)p + \sin(\|v+tw\|)\frac{v+tw}{\|v+tw\|}\right) \\
&= -\frac{v\cdot w + t\|w\|^2}{\|v+tw\|}\sin(\|v+tw\|)p + \frac{v\cdot w + t\|w\|^2}{\|v+tw\|^2}\cos(\|v+tw\|)(v+tw) \\
&\quad \left. + \frac{\sin(\|v+tw\|)}{\|v+tw\|}w - \frac{v\cdot w + t\|w\|^2}{\|v+tw\|^3}\sin(\|v+tw\|)(v+tw)\right|_{t=0} \\
&= -\frac{v\cdot w}{\|v\|}\sin\|v\|p + \frac{v\cdot w}{\|v\|^2}\cos\|v\|v + \frac{\sin\|v\|}{\|v\|}w - \frac{v\cdot w}{\|v\|^3}\sin\|v\|v
\end{aligned}$$

Now it is clear that $\operatorname{rank}(d\exp_p)_v = n$ for all $v \in T_p\mathbf{S}^n$ with $\|v\| \neq k\pi$ for $k \in \mathbf{Z}$. Also, we have that $\operatorname{rank}(d\exp_p)_v = 1$ for $v \in T_p\mathbf{S}^n$ with $\|v\| = k\pi$ for $k \in \mathbf{Z}$ since the map above simply reduces to multiplication by a scalar. ■

Definition 3. Let $w_1, \ldots, w_n \in \mathbf{R}^n$ be linearly independent. Consider the equivalence relation, $\sim$, on $\mathbf{R}^n$ where we say that $z_1 \sim z_2$ if there are $m_1, m_2, \ldots, m_n \in \mathbf{Z}$ with

$$z_1 - z_2 = \sum_{i=1}^n m_i w_i.$$

Now define the **flat torus** generated by $\{w_1, \ldots, w_n\}$ to be $\mathbf{T}^n := \mathbf{R}^n/\sim$. We make $\mathbf{T}^n$ a smooth n-manifold as follows: Suppose $U_\alpha \subseteq \mathbf{R}^n$ is open and does not contain any pair of equivalent points. We then put $V_\alpha := \pi(U_\alpha)$ and $\varphi_i\alpha = (\pi|_{U_\alpha})^{-1}$. Then $(V_\alpha, \varphi_\alpha)$ form a smooth atlas on $\mathbf{T}^n$. ⋄

Exercise 9. Same as Exercise 8 for the flat torus generated by $(1,0)$ and $(0,1)$ in $\mathbf{R}^2$.

Let $\mathbf{T}^2$ be the flat torus generated by $(1,0)$ and $(0,1)$. Note that $\mathbf{T}^2$ inherits the Riemannian metric from $\mathbf{R}^2$, and is in particular locally isometric to $\mathbf{R}^2$. A trivial computation shows that the Christoffel symbols are

$$\Gamma^i_{j,k} = 0.$$

Hence, the geodesic equation becomes $\ddot{x}(t) = 0$. Now fix $x \in \mathbf{T}^2$ and then all geodesics are given by

$$\gamma^v_x(t) = (x + tv)/\sim,$$

for $v \in T_x\mathbf{T}^2$. In particular, we have that $\exp_x : T_x\mathbf{T}^2 \cong \mathbf{R}^2 \to \mathbf{T}^2$ is given by

$$\exp_x(v) = [x + v]_\sim.$$

Now we compute

$$(d\exp_x)_v(w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_x(v + tw) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} [x + v + tw] = [w].$$

In particular, we see that $\mathrm{rank}(d\exp_x)_v$ is maximal for all $v \in T_x\mathbf{T}^2$. Finally, we see that the injectivity radius of $\exp_x$ is exactly $1/2$. ∎

Exercise 10. What is the transformation behavior of the Christoffel symbols under coordinate changes? Do they define a tensor?

Let $(x^i : i \in I)$ and $(y^\alpha : \alpha \in \Lambda)$ denote two coordinate systems on some neighborhood of a manifold M. Note that since g is a $(0,2)$-type tensor we have that

$$\widetilde{g}_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} g_{ij}.$$

Similarly, we have that

$$\widetilde{g}^{\alpha\beta} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} g^{ij}.$$

Now we compute

$$\begin{aligned}
\widetilde{g}_{\alpha\beta,\gamma} &= \frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} g_{ij}\right) \\
&= \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma} g_{ij,k} + g_{ij}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\right).
\end{aligned}$$

Now substituting this back into the definition of Christoffel symbols in the x-coordinates gives us that

$$\begin{aligned}
\widetilde{\Gamma}^\gamma_{\alpha\beta} &= \frac{1}{2}\widetilde{g}^{\gamma\delta}(\widetilde{g}_{\alpha\delta,\beta} + \widetilde{g}_{\beta\delta,\alpha} - \widetilde{g}_{\alpha\beta,\delta}) \\
&= \frac{1}{2}\left(\frac{\partial y^\gamma}{\partial x^{i_1}}\frac{\partial y^\delta}{\partial x^{j_1}} g^{i_1 j_1}\right)\Bigg[\left(\frac{\partial x^{i_2}}{\partial y^\alpha}\frac{\partial x^{j_2}}{\partial y^\delta}\frac{\partial x^{k_2}}{\partial y^\beta} g_{i_2 j_2,k_2} + g_{i_2 j_2}\frac{\partial}{\partial y^\beta}\left(\frac{\partial x^{i_2}}{\partial y^\alpha}\frac{\partial x^{j_2}}{\partial y^\delta}\right)\right) \\
&\qquad + \left(\frac{\partial x^{i_3}}{\partial y^\beta}\frac{\partial x^{j_3}}{\partial y^\delta}\frac{\partial x^{k_3}}{\partial y^\alpha} g_{i_3 j_3,k_3} + g_{i_3 j_3}\frac{\partial}{\partial y^\alpha}\left(\frac{\partial x^{i_3}}{\partial y^\beta}\frac{\partial x^{j_3}}{\partial y^\delta}\right)\right) \\
&\qquad - \left(\frac{\partial x^{i_4}}{\partial y^\alpha}\frac{\partial x^{j_4}}{\partial y^\beta}\frac{\partial x^{k_4}}{\partial y^\delta} g_{i_4 j_4,k_4} + g_{i_4 j_4}\frac{\partial}{\partial y^\delta}\left(\frac{\partial x^{i_4}}{\partial y^\alpha}\frac{\partial x^{j_4}}{\partial y^\beta}\right)\right)\Bigg] \\
&= \frac{\partial y^\gamma}{\partial x^i}\left(\Gamma^i_{jk}\frac{\partial x^j}{\partial y^\alpha}\frac{\partial x^k}{\partial y^\beta} + \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta}\right).
\end{aligned}$$

The last equality follows from simple algebraic manipulations.

We now provide another (more elegant) method to derive the transformation behavior. Recall that in local coordinates $x = (x^1, \dots, x^n)$ the Christoffel symbols are given by

$$(\nabla_X Y)_i = XY^i + \Gamma^i_{jk} X^j Y^k.$$

Now consider a coordinate transformation $y = y(x) = (y^1, \dots, y^n)$. Let $\partial_i = \frac{\partial}{\partial x^i}$ and $\widetilde{\partial}_j = \frac{\partial}{\partial y^j}$. We compute

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k.$$

We now see that

$$\begin{aligned}
\nabla_{\widetilde{\partial}_i} \widetilde{\partial}_j &= \widetilde{\Gamma}^k_{ij} \widetilde{\partial}_k \\
&= \frac{\partial x^\alpha}{\partial y^i} \nabla_{\partial_\alpha} \left(\frac{\partial x^\beta}{\partial y^j} \partial_\beta \right) \\
&= \frac{\partial x^\alpha}{\partial y^i} \left(\frac{\partial x^\beta}{\partial y^j} \nabla_{\partial_\alpha} \partial_\beta + \partial_\alpha \left(\frac{\partial x^\beta}{\partial y^j} \right) \partial_\beta \right) \\
&= \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \frac{\partial y^k}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} \partial_\gamma + \frac{\partial^2 x^\beta}{\partial y^i \partial y^j} \partial_\gamma \partial_\beta.
\end{aligned}$$

Now we conclude that

$$\widetilde{\Gamma}^k_{ij} = \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \frac{\partial y^k}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} + \frac{\partial y^k}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial y^i \partial y^j},$$

which is exactly the same result as the first method (up to relabeling).

The important thing to note is that the Christoffel symbols are *not* tensors since their transformation laws are not the same as those of any (r,s)-type tensor. ∎

Exercise 11. Let $c_0, c_1 : [0,1] \to M$ be smooth curves in a Riemannian manifold. If $d(c_0(t), c_1(t)) < i(c_0(t))$ for all t, there exists a smooth map $c : [0,1] \times [0,1] \to M$ with $c(t,0) = c_0(t)$, $c(t,1) = c_1(t)$ for which the curves $c(t,\cdot)$ are geodesics for all t.

Fix $t \in [0,1]$. Note that since $d(c_0(t), c_1(t)) < i(c_0(t))$ we have that the $c_1(t) \in \exp_{c_0(t)}(i(c_0(t)))$. In particular, since $\exp_{c_0(t)}$ is injective on a ball (in the tangent space) with radius $i(c_0(t))$ there exists a unique vector $v_t \in T_{c_0(t)}M$ such that $c_1(t) = \exp_{c_0(t)} v_t$. Now we simply define $c : [0,1] \times [0,1] \to M$ via

$$c(t,s) = \exp_{c_0(t)}(s v_t).$$

Note that

$$c(t,0) = \exp_{c_0(t)}(0 \cdot v_t) = c_0(t) \qquad \text{and} \qquad c(t,1) = \exp_{c_0(t)}(v_t) = c_1(t),$$

by choice of v_t. Furthermore, by definition of the exponential map we see that $s \mapsto c(t,s) = \exp_{c_0(t)}(s v_t)$ is a geodesic for all $t \in [0,1]$. Finally, we have that c is smooth since the exponential map is a smooth function of both it's base point and vector argument. ∎

Exercise 12. Consider the surface S of revolution obtained by rotating the curve $(x, y = e^x, z = 0)$ in the plane, i.e. the graph of the exponential function, about the x-axis in Euclidean 3-space, equipped with the induced Riemannian metric from that Euclidean space. Show that S is complete and compute its injectivity radius.

First we show that S is a closed subset of $\mathbf{R}^3$. Recall that we can parameterize using coordinates in $\mathbf{R}^2$ via the function $\varphi : U \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ given by

$$\varphi(u,v) = (u, e^u \cos(v), e^u \sin(v)),$$

where

$$U := \{(u,v) \in \mathbf{R}^2 \,:\, v \in [0, 2\pi)\}.$$

Note that since $\partial_u(u) + \partial_u(e^u) = 1 + e^u \neq 0$ and since $e^u \neq 0$ we have that $\varphi : U \to \mathbf{R}^3$ is an immersion. To consider the entire surface of revolution we consider the chart $\widetilde{\varphi} : \widetilde{U} \to \mathbf{R}^3$ given by $\varphi(u,v)$, where $\widetilde{U} = \{(u,v) \in \mathbf{R}^2 \,:\, v \in [\pi, 5\pi/2)\}$; the transition functions are simply the identity map. Now consider a sequence $\{p_n\}_{n\in\mathbf{N}} \subseteq S$ that converges (with respect to the Euclidean norm) to some point $y \in \mathbf{R}^3$. Now for every $n \in \mathbf{N}$ let $(u_n, v_n) \in U$ be the coordinates of the point $p_n \in S$. Since p_n converges, we see that $(u_n, v_n) \to (u,v) \in U$. By continuity of φ, we see that $y = \varphi(u,v) \in S$, and so S is closed. In particular, we see that $S \hookrightarrow \mathbf{R}^3$ is a proper embedding.

Let g denote the induced metric on S. We claim that $d_S(p,q) \geq d_{\mathbf{R}^3}(p,q)$ for all $p,q \in S$. Let $\iota : S \hookrightarrow \mathbf{R}^3$ be the inclusion map. If $\gamma : [a,b] \to S$ is a piecewise smooth curve from p to q in S then $\iota \circ \gamma$ is a piecewise smooth curve from p to q in $\mathbf{R}^3$. Since

$$L_g(\gamma) = L_{\mathbf{R}^3}(\iota \circ \gamma) \geq d_{\mathbf{R}^3}(p,q)$$

holds for all such γ, we can take the infimum over all admissible curves on the left hand side to deduce that

$$d_S(p,q) \geq d_{\mathbf{R}^3}(p,q). \tag{1}$$

Now we use this to show that (S, d_S) is complete as a metric space. Let $\{p_n\}_{n\in\mathbf{N}}$ be a Cauchy sequence in S. By (1) we see that $\{p_n\}_{n\in\mathbf{N}}$ is also a Cauchy sequence in $\mathbf{R}^3$ with the usual metric. In particular, $p_n \to x$ for some $p \in \mathbf{R}^3$. Since S is closed, we deduce that $p \in S$, and so (S, d_S) is complete. By the Hopf-Rinow theorem we deduce that (S, g) is geodesically complete.

Now we compute the injectivity radius of S. Visually, we see that S pinches off as $x \to \infty$, and so we should expect the injectivity radius of S to be zero. Note that since S is geodesically complete, the exponential map is defined on all of T_pS for all $p \in S$; therefore, computing the injectivity radius comes down to computing where $\exp_p$ is injective. Recall that $\|w\| = L(\exp_p tw|_{t\in[0,1]})$ for $v \in T_pS$. Now fix a point $x \in \mathbf{R}$ and consider the point $p = (x, e^x, 0) \in S$. Note that in the direction $w = (0,0,1) \in T_pS$ the curve traversed is a circle with radius e^x. So we see that $\iota_S(p) \leq e^x$. Now by considering a sequence $x_n \to -\infty$ we see that

$$0 \leq \iota(S) \leq \lim_{n\to\infty} \iota_S(p_n) = 0.$$

So we see that the injectivity radius is zero. ∎

Exercise 13. Show that the structure group of the tangent bundle of an oriented d-dimensional Riemannian manifold can be reduced to $\mathbf{SO}(d)$.

Let M be a d-dimensional *oriented* Riemannian manifold. First we show that the Riemannian metric allows us to reduce the structure group to $\mathbf{O}(d)$. Let $\{U_\alpha\}$ be an open cover of M which trivializes the tangent bundle TM. In particular, over U_α we have sections $s_1, \ldots, s_d : U_\alpha \to TM$ such that $\{s_i(p) \,:\, i = 1, \ldots, d\}$ is a basis of T_pM for every $p \in U_\alpha$. Now consider a point $p \in U_\alpha \cap U_\beta$ and let $t_1, \ldots, t_d$ be the corresponding sections over the coordinates in U_β. In particular, we have that $\{t_i(p) \,:\, i = 1, \ldots, d\}$ is another basis for T_pM. Now we have a change of basis matrix $g_{\alpha\beta}(p)$ which transforms $\{t_i(p)\}$ to $\{s_i(p)\}$, i.e. $s_i(p) = g_{\alpha\beta}(p)t_i(p)$ for $i = 1, \ldots, d$.

Now we can use the Riemannian metric, which is simply a inner product on T_pM, to apply the Gram-Schmidt process to obtain an orthonormal basis of T_pM. So without loss of generality, we can assume that both $\{s_i(p)\}$ and $\{t_i(p)\}$ are orthonormal basis of T_pM. Now we see that

$$\delta_i^j = \langle s_i(p), s_j(p)\rangle_p = \langle g_{\alpha\beta}(p)t_i(p), g_{\alpha\beta}(p)t_j(p)\rangle_p,$$

where $\langle \cdot, \cdot \rangle_p$ is the Riemannian metric on T_pM. Furthermore, we have that

$$\delta_i^j = \langle t_i(p), t_j(p) \rangle_p,$$

and so we deduce that

$$\langle g_{\alpha\beta}(p) t_i(p), g_{\alpha\beta}(p) t_j(p) \rangle_p = \langle t_i(p), t_j(p) \rangle_p.$$

That is to say that $g_{\alpha\beta}$ preserves the Riemannian metric, and is in particular an element of $\mathbf{O}(d)$.

Now since M is also oriented, we choose that sections corresponding to each atlas in a unique way such that the transition functions satisfy $\det g_{\alpha\beta}(p) > 0$. Since the Gram-Schmidt procedure above preserves the sign of the determinant, we can apply the above process to reduce the structure group of the tangent bundle to $\mathbf{O}(d) \cap \mathbf{GL}^+(d, \mathbf{R}) = \mathbf{SO}(d)$. ∎

Exercise 14. Can one define the normal bundle of a differentiable submanifold of a differentiable manifold in a meaningful way without introducing a Riemannian metric?

Yes, one can define the normal bundle of smooth submanifold of smooth manifold in a meaningful way without introducing a Riemannian metric. More generally, one can define the normal bundle of an immersed submanifold.

Let $i : M \to N$ be an immersion. We now define the **normal bundle** via the short exact sequence

$$0 \to TM \to i^*TN \to N_{N/M} \to 0,$$

where $N_{N/M} := i^*TN/TM$. Here i^*TN is the pullback of the tangent bundle of N back onto M; this is simply the restriction of the tangent bundle of N to $i(M)$ if $M \subseteq N$. Explicitly, if $M \subseteq N$ is an embedded submanifold, then at some point $p \in M$ the fiber of the normal bundle $(N_{N/M})_p$ is simply the quotient vector space T_pN/T_pM. ∎

Exercise 15. Let M be a differentiable submanifold of the Riemannian manifold N. M then receives an induced Riemannian metric, and this metric defines a distance function and a topology on M. Show that this topology coincides with the topology on M that is induced from the topology of N.

Let g denote the Riemannian metric on N. Let $i : M \to N$ be the embedding of M into N. Let $\widetilde{g} := i^*g$ denote the induced Riemannian metric on M. Recall the definition of the distance functions:

$$\begin{aligned} d_N(p, q) &= \inf\{L(\gamma) \,:\, \gamma : [a, b] \to N \text{ piecewise smooth }, \gamma(a) = p, \gamma(b) = q\} \\ d_M(x, y) &= \inf\{L(\alpha) \,:\, \alpha : [a, b] \to M \text{ piecewise smooth }, \alpha(a) = x, \alpha(b) = y\}. \end{aligned}$$

Now consider some local coordinates $(x^1, \ldots, x^m)$ on some open neighborhood $U \subseteq M$ where $m = \dim m$. Now fix some point $p \in U$ and let $0 < \varepsilon < 1$ be small enough such that $B := B_{\mathbf{R}^m}(\boldsymbol{x}(p), \varepsilon) \subseteq U$. Now let $y \in B$ and $v \in \mathbf{R}^m$ and note that

$$\widetilde{g}(v, v) = i^*g_q(v, v) = g_q(di_p(v), di_p(v)),$$

and so there exists a positive constant $\lambda > 0$ such that

$$\frac{1}{\lambda^2}\|di(v)\|^2 \leq \widetilde{g}_{ij}(y)v^iv^j \leq \lambda^2 \|di_p(v)\|^2.$$

Now since $i : M \to N$ is an embedding we have that di is injective for all $p \in M$ and that the topology on $i(M) = M \subseteq N$ coincides with the induced topology of N. Since di_p is nonsingular (and since i is a diffeomorphism), we have that that there exists some constant $\mu > 0$ such that for all point $y \in B$,

$$\frac{1}{\mu^2}\|v\|^2 \leq \|di(v)\|^2 \leq \mu^2 \|v\|^2.$$

So we deduce that
$$\frac{1}{\lambda^2\mu^2}\|v\|^2 \leq \widetilde{g}_{ij}v^iv^j \leq \lambda^2\mu^2\|v\|^2$$
in B. Now fix $x, y \in B$ and let $\gamma : [a,b] \to B$ be a piecewise smooth curve (into the parameterization of M) with $\gamma(a) = x$ and $\gamma(b) = y$. In local coordinates we have that

$$L(\gamma) = \int_a^b \sqrt{\widetilde{g}_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)}\,\mathrm{d}t \geq \frac{1}{\lambda\mu}\int_a^b \|\dot{\gamma}(t)\|\,\mathrm{d}t \geq \frac{1}{\lambda\mu}\|x-y\|.$$

So by taking the infimum over all such admissible curves in the definition of $d(x,y)$ we see that

$$\frac{1}{\lambda\mu}\|x-y\| \leq d(x,y) \leq L(\widetilde{\gamma}) \leq \lambda\mu\|x-y\|,$$

where $\widetilde{\gamma}$ is simply the straight line between x and y in B. Since this holds we have that

$$B_{\widetilde{g}}\left(x, \frac{\delta}{\lambda\mu}\right) \subseteq B_{\mathbf{R}^m}(x,\delta) \subseteq B_{\widetilde{g}}(x, \lambda\mu\delta)$$

for all $\delta \leq \varepsilon/(\mu\lambda)$. In particular, we have that the topology on M coincides with the topology on N. Furthermore, since i is a homeomorphism we conclude by noting that the topology on $M = i(M)$ also coincides with the induced topology from N. ■

Exercise 16. We consider the constant vector field $X(x) = a$ for all $x \in \mathbf{R}^{n+1}$. We obtain a vector field $\widetilde{X}(x)$ on $\mathbf{S}^n$ by projecting $X(x)$ onto $T_x\mathbf{S}^n$ for $x \in \mathbf{S}^n$. Determine the corresponding flow on $\mathbf{S}^n$.

Let $\mathbf{1}_{n+1}$ be the $(n+1)\times(n+1)$ identity matrix. Note that the projection onto the tangent space $T_x\mathbf{S}^n$ is simply given by
$$P_x := \mathbf{1}_{n+1} - xx^\top : \mathbf{R}^{n+1} \to T_x\mathbf{S}^n.$$
Now we see that
$$\widetilde{X}(x) = P_x(X(x)) = P_x(a) = (\mathbf{1}_{n+1} - xx^\top)a = a - xx^\top a = a - \langle a, x\rangle x.$$
For any $p \in \mathbf{S}^n$, the flow associated to $\widetilde{X}$ is given by the solution to the following ODE

$$\begin{cases} \dot{\gamma}(t) = \widetilde{X}(\gamma(t)) \\ \gamma(0) = p. \end{cases}$$

Expanding this out we see that $\gamma : I \to \mathbf{S}^n$ must satisfy $\dot{\gamma}(t) = a - \langle a, \gamma(t)\rangle\gamma(t)$. Note that there are two fixed point solutions given by the initial conditions satisfied by $\gamma_0(t) = \pm a/\|a\|$.

Fix $a \in \mathbf{R}^{n+1}$. By considering an $\mathbf{SO}(n+1,\mathbf{R})$ action on the usual coordinates of $\mathbf{R}^{n+1}$ we can assume that we have Cartesian coordinates such that $a = (a_0, 0, 0, \ldots, 0) \in \mathbf{R}^{n+1}$. We begin by solving the flow map for any initial condition on the circle $C = \{\boldsymbol{x} = (x_0, x_1, 0, \ldots, 0) : x_0^2 + x_1^2 = 1,\ x_0 \geq 0\} \subseteq \mathbf{S}^n$. The ODE above reduces to the following two real valued ODEs:

$$\begin{cases} \dot{\gamma}_0(t) &= a_0 - a_0\gamma_0(t)^2, \\ \dot{\gamma}_1(t) &= -a_0\gamma_0(t)\gamma_1(t). \end{cases}$$

It isn't hard to find that solutions are given by $\gamma_0(t) = \tanh(at - c_0)$ and $\gamma_1(t) = c_1\,\mathrm{sech}(at - c_0)$. Solving for the constants c_0 and c_1 in terms of the initial conditions $\boldsymbol{x} = (x_0, x_1, 0, \ldots, 0) \in C$ we find that $c_0 = -\operatorname{arctanh}(x_0)$ and $c_1 = \frac{x_1}{\sqrt{1-x_0^2}}$.

We can now take the solutions on C and act on them using an arbitrary $\mathbf{SO}(n+1,\mathbf{R})$ group action which preserves the a-axis, i.e. any rotation around a, and we will generate all of the integral curves starting from any initial point on $\mathbf{S}^n$. So given any point $p \in \mathbf{S}^n$ the solution is given up to an $\mathbf{SO}(n+1,\mathbf{R})$ action by the solutions on C. This of course hits all initial points since $\mathbf{SO}(n+1,\mathbf{R})$ acts transitively (and isometrically) on $\mathbf{S}^n$. ∎

Exercise 17. Let $\mathbf{T}$ be the flat torus generated by $(1,0)$ and $(0,1) \in \mathbf{R}^2$, with projection $\pi : \mathbf{R}^2 \to \mathbf{T}$. For which vector fields X on $\mathbf{R}^2$ can one define a vector field $\pi_* X$ on $\mathbf{T}$ in a meaningful way? Determine the flow of $\pi_* X$ on $\mathbf{T}$ for a constant vector field X.

Since $\pi : \mathbf{R}^2 \to \mathbf{T}$ is a local diffeomorphism with $d\pi = \text{id}$ locally, we see that a vector field $X \in \mathfrak{X}(\mathbf{R}^2)$ must be 1-periodic for $\pi_* X \in \mathfrak{X}(\mathbf{T})$ to make sense.

Let $X = (a_1, a_2)$ be a constant vector field on $\mathbf{R}^2$. This induces (via the pullback of π) a constant vector field

$$X|_p = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2}$$

on $\mathbf{T}$, where (x_1, x_2) represent the local coordinates of p. The trajectories in $\mathbf{R}^2$ are simply straight lines in the direction (a_1, a_2). When we factor the flow map of X in $\mathbf{R}^2$ by $\mathbf{Z}^2$ we see that the effect on these straight lines is as follows: when a trajectory hits the upper edge of the square $[0,1]^2$ it continues from the corresponding points on the lower edge; when it hits the right edge it continues from the corresponding point on the left edge, and so on. So the flow map $\Phi : \mathbf{T} \times \mathbf{R} \to \mathbf{T}$ is given by,

$$\Phi((x_1, x_2), t) = (x_1 + a_1 t, x_2 + a_2 t) \mod 1.$$

Note that if (a_1, a_2) is irrational then the flow map is dense in $\mathbf{T}$ for any initial condition. Otherwise, the flow map hits each edge at a finite number of points, and is therefore, in this case, periodic for any initial condition. ∎

Exercise 18. Compute the formula for the Lie derivative (in the direction of a vector field) for a p-times contravariant and q-times covariant tensor.

Fix a smooth vector field X on M, and let Φ denote its flow map. Let $S \in \Gamma(T^s_r M)$ and $T \in \Gamma(T^q_p M)$. Then,

$$(\Phi_t)^*(S \otimes T)_p = ((\Phi_t)^* S)_p \otimes ((\Phi_t)^* T)_p$$

for any point $p \in M$. Now by differentiating at $t = 0$ we have that

$$\mathscr{L}_X(S \otimes T) = \mathscr{L}_X S \otimes T + S \otimes \mathscr{L}_X T.$$

In particular, consider a monomial tensor field written locally as $T = X_1 \otimes \cdots \otimes \cdots X_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s$. Then we see that

$$\mathscr{L}_X T = \sum_{i=1}^{r} X_1 \otimes \cdots \otimes \mathscr{L}_X X_i \otimes \cdots \otimes X_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s + \sum_{j=1}^{s} X_1 \otimes \cdots \otimes X_r \otimes \alpha^1 \otimes \cdots \mathscr{L}_X \alpha^j \otimes \cdots \otimes \alpha^s.$$

Now by considering the (r,s)-type tensor field $\mathscr{L}_X T$ as a multilinear map over the $\mathscr{C}^\infty(M)$ module $\Omega^1(M)^r \times \mathfrak{X}(M)^s$ we see that for any smooth 1-forms ($(0,1)$-type tensors) $\alpha^1, \ldots, \alpha^r$ and smooth vector fields $Y_1, \ldots, Y_s$ that for any (r,s)-type tensor T that

$$(\mathscr{L}_X T)(\alpha^1, \ldots, \alpha^r, Y_1, \ldots, Y_s) = X(T(\alpha^1, \ldots, \alpha^r, Y_1, \ldots, Y_s)) - \sum_{i=1}^{r} T(\alpha^1, \ldots, \mathscr{L}_X \alpha^i, \ldots, \alpha^r, Y_1, \ldots, Y_s)$$
$$- \sum_{j=1}^{s} T(\alpha^1, \ldots, \alpha^r, Y_1, \ldots, \mathscr{L}_X Y_j, \ldots, Y_s).$$

This is fairly easy to compute in coordinates since we know that $\mathscr{L}_X Y_j = [X, Y_j]$ and for any 1-form $\alpha = \alpha_j dx^j$ we have that $\mathscr{L}_X \alpha = \left(\frac{\partial \alpha_j}{\partial x^i} X^i + \frac{\partial X^i}{\partial x^j}\omega_i\right) dx^j$. ■

Theorem 4. *Let $X \in \mathfrak{X}(M)$ be a vector field on M. Then the Lie derivative $\mathscr{L}_X$ is the unique derivation of the tensor algebra $T(V)$ with the following properties:*

- $\mathscr{L}_X f = \langle df, X\rangle = Xf$ *for all* $f \in \mathscr{C}^\infty(M)$,
- $\mathscr{L}_X Y = [X,Y]$ *for all* $X, Y \in \mathfrak{X}(M)$
- $\mathscr{L}_X$ *commutes with the contraction operator* $\operatorname{tr} : T^{s+1}_{r+1}M \to T^s_r M$.

Exercise 19. Show that for arbitrary vector fields X, Y, the Lie derivative satisfies

$$\mathscr{L}_X \circ \mathscr{L}_Y - \mathscr{L}_Y \circ \mathscr{L}_X = \mathscr{L}_{[X,Y]}.$$

First, we show that the Lie bracket of vector fields satisfies the Jacobi identity. We have

$$[[X,Y],Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX.$$

On the other hand, we have that

$$\begin{aligned}[X,[Y,Z]] + [Y,[X,Z]] = XYZ - XZY - YZX + ZYX \\ + YZX - YXZ - ZXY + XZY.\end{aligned}$$

So we have that

$$[[X,Y],Z] = [X,[Y,Z]] + [Y,[Z,X]].$$

Now by using the anticommutativity of the Lie bracket we have the desired Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

Now note that for any vector field $Z \in \mathfrak{X}(M)$ we have that

$$\mathscr{L}_{[X,Y]}Z = [[X,Y],Z] \qquad \text{and} \qquad (\mathscr{L}_X \circ \mathscr{L}_Y)Z - (\mathscr{L}_Y \circ \mathscr{L}_Y)Z = [X,[Y,Z]] - [Y,[X,Z]].$$

So in particular, using the Jacobi identity and the anticommutativity of the Lie bracket we have

$$\begin{aligned}\mathscr{L}_{[X,Y]}Z - (\mathscr{L}_X \circ \mathscr{L}_Y)Z + (\mathscr{L}_Y \circ \mathscr{L}_Y)Z &= [[X,Y],Z] - [X,[Y,Z]] + [Y,[X,Z]] \\ &= [[X,Y],Z] + [[Y,Z],X] - [[X,Z],Y] \\ &= [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \\ &= 0.\end{aligned}$$

So we have that $[\mathscr{L}_X, \mathscr{L}_Y]Z = \mathscr{L}_{[X,Y]}Z$. Now note that $[\mathscr{L}_X, \mathscr{L}_Y]$ is a derivation the tensor algebra since it is the commutator of derivations. Finally, since the contraction commutes with both $\mathscr{L}_X$ and $\mathscr{L}_Y$ we have that it commutes with $\mathscr{L}_{[X,Y]}$. So by Theorem 4 we have that $[\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$ for all (r,s)-type tensor fields. ■

Exercise 20. Prove Corollaries 4.2.3 and 4.2.4 below with the arguments used in the proofs of Theorem 1.4.5 and Corollary 1.4.2.

Corollary 4.2.3 (Gauss Lemma). *Let $p \in M$, $v \in T_pM$, $c(t) := \exp_p tv$ the geodesic with $c(0) = p$, $\dot{c}(0) = v$ ($t \in [0,1]$), assuming that v is contained in the domain of the definition of $\exp_p$. Then for any $w \in T_pM$*

$$\langle v, w\rangle = \langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle,$$

where $(d\exp_p)_v$, the differential of $\exp_p$ at the point v, is applied to the vectors v and w considered as vectors tangent to T_pM at the point v.

Proof. Let $p \in M$ be fixed. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame of the tangent spaces in a neighborhood of p. Note that for every point $x \in \mathrm{im}(\exp_p)$ we can uniquely write

$$x = \exp_p\left(x^i e_i\right).$$

Now we see that the family of functions $\{x^i\}_{i=1}^n$ for a local coordinate system in a neighborhood of p. Clearly, in these coordinates we have that the geodesics are given by $\gamma(t) = tv$ for any $v \in T_pM$. In particular, by taking $v = e_i$ we deduce that

$$g_{ij}(p) = \left\langle \frac{\partial}{\partial x^i}(0), \frac{\partial}{\partial x^j}(0)\right\rangle = \langle e_i, e_j\rangle = \delta_{ij}.$$

This equality only holds at the point $p \in M$. Note that the geodesic equations are given by

$$\Gamma_{ij}^k(\gamma(t))v^i v^j = 0.$$

If we multiply this equation by t^2 we deduce that

$$\Gamma_{ij}^k(\gamma(t))x^i x^j = 0. \tag{2}$$

Since the tangent vector has a constant length along $\gamma(t)$, i.e. $g_{ij}(\gamma(t))v^i v^j = g_{ij}(p)v^i v^j = v^i v^j$, we can multiply the above by t^2 to obtain

$$g_{ij}x^i x^j = x^i x^j$$

along $\gamma(t)$. Now by expanding out the definition of the Christoffel symbols in (2) we deduce that

$$\frac{1}{2}\left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\right)x^i x^j = 0.$$

Equivalently, that is to say

$$\partial_j g_{ik} x^i x^j = \frac{1}{2}\partial_k g_{ij} x^i x^j = \frac{1}{2}\partial_k\left(g_{ij}x^i x^j\right) - g_{kj}x^j = x^k - g_{kj}x^j.$$

Note that on the left hand side we also have

$$\partial_j g_{ik} x^i x^j = \partial_j(g_{ij}x^i)x^j - g_{ik}x^i,$$

and so we deduce $\partial_j(g_{ik}x^i)x^j = x^k$, which implies

$$\partial_j(g_{ik}x^i - x^k)x^j = 0.$$

Hence, along $\gamma(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(g_{ik}x^i - x^k) = 0.$$

Since at p we have $g_{ik}x^i - x^k = 0$ we deduce that in the domain of $\exp_p$ that

$$g_{ik}x^i = x^k.$$

Now the result immediately follows:

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle = g_{ij}(v^1, \ldots, v^n)v^i w^j = \langle v, w\rangle.$$

$\square$

We provide a second proof that is also based on the ideas in Theorem 1.4.5 and Corollary 1.4.2

Proof. Let $s \mapsto v(s)$ be a smooth curve in the domain of $\exp_p$. Now define $\gamma(t,s) := \exp_p(tv(s))$. We regard γ as a variation of the geodesic $\gamma(t,0) = \exp_p(tv(0))$. Now we compute the energy of the geodesic $t \mapsto \gamma(t,s)$ for all s in the domain of v:

$$\begin{aligned} E(\gamma(\cdot,s)) &= \frac{1}{2}\int_0^1 g\left(\frac{\partial}{\partial t}\gamma(t,s), \frac{\partial}{\partial t}\gamma(t,s)\right)\,\mathrm{d}t \\ &= \frac{1}{2} g\left(\frac{\partial \gamma}{\partial t}(0,s), \frac{\partial \gamma}{\partial t}(0,s)\right) \\ &= \frac{1}{2}\|v(s)\|^2. \end{aligned}$$

Now consider the first variation of the energy,

$$\frac{\partial}{\partial s}\bigg|_{s=0} E(\gamma(\cdot,s)) = \int_0^1 0\,\mathrm{d}t + g_{\gamma(1,0)}\left(\partial_t\gamma(t,0), \partial_s\gamma(1,s)\right) - g_{\gamma(0,0)}\left(\partial_t\gamma(0,0), 0\right).$$

From our first computation we note that the energy is constant, and so the first variation is identically zero; hence,

$$0 = g_{\gamma(1,0)}(\partial_t\gamma(1,0), \partial_s\gamma(1,0)),$$

where $\partial_s\gamma(1,0)$ is an arbitrary tangent vector of $\exp_p(\|v(0)\|\partial B(0,1))$. So we have Gauss's lemma for $w \perp v$. Now let $w \in T_pM$ be any vector. Write $w = w^\top + w^\perp$, where $w^\perp$ is orthogonal to v. Now we see from the definition of $\exp_p$ that

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w^\top)\rangle = \langle v, w^\top\rangle.$$

Since we proved the result for $w^\perp$, by the linearity of $d\exp_p$ we are done. □

Corollary 4.2.4. *Let $p \in M$, and let $v \in T_pM$ be contained in the domain of the definition of $\exp_p$, and let $c(t) = \exp_p(tv)$. Let the piecewise smooth curve $\gamma : [0,1] \to T_pM$ be likewise contained in the domain of the definition of $\exp_p$, and assume $\gamma(0) = 0$, $\gamma(1) = v$. Then*

$$\|v\| = L(\exp_p(tv)|_{t\in[0,1]}) \le L(\exp_p \circ \gamma),$$

and equality holds if and only if γ differs from the curve tv, $t \in [0,1]$ only by reparameterization.

Proof. Let $\gamma : [0,1] \to T_pM$ be a smooth curve, whose image is contained in the domain of $\exp_p$, with $\gamma(0) = 0$ and $\gamma(1) = v$. So we see that $\exp_p \circ \gamma$ is a curve connecting the points $c(0)$ and $c(1)$. Now consider the unit normal along $\gamma(t)$,

$$\alpha(t) := \frac{\gamma(t)}{\|\gamma(t)\|}.$$

Now we see that

$$\dot\gamma(t) = \langle \dot\gamma(t), \alpha(t)\rangle\alpha(t) + \beta(t),$$

where $\beta(t) \perp \alpha(t)$ for all $t \in [0,1]$. Now by Gauss's lemma we have

$$\begin{aligned} \dot\gamma(t) &= (d\exp_p)_{\gamma(t)}\dot\gamma(t) \\ &= (d\exp_p)_{\gamma(t)}\left(\langle \dot\gamma(t), \alpha(t)\rangle\alpha(t) + \beta(t)\right) \\ &= \langle \dot\gamma(t), \alpha(t)\rangle (d\exp_p)_{\gamma(t)}\alpha(t) + (d\exp_p)_{\gamma(t)}\beta(t). \end{aligned}$$

Now we can compute the length of $\exp_p \circ \gamma$ as follows:

$$L(\exp_p \circ \gamma) = \int_0^1 \|\dot{\gamma}(t)\| \geq \int_0^1 \|\langle \dot{\gamma}(t), \alpha(t)\rangle (d\exp_p)_{\gamma(t)}\alpha(t)\| \,\mathrm{d}t = \int_0^1 |\langle \dot{\gamma}(t), \alpha(t)\rangle|$$
$$\geq \int_0^1 \langle \dot{\gamma}(t), \alpha(t)\rangle \,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t), \alpha(t)\rangle \,\mathrm{d}t = \langle \gamma(1), \alpha(1)\rangle.$$

In the above we used the fact that $\|\alpha(t)\| = 1$, which implies that $\langle \alpha(t), \dot{\alpha}(t)\rangle = 0$ and in turn $\langle \gamma(t), \dot{\alpha}(t)\rangle = 0$. Now observe $\langle \gamma(1), \alpha(1)\rangle = \|\gamma(1)\|$ is the length of $\exp_p(tv)|_{t\in[0,1]}$. So we have shown

$$\|v\| = L(\exp_p(tv)|_{t\in[0,1]}) \leq L(\exp_p \circ \gamma).$$

Since the length functional is invariant under reparameterizations, we see that if $\gamma(t)$ is a reparameterization of tv then equality holds. On the other hand, if $\gamma(t)$ differs then we see that $\beta(t)$ in the above will be nonzero for some $t \in [0,1]$ which will make the above inequality strict. So we see that the remark about equality is true. $\square$

$\blacksquare$

Exercise 1. Compute the Laplace operator of $\mathbf{S}^n$ on p-forms ($0 \leq p \leq n$) in the coordinates given in Section 1.1.

The coordinates on $\mathbf{S}^n$ given in Section 1.1 are those given by stereographic projection. Explicitly, on $U_1 := \mathbf{S}^n \setminus \{(0, \dots, 0, 1)\}$ we put

$$\begin{aligned} f_1(\xi, s) &:= \left(f_1^1(\xi, s), \dots, f_1^n(\xi, s)\right) \\ &:= \left(\frac{\xi_1}{1-s}, \dots, \frac{\xi_n}{1-s}\right), \end{aligned}$$

and on $U_2 := \mathbf{S}^n \setminus \{(0, \dots, 0, -1)\}$ we put

$$\begin{aligned} f_2(\xi, s) &:= \left(f_2^1(\xi, s), \dots, f_2^n(\xi, s)\right) \\ &:= \left(\frac{\xi_1}{1+s}, \dots, \frac{\xi_n}{1+s}\right), \end{aligned}$$

First we compute the metric tensor in U_1. Note that the inverse of f_1 is $h_1 : \mathbf{R}^n \to \mathbf{S}^n$ given by

$$h_1(\boldsymbol{x}) = (y_1(\boldsymbol{x}), \dots, y_{n+1}(\boldsymbol{x})) = \left(\frac{2\boldsymbol{x}}{\|\boldsymbol{x}\|^2 + 1}, \frac{\|\boldsymbol{x}\|^2 - 1}{\|\boldsymbol{x}\|^2 + 1}\right),$$

where $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$. Now we compute for $i = 1, \dots, n$,

$$dy_i(\boldsymbol{x}) = 2\frac{1 + \|\boldsymbol{x}\|^2 - 2x_i^2}{(1 + \|\boldsymbol{x}\|^2)^2} dx_i - 4\sum_{j \neq i} \frac{x_i x_j}{(1 + \|x\|^2)^2} dx_j.$$

Similarly, we have that

$$dy_{n+1}(\boldsymbol{x}) = \frac{4}{(1 + \|x\|^2)^2} \sum_{i=1}^{n} x_i dx_i.$$

Now we see that since $(dy_i)_{\boldsymbol{x}}(\partial_i)$ forms an orthonormal basis of $T_{h_1(x)}\mathbf{S}^n$ that the induced Euclidean metric on U_1 is simply given by

$$g = \sum_{i=1}^{n+1} dy_i \otimes dy_i = \frac{4}{(1 + \|x\|^2)^2} \sum_{i=1}^{n} dx_i \otimes dx_i.$$

Now we compute the exterior derivative in these x-coordinates. First we see that for $f \in \mathscr{C}^\infty(\mathbf{S}^n)$ that

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i,$$

where of course by $\frac{\partial f}{\partial x_i}$ we mean $\frac{\partial (f \circ h_1)}{\partial x_1}$. Similarly, if we consider a monomial p-form $\alpha = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ we have that

$$d\alpha = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots dx_{i_p}.$$

Now we compute the Laplace-Beltrami operator ($\Delta = dd^* + d^*d$) on functions in these coordinates. We see that

$$\begin{aligned}
\star df &= \star \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \\
&= \sum_{i=1}^{n} (-1)^{i+1} \sqrt{\det(g_{ij})} g^{ii} \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{2}{1+\|x\|^2} \right)^n \left(\frac{1+\|x\|^2}{2} \right)^2 \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{2}{1+\|x\|^2} \right)^{n-2} \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n
\end{aligned}$$

and now

$$\begin{aligned}
d \star df &= d \left(\sum_{i=1}^{n} (-1)^{i+1} \left(\frac{2}{1+\|x\|^2} \right)^{n-2} \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(\frac{2}{1+\|x\|^2} \right)^{n-2} \frac{\partial f}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \left(\left(\frac{2}{1+\|x\|^2} \right)^{n-2} \frac{\partial^2 f}{\partial x_i} - (n-2) x_i \left(\frac{2}{1+\|x\|^2} \right)^{n-1} \frac{\partial f}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n.
\end{aligned}$$

By applying the Hodge star once more, we obtain

$$\begin{aligned}
\star d \star df &= \star \left(\sum_{i=1}^{n} \left(\left(\frac{2}{1+\|x\|^2} \right)^{n} \frac{\partial^2 f}{\partial x_i} - n x_i \left(\frac{2}{1+\|x\|^2} \right)^{n+1} \frac{\partial f}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n \right) \\
&= \left(\frac{1+\|x\|^2}{2} \right)^n \left(\sum_{i=1}^{n} \left(\frac{2}{1+\|x\|^2} \right)^{n-2} \frac{\partial^2 f}{\partial x_i} - (n-2) x_i \left(\frac{2}{1+\|x\|^2} \right)^{n-1} \frac{\partial f}{\partial x_i} \right) \\
&= \sum_{i=1}^{n} \left(\frac{1+\|x\|^2}{2} \right)^2 \frac{\partial^2 f}{\partial x_i} - (n-2) x_i \left(\frac{1+\|x\|^2}{2} \right) \frac{\partial f}{\partial x_i}.
\end{aligned}$$

So we see that the scalar Hodge Laplacian in these stereographic coordinates on the sphere is given by the above:

$$\Delta_0 f = \sum_{i=1}^{n} \left(\frac{1+\|x\|^2}{2} \right)^2 \frac{\partial^2 f}{\partial x_i} - (n-2) x_i \left(\frac{1+\|x\|^2}{2} \right) \frac{\partial f}{\partial x_i}.$$

Now we move onto the more general case of p-forms. First note that since g is conformally equivalent to the standard Euclidean metric $d\boldsymbol{x}$ we see that on p-forms that

$$\star_g = \left(\frac{2}{1+\|x\|^2} \right)^{n-2p} \star_{d\boldsymbol{x}} .$$

Let $\alpha = f dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ be a p-form. Write $I = \{i_1 \leq \cdots \leq i_p\}$ and $I^c := \{1 \leq 2 \leq \cdots \leq n\} \setminus I = \{k_1 \leq \cdots \leq k_r\}$. Now we compute

$$\begin{aligned}
\star d\alpha &= \star \sum_{j \notin I} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots dx_{i_p} \\
&= \sum_{j \notin I} \left(\frac{\partial f}{\partial x_j} \left(\frac{2}{1+\|x\|^2} \right)^{n-2(p+1)} \right) \operatorname{sgn}(\{j, k_1, \ldots, k_s\}) \widehat{dx_j} \wedge dx_{k_1} \wedge \cdots \wedge dx_{k_s},
\end{aligned}$$

$$
\begin{aligned}
d \star d\alpha &= d\left(\sum_{j\notin I}\left(\frac{\partial f}{\partial x_j}\left(\frac{2}{1+\|x\|^2}\right)^{n-2(p+1)}\right)\operatorname{sgn}(\{j,k_1,\dots,k_s\})\widehat{dx_j}\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_s}\right)\\
&=\sum_{j\notin I}\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}\left(\frac{2}{1+\|x\|^2}\right)^{n-2(p+1)}\right)d\boldsymbol{x}_{I^c}\\
&=\sum_{j\notin I}\left(\frac{\partial^2 f}{\partial x_j^2}\left(\frac{2}{1+\|x\|^2}\right)^{n-2(p+1)}-(n-2(p+1))x_j\left(\frac{2}{1+\|x\|^2}\right)^{n-2p-1}\frac{\partial f}{\partial x_j}\right)d\boldsymbol{x}_{I^c}
\end{aligned}
$$

Finally, we compute

$$
\begin{aligned}
\star d \star d\alpha &= \star\left(\sum_{j\notin I}\left(\frac{\partial^2 f}{\partial x_j^2}\left(\frac{2}{1+\|x\|^2}\right)^{n-2(p+1)}-(n-2(p+1))x_j\left(\frac{2}{1+\|x\|^2}\right)^{n-2p-1}\frac{\partial f}{\partial x_j}\right)d\boldsymbol{x}_{I^c}\right)\\
&=\left(\frac{2}{1+\|x\|^2}\right)^{n-2(n-p)}\sum_{j\notin I}\left(\frac{\partial^2 f}{\partial x_j^2}\left(\frac{2}{1+\|x\|^2}\right)^{n-2(p+1)}-(n-2(p+1))x_j\left(\frac{2}{1+\|x\|^2}\right)^{n-2p-1}\frac{\partial f}{\partial x_j}\right)d\boldsymbol{x}_I\\
&=\left(\frac{1+\|x\|^2}{2}\right)^{n-2p-2}\sum_{j\notin I}\left(\frac{\partial^2 f}{\partial x_j^2}\left(\frac{2}{1+\|x\|^2}\right)^{n-2p}-(n-2(p+1))x_j\left(\frac{2}{1+\|x\|^2}\right)^{n-2p-1}\frac{\partial f}{\partial x_j}\right)d\boldsymbol{x}_I\\
&=\sum_{j\notin I}\left(\frac{\partial^2 f}{\partial x_j^2}\left(\frac{1+\|x\|^2}{2}\right)^2-(n-2(p+1))x_j\left(\frac{1+\|x\|^2}{2}\right)\frac{\partial f}{\partial x_j}\right)d\boldsymbol{x}_I
\end{aligned}
$$

Now we see that

$$
\begin{aligned}
\delta d\alpha &= (-1)^{n(p+2)+1}\sum_{j\notin I}\left(\frac{\partial^2 f}{\partial x_j^2}\left(\frac{1+\|x\|^2}{2}\right)^2-(n-2(p+1))x_j\left(\frac{1+\|x\|^2}{2}\right)\frac{\partial f}{\partial x_j}\right)d\boldsymbol{x}_I\\
&=(-1)^{np+1}\sum_{j\notin I}\left(\frac{\partial^2 f}{\partial x_j^2}\left(\frac{1+\|x\|^2}{2}\right)^2-(n-2(p+1))x_j\left(\frac{1+\|x\|^2}{2}\right)\frac{\partial f}{\partial x_j}\right)d\boldsymbol{x}_I.
\end{aligned}
$$

The computations for $d\delta$ are very similar. We compute for $\alpha\in\Omega^p(M)$:

$$
\begin{aligned}
\star\alpha &= \star f\ dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
&=\left(\frac{2}{1+\|x\|^2}\right)^{n-2p} f\ dx_{k_1}\wedge\cdots\wedge dx_{k_r},
\end{aligned}
$$

and

$$
\begin{aligned}
d\star\alpha &= d\left(\left(\frac{2}{1+\|x\|^2}\right)^{n-2p} f\ dx_{k_1}\wedge\cdots\wedge dx_{k_r}\right)\\
&=\sum_{i\in I}\left(\frac{\partial f}{\partial x_i}\left(\frac{2}{1+\|x\|^2}\right)^{n-2p}-(n-2p)x_i\left(\frac{2}{1+\|x\|^2}\right)^{n-2p+1} f\right)dx_i\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_r},
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\star d \star \alpha &= \star\left(\sum_{i\in I}\left(\frac{\partial f}{\partial x_i}\left(\frac{2}{1+\|x\|^2}\right)^{n-2p} - (n-2p)x_i\left(\frac{2}{1+\|x\|^2}\right)^{n-2p+1} f\right) dx_i \wedge dx_{k_1}\wedge\cdots\wedge dx_{k_r}\right)\\
&= \left(\frac{2}{1+\|x\|^2}\right)^{n-2(n-p+1)} \sum_{i\in I}\left(\frac{\partial f}{\partial x_i}\left(\frac{2}{1+\|x\|^2}\right)^{n-2p} - (n-2p)x_i\left(\frac{2}{1+\|x\|^2}\right)^{n-2p+1} f\right)\widehat{dx_i}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
&= \left(\frac{1+\|x\|^2}{2}\right)^{n-2(p-1)} \sum_{i\in I}\left(\frac{\partial f}{\partial x_i}\left(\frac{2}{1+\|x\|^2}\right)^{n-2p} - (n-2p)x_i\left(\frac{2}{1+\|x\|^2}\right)^{n-2p+1} f\right)\widehat{dx_i}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
&= \sum_{i\in I}\left(\frac{\partial f}{\partial x_i}\left(\frac{1+\|x\|^2}{2}\right)^{2} - (n-2p)x_i\left(\frac{1+\|x\|^2}{2}\right) f\right)\widehat{dx_i}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_p}
\end{aligned}
$$

$$
\begin{aligned}
d \star d \star \alpha &= d\left(\sum_{i\in I}\left(\frac{\partial f}{\partial x_i}\left(\frac{1+\|x\|^2}{2}\right)^{2} - (n-2p)x_i\left(\frac{1+\|x\|^2}{2}\right) f\right)\widehat{dx_i}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_p}\right)\\
&= \sum_{i\in I}\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\left(\frac{1+\|x\|^2}{2}\right)^{2} - (n-2p)x_i\left(\frac{1+\|x\|^2}{2}\right) f\right) dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
&= \sum_{i\in I}\left(\frac{\partial^2 f}{\partial x_i^2}\left(\frac{1+\|x\|^2}{2}\right)^{2} + 2\frac{\partial f}{\partial x_i}\left(\frac{1+\|x\|^2}{2}\right) - (n-2p)x_i\left(\frac{1+\|x\|^2}{2}\right)\frac{\partial f}{\partial x_i}\right.\\
&\qquad\left. -(n-2p)\left(\frac{1+\|x\|^2+2x_i^2}{2}\right) f\right) dx_{i_1}\wedge\cdots\wedge dx_{i_p}.
\end{aligned}
$$

So we have shown that

$$
\begin{aligned}
d\delta\alpha &= (-1)^{n(p+1)+1}\sum_{i\in I}\left(\frac{\partial^2 f}{\partial x_i^2}\left(\frac{1+\|x\|^2}{2}\right)^{2} + 2\frac{\partial f}{\partial x_i}\left(\frac{1+\|x\|^2}{2}\right) - (n-2p)x_i\left(\frac{1+\|x\|^2}{2}\right)\frac{\partial f}{\partial x_i}\right.\\
&\qquad\left. -(n-2p)\left(\frac{1+\|x\|^2+2x_i^2}{2}\right) f\right) dx_{i_1}\wedge\cdots\wedge dx_{i_p}.
\end{aligned}
$$

Concluding, we see that the p-form Laplacian on $\mathbf{S}^n$ in stereographic coordinates is given by

$$
\begin{aligned}
\Delta_p\alpha &= (-1)^{np+1}\sum_{j\notin I}\left(\frac{\partial^2 f}{\partial x_j^2}\left(\frac{1+\|x\|^2}{2}\right)^{2} - (n-2(p+1))x_j\left(\frac{1+\|x\|^2}{2}\right)\frac{\partial f}{\partial x_j}\right) d\boldsymbol{x}_I\\
&\quad + (-1)^{n(p+1)+1}\sum_{i\in I}\left(\frac{\partial^2 f}{\partial x_i^2}\left(\frac{1+\|x\|^2}{2}\right)^{2} + 2\frac{\partial f}{\partial x_i}\left(\frac{1+\|x\|^2}{2}\right) - (n-2p)x_i\left(\frac{1+\|x\|^2}{2}\right)\frac{\partial f}{\partial x_i}\right.\\
&\qquad\left. -(n-2p)\left(\frac{1+\|x\|^2+2x_i^2}{2}\right) f\right) d\boldsymbol{x}_I.
\end{aligned}
$$

It is easy to check that when $p = 0$ that this collapses to what we found at the beginning of this exercise.

Note that in these coordinates we have that

$$
\Delta_{p,\mathbf{S}^n}\alpha = \left(\frac{1+\|x\|^2}{2}\right)^2\left(\Delta_{p,\mathbf{R}^n}\alpha - (n-2p)d(\iota_{\nabla\varphi}\alpha) - (n-2p-2)\iota_{\nabla\varphi}d\alpha + 2(n-2p)\nabla\varphi\wedge\iota_{\nabla\varphi}\alpha - 2\nabla\varphi\wedge\delta\alpha\right),
$$

where $\Delta_{p,\mathbf{R}^n}$ is the Hodge Laplacian on p-forms in standard Euclidean coordinates, and

$$
\varphi(\boldsymbol{x}) = \log\left(\frac{2}{1+\|x\|^2}\right).
$$

∎

Exercise 2. Let $\omega \in \Omega^1(\mathbf{S}^2)$ be a 1-form on $\mathbf{S}^2$. Suppose that

$$\varphi^*\omega = \omega$$

for all $\varphi \in \mathbf{SO}(3)$. Show that $\omega \equiv 0$.

Let $\omega \in \Omega^1(\mathbf{S}^2)$ be such that $\varphi^*\omega = \omega$ for all $\varphi \in \mathbf{SO}(3)$.

First, we claim that ω has at least one zero. Assume, for the sake of contradiction, that $\omega_p \neq 0$ for all $p \in M$. Now consider the vector field $\omega^\# \in \mathfrak{X}(\mathbf{S}^2)$; that is if $\omega = \omega_i dx^i$ in local coordinates then $\omega^\# := g^{ij}\omega_i\partial_j$. Note that given this definition we have for any other vector field $X \in \mathfrak{X}(M)$ that

$$\langle \omega^\#, X\rangle = \omega(X).$$

Since ω is nowhere zero, we have that $\omega^\#$ is nowhere zero as well. However, this is a contradiction with the Poincare-Hopf theorem since

$$\sum_{p\in\mathbf{S}^2} \mathrm{ind}_p(\omega^\#) = \chi(\mathbf{S}^2) = 2,$$

and so the sum cannot be empty, i.e. $\omega^\#$ must vanish somewhere on $\mathbf{S}^2$.

Now let $p \in \mathbf{S}^2$ be any point such that $\omega_p = 0$ (at least one such point exists by the above argument). Recall that $\mathbf{SO}(3)$ acts transitively on $\mathbf{S}^2$; that is to say that for every $x, y \in \mathbf{S}^2$ there exists some $A \in \mathbf{SO}(3)$ such that $A \cdot x = y$. Now fix any point $q \in \mathbf{S}^2$ and let $\varphi \in \mathbf{SO}(3)$ be the group action such that $\varphi(q) = p$. Let $X, Y \in \mathfrak{X}(\mathbf{S}^2)$. We compute

$$(\varphi^*\omega)_q(X,Y) = \omega_{\varphi(q)}(d\varphi(X), d\varphi(Y)) = \omega_p(d\varphi(X), d\varphi(Y)) = 0,$$

where in the last step we used that $\omega_p = 0$. The result follows since $\varphi^*\omega = \omega$, and so

$$\omega_q(X,Y) = (\varphi^*\omega)_q(X,Y) = 0.$$

Since $q \in \mathbf{S}^2$ was arbitrary we deduce that ω is identically zero. ∎

Exercise 3. Give a detailed proof of the formula

$$\star\Delta = \Delta\star.$$

Throughout this problem we write δ for d^*, this is both a common symbol for the codifferential and helps differentiate it from the exterior derivative.

Let $1 \le p \le n$. We see that

$$\begin{aligned}
\star_p\Delta_p &= \star_p\delta_{p+1}d_p + \star_p d_{p-1}\delta_p\\
&= (-1)^{n(p+1)+1}\star_p\star_{n-p}d_{n-p-1}\star_{p+1}d_p + (-1)^{n(p+1)+1}\star_p d_{p-1}\star_{n-p+1}d_{n-p}\star_p\\
&= (-1)^{n(n+p)+1}(-1)^{(n-p)p}d_{n-p-1}\star_{p+1}d_p + (-1)^{n(p+1)+1}\star_p d_{p-1}\star_{n-p+1}d_{n-p}\star_p\\
&= (-1)^{n(p+1)+1}\star_p d_{p-1}\star_{n-p+1}d_{n-p}\star_p + (-1)^{n(p+1)+1}d_{n-p+1}\star_{p+1}d_p(-1)^{(n-p)p}\\
&= (-1)^{n(p+1)+1}\star_p d_{p-1}\star_{n-p+1}d_{n-p}\star_p + (-1)^{n(p+1)+1}d_{n-p+1}\star_{p+1}d_p\star_{n-p}\star_p\\
&= \delta_{n-p+1}d_{n-p}\star_p + d_{n-p-1}\delta_{n-p}\star_p\\
&= (\delta_{n-p+1}d_{n-p} + d_{n-p-1}\delta_{n-p})\star_p\\
&= \Delta_{n-p}\star_p.
\end{aligned}$$

∎

Exercise 4. Let M be a two dimensional Riemannian manifold. Let the metric be given by $g_{ij}(x)dx^i \otimes dx^j$ in local coordinates (x^1, x^2). Compute the Laplace operator on 1-forms in these coordinates. Discuss the case where

$$g_{ij}(x) = \lambda^2(x)\delta_{ij}$$

with a positive function $\lambda^2(x)$.

First we compute the Hodge star on all basis forms. We know that

$$\star(1) = \sqrt{\det(g_{ij})}dx^1 \wedge dx^2,$$

and so

$$\star(dx^1 \wedge dx^2) = \frac{1}{\sqrt{\det(g_{ij})}}.$$

Recall the classifying property of the Hodge star: $\omega \wedge \star\eta = g_{\bigwedge^p(T^*M)}(\omega, \eta) \star (1)$, where the metric on $\bigwedge^p(T^*M)$ is induced by $g^{-1} = (g^{ij})$. Hence,

$$dx^1 \wedge \star dx^1 = g^{11}\sqrt{\det(g_{ij})}dx^1 \wedge dx^2, \qquad dx^2 \wedge \star dx^1 = g^{12}\sqrt{\det(g_{ij})}dx^2 \wedge dx^2.$$

So we deduce that

$$\star dx^1 = \sqrt{\det(g_{ij})}\left(g^{11}dx^2 - g^{12}dx^1\right) = \frac{1}{\sqrt{\det(g_{ij})}}\left(g_{12}dx^1 + g_{22}dx^2\right).$$

Similarly, we find that

$$\star dx^2 = \sqrt{\det(g_{ij})}\left(g^{12}dx^2 - g^{22}dx^1\right) = -\frac{1}{\sqrt{\det(g_{ij})}}\left(g_{11}dx^1 + g_{12}dx^2\right).$$

Now we can easily compute the Laplace operator on 1-forms. Consider a simple 1-form $\alpha \in \Omega^1(M)$. In these local coordinates write $\alpha = f dx^1 + h dx^2$, where $f, h \in \mathscr{C}^\infty(M)$. Now we compute

$$\begin{aligned}
\delta d\alpha &= - \star d \star d(f dx^1 + h dx^2) \\
&= - \star d \star \left(\frac{\partial f}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial h}{\partial x^1} dx^1 \wedge dx^2 \right) \\
&= - \star d \star \left(\frac{\partial h}{\partial x^1} - \frac{\partial f}{\partial x^2} \right) dx^1 \wedge dx^2 \\
&= - \star d \left(\frac{1}{\sqrt{\det(g_{ij})}} \left(\frac{\partial h}{\partial x^1} - \frac{\partial f}{\partial x^2} \right) \right) \\
&= - \star \left(\left(\frac{\partial}{\partial x_1} \left(\frac{1}{\sqrt{\det(g_{ij})}} \right) \left(\frac{\partial h}{\partial x^1} - \frac{\partial f}{\partial x^2} \right) + \frac{1}{\sqrt{\det(g_{ij})}} \left(\frac{\partial^2 h}{\partial (x^1)^2} - \frac{\partial^2 f}{\partial x^2 \partial x^1} \right) \right) dx^1 \right. \\
&\qquad \left. + \left(\frac{\partial}{\partial x_2} \left(\frac{1}{\sqrt{\det(g_{ij})}} \right) \left(\frac{\partial h}{\partial x^1} - \frac{\partial f}{\partial x^2} \right) + \frac{1}{\sqrt{\det(g_{ij})}} \left(\frac{\partial^2 h}{\partial x^1 \partial x^2} - \frac{\partial^2 f}{\partial (x^2)^2} \right) \right) dx^2 \right)
\end{aligned}$$

Now expanding this out give

$$\begin{aligned}
\delta d\alpha = -\Bigg[&\sqrt{\det(g_{ij})} g^{11}\left(\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{\det(g_{ij})}}\right)\left(\frac{\partial h}{\partial x^1}-\frac{\partial f}{\partial x^2}\right)+\frac{1}{\sqrt{\det(g_{ij})}}\left(\frac{\partial^2 h}{\partial (x^1)^2}-\frac{\partial^2 f}{\partial x^2\partial x^1}\right)\right)dx^2 \\
&-\sqrt{\det(g_{ij})} g^{12}\left(\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{\det(g_{ij})}}\right)\left(\frac{\partial h}{\partial x^1}-\frac{\partial f}{\partial x^2}\right)+\frac{1}{\sqrt{\det(g_{ij})}}\left(\frac{\partial^2 h}{\partial (x^1)^2}-\frac{\partial^2 f}{\partial x^2\partial x^1}\right)\right)dx^1 \\
&+\sqrt{\det(g_{ij})} g^{12}\left(\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{\det(g_{ij})}}\right)\left(\frac{\partial h}{\partial x^1}-\frac{\partial f}{\partial x^2}\right)+\frac{1}{\sqrt{\det(g_{ij})}}\left(\frac{\partial^2 h}{\partial x^1\partial x^2}-\frac{\partial^2 f}{\partial (x^2)^2}\right)\right)dx^2 \\
&-\sqrt{\det(g_{ij})} g^{22}\left(\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{\det(g_{ij})}}\right)\left(\frac{\partial h}{\partial x^1}-\frac{\partial f}{\partial x^2}\right)+\frac{1}{\sqrt{\det(g_{ij})}}\left(\frac{\partial^2 h}{\partial x^1\partial x^2}-\frac{\partial^2 f}{\partial (x^2)^2}\right)\right)dx^1\Bigg].
\end{aligned}$$

For simplicity we write $|g| := \sqrt{\det(g_{ij})}$. Similarly, we compute

$$\begin{aligned}
d\delta\alpha &= -d\star d\star\alpha = -d\star d\left(\sqrt{\det(g_{ij})}\left(fg^{11}dx^1-(f+h)g^{12}dx^2-hg^{22}dx^1\right)\right) \\
&= d\star\Big[\left(fg^{11}\partial_2|g|+g^{11}|g|\partial_2 f+f|g|\partial_2 g^{11}-hg^{22}\partial_2|g|+g^{22}|g|\partial_2 h+h|g|\partial_2 g^{22}\right)dx^1\wedge dx^2 \\
&\quad+\left((f+h)g^{12}\partial_1|g|+g^{12}|g|(\partial_1 f+\partial_1 h)+(f+h)|g|\partial_1 g^{12}\right)dx^1\wedge dx^2\Big] \\
&= d\Bigg(\frac{1}{|g|}\big(fg^{11}\partial_2|g|+g^{11}|g|\partial_2 f+f|g|\partial_2 g^{11}-hg^{22}\partial_2|g|+g^{22}|g|\partial_2 h+h|g|\partial_2 g^{22} \\
&\qquad+(f+h)g^{12}\partial_1|g|+g^{12}|g|(\partial_1 f+\partial_1 h)+(f+h)|g|\partial_1 g^{12}\big)\Bigg)
\end{aligned}$$

After expanding out this (very long) expression we obtain that that the Hodge Laplacian on 1-forms given in local coordinates is:

$$\begin{aligned}
\Delta_1\alpha = -\frac{1}{(\det(g_{ij}))^2}\Bigg[&\left(g_{12}g_{22}\frac{\partial g_{11}}{\partial x^1}-g_{11}g_{22}\frac{\partial g_{11}}{\partial x^2}-2g_{12}^2\frac{\partial g_{12}}{\partial x^1}+2g_{11}g_{12}\frac{\partial g_{12}}{\partial x^2}+g_{11}g_{12}\frac{\partial g_{22}}{\partial x^1}-g_{11}^2\frac{\partial g_{22}}{\partial x^2}\right)\left(\frac{\partial f}{\partial x^2}-\frac{\partial h}{\partial x^1}\right) \\
&+2\left(g_{12}^3-g_{11}g_{12}g_{22}\right)\frac{\partial^2 f}{\partial x^1\partial x^2}-2\left(g_{11}g_{12}^2-g_{11}^2g_{22}\right)\frac{\partial^2 f}{\partial (x^2)^2} \\
&-2\left(g_{12}^3-g_{11}g_{12}g_{22}\right)\frac{\partial^2 h}{\partial (x^1)^2}+2\left(g_{11}g_{12}^2-g_{11}^2g_{22}\right)\frac{\partial^2 h}{\partial x^1\partial x^2}\Bigg]\,dx^1 \\
-\frac{1}{(\det(g_{ij}))^2}\Bigg[&\left(g_{22}^2\frac{\partial g_{11}}{\partial x^1}-g_{12}g_{22}\frac{\partial g_{11}}{\partial x^2}-2g_{12}g_{22}\frac{\partial g_{12}}{\partial x^1}+2g_{12}^2\frac{\partial g_{12}}{\partial x^2}+g_{11}g_{22}\frac{\partial g_{22}}{\partial x^1}-g_{11}g_{12}\frac{\partial g_{22}}{\partial x^2}\right)\left(\frac{\partial f}{\partial x^2}-\frac{\partial h}{\partial x^1}\right) \\
&+2\left(g_{12}^2g_{22}-g_{11}g_{22}^2\right)\frac{\partial^2 f}{\partial x^1\partial x^2}-2\left(g_{12}^3-g_{11}g_{12}g_{22}\right)\frac{\partial^2 f}{\partial (x^2)^2} \\
&-2\left(g_{12}^2g_{22}-g_{11}g_{22}^2\right)\frac{\partial^2 h}{\partial (x^1)^2}+2\left(g_{12}^3-g_{11}g_{12}g_{22}\right)\frac{\partial^2 h}{\partial x^1\partial x^2}\Bigg]\,dx^2
\end{aligned}$$

Now we specialize to the case when the metric is conformally equivalent to the Euclidean metric, i.e.

$$g_{ij}(x) = \lambda^2(x)\delta_{ij},$$

where λ^2 is a sufficiently smooth positive function. We have that all of the diagonal terms drop out since $g_{12} = g_{21} = 0$. Furthermore, we see that

$$\det(g_{ij}) = (\lambda^2)^2.$$

Just by removing all of the nonzero terms we have

$$\Delta_1\alpha = -\frac{1}{\lambda^4}\left[\left(-g_{11}g_{22}\frac{\partial g_{11}}{\partial x^2} - g_{11}^2\frac{\partial g_{22}}{\partial x^2}\right)\left(\frac{\partial f}{\partial x^2} - \frac{\partial h}{\partial x^1}\right) + 2g_{11}^2 g_{22}\frac{\partial^2 f}{\partial (x^2)^2} - 2g_{11}^2 g_{22}\frac{\partial^2 h}{\partial x^1 \partial x^2}\right] dx^1$$
$$- \frac{1}{\lambda^4}\left[\left(g_{22}^2\frac{\partial g_{11}}{\partial x^1} + g_{11}g_{22}\frac{\partial g_{22}}{\partial x^1}\right)\left(\frac{\partial f}{\partial x^2} - \frac{\partial h}{\partial x^1}\right) - 2g_{11}g_{22}^2\frac{\partial^2 f}{\partial x^1 \partial x^2} + 2g_{11}g_{22}^1\frac{\partial^2 h}{\partial (x^1)^2}\right] dx^2.$$

By using the fact that $g_{11} = g_{22} = \lambda^2$ we further simplify to see that

$$\Delta_1\alpha = \frac{4}{\lambda^3}\left[\frac{\partial \lambda}{\partial x^2}\left(\frac{\partial f}{\partial x^2} - \frac{\partial h}{\partial x^2}\right) - \frac{\lambda}{2}\left(\frac{\partial^2 f}{\partial (x^2)^2} - \frac{\partial^2 h}{\partial x^1 \partial x^2}\right)\right] dx^1$$
$$+ \frac{4}{\lambda^3}\left[\frac{\partial \lambda}{\partial x^1}\left(\frac{\partial h}{\partial x^1} - \frac{\partial f}{\partial x^2}\right) + \frac{\lambda}{2}\left(\frac{\partial^2 f}{\partial x^1 \partial x^2} - \frac{\partial^2 h}{\partial (x^1)^2}\right)\right] dx^2.$$

∎

Exercise 5. Suppose that $\alpha \in H_p^{1,2}(M)$ satisfies

$$(d^*\alpha, d^*\varphi) + (d\alpha, d\varphi) = (\eta, \varphi) \quad \text{for all} \quad \varphi \in \Omega^p(M),$$

with some given $\eta \in \Omega^p(M)$. Show that $\alpha \in \Omega^p(M)$, i.e. smoothness of α.

Note that since d and d^* are formal adjoints over $H_p^{1,2}(M)$ this question says that α is a weak-solution to the p-form Laplace equation, i.e. $\Delta_p\alpha = \eta$ in the sense of distributions, where $\alpha \in H_p^{1,2}(M)$.

We prove the following elliptic regularity estimate:

Theorem 5. *Let $f \in H^{k-1}(M)$, and $u \in H^1(M)$ a weak solution to $Lu = f$, where $L = \Delta + X$ for some $X \in \mathbf{PDO}_1(M)$. Then $u \in H^{k+1}(M)$, and*

$$\|u\|_{H^{k+1}}^2 \le C\|\Delta u\|_{H^{k-1}}^2 + C\|u\|_{H^k}^2, \tag{3}$$

for all $u \in H^{k+1}(M) \cap H^1(M)$.

Proof. First we prove (3) for $k = 0$. Note that for $u \in H^1(M)$ that $(\Delta u, u) \ge C\|u\|_{H^1(M)}^2$ for some constant only depending on M. We also have

$$|(Xu, u)| \le C\|u\|_{H^1}\|u\|_{L^2} \le \frac{C}{2}\left(\varepsilon\|u\|_{H^1}^2 + \frac{1}{\varepsilon}\|u\|_{L^2}^2\right).$$

So we have that

$$(Lu, u) \ge C\|u\|_{H^1}^2 - \widetilde{C}\|u\|_{L^2}^2, \qquad \text{for } u \in H^1(M).$$

Hence,

$$\|u\|_{H^1}^2 \le C(Lu, u) + \widetilde{C}\|u\|_{L^2}^2.$$

By Cauchy's inequality we have

$$(Lu, u) \le C\|Lu\|_{H^{-1}}\|u\|_{H^1} \le C\varepsilon\|u\|_{H^1}^2 + \frac{C}{\varepsilon}\|\Delta u\|_{H^{-1}}^2.$$

Now by taking ε small enough, we can absorb the $\|u\|_{H^1}^2$ term to obtain

$$\|u\|_{H^1}^2 \le C\|Lu\|_{H^{-1}}^2 + C\|u\|_{L^2}^2.$$

Now we prove the result by induction on k. Given that $u \in H^1(M)$ and $Lu = f \in H^{k-1}(M)$ implies $u \in H^{k+1}(M)$ and that (3) holds, suppose that $u \in H^1(M)$ and $\Delta u \in H^k(M)$. We already know that $u \in H^{k+1}(M)$, and we want to prove that $u \in H^{k+2}(M)$. First note that for any $\varphi \in \mathscr{C}^\infty(M)$,

$$L(\varphi u) = \varphi(Lu) + [L, \varphi]u,$$

since the commutator $[L, \varphi]$ is a first-order differential operator, the inductive hypothesis together with the observation that $u \in H^{k+1}(M)$ implies $L(\varphi u) \in H^k(M)$. So we can localize our analysis.

Suppose that $u \in H^{k+1}(M)$ satisfies $Lu = f \in H^k(M)$ is supported on a coordinate neighborhood U. Now we can apply (3) with u replaced by

$$\delta_{j,h}u(x) = \frac{1}{h}\left(\tau_{j,h}u(x) - u(x)\right) = \frac{1}{h}\left(u(x + he_j) - u(x)\right),$$

where e_i are the standard coordinate vectors in $\mathbf{R}^n$. Now take any $1 \le j \le n$, and we have

$$\|\delta_{j,h}u\|^2_{H^{k+1}} \le C\|L\delta_{j,h}u\|^2_{H^{k-1}} + C\|u\|^2_{H^{k+1}} \le C\|\delta_{j,h}Lu\|^2_{H^{k-1}} + C\|[L, \delta_{j,h}]u\|^2_{H^{k-1}} + C\|u\|^2_{H^{k+1}}.$$

Now we estimate the commutator in the above inequality. For a function $a \in \mathscr{C}^\infty(M)$ consider the multiplication operator $M_a : H^s(M) \to H^s(M)$ given by $M_a\varphi(x) = a(x)f(x)$. Now we see that

$$[M_\varphi, \delta_{j,h}]v = -M_{(\delta_{j,h}\varphi)} \circ \tau_{j,h}v,$$

so in turn

$$\|[M_\varphi, \delta_{j,h}]v\|_{H^k} \le C\|v\|_{H^k},$$

in turn

$$\|[L, \delta_{j,h}]u\|_{H^{k-1}} \le C\|u\|_{H^{k+1}}.$$

Using this inequality, we deduce that

$$\|\delta_{j,h}u\|^2_{H^{k+1}} \le C\|Lu\|^2_{H^k} + C\|u\|^2_{H^{k+1}}.$$

Passing to the limit as $h \to 0$ gives

$$\frac{\partial u}{\partial x_j} \in H^{k+1}(M).$$

Since this holds for all $1 \le j \le n$ we have that $u \in H^{k+2}(M)$. So the desired result is shown. $\square$

Using the above theorem, we immediately deduce (using a bootstrapping procedure) the regularity when $p = 0$, i.e. when we are dealing with the scalar Laplacian on M.

To use the above result for the Hodge Laplacian acting on p-forms, we first establish a coercivity type condition on Δ and prove a decomposition which will allow us to use the above elliptic regularity proof.

In local coordinates, we can write the Hodge Laplacian on p-forms as

$$\Delta\eta = g^{j\ell}(x)\partial_j\partial_\ell\eta + Y_j\eta, \tag{4}$$

where Y_k are first order differential operators. This decomposition follows since the symbol of the Hodge Laplacian is given by $\sigma_\Delta(x, \xi) = \|\xi\|^2\,\mathrm{id}$. Now let η be a p-form in $H^1(M; \bigwedge^p M)$. Cover M with coordinate patches U_j, and let $\varphi_j \in \mathscr{C}_0^\infty(U_j)$ such that $\sum \varphi_j^2 = 1$. So

$$(\Delta\eta, \eta) = \sum_j (\Delta(\varphi_j^2\eta), \eta) = \sum_j (\Delta(\varphi_j\eta), \varphi_j\eta) + (Y\eta, \eta),$$

where Y is a first order differential operator given by $Y = \sum[\Delta, \varphi_j]$. The local coordinate expression (4) and integration by parts yields

$$(\Delta(\varphi_j \eta), \varphi_j \eta) \geq C\|\varphi_j \eta\|_{H^1}^2 - \widetilde{C}\|\varphi_j \eta\|_{L^2}^2.$$

Summing this inequality gives

$$(\Delta \eta, \eta) \geq C_2\|\eta\|_{H^1}^2 - C_3\|\eta\|_{L^2}^2 - C_4\|Yu\|_{L^2}\|u\|_{L^2}.$$

Now the product in the last term is dominated by $\varepsilon\|u\|_{H^1}^2 + (C/\varepsilon)\|u\|_{L^2}^2$, and so we can absorb $\varepsilon\|u\|_{H^1}^2$ into the first term on the right hand side to obtain

$$(\Delta \eta, \eta) \geq C_0\|\eta\|_{H^1}^2 - C_1\|\eta\|_{L^2}^2, \qquad \eta \in \Omega^p(M). \tag{5}$$

Now since we have this estimate, the proof of Theorem 5 follows through. The only thing to note is that nothing changes if the first order differential operator X is matrix valued in local coordinates. So we see that the Laplacian forces an elliptic regularity on sections between vector bundles as well. ∎

Exercise 6. Compute a relation between the Laplace operator on functions on $\mathbf{R}^{n+1}$ and the one on $\mathbf{S}^n \subseteq \mathbf{R}^{n+1}$.

First we compute the metric of $\mathbf{R}^{n+1}$ (more precisely the induced metric on $\mathbf{R}^{n+1} \setminus \{0\}$) in spherical coordinates. Let $(s_1, \dots, s_n)$ be any set of coordinates on $\mathbf{S}^n$. We then obtain a parameterization of $\mathbf{R}^{n+1} \setminus \{0\}$ over $\mathbf{R}^+ \times \mathbf{S}^n$ via the map:

$$(r, \sigma) \mapsto \boldsymbol{x}(r, \sigma) := r\sigma.$$

Let $G_{\mathbf{S}^n}$ be the $n \times n$ matrix representing the metric on $\mathbf{S}^n$ in the coordinates $(s_1, \dots, s_n)$. Note that if $\sigma \in \mathbf{S}^n$ then $\sigma \cdot \sigma = 1$ and so,

$$\frac{\partial \sigma}{\partial s_i} \cdot \sigma = \frac{1}{2}\frac{\partial}{\partial s_i}(\sigma \cdot \sigma) = 0.$$

Note that $\partial_{s_i}\sigma$ form a basis of $T_p\mathbf{S}^n$. Now we compute for $1 \leq i \leq n$,

$$\frac{\partial \boldsymbol{x}}{\partial s_i} = r\frac{\partial \sigma}{\partial s_i}, \qquad \text{and} \qquad \frac{\partial \boldsymbol{x}}{\partial r} = \sigma.$$

Since the partial derivatives of $\boldsymbol{x}$ form a basis of the tangents space of $\mathbf{R}^{n+1}$ we find that in these coordinates we have

$$g_{ij} = \frac{\partial \boldsymbol{x}}{\partial s_i} \cdot \frac{\partial \boldsymbol{x}}{\partial s_j} = r^2 \frac{\partial \sigma}{\partial s_i} \cdot \frac{\partial \sigma}{\partial s_j} = r^2 g_{ij}^{\mathbf{S}^n}.$$

We also have that

$$g_{ir} = \frac{\partial \boldsymbol{x}}{\partial s_i} \cdot \frac{\partial \boldsymbol{x}}{\partial r} = r\frac{\partial \sigma}{\partial s_i} \cdot \sigma = 0,$$

and

$$g_{rr} = \frac{\partial \boldsymbol{x}}{\partial r} \cdot \frac{\partial \boldsymbol{x}}{\partial r} = \sigma \cdot \sigma = 1.$$

So we see that the metric is given by

$$G_{\mathbf{R}^{n+1}} = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & r^2 G_{\mathbf{S}^n} & \\ 0 & & & \end{array}\right)$$

It should now be clear (in light of the Laplace expansion and multilinearity of the determinant) that

$$\det(g_{ij}) = r^{2n} \det(g_{ij}^{\mathbf{S}^n}).$$

Now since the Hodge Laplacian on functions, given by $\Delta = -\delta d$, in coordinates is

$$\Delta f = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{n+1} \frac{\partial}{\partial x_i}\left(\sqrt{\det(g_{ij})} g^{ij} \frac{\partial f}{\partial x_j}\right),$$

we immediately compute for smooth functions $f \in \mathscr{C}^\infty(M)$:

$$\begin{aligned}
\Delta_{\mathbf{R}^n} f &= \frac{1}{r^n \sqrt{\det(g_{ij}^{\mathbf{S}^n})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(r^n \sqrt{\det(g_{ij}^{\mathbf{S}^n})} \frac{g_{\mathbf{S}^n}^{ij}}{r^2} \frac{\partial f}{\partial x_j}\right) + \frac{1}{r^n \sqrt{\det(g_{ij}^{\mathbf{S}^n})}} \frac{\partial}{\partial r}\left(r^n \sqrt{\det(g_{ij}^{\mathbf{S}^n})} \frac{\partial f}{\partial r}\right) \\
&= \frac{1}{r^2 \sqrt{\det(g_{ij}^{\mathbf{S}^n})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det(g_{ij}^{\mathbf{S}^n})} g_{\mathbf{S}^n}^{ij} \frac{\partial f}{\partial x_j}\right) + \frac{1}{r^n} \frac{\partial}{\partial r}\left(r^n \frac{pf}{\partial r}\right) \\
&= \frac{1}{r^2} \Delta_{\mathbf{S}^n} f + \frac{1}{r^n} \frac{\partial}{\partial r}\left(r^n \frac{\partial f}{\partial r}\right).
\end{aligned}$$

■

Exercise 7 ***(Eigenvalues of the Laplace operator).*** Let M be a compact oriented Riemannian manifold, and let Δ be the Laplace operator on $\Omega^p(M)$. $\lambda \in \mathbf{R}$ is called an eigenvalue if there exists some $u \in \Omega^p(M)$, $u \neq 0$, with

$$\Delta u = \lambda u.$$

Such a u is called an eigenform or eigenvector corresponding to λ. The vector space spanned by the eigenforms for λ is denoted by V_λ and is called the eigenspace for λ. Show:

(a) All eigenvalues of Δ are nonnegative.

(b) All eigenspaces are finite dimensional.

(c) The eigenvalues have no finite accumulation point.

(d) Eigenvectors for different eigenvalues are orthogonal.

The next results need a little more analysis

(e) There exist infinitely many eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

(f) All eigenvectors of Δ are smooth.

(g) The eigenvectors of Δ constitute an L^2-orthonormal basis for the space of p-forms of class L^2.

(a) Recall that we endow the vector space of p-forms with the L^2-inner product defined as

$$\langle\!\langle \alpha, \beta \rangle\!\rangle := \int_M \alpha \wedge \star\beta, \qquad \alpha, \beta \in \Omega^p(M).$$

Since M is compact and orientable the above expression makes sense and is finite for any smooth p-forms.

First we show that all eigenvalues of the Laplacian are real. This follows immediately since Δ is formally self-adjoint. Let $\alpha \in \Omega^p(M)$ for some $1 \leq p \leq n$ be an eigenform of the Laplacian. Then we see that

$$\lambda \langle\!\langle \alpha, \alpha \rangle\!\rangle = \langle\!\langle \lambda\alpha, \alpha \rangle\!\rangle = \langle\!\langle \Delta\alpha, \alpha \rangle\!\rangle = \langle\!\langle \alpha, \Delta\alpha \rangle\!\rangle = \langle\!\langle \alpha, \lambda\alpha \rangle\!\rangle = \overline{\lambda}\langle\!\langle \alpha, \alpha \rangle\!\rangle,$$

and so $\lambda \in \mathbf{R}$.

We now show the result for the scalar Laplacian ($p = 0$). Let $f \in \Omega^0(M) = \mathscr{C}^\infty(M)$ be an eigenfunction for the Laplacian with eigenvalue λ; that is $\Delta_0 f = \lambda f$. We have that

$$\lambda \int_M |f|^2 \, \mathrm{d}\mu = \int_M (\lambda f) f \, \mathrm{d}\mu = \int_M (\Delta_0 f) f \, \mathrm{d}\mu = \int_M g(\nabla f, \nabla f) \, \mathrm{d}\mu \geq 0,$$

so we see that $\lambda \geq 0$, and so all eigenvalues of Δ_0 are non-negative.

Now let $\alpha \in \Omega^p(M)$ be an eigenform of the p-form Laplacian; i.e. $\Delta_p \alpha = \lambda\alpha$ for some $\lambda \in \mathbf{R}$. It now follows since d^* is the formal adjoint of d that

$$\langle\!\langle \lambda\alpha, \alpha \rangle\!\rangle = \langle\!\langle \Delta_p\alpha, \alpha \rangle\!\rangle = \langle\!\langle dd^*\alpha, \alpha \rangle\!\rangle + \langle\!\langle d^*d\alpha, \alpha \rangle\!\rangle = \langle\!\langle d^*\alpha, d^*\alpha \rangle\!\rangle + \langle\!\langle d\alpha, d\alpha \rangle\!\rangle \geq 0.$$

Since $\langle\!\langle \alpha, \alpha \rangle\!\rangle \geq 0$, this implies $\lambda \geq 0$.

(b) Again, we consider the scalar Hodge-Laplacian Δ_0 first. Consider some $\lambda \geq 0$ and let $\{f_i\}_{i\in I}$ be an orthonormal set of eigenfunctions in V_λ. Note that by integration by parts we have

$$\int_M g(\nabla f_i, \nabla f_i) \, \mathrm{d}\mu = \int_M f_i \Delta_0 f_i \, \mathrm{d}\mu = \lambda \int_M |f_i|^2 \, \mathrm{d}\mu = \lambda.$$

So we see that $\{f_i\}_{i\in I}$ is bounded in $W^{2,1}(M)$. Since $W^{1,2}(M)$ is compactly embedded in $L^2(M)$ we see that $\{f_i\}$ is relatively compact in $L^2(M)$. Now assume, for the sake of contradiction, that $|I| = +\infty$, i.e. there exists an infinite orthonormal system in V_λ. Then by the precompactness of $\{f_i\}$ in $L^2(M)$ there exists a subsequence $\{f_{i_j}\}_{j=1}^\infty$ such that $f_{i_j} \to f$ in $L^2(M)$. On the other hand, since $f_{i_j} \perp f_{i_k}$ for all $j \neq k$ we see that $f_{i_j} \rightharpoonup 0$ in $L^2(M)$, and so we have a contradiction. Hence $\dim V_\lambda < +\infty$.

For the general case of the p-form Laplacian we begin in the same way. Let $\{\alpha_i\}_{i\in I} \subseteq \Omega^p(M)$ be an orthonormal system in V_λ with respect to the L^2-inner product on p-forms. Following the same argument in part (a) we have that

$$\langle\!\langle \delta\alpha_i, \delta\alpha_i \rangle\!\rangle + \langle\!\langle d\alpha_i, d\alpha_i \rangle\!\rangle = \lambda\langle\!\langle \alpha_i, \alpha_i \rangle\!\rangle = \lambda, \qquad i \in I.$$

So in particular, we have that

$$\|\alpha_i\|_{W^{1,2}(\bigwedge^p M)} := \langle\!\langle \alpha_i, \alpha_i \rangle\!\rangle + \langle\!\langle \delta\alpha_i, \delta\alpha_i \rangle\!\rangle + \langle\!\langle d\alpha_i, d\alpha_i \rangle\!\rangle = 1 + \lambda.$$

Since we have the same compactness results for Sobolev spaces over vector bundles (cf. Lemma 2.2.2) we can apply the Rellich-Kondrachov compactness theorem to find that $\{\alpha_i\}_{i\in I}$ is sequentially precompact in $L^2(\bigwedge^p M)$. Now we conclude in an identical manner as in the scalar case: if $\{f_i\}_{i\in I}$ had an infinite number of eigenforms then there would be a strongly converging subsequence in $L^2(\bigwedge^p M)$; however, the existence of a strongly converging subsequence of an orthonormal sequence is a contradiction. So we deduce that $\{f_i\}_{i\in I}$ only has a finite number of elements; in particular, $\dim V_\lambda < +\infty$.

(c) Assume, for the sake of contradiction that the eigenvalues of the p-form Laplacian had a finite accumulation point at $\mu < +\infty$. Let $\{\lambda_j\}_{j\in\mathbf{N}}$ be a sequence of eigenvalues converging to μ. Up to a subsequence (not relabeled) we have that $\lambda_j < 2\mu$ for all $j \in \mathbf{N}$. Now let

$$\mathscr{E}_j := \{\alpha_{i,j}\}_{i\in I}$$

be a basis for V_{λ_j}. Observe that by part (d), $V_{\lambda_j} \perp V_{\lambda_k}$ for $j \neq k$, and so we consider

$$\mathscr{E} := \bigcup_{j\in\mathbf{N}} \mathscr{E}_j.$$

Now we see that for any p-form $\omega \in \mathscr{E}$ that $\|\omega\|_{W^{1,2}(\bigwedge^p M)} < 2\mu$. Hence, by the Rellich-Kondrachov compactness theorem we have that there is a subsequence $\{\omega_k\}_{k\in\mathbf{N}} \subseteq \mathscr{E}$ such that $\omega_k \to \omega$ for some $\omega \in L^2(\bigwedge^p M)$. However, since all of the elements in the sequence $\{\omega_k\}_{k\in\mathbf{N}}$ are orthogonal we have a contradiction. Hence, there is no finite accumulation point.

(d) This immediately follows from the fact that Δ is formally self-adjoint. Let λ and μ be distinct eigenvalues of the p-form Hodge Laplacian. Let $\alpha \in V_\lambda$ and $\beta \in V_\mu$. Then,

$$\lambda\langle\!\langle \alpha, \beta\rangle\!\rangle = \langle\!\langle \lambda\alpha, \beta\rangle\!\rangle = \langle\!\langle \Delta\alpha, \beta\rangle\!\rangle = \langle\!\langle \alpha, \Delta\beta\rangle\!\rangle = \langle\!\langle \alpha, \mu\beta\rangle\!\rangle = \mu\langle\!\langle \alpha, \beta\rangle\!\rangle.$$

So we have that

$$(\lambda - \mu)\langle\!\langle \alpha, \beta\rangle\!\rangle = 0.$$

Since $\lambda \neq \mu$ this implies that $\langle\!\langle \alpha, \beta\rangle\!\rangle = 0$, i.e. that α is orthogonal to β.

(e) Note that in the case when $p = 0$ that $\ker \Delta_0$ is nonempty since all constant functions are scalar harmonics. Now for $p \geq 1$ we claim that Δ_p has at least one eigenvalue. Let $D_p : \Omega^p(M) \to \Omega^p(M)$ given by $D_p(\alpha) = d_{p-1}\delta_{p-1}$ and $\widetilde{D}_p : \Omega^p(M) \to \Omega^p(M)$. Then we see that $\sigma_p(\Delta_p) \setminus \{0\} = \sigma_p(D_p) \cup \sigma_p(\widetilde{D}_p) \setminus \{0\}$. In particular, since D_p is the product of two differential operators, we find that $\sigma_p(D_p)\setminus\{0\} = \sigma_p(\widetilde{D}_{p-1})\setminus\{0\}$. Concluding, we see that if we know the point spectrum of Δ_{p-1} and Δ_{p+1} then we also not the point spectrum of Δ_p. In particular, we will see that the existence of infinitely many eigenfunctions of the scalar Hodge Laplacian will imply the existence of *some* eigenvalue of the the p-form Laplacian.

We show the existence of countably infinitely many eigenvalues inductively. Let $\lambda_1 \leq \cdots \leq \lambda_k$ be the first k eigenvalues repeated according to their multiplicity, and let $\mathscr{E}_k := \{\alpha_i : i = 1, \ldots, k\}$ be the corresponding eigenfunctions. Now we consider the Rayleigh-Ritz quotient $\mathscr{R} : W^{1,2}(\bigwedge^p M) \to \mathbf{R}$ given by

$$\mathscr{R}(\alpha) := \frac{\langle\!\langle \alpha, \Delta_p\alpha\rangle\!\rangle_{L^2(\bigwedge^p M)}}{\langle\!\langle \alpha, \alpha\rangle\!\rangle_{L^2(\bigwedge^p M)}} = \frac{\langle\!\langle \delta\alpha, \delta\alpha\rangle\!\rangle + \langle\!\langle \alpha, \alpha\rangle\!\rangle}{\|\alpha\|^2}.$$

We now leave the subscript out denoting that the inner product is taken in $L^2(\bigwedge^p M)$. Now let $\mathscr{E}_k^\perp$ be the space of p-forms in $W^{1,2}(\bigwedge^p M)$ that are $L^2(\bigwedge^p M)$-orthogonal to $\mathscr{E}_k$. Now let

$$\lambda := \inf_{\alpha\in\mathscr{E}_k^\perp} \mathscr{R}(\alpha) = \inf_{\alpha\in\mathscr{E}_k^\perp} \frac{\langle\!\langle \delta\alpha, \delta\alpha\rangle\!\rangle + \langle\!\langle \alpha, \alpha\rangle\!\rangle}{\|\alpha\|^2}.$$

We claim that $\lambda = \lambda_{k+1}$. Note that since $\mathscr{E}_{k-1} \subseteq \mathscr{E}_k$ we have that $\lambda \geq \lambda_k$. Now consider an infimizing sequence of unit norm p-forms $\{\alpha_j\}_{j\in\mathbf{N}} \subseteq \mathscr{E}_k^\perp$, i.e. $\|\alpha_j\|_{L^2(\bigwedge^p M)} = 1$ and

$$\langle\!\langle d\alpha_j, d\alpha_j\rangle\!\rangle + \langle\!\langle \delta\alpha_j, \delta\alpha_j\rangle\!\rangle \to \lambda \quad \text{as} \quad j \to +\infty.$$

Note that since $\langle\!\langle \alpha_j, \Delta_p\alpha_j\rangle\!\rangle$ converges we see that $\{\alpha_j\}_{j\in\mathbf{N}}$ is bounded in $W^{1,2}(\bigwedge^p M)$. Since $W^{1,2}(\bigwedge^p M)$ is a Hilbert space it is reflexive, and so bounded sets are weakly compact. So up to a subsequence (not relabeled) we can find some $\alpha \in W^{1,2}(\bigwedge^p M)$. Note that we still have that $\alpha \in \mathscr{E}_k^\perp$. Now we see that since the embedding from $W^{1,2}(\bigwedge^p M)$ into $L^2(\bigwedge^p M)$ is compact we have that the L^2-norm of α_j converges to the norm of α. So we see that $\|\alpha\| = 1$. Now by the lower-semicontinuity of the $W^{1,2}$-norm with respect to weak convergence we see that

$$\langle\!\langle d\alpha, d\alpha\rangle\!\rangle + \langle\!\langle \delta\alpha, \delta\alpha\rangle\!\rangle \leq \liminf_{j\to\infty}\big(\langle\!\langle d\alpha_j, d\alpha_j\rangle\!\rangle + \langle\!\langle \delta\alpha_j, \delta\alpha_j\rangle\!\rangle\big) = \lambda.$$

So we see that $\alpha \in \mathscr{E}_k^\perp$ attains the infimum which defines λ. Now we claim that α is an eigenform (in the sense of distributions – we show the smoothness in the next part of the problem). Let $\beta \in W^{1,2}(\bigwedge^p M)$ be any p-form. Note that by the weak convergence in $W^{1,2}(\bigwedge^p M)$ we have that

$$0 = \lim_{j\to\infty} \left(\langle\!\langle d\beta, d\alpha_j \rangle\!\rangle + \langle\!\langle \delta\beta, \delta\alpha_j \rangle\!\rangle - \lambda \langle\!\langle \beta, \alpha_j \rangle\!\rangle \right) = \langle\!\langle d\beta, d\alpha \rangle\!\rangle + \langle\!\langle \delta\beta, \delta\alpha \rangle\!\rangle - \lambda \langle\!\langle \beta, \alpha \rangle\!\rangle$$
$$= \int_M \beta \wedge \star \Delta_p \alpha - \lambda \beta \wedge \star \alpha \, \mathrm{d}\mu,$$

which implies that $\Delta_p \alpha = \lambda\alpha$ weakly in $W^{1,2}(\bigwedge^p M)$. So we have that $\lambda_{k+1} = \lambda$ and $\alpha_{k+1} = \alpha$.

(f) Consider the differential operator $L : W^{1,2}(\bigwedge^p M) \to W^{1,2}(\bigwedge^p M)$ given by

$$L(\omega) = \Delta_p \omega + \omega.$$

Note that $\ker(L)$ is zero, and so by the Fredholm alternative, since $\alpha \in W^{1,2}(\bigwedge^p M)$ (as in the previous part) there is a unique solution $\eta \in W^{3,2}(\bigwedge^p M)$ of $L\eta = (1+\lambda)\alpha$. Now we claim that $\eta = \alpha$. Consider $\widetilde{\eta} := \eta - \alpha \in W^{1,2}(\bigwedge^p M)$. Since α is a weak solution to $L(\omega) = (1+\lambda)\alpha$, we see that $\widetilde{\eta}$ is a weak solution of $L\widetilde{\eta} = 0$. That is to say that for all $\beta \in W^{1,2}(\bigwedge^p M)$ that

$$\langle\!\langle d\beta, d\widetilde{\eta} \rangle\!\rangle + \langle\!\langle \delta\beta, \delta\widetilde{\eta} \rangle\!\rangle + \langle\!\langle \beta, \alpha \rangle\!\rangle = 0.$$

Now by taking $\beta = \widetilde{\eta}$ we see that $\widetilde{\eta} = 0$. This shows that $\alpha \in W^{3,2}(\bigwedge^p M)$. By repeating this procedure we see that $\alpha \in W^{s,2}(\bigwedge^p M)$ for all $s \in \mathbf{N}$. Now by taking s sufficiently large (say $s \geq 2\dim M$) we can use Morrey's inequality to deduce that $\alpha \in \mathscr{C}^{1,\gamma}(\bigwedge^p M)$ for some $0 < \gamma < 1$. Now since Δ_p is an elliptic operator we deduce that in fact $\alpha \in \mathscr{C}^{3,\gamma}(\bigwedge^p M)$. By repeating this procedure we have that $\alpha \in \mathscr{C}^{s,\gamma}(\bigwedge^p M)$ for all $s \in \mathbf{N}$. Hence, we have shown that $\alpha \in \mathscr{C}^\infty(\bigwedge^p M) = \Omega^p(M)$.

(g) Let Π_k be the L^2-projection operator onto the eigenspace spanned by the first k eigenforms. Note that Π_k is self-adjoint. Explicitly, we write

$$\Pi_k \omega = \sum_{1 \leq i \leq k} \langle\!\langle \omega, \alpha_i \rangle\!\rangle \alpha_i.$$

We want to show that $\omega - \Pi_k\omega \to 0$ as $k \to \infty$ for any $\omega \in L^2(\bigwedge^p M)$. First, we show this for any $\omega \in W^{1,2}(\bigwedge^p M)$. Note that by the definition of λ_{k+1} using the Rayleigh-Ritz quotient we see that

$$\|\omega - \Pi_k\omega\|^2 \leq \frac{1}{\lambda_{k+1}} \langle\!\langle \Delta_p(\omega - \Pi_k\omega), \omega - \Pi_k\omega \rangle\!\rangle.$$

A direct computation shows that $\Delta_p \Pi_k = \Pi_k \Delta_p$, and so we derive

$$\langle\!\langle \Delta_p \Pi_k \omega, \omega - \Pi_k\omega \rangle\!\rangle = 0.$$

In particular, we have that

$$\begin{aligned}
\langle\!\langle \Delta_p(\omega - \Pi_k\omega), \omega - \Pi_k\omega \rangle\!\rangle &= \langle\!\langle \Delta_p \omega, \omega - \Pi_k\omega \rangle\!\rangle \\
&= (d\omega, d\omega) + (\delta\omega, \delta\omega) - [\langle\!\langle d(\Pi_k\omega), d(\Pi_k\omega) \rangle\!\rangle + \langle\!\langle \delta(\Pi_k\omega), \delta(\Pi_k\omega) \rangle\!\rangle] \\
&\leq (d\omega, d\omega) + (\delta\omega, \delta\omega).
\end{aligned}$$

Now by plugging this into our bound on $\|\omega - \Pi_k\omega\|^2$ and using the fact that $\lambda_k \to +\infty$ as $k \to +\infty$ we find that

$$\|\omega - \Pi_k\omega\|^2 \to 0$$

for $\omega \in W^{1,2}(\bigwedge^p M)$. This shows the density of the eigenbasis of the p-form Laplacian in $W^{1,2}(\bigwedge^p M)$. Now let $\omega \in L^2(\bigwedge^p M)$. Recall that smooth functions (and in particular smooth p-forms) are dense in L^2 (of course with respect to the L^2-norm). So fix $\varepsilon > 0$, and find some smooth p-form $\eta \in \Omega^p(M)$ such that $\|\omega - \eta\| < \varepsilon/2$. Now let k be large enough such that $\|\omega - \Pi_k \omega\| < \varepsilon/2$. Since $(\omega - \Pi_k \eta) \perp \Pi_k(\omega - \eta)$ for any $\eta \in \Omega^p(M)$, we see that

$$\|\omega - \Pi_k \omega\|^2 = \|\omega - \Pi_k \eta\|^2 - \|\Pi_k(\omega - \eta)\|^2 \leq \|\omega - \Pi_k \eta\|^2.$$

Concluding, we see that

$$\|\omega - \Pi_k \omega\|^2 \leq \|\omega - \Pi_k \eta\|^2 \leq \|\omega - \eta\| + \|\eta - \Pi_k \eta\| < \varepsilon.$$

Now by taking $\varepsilon \to 0$ we see that the eigenvectors of the p-form Hodge Laplacian form an L^2-orthonormal basis of $L^2(\bigwedge^p M)$, the space of p-forms of class L^2.

■

Theorem 6. *Let M be a smooth compact manifold with boundary, and P a first order differential operator acting on sections of a vector bundle, then*

$$(Pu, v) - (u, P^\top v) = \frac{1}{i} \int_{\partial M} \langle \sigma_P(x, \nu) u, v \rangle \, \mathrm{d}\ell.$$

Exercise 8 ***(Another long exercise).*** Let M be a compact oriented Riemannian manifold with boundary $\partial M \neq \emptyset$. For $x \in \partial M$, $V \in T_x M$ is called tangential if it is contained in $T_x \partial M \subseteq T_x M$ and $W \in T_x M$ is called normal if

$$\langle V, W \rangle = 0 \quad \text{for all tangential } V.$$

An arbitrary $Z \in T_x M$ can then be decomposed into a tangential and normal component:

$$Z = Z_{\tan} + Z_{\mathrm{nor}}.$$

Analogously, $\eta \in \Gamma^p(TM^*)$ can be decomposed into

$$\eta = \eta_{\tan} + \eta_{\mathrm{nor}}$$

where $\eta_{\tan}$ operates on tangential p-vectors and η_{nor} on normal ones. For p-forms ω on M, we may impose the so-called absolute boundary conditions

$$\omega_{\tan} = 0$$
$$(\delta \omega)_{\mathrm{nor}} = 0$$

on ∂M or the relative boundary conditions

$$\omega_{\mathrm{nor}} = 0$$
$$(d \omega)_{\mathrm{nor}} = 0$$

on ∂M. (These two boundary conditions are interchanged by the $\star$-operator.) Develop a Hodge theory under either set of boundary conditions.

We begin by providing an alternative formalization of the notion of tangent and normal differential forms and vector fields. Let $i : \partial M \to M$ be the inclusion map. We say that $\omega \in \Omega^p(M)$ is **tangent** to ∂M if the normal part

$$n\omega = i^*(\star \omega)$$

is zero. Analogously, ω is **normal** to ∂M if the tangent part defined by

$$t\omega = i^*(\omega)$$

is zero. Now if X is a vector field on M, we can simply use the metric to determine when X is tangent or normal to ∂M. It isn't hard to check that X is tangent to ∂M if and only if X^β is tangent to ∂M (which of course happens precisely when ι_X vol is normal to ∂M). We have the analogous result for characterizing when X is normal to ∂M.

This presentation clearly shows that these boundary conditions are interchanged by the Hodge star operator.

Now we prove two *distinct* Hodge decomposition theorems for smooth manifolds with boundary.

Theorem 7 (Hodge Decomposition for Manifolds with Boundary). *Let M be a smooth compact oriented Riemannian manifold with boundary. We have the following decomposition*

$$\Omega^p(M) = d\Omega^{p-1}(M)^\top \oplus \delta\Omega^{p+1}(M)^\perp \oplus \mathscr{H}_p(M),$$

where we define the relevant function spaces as

$$\begin{aligned}\Omega^p(M)^\top &= \{\alpha \in \Omega^p(M) \,:\, \alpha \textit{ is tangent to } \partial M\},\\ \Omega^p(M)^\perp &= \{\alpha \in \Omega^p(M) \,:\, \alpha \textit{ is normal to } \partial M\},\\ \mathscr{H}_p(M) &= \{\alpha \in \Omega^p(M) \,:\, d\alpha = \delta\alpha = 0\}.\end{aligned}$$

Proof. Note that the condition $d\alpha = \delta\alpha = 0$ is stronger than $\Delta\alpha = 0$ in the case when M has a boundary. We call the elements of $\mathscr{H}_p$ **harmonic fields**. □

Theorem 8 (Hodge Decomposition with Specified Boundary Conditions). *Let M be a smooth compact oriented Riemannian manifold with boundary. We have the following decomposition*

$$\Omega^k(M) = d\Omega^{k-1}(M)^{\mathscr{R}} \oplus \delta\Omega^{k+1}(M)^{\mathscr{R}} \oplus \mathscr{H}_k^{\mathscr{R}}(M),$$

where $\mathscr{R}$ denotes the relative boundary conditions. Analogously, we have another decomposition given by

$$\Omega^k(M) = d\Omega^{k-1}(M)^{\mathscr{A}} \oplus \delta\Omega^{k+1}(M)^{\mathscr{A}} \oplus \mathscr{H}_k^{\mathscr{A}}(M),$$

where $\mathscr{A}$ denotes the absolute boundary conditions.

Proof. The main idea is that we use the Freedholm alternative to obtain an L^2 elliptic splitting of the space of differential k-forms, and then we use the elliptic regularity of the Hodge Laplacian to force a $\mathscr{C}^\infty$-splitting of $\Omega^k(M)$.

We claim that

$$(\Delta u, v) = (du, dv) + (\delta u, \delta v) + \frac{1}{i}\int_{\partial M} (\langle \sigma_d(x,\nu)\delta u, v\rangle + \langle du, \sigma_d(x,\nu)v\rangle)\ d\ell. \tag{6}$$

To show this we want to use Theorem 6 in the case when $P = d$ and $P = \delta$. First we compute the principal symbols of d and δ. Since for a k-form u,

$$d(ue^{i\lambda\psi}) = i\lambda e^{i\lambda\psi}(d\psi)\wedge u + e^{i\lambda\psi}du,$$

we see that $\frac{1}{i}\sigma_d(x,\xi)u = \xi \wedge u$. Observe that $\sigma_\delta(x,\xi) = \sigma_d(x,\xi)^\top$, and that the adjoint of the map σ_d from $\bigwedge^k T_x^* \to \bigwedge^{k+1} T_x^*$ is given by the interior product $\iota_\xi u$. Consequently, we see that $\frac{1}{i}\sigma_\delta(x,\xi)u = -\iota_\xi u$. Now Theorem 6 implies, for M a compact Riemannian manifold with boundary,

$$(du, v) = (u, \delta v) + \frac{1}{i}\int_{\partial M} \langle \sigma_d(x,\nu)u, v\rangle \, \mathrm{d}\ell = (u,\delta v) + \int_{\partial M} \langle \nu \wedge u, v\rangle \, \mathrm{d}\ell,$$

and

$$(\delta u, v) = (u, dv) + \frac{1}{i}\int_{\partial M} \langle \sigma_\delta(x,\nu)u, v\rangle \, d\ell = (u, dv) - \int_{\partial M} \langle \iota_\nu u, v\rangle \, \mathrm{d}\ell,$$

where ν is the outward-pointing unit norm to ∂M. So we see that

$$\begin{aligned}(\Delta u, v) &= (du, dv) + (\delta u, \delta v) + \int_{\partial M} \left(\langle \nu \wedge (\delta u), v\rangle - \langle \iota_\nu (du), v\rangle\right) \, \mathrm{d}\ell \\ &= (du, dv) + (\delta u, \delta v) + \int_{\partial M} \left(\langle \delta u, \iota_\nu v\rangle - \langle du, \nu \wedge v\rangle\right) \, \mathrm{d}\ell,\end{aligned}$$

which is exactly Equation 6. Note that if u and v satisfy absolute boundary conditions, then the boundary integral in Equation 6 vanishes. For the rest of this proof we introduce the following function spaces. It is easy to see that these are closed subspaces (and hence Banach subspaces) of the Sobolev space of k-forms:

$$\begin{aligned}
H^1_{\mathscr{R}}(M, \textstyle\bigwedge^k TM^*) &= \{u \in H^1(M, \textstyle\bigwedge^k TM^*) \,:\, \sigma_d(x,\nu)u|_{\partial M} = 0\}, \\
H^1_{\mathscr{A}}(M, \textstyle\bigwedge^k TM^*) &= \{u \in H^1(M, \textstyle\bigwedge^k TM^*) \,:\, \sigma_\delta(x,\nu)u|_{\partial M} = 0\}, \\
H^2_{\mathscr{R}}(M, \textstyle\bigwedge^k TM^*) &= \{u \in H^2(M, \textstyle\bigwedge^k TM^*) \,:\, u \text{ satisfies the relative boundary conditions}\}, \\
H^2_{\mathscr{A}}(M, \textstyle\bigwedge^k TM^*) &= \{u \in H^2(M, \textstyle\bigwedge^k TM^*) \,:\, u \text{ satisfies the absolute boundary conditions}\}.
\end{aligned}$$

Note that the elliptic regularity estimates from Exercise 3.5 (specifically Theorem 5) we deduce that in the case of a compact manifold with smooth boundary that for all $u \in H^1_{\mathscr{R}}(M, \bigwedge^k TM^*) \cup H^1_{\mathscr{A}}(M, \bigwedge^k TM^*)$ that

$$\|u\|^2_{H^1} \le C\|du\|^2_{L^2} + C\|\delta u\|^2_{L^2(M)} + C\|u\|^2_{L^2}. \tag{7}$$

This follows by considering an isometric dual of M, i.e. a manifold N such that $\partial N = \emptyset$ and $M \hookrightarrow N$ is an isometric embedding, and using the previous elliptic regularity results. Similarly, we deduce that if $u \in H^1_{\mathscr{R}}(M, \bigwedge^k TM^*)$ satisfies

$$(du, dv) + (\delta u, \delta v) \le C\|v\|_{L^2(M)}, \qquad \text{for all } v \in H^1_{\mathscr{R}}(M, \textstyle\bigwedge^k TM^*),$$

then $u \in H^2_{\mathscr{R}}(M, \bigwedge^k TM^*)$. We have the analogous results for the absolute boundary conditions as well. We can rewrite (7) as the following pair of estimates:

$$\|u\|^2_{H^1} \le C\|du\|^2_{L^2} + C\|\delta u\|^2_{L^2} + C\|\sigma_d(x,\nu)u\|^2_{H^{1/2}(\partial M)} + C\|u\|^2_{L^2} \tag{8}$$

$$\|u\|^2_{H^1} \le C\|du\|^2_{L^2} + C\|\delta u\|^2_{L^2} + C\|\sigma_\delta(x,\nu)u\|^2_{H^{1/2}(\partial M)} + C\|u\|^2_{L^2}. \tag{9}$$

In particular, these two estimates hold for all $u \in H^1(M, \bigwedge^k TM^*)$ regardless of boundary behavior.

Now we consider the linear operator

$$L_{\mathscr{R}} : H^1_{\mathscr{R}}(M, \textstyle\bigwedge^k TM^*) \to H^1_{\mathscr{R}}(M, \textstyle\bigwedge^k TM^*)^*$$

defined via

$$(L_{\mathscr{R}} u, v) = (du, dv) + (\delta u, \delta v), \qquad u, v \in H^1_{\mathscr{R}}(M, \textstyle\bigwedge^k TM^*).$$

Similarly, we define $L_{\mathscr{A}} : H^1_{\mathscr{A}}(M, \bigwedge^k TM^*) \to H^1_{\mathscr{A}}(M, \bigwedge^k TM^*)^*$ in the same way (we just change the domain). Now we see for $\mathscr{T} \in \{\mathscr{A}, \mathscr{R}\}$ that the estimates (8) and (9) imply for some $C_0 > 0$,

$$((L_{\mathscr{T}} + C_0)u, u) \geq C\|u\|^2_{H^1}, \qquad u \in H^1_{\mathscr{T}}(M, \bigwedge\nolimits^k TM^*).$$

This coercivity implies that the operator is elliptic, and so the maps $L_{\mathscr{T}} + C_0 : H^1_{\mathscr{T}}(M, \bigwedge^k TM^*) \to H^1_{\mathscr{T}}(M, \bigwedge^k TM^*)^*$ are bijective. In particular, the maps $S_{\mathscr{T}} : H^1_{\mathscr{T}}(M, \bigwedge^k TM^*)^* \to H^1_{\mathscr{R}}(M, \bigwedge^k TM^*)$ giving the two sided inverses of $(L_{\mathscr{T}} + C_0)$ are compact, self-adjoint operators on $L^2(M, \bigwedge^k TM^*)$. So we have orthonormal bases $\{u_{j,k,\mathscr{T}}\}$ of $L^2(M, \bigwedge^k TM^*)$ satisfying

$$S_{\mathscr{T}} u_{j,k,\mathscr{T}} = \lambda_{j,k,\mathscr{T}} u_{j,k,\mathscr{T}}, \qquad u_{j,k,\mathscr{T}} \in H^1_{\mathscr{T}}(M, \bigwedge\nolimits^k TM^*).$$

Since $((L_{\mathscr{T}} + 1)u, u) \geq \|u\|^2_{L^2}$, we can clearly take $C_0 = 1$. So the magnitude of all of the eigenvalues of $S_{\mathscr{T}}$ are bounded above by 1. Furthermore, we can order them such that the eigenvalues are decreasing to zero as $j \to \infty$. So for all k we have the eigenvalues of $L_{\mathscr{T}}$ increase to infinity (i.e. they have no finite accumulation point). Now we can prove in the same way as Theorem 5 the following generalization for manifolds with boundary.

Theorem 9. *Given $f \in H^j(M, \bigwedge^k TM^*)$ for $j = 1, 2, \ldots$, a k-form $u \in H^{j+1}(M, \bigwedge^k TM^*)$ satisfying*

$$\Delta u = f_1 \qquad \text{on } M$$

and either the relative or absolute boundary conditions, belongs to $H^{j+2}(M, \bigwedge^k TM^*)$. Furthermore, we have the following elliptic estimates

$$\|u\|^2_{H^{j+2}} \leq C\|\Delta u\|^2_{H^j} + C\|\sigma_d(x,\nu)u\|^2_{H^{j+3/2}(\partial M)} + C\|\sigma_d(x,\nu)\delta u\|^2_{H^{j+1/2}(\partial M)} + C\|u\|^2_{H^{j+1}}$$

in the case of relative boundary conditions, and

$$\|u\|^2_{H^{j+2}} \leq C\|\Delta u\|^2_{H^j} + C\|\sigma_\delta(x,\nu)u\|^2_{H^{j+3/2}(\partial M)} + C\|\sigma_\delta(x,\nu)du\|^2_{H^{j+1/2}(\partial M)} + C\|u\|^2_{H^{j+1}}$$

in the case of absolute boundary conditions.

Now let $\mathscr{H}^{\mathscr{T}}_k(M)$ denote the 0-eigenspace of $L_{\mathscr{T}}$ be the space of **harmonic k-forms satisfying $\mathscr{T}$-based boundary conditions**. Let $\Pi^{\mathscr{T}}$ denote the orthogonal projections of $L^2(M, \bigwedge^k TM^*)$ onto $\mathscr{H}^{\mathscr{T}}_k$. Parallel to the case of the Neumann boundary problem we have continuous linear maps

$$G^{\mathscr{T}} : L^2(M, \bigwedge\nolimits^k TM^*) \to H^2_{\mathscr{T}}(M, \bigwedge\nolimits^k TM^*),$$

such that $G^{\mathscr{T}}$ annihilates $\mathscr{H}^{\mathscr{T}}_k$ and inverts Δ on the orthogonal complement $\left(\mathscr{H}^{\mathscr{T}}_k\right)^\perp$:

$$\Delta G^{\mathscr{T}} u = (\mathbf{1} - \Pi^{\mathscr{T}})u, \qquad \text{for } u \in L^2(M, \bigwedge\nolimits^k TM^*). \tag{10}$$

Furthermore, for $j \geq 0$,

$$G^{\mathscr{T}} : H^j(M, \bigwedge\nolimits^k TM^*) \to H^{j+2}(M, \bigwedge\nolimits^k TM^*).$$

Now we obtain from (10) candidate Hodge decompositions for a compact Riemannian manifold with boundary:

$$u = d\delta G^{\mathscr{R}} u + \delta d G^{\mathscr{R}} u + \Pi^{\mathscr{R}} u,$$

and analogously for the case with absolute boundary conditions imposed

$$u = d\delta G^{\mathscr{A}} u + \delta d G^{\mathscr{A}} u + \Pi^{\mathscr{A}} u.$$

It remains only to show that this decomposition is an orthogonal decomposition. By continuity, it suffices to check orthogonality for $u \in \mathscr{C}^\infty(\overline{M}, \bigwedge^k TM^*)$. We will use the identity $(du, v) = (u, \delta v) + \gamma(u, v)$ for $u \in \bigwedge^{j-1}(\overline{M})$ and $v \in \bigwedge^j(\overline{M})$, with

$$\gamma(u,v) = \frac{1}{i}\int_{\partial M} \langle \sigma_d(x,\nu)u, v\rangle \, \mathrm{d}\ell = \frac{1}{i}\int_{\partial M} \langle u, \sigma_\delta(x,\nu)v\rangle \, \mathrm{d}\ell.$$

Note that $\gamma(u,v) = 0$ if either $u \in H^1_{\mathscr{R}}(M, \bigwedge^{j-1} TM^*)$ or $v \in H^1_{\mathscr{A}}(M, \bigwedge^j TM^*)$. In particular, we see that

$$u \in H^1_{\mathscr{R}}(M, \textstyle\bigwedge^{j-1} TM^*) \implies du \perp \ker\delta \cap H^1(M, \textstyle\bigwedge^{j-1} TM^*)$$
$$v \in H^1_{\mathscr{A}}(M, \textstyle\bigwedge^{j} TM^*) \implies \delta v \perp \ker d \cap H^1(M, \textstyle\bigwedge^{j-1} TM^*).$$

Now from our definitions, we have

$$\delta : H^2_{\mathscr{R}}(M, \textstyle\bigwedge^j TM^*) \to H^1_{\mathscr{R}}(M, \textstyle\bigwedge^{j-1} TM^*),$$
$$d : H^1_{\mathscr{A}}(M, \textstyle\bigwedge^j TM^*) \to H^1_{\mathscr{A}}(M, \textstyle\bigwedge^{j+1} TM^*),$$

and so in particular

$$d\delta H^2_{\mathscr{R}}(M, \textstyle\bigwedge^k TM^*) \perp \ker\delta \cap H^1(M, \textstyle\bigwedge^k TM^*),$$
$$\delta d H^2_{\mathscr{A}}(M, \textstyle\bigwedge^k TM^*) \perp \ker d \cap H^1(M, \textstyle\bigwedge^k TM^*).$$

This orthogonality implies

$$\mathrm{range}(\Pi^{\mathscr{R}}_d) \perp \mathrm{range}(\Pi^{\mathscr{R}}_\delta) + \mathrm{range}(\Pi^{\mathscr{R}}), \qquad \mathrm{range}(\Pi^{\mathscr{A}}_\delta) \perp \mathrm{range}(\Pi^{\mathscr{A}}_d) + \mathrm{range}(\Pi^{\mathscr{A}}).$$

Furthermore, if $u \in \mathscr{H}^{\mathscr{R}}_k$ and $v = dG^{\mathscr{R}}w$, then $\gamma(u,v) = 0$, and so $(u, \delta v) = (du, v) = 0$. Similarly, if $v \in \mathscr{H}^{\mathscr{A}}_k$ and $u = \delta G^A w$, then $\gamma(u,v) = 0$, so $(du, v) = (u, \delta v) = 0$. This implies the desired orthogonality conditions, and so the theorem is proved. $\square$

Now as in the general Hodge theory for compact manifolds, we would like to relate the cohomology groups to the spaces of $\mathscr{T}$-harmonic forms, i.e. $\mathscr{H}^{\mathscr{R}}_k(M)$ or $\mathscr{H}^{\mathscr{A}}_k(M)$. First we consider the case of relative boundary conditions. Consider the function space $\mathscr{C}^\infty(\overline{M}, \bigwedge^k TM^*) = \{u \in \mathscr{C}^\infty(\overline{M}, \bigwedge^k TM^*) \,:\, i^*u = 0\}$. Since $d \circ i^* = i^* \circ d$, we see that $d : \mathscr{C}^\infty(\overline{M}, \bigwedge^k TM^*) \to \mathscr{C}^\infty(\overline{M}, \bigwedge^{k+1} TM^*)$. Now we consider the space of *relatively closed* and *relatively exact* forms as

$$\mathcal{C}^{\mathscr{R}}_k(\overline{M}) = \{u \in \mathscr{C}^\infty(\overline{M}, \textstyle\bigwedge^k TM^*) \,:\, du = 0\},$$
$$\mathcal{E}^{\mathscr{R}}_k(\overline{M}) = d\mathscr{C}^\infty(\overline{M}, \textstyle\bigwedge^{k-1} TM^*).$$

Now we define

$$H_k(\overline{M}, \partial M) = \mathcal{C}^{\mathscr{R}}_k(\overline{M}) / \mathcal{E}^{\mathscr{R}}_k(\overline{M}).$$

Now we prove that there is a natural isomorphism $H_k(\overline{M}, \partial M) \cong \mathscr{H}^{\mathscr{R}}_k(M)$. To see this, note that there is an injection

$$j : \mathscr{H}^{\mathscr{R}}_k(M) \to \mathcal{C}^{\mathscr{R}}_k(\overline{M}),$$

which yields a map (by postcomposing with the projection map) $J : \mathscr{H}^{\mathscr{R}}_k(M) \to H_k(\overline{M}, \partial M)$. The orthogonality of the terms in the Hodge decomposition theorem imply that $\mathrm{image}(j) \cap \mathcal{E}^{\mathscr{R}}_k(\overline{M}) = 0$, and so J is injective. Furthermore, if $u \in \mathcal{C}^{\mathscr{R}}_k(\overline{M})$, then u is orthogonal to δv for any $v \in \mathscr{C}^\infty(\overline{M}, \bigwedge^{k+1} TM^*)$, and so the term $\delta(dG^{\mathscr{R}}u)$ in the Hodge decomposition vanishes, and so J is surjective. Hence, J is a natural isomorphism as desired. We have an analogous relationship for the absolute harmonic forms, the proof is identical. $\blacksquare$

Exercise 1. Compute the transformation behavior of the Christoffel symbols of a connection under coordinate transformations.

Let $\nabla : \Gamma(TM) \to \Gamma(E) \otimes \Gamma(T^*M)$ be a connection on a Riemannian manifold M (where E is some vector bundle over M). Let Γ^i_{km} be the Christoffel symbols of ∇ in local coordinates $\{x^i\}$ in some neighborhood $U \subseteq M$. By shrinking U if necessary we can assume that U is also a bundle chart (i.e. local trivialization) of E. Now identify $E|_U \cong U \times \mathbf{R}^n$, where $n = \dim E_x$ is the dimension of the fibers of E. Under this isomorphism we see that a basis for $\mathbf{R}^n$ induces a basis $\mu_1, \dots, \mu_n$ of sections of $E|_U$. Now recall that the Christoffel symbols are simply defined via

$$\nabla_{\frac{\partial}{\partial x^i}} \mu_j = \Gamma^k_{ij} \mu_k.$$

Now let $\{y^\alpha\}$ be another set of local coordinates over $V \subseteq M$ (such that the domains have nonempty intersection). Let $\nu_1, \dots, \nu_n$ be a basis of sections over $E|_V$ by following the same procedure above. Then we see that

$$\nabla_{\frac{\partial}{\partial y^\alpha}} \nu_\beta = \Gamma^\gamma_{\alpha\beta} \nu_\gamma.$$

Now let $g_{UV} \in \mathscr{C}^\infty(U \cap V, \mathbf{GL}(n, \mathbf{R}))$ be the vector bundle transition functions of E on $U \cap V$. That is to say for $x \in U \cap V$ any any section of $s \in \Gamma(E|_{U\cap V})$ written as

$$s = \sum_{i=1}^n s^i \mu_i = \sum_{\alpha=1}^n \widehat{s}^\alpha \nu_\alpha \qquad \text{that we have} \qquad [s^i] = g_{UV}[\widehat{s}^\alpha],$$

where $[\cdot]$ is the column vector with entries specified above. In particular, we can find some smooth functions g^α_i such that

$$\nu_\alpha = g^i_\alpha \mu_i.$$

Now by using the fact that ∇ is tensorial over TM and an $\mathbf{R}$-derivation over the sections we see that

$$\begin{aligned}
\Gamma^\gamma_{\alpha\beta} \nu_\gamma = \nabla_{\frac{\partial}{\partial y^\alpha}} \nu_\beta = \nabla_{\frac{\partial}{\partial y^\alpha}} \left(g^j_\beta \mu_j \right) &= g^j_\beta \nabla_{\frac{\partial}{\partial y^\alpha}} \mu_j + \frac{\partial}{\partial y^\alpha}(g^j_\beta)\mu_j \\
&= g^j_\beta \nabla_{\frac{\partial x^i}{\partial y^\alpha}\frac{\partial}{\partial x^i}} \mu_j + \frac{\partial g^j_\beta}{\partial y^\alpha}\mu_j = g^j_\beta \frac{\partial x^i}{\partial y^\alpha} \nabla_{\frac{\partial}{\partial x^i}} + \frac{\partial g^j_\beta}{\partial y^\alpha}\mu_j \\
&= g^j_\beta \frac{\partial x^i}{\partial y^\alpha} \Gamma^k_{ij} \mu_k + \frac{\partial g^j_\beta}{\partial y^\alpha}\mu_j = g^j_\beta \frac{\partial x^i}{\partial y^\alpha} \Gamma^k_{ij} g^\gamma_k \nu_\gamma + \frac{\partial g^j_\beta}{\partial y^\alpha} g^\gamma_j \nu_\gamma.
\end{aligned}$$

Now by comparing the coefficients of the first and last terms we find the following transformation behavior:

$$\Gamma^\gamma_{\alpha\beta} = g^j_\beta \frac{\partial x^i}{\partial y^\alpha} g^\gamma_k \Gamma^k_{ij} + \frac{\partial g^\ell_\beta}{\partial y^\alpha} g^\gamma_\ell.$$

Note that in particular, if we consider a connection over TM we find that the transformation laws are simply given by

$$\Gamma^\gamma_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma} \Gamma^k_{ij} + \frac{\partial^2 x^\ell}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^\ell}.$$

Note that the transformation law is nonlinear, and so in particular we deduce that the Christoffel symbols do not transform like a tensor at all. Note that if we choose some coordinates where the transformation is linear (i.e. the second partials are zero) then the transformation rules of the Christoffel symbols are that of $(1,2)$-type tensors. For the same reason, we see that the difference of Christoffel symbols associated to different linear connections is a tensor. ∎

Exercise 2. Let E be a vector bundle with fiber $\mathbf{C}^n$ and a Hermitian bundle metric. Develop a theory of unitary connections, i.e. of connections respecting the bundle metric.

Let $E \to M$ be a complex vector bundle with fiber $\mathbf{C}^n$. A Hermitian bundle metric is a smooth section $h : M \to (E^*)^{\otimes 2} \cong (E \otimes_{\mathbf{R}} E)^*$ such that (by using the natural duality pairing of the tensor product and multilinear maps) $h_p : E_p \times E_p \to \mathbf{C}$ is a Hermitian inner product. Note that with respect to a local frame $\{\sigma_1, \ldots, \sigma_n\}$ of E the Hermitian metric is given by a Hermitian matrix-valued function $H = (H_{ij})$ given by $H_{ij} = h(\sigma_i, \sigma_j)$, and transforms according to $\widetilde{H} = BH\overline{B}^{\top}$, where B is a complex $n \times n$ matrix.

Now recall that a linear connection over $E \to M$ is a map $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M) \cong \Gamma(E) \otimes \Omega^1(M)$ that satisfies certain properties. Observe that we can more concisely described a linear connection as a $\mathbf{K}$-linear map $\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$ satisfying the Leibniz product rule

$$\nabla(f\sigma) = df \otimes \sigma - f(\nabla\sigma),$$

where $df \otimes \sigma \in \Omega^1(M; E)$, when evaluated at $p \in M$, is the $\mathbf{R}$-linear map $T_pM \to E_p$ given by $v \mapsto (df)_p(v) \cdot \sigma(p)$. Similarly, $f\nabla\sigma$ is the element of $\Omega^1(M; E)$, when evaluated at $p \in M$, is the $\mathbf{R}$-linear map $T_pM \to E_p$ given by $v \mapsto f(p)(\nabla\sigma)_p(v)$. This is easily seen to coincide with the definition provided in the text since $df(V) = V(f)$.

A linear connection ∇_E is a **unitary connection** if for any smooth sections $\sigma_1, \sigma_2 \in \Gamma(E)$ we have

$$d(h(\sigma_1, \sigma_2)) = h(\nabla_E\sigma_1, \nabla_E\sigma_2) + h(\sigma_1, \nabla_E\sigma_2).$$

Here the exterior derivative, d, is applied to the complex valued function $h(\sigma_1, \sigma_2)$ and on the right hand side we use the following definition $h(\alpha \otimes \sigma, \tau) := \alpha h(\sigma, \tau)$ for a complex valued 1-form $\alpha \in \Omega^1(M; \mathbf{C})$ and smooth sections $\sigma, \tau \in \Gamma(E)$. Analogously, we define $h(\sigma, \alpha \otimes \tau) = \overline{\alpha} h(\sigma, \tau)$. Now we develop some theory regarding unitary connections:

- *Algebraic structure of linear connections:* We show that the space of complex linear connections is an affine space, whose group of translations is simply the vector space $\Omega^1(M; \mathrm{End}(E))$. Let ∇ and $\widetilde{\nabla}$ be linear connections. It suffices to show that $(\nabla - \widetilde{\nabla})$ is an element of $\Omega^1(M; E)$. Let $f \in \mathscr{C}^\infty(M; \mathbf{C})$ and $\sigma \in \Gamma(E)$. Now we see that

$$\begin{aligned}(\nabla - \widetilde{\nabla})(f\sigma) &= \nabla(f\sigma) - \widetilde{\nabla}(f\sigma)\\ &= df \otimes \sigma + f(\nabla\sigma) - df \otimes \sigma - f(\widetilde{\nabla}\sigma)\\ &= f(\nabla - \widetilde{\nabla})\sigma.\end{aligned}$$

  So we see that $(\nabla - \widetilde{\nabla})$ is a $\mathscr{C}^\infty(M; \mathbf{C})$ linear map from $\Omega^0(M; E) \cong \Gamma(E)$ to $\Omega^1(M; E)$. Hence $(\nabla - \widetilde{\nabla})$ is an $\mathrm{End}(E)$-valued 1-form on M. Furthermore, by an identical argument we find that the group of unitary connections is an affine space, with translation group given by the vector space $\Omega^1(M; \mathfrak{u}_h(E))$, where $\mathfrak{u}_h(E)$ is the Lie algebra associated to the Lie group $\mathbf{U}(E)$.

- *Reduction of the structure group:* The structure group of a complex vector bundle with a Hermitian connection reduces to $\mathbf{U}(n)$. To see this, we simply use the Gram-Schmidt orthogonalization procedure to obtain a h-unitary local frame out of any given local frame of E. By localizing, we can identify $E|_U$ with $U \times \mathbf{C}^n$, where $\mathbf{C}^n$ is endowed with the usual Hermitian metric. The transition functions of such an atlas have an associated 1-cocycle of transition maps which preserve the Hermitian product; hence, this transition functions are $\mathbf{U}(n)$-valued.

- *Existence of a unitary connection:* Let ∇^0 be an arbitrary connection on E. We define its adjoint $(\nabla^0)^*$ by

$$h\big(\psi_1, (\nabla^0)^*_X\psi_2\big) = X \cdot h(\psi_1, \psi_2) - h(\nabla^0_X\psi_1, \psi_2).$$

Since h is non-degenerate this suffices to define $(\nabla^0)^*_X$ as a $\mathscr{C}^\infty(M)$-linear function in X. In ψ_2 we see that

$$\begin{aligned} h(\psi_1, (\nabla^0)^*_X(f\psi_2)) &= X\cdot h(\psi_1,\psi_2) - h(\nabla^0_X\psi_1, f\psi_2) \\ &= (X\cdot f)h(\psi_1,\psi_2) + fX\cdot h(\psi_1,\psi_2) - fh(\nabla^0_X\psi_1,\psi_2) \\ &= h(\psi_1, (Xf)\psi_2 + f(\nabla^0)^*_X\psi_2). \end{aligned}$$

Since this holds for all ψ_1 and ψ_2, we have

$$(\nabla^0)^*_X(f\psi_2) = (X\cdot f)\psi_2 + f(\nabla^0)^*_X\psi_2,$$

for all ψ_2< and so $(\nabla^0)^*$ is a connection.

The defining property of the adjoint implies that ∇ is unitary if and only if $\nabla = \nabla^*$ and $(\nabla^*)^* = \nabla$. Since any convex combination of connections is a connection, we take

$$\nabla = \frac{1}{2}(\nabla^0 + (\nabla^0)^*).$$

We clearly have that ∇ is equal to its own adjoint, and hence is a unitary connection.

- *The $\overline{\partial}$ operator:* From here on out we will assume that M is a complex manifold and $E\to M$ is a holomorphic vector bundle. Note that $\Omega^1(M;E)$ splits as $\Omega^{1,0}(M;E)\oplus\Omega^{0,1}(M;E)$, where $\Omega^{1,0}(M;E)$ is the complex vector space of $\mathbf{C}$-linear 1-forms $\alpha : TM \to E$, and $\Omega^{0,1}(M;E)$ is the complex vector space of $\mathbf{C}$-antilinear forms. This splitting follows in the same way as the splitting of the complexification of the tangent bundle:

  $$TM\otimes_{\mathbf{R}}\mathbf{C} = T^{1,0}M\oplus T^{0,1}M.$$

  Note that we can split a unitary connection as $\nabla = \nabla^{1,0}+\nabla^{0,1}$, where $\nabla^{0,1}:\Omega^0(M;E)\to\Omega^{0,1}(M;E)$ satisfies

  $$\nabla^{0,1}(f\sigma) = \overline{\partial}f\otimes\sigma + f\nabla^{0,1}\sigma.$$

  We show that there exists a canonical $\overline{\partial}$-operator. That is, there is a canonical $\overline{\partial}^E : \mathscr{C}^\infty(M;E)\to\Omega^{0,1}(E)$ operator satisfying the following Leibniz rule: for all $f\in\mathscr{C}^\infty(M;\mathbf{C})$ and $\sigma\in\Omega^0(M;E)$ we have

  $$\overline{\partial}^E(f\sigma) = (\overline{\partial}f)\sigma + f\overline{\partial}^E\sigma.$$

  The canonical $\overline{\partial}^E$ operator over the tangent bundle is simply given by the splitting of the exterior derivative $d = \partial+\overline{\partial}$. This satisfies $\overline{\partial}^2 = 0$ and that $\overline{\partial}f = 0$ if and only if f is holomorphic. To generalize this to arbitrary complex vector bundles $E\to M$ we simply define $\overline{\partial}^E$ locally as

  $$\overline{\partial}^E(f\sigma) = \overline{\partial}(f)\otimes\sigma.$$

- *The canonical unitary connection:* Let $\overline{\partial}$ be any $\overline{\partial}$-operator as above. We claim there is a unique unitary connection on (E,h) satisfying $\nabla^{0,1} = \overline{\partial}$. Let $\{e_i\}$ be a local holomorphic frame for E. Then any local section σ can be written as $\sigma = \sigma^i e_i$. Then we see that

  $$\nabla\sigma = (d\sigma_i + \sigma_j\theta_{ji})\otimes e_i,$$

  where θ_{ji} is the matrix of 1-forms given by $\nabla e_j = \theta_{ji}\otimes e_i$. Now if we assume that ∇ has $(0,1)$-part given by $\overline{\partial}$, then $\nabla^{0,1}e_i = 0$, and so θ_{ji} has type $(1,0)$. We se $h_{ij} = h(e_i,e_j)$. Since ∇ is a unitary connection,

  $$dh_{ij} = \partial h_{ij} + \overline{\partial}h_{ij} = h(\nabla e_i, e_j) + h(e_i,\nabla e_j) = \theta_{ik}h_{kj} + h_{ik}\overline{\theta}_{jk}.$$

  By equating the $(1,0)$ types we see that $\theta = \partial h\cdot h^{-1}$, which is entirely determined locally. A direct computation shows that θ transforms in a way that makes ∇ globally a connection on $E\to M$.

  Now by taking $\overline{\partial} = \overline{\partial}^E$, the canonical $\overline{\partial}$-operator coinciding with the holomorphic structure on E, we see that there is a unique unitary connection which coincides with the holomorphic structure of the complex vector bundle.

■

Exercise 3. Show that each vector bundle with a bundle metric admits a metric connection.

Let $E \to M$ be a vector bundle with bundle metric g_E. Recall that a connection ∇ is said to be a metric connection if and only if for all $X \in \mathfrak{X}(M)$ and $\sigma, \tau \in \Gamma(E)$ we have

$$g_E(\nabla_X \sigma, \tau) + g_E(\sigma, \nabla_X \tau) = X(g_E(\sigma, \tau)).$$

So in local coordinates, we see that the connection is a metric connection if and only if the associated Christoffel symbols satisfy

$$\frac{\partial}{\partial x^i}(g_E)_{jk} = \Gamma_{ij}^{\ell}(g_E)_{\ell k} + \Gamma_{ik}^{\ell}(g_E)_{j\ell}.$$

Now given a matrix function g_E, we can always find functions Γ_{ij}^k that satisfy these relations. In particular, if we consider a local trivialization of E, then we can find the associated Christoffel symbols. Now by covering M with a family of open sets $\{U_\alpha\}_{\alpha \in \Lambda}$ such that $E \to U_\alpha$ is a trivial bundle we can find the desired Christoffel symbols, and then patch them together using a partition of unity. ■

Exercise 4. Let $x_0 \in M$, D a flat metric connection on a vector bundle E over M. Show that D induces a map $\pi_1(M, x_0) \to \mathbf{O}(n)$, considering $\mathbf{O}(n)$ as the isometry group of the fiber E_{x_0}.

By starting with a flat metric connection we easily obtain a representation of the fundamental group. Let $\mathscr{P}(D, \gamma, v)$ be the parallel transport of the vector $v \in T_{\gamma(0)}E$ along the closed curve γ with respect to the connection D. First let $\Omega(M, x_0)$ denote the *loop space* of M with basepoint x_0. Now we simply define the representation $\rho : \Omega(M, x_0) \to \mathrm{End}(E_{x_0})$ given by

$$\rho(\gamma) = (v \mapsto \mathscr{P}(D, \gamma, v)), \qquad v \in E_{x_0}.$$

Note that since $\mathscr{P}(D, \gamma, \cdot)$ is obtained by a linear ordinary differential equation we see that the flow map is a linear map as well. Furthermore, by the same reasoning we deduce that $\mathscr{P}(D, \gamma, \cdot)$ is a linear isomorphism of vector spaces. In particular, we have $\rho : \Omega(M, x_0) \to \mathbf{GL}(E_{x_0})$.

Now we use the fact that D is a metric connection to show that $\rho(\gamma)$ is an isometry for any $\gamma \in \Omega(M, x_0)$. Recall that D is a metric connection if for any sections $\mu, v \in \Gamma(E)$ we have

$$d\langle \mu, v\rangle = \langle D\mu, v\rangle + \langle \mu, Dv\rangle,$$

in $\Omega^1(M)$. In the above, the $\mathscr{C}^\infty(M)$-valued pairings on the right hand side between $\Gamma(E)$ and $(TM^* \otimes E)$ are defined in the obvious way by taking the 1-form on the outside. Equivalently, we have that

$$X\langle \mu, v\rangle = \langle D_X\mu, v\rangle + \langle \mu, D_X v\rangle.$$

Now fix $\gamma : \mathbf{S}^1 \to M$ in the loop space, and let $\mu : \mathbf{S}^1 \to E$ and $v : \mathbf{S}^1 \to E$ be smooth curves along γ, i.e. $\mu, v \in \Gamma(\gamma(\mathbf{S}^1))$ and $\mu(t), v(t) \in E_{\gamma(t)}$. Let $\mathbf{S}^1$ be parameterized by $\theta \in [0, 2\pi)$ with the obvious meaning. The metric compatibility gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu(t), v(t)\rangle = \left\langle \frac{D\mu}{\mathrm{d}t}, v\right\rangle + \left\langle \mu, \frac{Dv}{\mathrm{d}t}\right\rangle.$$

Now fix $v, w \in E_{x_0}$ and define $\mu(t) = \mathscr{P}(D, \gamma|_{[0,t]}, v)$ and $v(t) = \mathscr{P}(D, \gamma|_{[0,t]}, w)$. Then we see that since μ and v are parallel along γ that

$$\frac{D\mu}{\mathrm{d}t} = \frac{Dv}{\mathrm{d}t} = 0,$$

and in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu(t), \nu(t)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{P}(D,\gamma|_{[0,t]},\nu), \mathscr{P}(D,\gamma|_{[0,t]},w)\rangle = 0.$$

So we see that parallel transport is an isometry between the corresponding fibers for all $t \in [0,2\pi)$. Now by fully traversing the curve (taking $t = 2\pi$) we see that

$$\langle \rho(\gamma)\nu, \rho(\gamma)w\rangle = \langle \nu, w\rangle.$$

Hence $\rho(\gamma) \in \mathbf{O}(E_{x_0}) = \mathbf{O}(n)$.

It now remains to show that $\rho(\gamma)$ only depends on the path-homotopy class of γ – this will follow from the fact that D is a flat connection. Let $\gamma_0, \gamma_1 \in \Omega(M, x_0)$ be path-homotopic, where $\gamma : \mathbf{S}^1 \times [0,1] \to M$ is a smooth function such that

$$\gamma(t,0) = \gamma_0(t), \qquad \gamma(t,1) = \gamma_1(t), \qquad t \in \mathbf{S}^1.$$

Now fix $\nu_0 \in E_{x_0}$. We claim that $\rho(\gamma_0)\nu_0 = \rho(\gamma_1)\nu_0$. To do this, consider $\nu : \mathbf{S}^1 \times [0,1] \to E$ be a smooth section over γ defined as

$$\nu(t,s) = \mathscr{P}(D, \gamma(\cdot,s)|_{[0,t]}, \nu_0).$$

By the definition of parallel transport we have that $\frac{D\nu}{\mathrm{d}t} = 0$. Now since D is a linear connection, a direct computation shows

$$\frac{D^2\nu}{\mathrm{d}t\mathrm{d}s} - \frac{D^2\nu}{\mathrm{d}s\mathrm{d}t} = T_D\left(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right),$$

where

$$T_D(X,Y)\sigma = D_X D_Y \sigma - D_Y D_X \sigma - D_{[X,Y]}\sigma$$

is the torsion tensor. Since D is flat we have that the torsion tensor is identically zero, and so by taking $u := \frac{Du}{\mathrm{d}t}$ we have

$$\frac{Du}{\mathrm{d}t} = 0.$$

But for $t = 0$, $s \in [0,1]$ we have that $\nu(0,s) = \nu_0$, and so $u(0,s) = 0$. In other words, for a fixed $s \in [0,1]$ the path $u(\cdot,s)$ is parallel in E with respect to D. Now by uniqueness of the parallel transport, we deduce that $u \equiv 0$. Now we use this at $t = 2\pi$. In this case we know that $\nu(1,s) \in E_{x_0}$ for all $s \in [0,1]$. Moreover, from the definition of covariant differentiation we have

$$\frac{D\nu}{\mathrm{d}s}(1,s) = \frac{\partial}{\partial s}\nu(1,s) \in T_{x_0}E_{x_0} \cong E_{x_0}.$$

So we deduce that $s \mapsto \nu(1,s)$ is constant. In particular, $\nu(2\pi,0) = \nu(2\pi,1)$, i.e. $\rho(\gamma_0)\nu_0 = \rho(\gamma_1)\nu_0$. Now since $\nu_0 \in E_{x_0}$ was arbitrary we see that $\rho(\gamma)$ only depends on the path-homotopy class of γ.

We now conclude: since $\pi_1(M,x_0)$ is given as the quotient space of $\Omega(M,x_0)$ with respect to the equivalence relation induced by path-homotopy equivalence we see that $\rho([\gamma])$ is well defined for any $[\gamma] \in \pi_1(M,x_0)$. By the above results we deduce that

$$\rho : \pi_1(M,x_0) \to \mathbf{O}(E_{x_0}).$$

By considering $\pi_1(M,x_0)$ as a group (with group operation induced by path concatenation) we see that ρ is a group homomorphism (this follows immediately from the fact that parallel transport is given by the solution to a linear ordinary differential equation). Hence, ρ is a representation of the group $\pi_1(M,x_0)$ over the vector space E_{x_0}. ■

Exercise 5. Let $\mathbf{S}^n_r := \{x \in \mathbf{R}^{n+1} : \|x\| = r\}$ be the sphere of radius r. Compute its curvature tensor and volume.

We endow $\mathbf{S}_r^n$ with the induced Riemannian metric from $\mathbf{R}^{n+1}$. Now let $\overline{\nabla}$ denote the Levi-Civita connection on $\mathbf{R}^{n+1}$. Now let $X, Y \in \Gamma(T\mathbf{S}_r^n)$, and let $\overline{X}, \overline{Y}$ be local extensions of X and Y to $\mathbf{R}^{n+1}$. Since $\overline{\nabla}_{\overline{X}}\overline{Y}$ at any point $p \in \mathbf{S}^n$ only depends on the vector $\overline{X}(p) = X(p)$ we simply write $\overline{\nabla}_X Y$ for $\overline{\nabla}_{\overline{X}}\overline{Y}$. Since $\mathbf{S}_r^n$ is a Riemannian submanifold of $\mathbf{R}^{n+1}$ we have that the Levi-Civita connection, ∇, on $\mathbf{S}_r^n$ is simply given by orthogonal projection, i.e.

$$\nabla_X Y = \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \nu \rangle \nu,$$

where $\nu = \nu(p)$ is the unit outward normal vector on $\mathbf{S}_r^n$. Note that for some $p \in \mathbf{S}_r^n$ we have that

$$\nu(p) = \frac{p}{\|p\|} = \frac{p}{r}.$$

Hence,

$$\nabla_X Y = \overline{\nabla}_X Y - \frac{1}{r^2}\langle \overline{\nabla}_X Y, p \rangle p = X(Y) - \frac{1}{r^2}\langle X(Y), p \rangle p,$$

where $p \in \mathbf{S}_r^n \subseteq \mathbf{R}^{n+1}$. Now we compute at some point $p \in \mathbf{S}_r^n$,

$$\begin{aligned}
\nabla_X \nabla_Y Z &= X(\nabla_Y Z) - \frac{1}{r^2}\langle X(\nabla_Y Z), p \rangle p \\
&= XY(Z) - \frac{1}{r^2}\langle XY(Z), p \rangle p - \frac{1}{r^2}\langle Y(Z), X \rangle p - \frac{1}{r^2}\langle Y(Z), p \rangle X \\
&\quad - \frac{1}{r^2}\langle XY(Z), p \rangle p + \langle XY(Z), p \rangle p + \langle Y(Z), X \rangle p + \frac{\langle Y(Z), p \rangle}{r^2}\langle X, p \rangle p \\
&= XY(Z) + \left(1 - \frac{2}{r^2}\right)\langle XY(Z), p \rangle p + \left(1 - \frac{1}{r^2}\right)\langle Y(Z), X \rangle p - \frac{1}{r^2}\langle Y(Z), p \rangle X \\
&= XY(Z) + \left(1 - \frac{2}{r^2}\right)\langle XY(Z), p \rangle p + \left(1 - \frac{1}{r^2}\right)\langle Y(Z), X \rangle p + \frac{1}{r^2}\langle Z, Y \rangle X,
\end{aligned}$$

where we have repeatedly used the fact that $\langle X, p \rangle = \langle Y, p \rangle = \langle Z, p \rangle = 0$. Now we see that since the Lie bracket of vector fields in $\mathbf{R}^{n+1}$ vanish,

$$\begin{aligned}
R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \\
&= XY(Z) + \left(1 - \frac{2}{r^2}\right)\langle XY(Z), p \rangle p + \left(1 - \frac{1}{r^2}\right)\langle Y(Z), X \rangle p + \frac{1}{r^2}\langle Z, Y \rangle X \\
&\quad - \left(YX(Z) + \left(1 - \frac{2}{r^2}\right)\langle YX(Z), p \rangle p + \left(1 - \frac{1}{r^2}\right)\langle X(Z), Y \rangle p + \frac{1}{r^2}\langle Z, X \rangle Y\right) \\
&= \frac{1}{r^2}\left(\langle Z, Y \rangle X - \langle Z, X \rangle Y\right).
\end{aligned}$$

So summarizing we have shown that

$$R(X,Y)Z = \frac{1}{r^2}\langle Y, Z \rangle X - \frac{1}{r^2}\langle X, Z \rangle Y.$$

Now we compute the volume of $\mathbf{S}_r^n$. Recall that the Gamma function is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, \mathrm{d}x, \qquad t > 0.$$

By the homogeneity of the $\mathcal{H}^n$-measure, we see that $\mathrm{vol}(\mathbf{S}_r^n) = r^n \,\mathrm{vol}(\mathbf{S}_1^n)$. We compute the following integral in two ways

$$\int_{\mathbf{R}^{n+1}} e^{-\|\boldsymbol{x}\|^2}\, \mathrm{d}\boldsymbol{x}.$$

First consider the polar coordinates change of variables $\mathbf{S}^n \times [0,\infty) \to \mathbf{R}^{n+1}$ given by $(\sigma, r) = r\sigma$. Now we can write

$$\int_{\mathbf{R}^{n+1}} e^{-\|\boldsymbol{x}\|^2} \, \mathrm{d}\boldsymbol{x} = \int_0^\infty r^n e^{-r^2} \, \mathrm{d}r \int_{\mathbf{S}^n} 1 \, \mathrm{d}\sigma = \mathrm{vol}(\mathbf{S}^n_1) \int_0^\infty r^n e^{-r^2} \, \mathrm{d}r.$$

Alternatively, by writing $\boldsymbol{x} = (x_0, \ldots, x_{n+1})$ we deduce that

$$\begin{aligned}
\int_{\mathbf{R}^{n+1}} e^{-\|\boldsymbol{x}\|^2} \, \mathrm{d}\boldsymbol{x} &= \int_{\mathbf{R}^{n+1}} e^{-x_1^2 - \cdots - x_{n+1}^2} \, \mathrm{d}x_1 \cdots \mathrm{d}x_{n+1} \\
&= \int_{\mathbf{R}} e^{-x_1^2} \, \mathrm{d}x_1 \int_{\mathbf{R}} e^{-x_2^2} \, \mathrm{d}x_2 \, \cdots \, \int_{\mathbf{R}} e^{-x_{n+1}^2} \, \mathrm{d}x_{n+1} \\
&= \left(\int_{\mathbf{R}} e^{-y^2} \, \mathrm{d}y \right)^{n+1}.
\end{aligned}$$

Now by considering the change of variables $u = -r^2$ we can compute

$$\int_0^\infty r e^{-r^2} \, \mathrm{d}r = -\frac{1}{2} \int_0^{-\infty} e^u \, \mathrm{d}u = \frac{1}{2}.$$

Since we know the circumference of the circle $\mathbf{S}^1 \subseteq \mathbf{R}^2$, we deduce

$$\int_{\mathbf{R}^2} e^{-\|\boldsymbol{x}\|^2} \, \mathrm{d}\boldsymbol{x} = \pi.$$

Hence,

$$\int_{\mathbf{R}^{n+1}} e^{-\|x\|^2} \, \mathrm{d}\boldsymbol{x} = \left(\int_{\mathbf{R}^2} e^{-\|\boldsymbol{x}\|^2} \, \mathrm{d}\boldsymbol{x} \right)^{\frac{n+1}{2}} = \pi^{\frac{n+1}{2}}.$$

In particular, we find that

$$\mathrm{vol}(\mathbf{S}^n_1) = \frac{\pi^{n+1}}{\int_0^\infty e^n e^{-r^2} \, \mathrm{d}r}.$$

Now by considering the change of variables $u = r^2$ we find that

$$\int_0^\infty r^n e^{-r^2} \, \mathrm{d}r = \frac{1}{2} \int_0^\infty u^{\frac{n-1}{2}} e^{-u} \, \mathrm{d}u = \frac{1}{2} \Gamma\left(\frac{n+1}{2} \right).$$

So we find that

$$\mathrm{vol}(\mathbf{S}^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}, \qquad \mathrm{vol}(\mathbf{S}^n_r) = \frac{2\pi^{\frac{n+1}{2}} r^n}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Where the last step follows from the homogeneity of the Hausdorff measure. ∎

Exercise 6. Consider the hyperboloid in $\mathbf{R}^3$ defined by the equation

$$x^2 + y^2 - z^2 = -1, \; z > 0,$$

and compute its curvature.

Denote the hyperboloid as

$$H := \{(x,y,z) \in \mathbf{R}^3 \ : \ x^2+y^2-z^2=-1, \ z>0\}.$$

Since we are only dealing with a 2-dimensional hyperboloid it is easy to do all of these computations in coordinates. We parameterize H by coordinates (θ,ϕ) as follows:

$$(\theta,\phi) \mapsto \begin{pmatrix} \sinh(\theta)\cos(\phi) \\ \sinh(\theta)\sin(\phi) \\ \cosh(\theta) \end{pmatrix}$$

Now we compute the orthogonal coordinate basis vectors of the tangent space

$$\begin{aligned} \partial_\theta &= (\cosh(\theta)\cos(\phi), \ \cosh(\theta)\sin(\phi), \ \sinh(\theta)), \\ \partial_\phi &= (-\sinh(\theta)\sin(\phi), \sinh(\theta)\cos(\phi), \ 0). \end{aligned}$$

Now we see that the metric is given by $g_{ij} = \partial_i \cdot \partial_j$:

$$(g_{ij}) = \begin{pmatrix} \cosh(2\theta) & 0 \\ 0 & \sinh^2(\theta) \end{pmatrix}.$$

Since this is a diagonal matrix we easily compute the inverse (g^{ij}):

$$(g^{ij}) = \begin{pmatrix} \frac{1}{\cosh(2\theta)} & 0 \\ 0 & \frac{1}{\sinh^2(\theta)} \end{pmatrix}.$$

Now we compute the derivatives of the basis vectors:

$$\begin{aligned} \partial_{\theta,\theta} &= (\sinh(\theta)\cos(\phi), \ \sinh(\theta)\sin(\phi), \ \cosh(\theta)) \\ \partial_{\theta,\phi} &= (-\cosh(\theta)\sin(\phi), \ \cosh(\theta)\cos(\phi), \ 0) \\ \partial_{\phi,\theta} &= (-\cosh(\theta)\sin(\phi), \ \cosh(\theta)\cos(\phi), \ 0) \\ \partial_{\phi,\phi} &= (-\sinh(\theta)\cos(\phi), \ -\sinh(\theta)\sin(\phi), \ 0). \end{aligned}$$

The associated dual basis is given by $d\theta = g^{\theta j}\partial_j$ and $d\phi = g^{\phi j}\partial_j$. Computing them gives us

$$\begin{aligned} d\theta &= \frac{1}{\cosh(2\theta)}\left(\cosh(\theta)\cos(\phi)dx + \cosh(\theta)\sin(\phi)dy + \sinh(\theta)dz\right), \\ d\phi &= \frac{1}{\sinh^2(\theta)}\left(-\sinh(\theta)\sin(\phi)dx + \sinh(\theta)\cos(\phi)dy\right). \end{aligned}$$

Now we compute the Christoffel symbols $\Gamma^i_{jk} = \Gamma^i_{kj} = di(\partial_{j,k})$:

$$\begin{aligned} &\Gamma^\theta_{\theta\theta} = d\theta(\partial_{\theta,\theta}) = \tanh(2\theta), && \Gamma^\theta_{\theta\phi} = d\theta(\partial_{\theta,\phi}) = 0, && \Gamma^\theta_{\phi\phi} = d\theta(\partial_{\phi,\phi}) = -\frac{1}{2}\tanh(2\theta) \\ &\Gamma^\phi_{\theta\theta} = d\phi(\partial_{\theta,\theta}) = 0, && \Gamma^\phi_{\theta\phi} = d\phi(\partial_{\theta,\phi}) = \coth(\theta), && \Gamma^\phi_{\phi\phi} = d\phi(\partial_{\phi,\phi}) = 0. \end{aligned}$$

We now write out the two Christoffel matrices

$$\Gamma_\theta = \begin{pmatrix} \Gamma^{\theta_{\theta\theta}} & 0 \\ 0 & \Gamma^\phi_{\theta\phi} \end{pmatrix} = \begin{pmatrix} \tanh(2\theta) & 0 \\ 0 & \coth(\theta) \end{pmatrix}, \qquad \Gamma_\phi = \begin{pmatrix} 0 & \Gamma^\theta_{\phi\phi} \\ \Gamma^\phi_{\phi\theta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\tanh(2\theta) \\ \coth(\theta) & 0 \end{pmatrix}.$$

Now we have

$$[\Gamma_\theta, \Gamma_\phi] = \begin{pmatrix} 0 & \cosh^2(\theta)\operatorname{sech}^2(2\theta) \\ \coth^2(\theta)\operatorname{sech}^2(2\theta) & 0 \end{pmatrix}.$$

Recall that the Riemann curvature tensor in coordinates is given by

$$R^i_{mnk} = (R_{nk})^i_m = [\partial_k + \Gamma_k, \partial_n + \Gamma_n]^i_m = (\Gamma_{n,k} - \Gamma_{k,n} + \Gamma_k\Gamma_n - \Gamma_n\Gamma_k)^i_m.$$

Now we have that the derivatives of the Christoffel symbols are given by

$$\Gamma_{\theta,\theta} = \begin{pmatrix} 2\operatorname{sech}^2(2\theta) & 0 \\ 0 & -\operatorname{csch}^2(\theta) \end{pmatrix}, \qquad \Gamma_{\phi,\theta} = \begin{pmatrix} 0 & -\operatorname{sech}^2(2\theta) \\ -\operatorname{csch}^2(\theta) & 0 \end{pmatrix},$$

and the rest are identically zero. Since $R^i_{m\theta\theta} = R^i_{m\phi\phi} = 0$ and $R^\theta_{\theta\theta\phi} = R^\phi_{\phi\theta\phi} = 0$ we have that the nonzero terms of the curvature tensor are

$$R^\phi_{\theta\theta\phi} = -(\Gamma_{\phi,\theta})^\theta_\phi + [\Gamma_\phi, \Gamma_\theta]^\phi_{\phi\phi}\theta = \operatorname{csch}^2(\theta) + \coth^2(\theta)\operatorname{sech}(2\theta)$$
$$R^\theta_{\phi\theta\phi} = -(\Gamma_{\phi,\theta})^\theta_\phi + [\Gamma_\phi, \Gamma_\theta]^\theta_\phi = \operatorname{sech}^2(2\theta) + \operatorname{sech}^2(2\theta)\cosh^2(\theta).$$

Now we compute the Ricci curvature, which is simply the contraction of the Riemann curvature tensor:

$$R_{\theta\theta} = R^\theta_{\theta\theta\theta} + R^\phi_{\phi\theta\phi} = \operatorname{csch}^2(\theta) + \coth^2(\theta)\operatorname{sech}(2\theta)$$
$$R_{\phi\phi} = R^\theta_{\phi\theta\phi} + R^\phi_{\phi\phi\phi} = -\operatorname{sech}^2(2\theta) - \operatorname{sech}^2(2\theta)\cosh^2(\theta).$$

Now we compute the scalar curvature, which is simply the contraction of the Ricci curvature tensor: $R = g^{km}R_{mk}$:

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 2\operatorname{sech}^2(2\theta).$$

Note that we can also express the scalar curvature implicitly as

$$R = \frac{2}{(1-2z^2)^2}.$$

Since we are working with a 2-manifold we see that the Guassian curvature, K, satisfies $R = 2K$, and so we deduce that

$$K = \operatorname{sech}^2(2\theta).$$

■

Exercise 7. Verify that the catenoid, the helicoid, and Enneper's surface are minimal surfaces.

- *Catenoid:* we parameterize the catenoid as follows:

  $$(u,v) \mapsto \begin{pmatrix} \cosh(u)\cos(v) \\ \cosh(u)\sin(v) \\ u \end{pmatrix}$$

  Now we compute the basis tangent vectors

  $$\partial_u = (\sinh(u)\cos(v),\ \sinh(u)\sin(v), 1), \qquad \partial_v = (-\cosh(u)\sin(v),\ \cosh(u)\cos(v),\ 0).$$

  Now we see that the normal vector is given by

  $$\partial_u \times \partial_v = (-\cosh(u)\cos(v),\ -\cosh(u)\sin(v),\ \sinh(u)\cosh(u)).$$

  We also compute $\|\partial_u \times \partial_v\| = \cosh^2(u)$, and so the unit normal vector field is given by

  $$\nu(u,v) = \left(-\frac{\cos(v)}{\cosh(u)},\ -\frac{\sin(v)}{\cosh(u)},\ \tanh(u)\right).$$

Furthermore,

$$\nabla^{\mathbf{R}^3}_{\partial_u} \nu = \partial_u(\nu) = \big(\cos(v)\operatorname{sech}(u)\tanh(u),\ \sin(v)\operatorname{sech}(u)\tanh(u),\ \operatorname{sech}^2(u)\big),$$
$$\nabla^{\mathbf{R}^3}_{\partial_v} \nu = \partial_v(\nu) = (\sin(v)\operatorname{sech}(u),\ -\cos(v)\operatorname{sech}(u),\ 0).$$

Now we compute the shape operator

$$S(\partial_u, \nu) = \partial_u(\nu) - \langle \partial_u(\nu), \nu\rangle \nu = \big(\cos(v)\operatorname{sech}(u)\tanh(u),\ \sin(v)\operatorname{sech}(u)\tanh(u),\ \operatorname{sech}^2(u)\big),$$
$$S(\partial_v, \nu) = \partial_v(\nu) - \langle \partial_v(\nu), \nu\rangle \nu = (\sin(v)\operatorname{sech}(u),\ -\cos(v)\operatorname{sech}(u),\ 0).$$

So we can write

$$S(\partial_u, \nu) = \operatorname{sech}^2(u)\partial_u, \qquad S(\partial_v, \nu) = -\operatorname{sech}^2(u)\partial_v.$$

Now we compute

$$H_\nu = \frac{1}{2}\operatorname{tr}(S_\nu) = \frac{1}{2}\big(\operatorname{sech}^2(u) - \operatorname{sech}^2(u)\big) = 0.$$

So we see that the mean curvature vector is identically zero; hence the catenoid is a minimal surface.

- *Helicoid:* we parameterize the helicoid as follows:

$$(u, v) \mapsto \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ v \end{pmatrix}$$

The basis tangent vectors are given by

$$\partial_u = (\cos(v),\ \sin(v), 0), \qquad \partial_v = (-u\sin(v),\ u\cos(v),\ 1).$$

The normal vector is given by

$$\partial_u \times \partial_v = (\sin(v),\ -\cos(v),\ u),$$

with norm $\|\partial_u \times \partial_v\| = \sqrt{1+u^2}$. So the unit normal vector field is given by

$$\nu(u, v) = \frac{1}{\sqrt{1+u^2}}(\sin(v),\ -\cos(v),\ u).$$

Now we compute the shape operator

$$S(\partial_u, \nu) = \frac{1}{(1+u^2)^{3/2}}(-u\sin(v),\ u\cos(v),\ 1) = \frac{1}{(1+u^2)^{3/2}}\partial_v,$$
$$S(\partial_v, \nu) = \frac{1}{\sqrt{1+u^2}}(\cos(v),\ \sin(v),\ 0) = \frac{1}{\sqrt{1+u^2}}\partial_u.$$

Now it is clear that $H_\nu = \frac{1}{2}\operatorname{tr}(S_\nu) = 0$, and so the helicoid is a minimal surface.

- *Enneper's surface:* we parameterize Enneper's surface, E, as follows:

$$(u, v) \mapsto \left(\frac{u}{2} - \frac{u^3}{6} + \frac{uv^2}{2},\ -\frac{v}{2} + \frac{v^3}{6} - \frac{u^2 v}{2},\ \frac{u^2}{2} - \frac{v^2}{2}\right)$$

As in the previous two examples, we now compute the basis tangent vectors

$$\partial_u = \left(\frac{1}{2} - \frac{u^2}{2} + \frac{v^2}{2},\ -uv,\ u\right), \qquad \partial_v = \left(uv,\ -\frac{1}{2} - \frac{u^2}{2} + \frac{v^2}{2},\ -v\right).$$

The normal vector is again given by the cross product:

$$\partial_u \times \partial_v = \frac{1}{2}\left(u(1+u^2+v^2),\ v(1+u^2+v^2),\ \frac{1}{2}(-1+u^4+2u^2v^2+v^4)\right), \qquad \|\partial_u \times \partial_v\| = \frac{1}{4}(1+u^2+v^2)^2,$$

and so

$$\nu(u,v) = \frac{1}{1+u^2+v^2}\left(2u,\ 2v,\ -1+u^2+v^2\right).$$

Now we compute the shape operator:

$$S(\partial_u, \nu) = \frac{1}{(1+u^2+v^2)^2}\left(2-2u^2+2v^2,\ -4uv,\ 4u\right) = \frac{4}{(1+u^2+v^2)^2}\partial_u,$$

and

$$S(\partial_v, \nu) = \frac{1}{(1+u^2+v^2)^2}\left(-4uv,\ 2(1+u^2-v^2),\ 4v\right) = -\frac{4}{(1+u^2+v^2)^2}\partial_v.$$

Now we see that

$$H_\nu = \frac{1}{2}\operatorname{tr}(S_\nu) = 0,$$

and so the Enneper's surface is a minimal submanifold of $\mathbf{R}^3$.

■

Exercise 8. Determine all surfaces of revolution in $\mathbf{R}^3$ that are minimal. (Answer: the catenoid is the only one.)

We claim that the only minimal surfaces of revolution in $\mathbf{R}^3$ are the catenoid and the plane. A surface of revolution is parameterized as $f : \mathbf{R} \times [0, 2\pi) \to \mathbf{R}^3$ given by

$$f(r(t)\cos(\theta),\ r(t)\sin(\theta),\ h(t)),$$

for some positive function $r : \mathbf{R} \to \mathbf{R}^+$ and $h : \mathbf{R} \to \mathbf{R}$. Now we compute the basis tangent vectors as

$$\partial_t = (r'\cos\theta,\ r'\sin\theta,\ h'), \qquad \partial_\theta = (-r\sin\theta,\ r\cos\theta,\ 0).$$

The metric in these coordinates is given by

$$(g_{ij}) = \begin{pmatrix} (r')^2 + (h')^2 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad \text{and} \qquad (g^{ij}) = \begin{pmatrix} \frac{1}{(r')^2+(h')^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}.$$

Now we see that the normal vector field is simply given by

$$\nu = \frac{1}{\sqrt{(r')^2 + (h')^2}}(-h'\cos\theta,\ -h'\sin\theta,\ r').$$

Rather than computing the shape operator, we begin by computing the principal curvatures. The second derivatives of f are simply given by

$$\partial_{tt}f = (r''\cos\theta,\ r''\sin\theta,\ h''), \qquad \partial_{\theta\theta})f = (-r\cos\theta,\ -r\sin\theta, 0), \qquad \partial_{t\theta} = (-r'\sin\theta,\ r'\cos\theta,\ 0).$$

Now we see that binormal matrix is given by

$$b_{tt} = \langle \partial_{tt}f, \nu\rangle = \frac{r'h'' - h'r''}{(r')^2 + (h')^2}, \qquad b_{\theta\theta} = \langle \partial_{\theta\theta}f, \nu\rangle = \frac{rh'}{\sqrt{(r')^2 + (h')^2}}, \qquad b_{t\theta} = \langle \partial_{t\theta}, \nu\rangle = 0.$$

Since both (b_{ij}) and (g_{ij}) are diagonal matrices we see that $g^{-1}b$ is also diagonal. Hence the basis vector directions are principal curvature directions, and we see that $\kappa_1 = g^{tt}b_{tt}$ and $\kappa_2 = g^{\theta\theta}b_{\theta\theta}$. Hence

$$H = \frac{\kappa_1 + \kappa_2}{2}.$$

In particular, we see that $f(\mathbf{R} \times [0, 2\pi))$ is a minimal submanifold of $\mathbf{R}^3$ if and only if

$$r(r'h'' - h'r'') + h'((r')^2 + (h')^2) = 0. \tag{11}$$

Now we see that if $h(t) = c$ for any fixed constant $c \in \mathbf{R}$ that Equation 11 is satisfied. Hence, this is a minimal submanifold. Geometrically, we see that the corresponding surface of revolution is given by $P = \{z = c\}$.

Now suppose that $h(t)$ is not constant. So we see that there is some point $t_0 \in \mathbf{R}$ such that $h'(t_0) \neq 0$, and in particular h is locally monotone. By a reparameterization of r and h we can assume that $h(t) = t$ locally. In this case Equation 11 becomes

$$rr'' = 1 + (r')^2, \qquad r > 0.$$

The solution of this ordinary differential equation is simply given by

$$r(t) = a\cosh\left(\frac{t - t_0}{a}\right),$$

for any $a > 0$. By the uniqueness of the Cauchy-Lipchitz-Picard-Lindelöf theorem these are all of the solutions locally. Furthermore, these solutions extend globally for all $t \in \mathbf{R}$. As a result, we see that $h'(t_0) \neq 0$ for some $t_0 \in \mathbf{R}$ that $h'(t) \neq 0$ for all $t \in \mathbf{R}$. Since these are all solutions, we deduce that the only other minimal surfaces are given by

$$f(t,\theta) = \left(a\cosh\left(\frac{t-t_0}{a}\right)\cos\theta,\ a\cosh\left(\frac{t-t_0}{a}\right)\sin\theta,\ t\right),$$

where $a > 0$ is a free parameter.

Of course since rotations and translations are isometries of $\mathbf{R}^3$ we have that all minimal surfaces of revolution are given by translated and/or rotated catenoids and planes. ■

Exercise 9. Let $F : M^m \to \mathbf{R}^{m+1}$ be an isometric immersion ($m = \dim M$). Give a complete derivation of the formula

$$\Delta F = m\eta,$$

where Δ is the Laplace-Beltrami operator of M and η is the mean curvature vector of $F(M)$.

Of course by ΔF we mean that the Laplace-Beltrami operator is applied coordinate wise. By postcomposing with the projections we can write $F = (F_1, \ldots, F_{m+1})$, where $F_i : M^m \to \mathbf{R}$.

Fix any vector $\boldsymbol{v} \in \mathbf{R}^{m+1}$; we will consider the real valued function $f = \langle F, \boldsymbol{v}\rangle : M^m \to \mathbf{R}$. Now fix $p \in M$ and let $\{e_1, \ldots, e_m\}$ be a geodesic frame in a neighborhood of p. Let $\vec{N}$ be the local normal frame in the same neighborhood of $F(p) \in F(M^m) \subseteq \mathbf{R}^{m+1}$. Now recall that the Laplace-Beltrami operator in geodesic coordinates is given by

$$\Delta h = \sum_{i=1}^{m} e_i(e_i(h)), \qquad h \in \mathscr{C}^\infty(M).$$

Note that for some vector $w \in T_pM$, that $df_p(w) = \langle dF_p(w), v\rangle$. We use this to compute

$$\Delta f = \Delta\langle F, \boldsymbol{v}\rangle = \sum_{i=1}^{m} e_i(e_i\langle F, \boldsymbol{v}\rangle) = \sum_{i=1}^{m} e_i(df(e_i)) = \sum_{i=1}^{m} e_i\langle dF(e_i), \boldsymbol{v}\rangle.$$

Since F is an isometric immersion we have a natural identification, $T_pM \cong T_pF(M)$, which respects the inner product. In particular, we have that $e_i\langle dF(e_i), \boldsymbol{v}\rangle = dF(e_i)\langle dF(e_i), \boldsymbol{v}\rangle$. Now we use the compatibility of the Levi-Civita connection, $\overline{\nabla}$, of $\mathbf{R}^{m+1}$ with the Euclidean metric $\langle\cdot,\cdot\rangle$ to obtain

$$\begin{aligned}\sum_{i=1}^m dF(e_i)\langle dF(e_i), \boldsymbol{v}\rangle &= \sum_{i=1}^m \langle \overline{\nabla}_{dF(e_i)} dF(e_i), \boldsymbol{v}\rangle + \langle dF(e_i), \overline{\nabla}_{dF(e_i)}\boldsymbol{v}\rangle \\ &= \sum_{i=1}^m \langle \overline{\nabla}_{dF(e_i)} dF(e_i), \boldsymbol{v}\rangle \\ &= \sum_{i=1}^m \langle dF(\nabla_{e_i} e_i), \boldsymbol{v}\rangle + \langle \mathrm{II}(dF(e_i), dF(e_i)), \boldsymbol{v}\rangle,\end{aligned}$$

where II is the vector valued second-fundamental form. Now since e_i was chosen to be a geodesic frame at p we have $\nabla_{e_i} e_i(p) = 0$, and so

$$\sum_{i=1}^m \langle dF(\nabla_{e_i} e_i), \boldsymbol{v}\rangle + \langle \mathrm{II}(dF(e_i), dF(e_i)), \boldsymbol{v}\rangle = \sum_{i=1}^m \langle \mathrm{II}(dF(e_i), dF(e_i)), \boldsymbol{v}\rangle = mH\langle N, \boldsymbol{v}\rangle.$$

So we have shown that

$$\Delta\langle F, \boldsymbol{v}\rangle = mH\langle N, \boldsymbol{v}\rangle.$$

Now by applying this result where $\boldsymbol{v} = x_i$ are the standard unit vectors of $\mathbf{R}^{m+1}$ we find that

$$\Delta F = mH\vec{N} = m\eta.$$

■

Exercise 10. Let $F : M^m \to \mathbf{S}^n \subseteq \mathbf{R}^{n+1}$ be an isometric immersion. Show that $F(M)$ is minimal in $\mathbf{S}^n$ if and only if there exists a function φ on M with $\Delta F = \varphi F$, and that in this case necessarily $\varphi \equiv m$.

Let $\overline{\nabla}$ denote the Levi-Civita connection on $\mathbf{R}^{n+1}$ and ∇ the Levi-Civita connection of $\mathbf{S}^n$. Now if $\overline{\eta}$ and η are the mean curvature vectors of $F(M)$ in $\mathbf{R}^{n+1}$ and $\mathbf{S}^n$, respectively, then

$$\eta = (\overline{\eta})^{\top \mathbf{S}^n} = \frac{1}{n}(\nabla F)^{\top \mathbf{S}^n},$$

where we used Exercise 3.9. Now we see that F is minimal in $\mathbf{S}^n$ if and only if there exists a function $\varphi : M \to \mathbf{R}$ with $\Delta F = \varphi F$. Note that in this case

$$\langle \Delta F, F\rangle = \varphi \|F\|^2 = \varphi.$$

Observe for $v \in T_pM$, that $\langle dF(v), F\rangle = 0$. Now let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of T_pM, and so we have $\langle dF(e_i), F\rangle = 0$ for all $i = 1, \ldots, m$. In particular,

$$0 = dF(e_i)\langle dF(e_i), F\rangle = \langle \overline{\nabla}_{dF(e_i)} dF(e_i), F\rangle + \langle dF(e_i), dF(e_i)\rangle,$$

where we used the comptability of the Levi-Civita connection with the Euclidean metric. Since

$$\langle \overline{\nabla}_{dF(e_i)} dF(e_i), F\rangle = d\overline{\nabla}_{dF(e_i)} dF(e_i)^{\perp},$$

we deduce that

$$0 = \langle \mathrm{tr}(\mathrm{II}), F\rangle + m = -\langle \nabla F, F\rangle + m.$$

So we deduce that $\varphi \equiv m$ in this case. ■

Exercise 11. Show that for $n \geq 4$, there exists no hypersurface (i.e. a submanifold of codimension 1) in $\mathbf{R}^n$ with negative sectional curvature.

Assume, for the sake of contradiction, that all of the sectional curvatures of a hypersurface $M \subseteq \mathbf{R}^n$ for some $n \geq 4$ are negative at some point $p \in M$. Now let $\{e_1, \dots, e_n\}$ be an orthonormal basis for T_pM where the vector valued second fundamental form, II, is diagonal, with corresponding eigenvalues (λ_i). From the Gauss theorem we know that

$$K(e_i, e_j) = \widetilde{K}(e_i, e_j) + \overline{K}(e_i, e_j) = \lambda_i \lambda_j,$$

since the ambient space is flat. Now by assumption, for $i \neq j$, we deduce that $\lambda_i \lambda_j < 0$. However, since $\dim M \geq 3$ this is impossible. To illustrate this contradiction, note that in the case $\dim M = 3$ this would mean that

$$\lambda_1 \lambda_2 < 0, \quad \lambda_1 \lambda_3 < 0, \quad \lambda_2 \lambda_3 < 0,$$

which is clearly nonsensical. The same thing generalizes to all dimensions. ∎

Exercise 12. Verify the formula $\partial\!\!\!/ = cl \circ \nabla$ given in Section 3.4.

Fix $p \in M$. Since TM is a locally trivial vector bundle, we can find an open neighborhood U of p such that $TM|_U \cong TU$ is trivial. Now let $\{e_1, \dots, e_n\}$ be a global frame as a vector bundle over U, and let $\{e^1, \dots, e^n\}$ be the associated dual frame for $TM^*|_U \cong TU^*$ as a vector bundle over U. Now we see that for any 1-form $\omega \in \Omega^1(M)$,

$$\omega = \sum_{i=1}^{n} \omega(e_i) e^i.$$

Similarly, if we consider a section $\sigma \in \Gamma(U, \widetilde{P})$, we can write

$$\nabla \sigma = \sum_{i=1}^{n} e^i \otimes \nabla_{e_i} \sigma \in \Omega^1(U, \widetilde{P}),$$

where this equality is valid over all of U. Now by postcomposing by the Clifford multiplication we see that over U,

$$cl \circ \nabla = cl\left(\sum_{i=1}^{n} e^i \otimes \nabla_{e_i} \sigma\right) \cong cl\left(\sum_{i=1}^{n} e_i \otimes \nabla_{e_i} \sigma\right) = \sum_{i=1}^{n} cl(e_i \otimes \nabla_{e_i} \sigma) = \sum_{i=1}^{n} e_i \nabla_{e_i} \sigma.$$

Since the above right hand side is exactly the "definition" of the Dirac operator in local coordinates, we see that

$$\partial\!\!\!/ = cl \circ \nabla.$$

∎

Exercise 1. Let M_1, M_2 be submanifolds of the Riemannian manifold M. Let the curve $c : [a,b] \to M$ satisfy $c(a) \in M_1$, $c(b) \in M_2$. A variation $c : [a,b] \times (-\varepsilon, \varepsilon) \to M$ is called a variation of $c(t)$ with respect to M_1, M_2 if $c(a,s) \in M_1$, $c(b,s) \in M_2$ for all $s \in (-\varepsilon, \varepsilon)$.

What are the conditions for c to be an extremal of L or E with respect to such variations?

Compute the second variation of E for such an extremal and express any boundary terms by the second fundamental forms of M_1 and M_2.

Let $c : [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a variation of $c(t)$ with respect to M_1, M_2. Note that in particular, $\frac{\partial c}{\partial s}(a,0) \in T_{c(a)} M_1$ and $\frac{\partial c}{\partial s}(b,0) \in T_{c(a)} M_2$. For $-\varepsilon < s < \varepsilon$ define $c_s(t) := c(t,s)$, which is a curve connecting M_1 to M_2 for all s.

We compute the first variation of E with respect to such a variation by a direct computation:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} E(c_s) &= \frac{1}{2} \int_a^b \frac{\partial}{\partial s} \left\langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial t} \right\rangle \mathrm{d}t \\
&= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \right\rangle \mathrm{d}t \\
&= \int_a^b \left(\frac{\partial}{\partial t} \left\langle \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \right\rangle - \left\langle \frac{\partial c}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial t} \right\rangle \right) \mathrm{d}t.
\end{aligned}$$

Now define $X(t) = \frac{\partial c}{\partial s}(t,0)$ and note that $\frac{\partial c}{\partial t}(t,0) = \dot{c}(t)$. We now use the fundamental theorem of calculus to find

$$\left. \frac{\mathrm{d}}{\mathrm{d}s} E(c_s) \right|_{s=0} = -\int_a^b \langle \nabla_{\frac{\partial}{\partial t}} \dot{c}(t), X(t) \rangle \, \mathrm{d}t + \langle X(b), \dot{c}(b) \rangle - \langle X(a), \dot{c}(a) \rangle.$$

Similarly, we compute

$$\left. \frac{\mathrm{d}}{\mathrm{d}s} L(c_s) \right|_{s=0} = -\int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} \left(\frac{\dot{c}(t)}{\|\dot{c}(t)\|} \right), X(t) \right\rangle \mathrm{d}t + \left\langle X(b), \frac{\dot{c}(b)}{\|\dot{c}(b)\|} \right\rangle - \left\langle X(a), \frac{\dot{c}(a)}{\|\dot{c}(a)\|} \right\rangle$$

Now suppose c is extremal for E. It is easy to check that c must be geodesic in M. Now for any $v \in T_{c(a)} M_1$ and $w \in T_{c(b)} M_2$ we can find a variation $c : [a,b] \times (-\varepsilon, \varepsilon) \to M$ of c with respect to M_1, M_2 satisfying $\partial_s c(a,0) = v$ and $\partial_s c(b,0) = w$. Then from out first variation formula, we find that

$$0 = \left. \frac{\partial}{\partial s} E(c_s) \right|_{s=0} = \langle w, \dot{c}(b) \rangle - \langle v, \dot{c}(a) \rangle.$$

So we deduce that $\dot{c}(a) \in T_{c(a)} M_1^{\perp}$ and $-\dot{c}(b) \in T_{c(b)} M_2^{\perp}$ for c to be extremal. A direct computation shows that the conditions that c is geodesic, $\dot{c}(a) \in T_{c(a)} M_1^{\perp}$, and $-\dot{c}(b) \in T_{c(b)} M_2^{\perp}$ are also sufficient for c to be extremal with respect to E. These conditions are also necessary and sufficent for c to be an extremal curve of L.

Finally, we compute the second variation of E for such an extremal curve c.

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} E(c_s) = \int_a^b \frac{\partial}{\partial s} \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \right\rangle \mathrm{d}t.$$

A direct computation shows

$$\begin{aligned}
\frac{\partial}{\partial s}\left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s},\frac{\partial c}{\partial t}\right\rangle &= \left\langle \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s},\frac{\partial c}{\partial t}\right\rangle + \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s},\nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial t}\right\rangle \\
&= \left\langle \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial s},\frac{\partial c}{\partial t}\right\rangle + \left\langle R\left(\frac{\partial c}{\partial s},\frac{\partial c}{\partial t}\right)\frac{\partial c}{\partial s},\frac{\partial c}{\partial t}\right\rangle + \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s},\nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s}\right\rangle \\
&= \frac{\partial}{\partial t}\left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial s},\frac{\partial c}{\partial t}\right\rangle - \left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial s},\nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial t}\right\rangle - \left\langle R\left(\frac{\partial c}{\partial s},\frac{\partial c}{\partial t}\right)\frac{\partial c}{\partial t},\frac{\partial c}{\partial s}\right\rangle + \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s},\nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s}\right\rangle.
\end{aligned}$$

Now by taking $s=0$ we see that $\nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial t}\equiv 0$ since c is a geodesic. Furthermore, $\dot{c}(t)$ is smooth and we get

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(c_s)\bigg|_{s=0} = \int_a^b \left(\langle \nabla X(t),\nabla X(t)\rangle - \langle R(X(t),\dot{c}(t))\dot{c}(t),X(t)\rangle\right)\mathrm{d}t + \left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial s}(b,0),\dot{c}(b)\right\rangle - \left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial s}(a,0),\dot{c}(a)\right\rangle.$$

Now since c is extremal we have $\dot{c}(a)\in T_{c(a)}M_1^{\perp}$ and $\dot{c}(b)\in T_{c(b)}M_2^{\perp}$, and so we can write the boundary terms in terms of the shape operator to obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(c_s)\bigg|_{s=0} = \int_a^b \left(\langle \nabla X(t),\nabla X(t)\rangle - \langle R(X(t),\dot{c}(t))\dot{c}(t),X(t)\rangle\right)\mathrm{d}t + \left\langle S_{-\dot{c}(b)}X(b),X(b)\right\rangle + \left\langle S_{\dot{c}(a)}X(a),X(a)\right\rangle.$$

∎

Exercise 2. Let M be a submanifold of the Riemannian manifold N, $c:[a,b]\to N$ geodesic with $c(a)\in M$, $\dot{c}(a)\in T_{c(a)}M^{\perp}$. For $\tau\in(a,b]$, $c(\tau)$ is called a **focal point** of M along c if there exists a nontrivial Jacobi field X along c with $X(a)\in T_{c(a)}M$, $X(\tau)=0$. Show:

(a) If M has no focal point along c, then for each $\tau\in(a,b)$, c is the unique shortest connection to $c(\tau)$ when compared with all sufficiently close curves with initial point in M.

(b) Beyond a focal point, a geodesic is no longer the shortest connection to M.

There is a typo in the definition of a focal point. We need to require $X(\tau)=0$, $X(a)\in T_{c(a)}M$, and $\dot{X}(a)+S_{\dot{c}(a)}(X(a))\in(T_{c(a)}M)^{\perp}$.

(a) Fix any $\tau\in(a,b]$. We want to show that any curve $\gamma:[a,b]\to N$ with $\gamma(a)\in U$ and $\gamma(\tau)=c(\tau)$ satisfies $L(\gamma)\geq L(c|_{[a,\tau]})$, where U is some open subset of $c(a)$ in M. Clearly, we only need to consider the case when γ is geodesic, since other curves cannot locally minimize the length between their endpoints.

Find a Jacobi field J along c satisfying

- $J(a)\in T_{c(a)}M$,
- $\|J(a)\|_g=1$,
- $J(\tau)=0$.

We choose J in such a way since we will associate it to a variation $c(t,s)=c_s(t)$, and we want to have $c_s(a)\in M$ for all $s\in(-\varepsilon,\varepsilon)$ for some sufficiently small $\varepsilon>0$. Define $J_s=dc_{(t,s)}\left(\frac{\partial}{\partial s}\right)=c'(t,s)$.

We now use the fact that $J\perp\dot{c}$ to compute the second variation of L with respect to J:

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}L(c_s)\bigg|_{s=0} &= \left\langle \frac{\partial}{\partial s}J_s(t),\dot{c}(t)\right\rangle\bigg|_{t=0,s=0}^{t=\tau,s=0} + \int_0^{\tau}\langle \dot{J},\dot{J}\rangle - R(J,\dot{c},J,\dot{c})\,\mathrm{d}t \\
&= -\left\langle \frac{\partial}{\partial s}J_s(a),\dot{c}(a)\right\rangle - \langle J(a),\dot{J}(a)\rangle.
\end{aligned}$$

Now we just need to show that this is positive. Since J is a Jacobi field with $\|J(a)\|_g = 1$ and $\|J(\tau)\|_g = 0$ we have that $\dot{J}(a) = -\frac{1}{\tau - a} J(a)$. Therefore,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} L(c_s)\bigg|_{s=0} = \frac{1}{\tau - a} - \left\langle \frac{\partial}{\partial s} J_s(a), \dot{c}(a) \right\rangle\bigg|_{s=0}$$

Now if $c(\tau)$ was a focal point of M along c then we would like to be able to say that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} L(c_s)\bigg|_{s=0} = 0.$$

This is since we should expect the length of the geodesics to be constant in a neighborhood of $t = 0$. Let $c(\lambda)$ be the first focal point of M along c, if one does not exist we set $\lambda = +\infty$, where $c(\lambda)$ is understood in the usual sense as a limit. So we can find a Jacobi field X that witnesses $c(\lambda)$ as a focal point. We normalize X such that $\|X(a)\|_g = 1$. Now we compute

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2} L(c_s)\bigg|_{s=0} &= -\left\langle \frac{\partial}{\partial s} X_s(a), \dot{c}_s(a) \right\rangle\bigg|_{s=0} - \langle X(a), \dot{X}(a) \rangle \\ &= -\left(\frac{\partial}{\partial s} \langle X_s(a), \dot{c}_s(a) \rangle - \left\langle X_s(a), \frac{\partial}{\partial s} \dot{c}_s(a) \right\rangle \right)\bigg|_{s=0} - \langle X(a), \dot{X}(a) \rangle.\end{aligned}$$

By the construction of X_s we have that $\langle X_s(a), \dot{c}_s(a) \rangle = 0$ for all s; we also have $\frac{\partial}{\partial s} \dot{c}_s(a) = -S_{\dot{c}(a)}(X(0))$. Hence,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} L(c_s)\bigg|_{s=0} = -\langle X(a), \dot{X}(a) \rangle + S_{\dot{c}(a)}(X(a)) = 0.$$

So we conclude that

$$\left\langle \frac{\partial}{\partial s} X_0(0), \dot{c}_0(0) \right\rangle = \frac{1}{\lambda - a}.$$

Note that this value depends only on Riemannian structure of M and the point $c(a)$, but not on the point $c(\tau)$. In particular, this justifies the computation using a different vector field X than J. Now we use this computation to compute the second variation for any variation that fixed $c(\tau)$ as

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} L(c_s)\bigg|_{s=0} = \frac{1}{\tau - a} - \frac{1}{\lambda - a}.$$

From the above we deduce that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} L(c_s)\bigg|_{s=0} > 0$$

for all $\tau < \lambda$. Now we deduce that in a neighborhood of $c(a)$ that there is no shortest geodesic to $c(\tau)$ expect c. By a simple compactness argument we can find a neighborhood of $c(a)$ in M such that the desired result holds. Explicitly, consider any $v \in T_{c(a)}M$ with $\|v\|_g = 1$ and find a Jacobi field X with $X(a) = v$. Now we find some $\varepsilon_v > 0$ such that $\frac{\mathrm{d}^2}{\mathrm{d}s^2} L(c_s) > 0$ for all $s \in (-\varepsilon_v, \varepsilon_v)$. But since $\mathbf{S}^n \subseteq T_{c(a)}M$ is compact, we can take the minimum of these call it ε, and take an actual neighborhood of $c(a)$ in which c is a local minimizer.

(b) By the above, we see that $\frac{\mathrm{d}^2}{\mathrm{d}s^2} L(c_s)\Big|_{s=0} < 0$ for any point after a focal point. So we deduce that a geodesic cannot be minimizing past it's focal point.

We also present an alternative proof for this fact. Expand the domain of the index form I to the set of *all* vector fields X along c satisfying $X(t) \perp \dot{c}(t)$ for all $t \in [a, b]$ and $X(a) \in T_{c(a)}M$. This set will be denotes at Γ, and Γ_0 will denote the set of $X \in \Gamma$ such that $X(b) = 0$ as well. Let $\tau \in (a, b)$ be such that $c(\tau)$ is a focal point of M along c. We claim that for any $t > \tau$ that $c|_{[a,t]}$ is not a minimal geodesic from M to $c(t)$, i.e. $\operatorname{dist}(M, c(t)) < L(c|_{[a,t]})$.

Since $c(\tau)$ is a focal point of M along c, we have a nonzero Jacobi field J along c with $J(\tau) = 0$ and $J(a) \in T_{c(a)}M$. Define

$$\widetilde{J}(t) := \begin{cases} J(t) & \text{if } t \in [a, \tau], \\ 0 & \text{else.} \end{cases}$$

Note that $\widetilde{J} \in \Gamma_0$, and $I(\widetilde{J}, \widetilde{J}) = 0$. Now we want to perturb $\widetilde{J}$ to produce some X satisfying $I(X, X) < 0$.

Note that $\dot{J}(\tau) \neq 0$. Now let $Z(t)$ be the parallel transport along c satisfying $Z(\tau) = -\dot{J}(\tau)$. Let $\varphi : [a, b] \to \mathbf{R}$ be a smooth function such that $\varphi(a) = \varphi(b) = 0$, and $\varphi(\tau) = 1$. Define

$$X_\lambda = \widetilde{J} + \lambda \varphi Z.$$

A direct computation shows that

$$\begin{aligned} I(X_\lambda, X_\lambda) &= I(\widetilde{J}, \widetilde{J}) + 2\lambda I(\widetilde{J}, \varphi Z) + \mathcal{O}(\lambda^2) \\ &= 2\lambda \int_a^\tau \left(\langle \dot{J}, (\varphi Z)^\cdot \rangle - \langle R(\dot{c}, J)\dot{c}, \varphi Z \rangle \right) \mathrm{d}t + \mathcal{O}(\lambda^2) \\ &= 2\lambda \langle \dot{J}, \varphi Z \rangle \Big|_{t=a}^{t=\tau} + \mathcal{O}(\lambda^2) \\ &= -2\lambda \|\dot{J}(\tau)\|^2 + \mathcal{O}(\lambda^2), \end{aligned}$$

which is clearly less than 0 for sufficiently small $\lambda > 0$. Since $I(X, X)$ is the second variation of L with respect to X, we see that c cannot be a minimal geodesic past $c(\tau)$

■

Exercise 3. Let $\mathbf{S}^{n-1} := \{(x^1, \dots, x^n, 0) \in \mathbf{R}^{n+1} : \sum x^i x^i = 1\} \subseteq \mathbf{S}^n$ be the equator sphere. Determine all focal points of $\mathbf{S}^{n-1}$ in $\mathbf{S}^n$, and also all focal points of $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$.

First note that for any geodesic $c : [0, 2\pi] \to \mathbf{S}^n$ that the Jacobi fields J along c that satisfy $\dot{J} = 0$ are of the form $J(t) = \cos(t)E(t)$ where E is any parallel vector field along c. I proved a completely analogous result in Exercise 4.11. So we see that every geodesic that starts normal to c hits a focal point exactly at distance $\pi/2$. In the case when c parameterizes the equator $\mathbf{S}^{n-1}$ we find that the set of focal points in $\mathbf{S}^n$ is exactly the set of poles orthogonal to $\mathbf{S}^{n-1}$, i.e. $N = (0, 0, \dots, 0, 1)$ and $S = (0, 0, \dots, 0, -1)$ are the only focal points of $\mathbf{S}^{n-1}$ in $\mathbf{S}^n$.

Now we compute the set of focal points of $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$. Consider any geodesic $c : [a, b] \to \mathbf{R}^{n+1}$ such that $c(a) \in \mathbf{S}^n$ and $\dot{c}(a) \in (T_{c(a)}\mathbf{S}^n)^\perp$. Since $\dim \mathbf{S}^n = n = \dim \mathbf{R}^{n+1} - 1$ we see that the normal bundle of $\mathbf{S}^n$ is simply a line bundle over $\mathbf{S}^n$, and since all geodesics in $\mathbf{R}^n$ are affine we see that c is of the form

$$c(t) = (1 + t\alpha)p,$$

where $\alpha \in \mathbf{R}$ is any constant. Again, since $\mathbf{R}^{n+1}$ is a constant curvature (flat) manifold we see that all Jacobi fields are of the form $J(t) = v$ where $v \in \mathbf{R}^{n+1}$ is any vector, and where we make the usual identification of $T_{c(t)}\mathbf{R}^{n+1} \cong \mathbf{R}^{n+1}$. Now we clearly see that the only focal point of $\mathbf{S}^n \subseteq \mathbf{R}^{n+1}$ is the origin $O = (0, 0, \dots, 0) \in \mathbf{R}^{n+1}$.

■

Exercise 4. Let p,q be relatively prime integers. We can represent $\mathbf{S}^3$ as

$$\mathbf{S}^3 = \{(z_1,z_2) \in \mathbf{C}^2 \ : \ |z_1|^2 + |z_2|^2 = 1\}.$$

$\mathbf{Z}_q$ operates on $\mathbf{S}^3$ via

$$(z_1,z_2) \mapsto \left(z_1 \exp\left(\frac{2\pi i m}{q}\right), z_2 \exp\left(\frac{2\pi i m p}{q}\right)\right) \quad \text{with } 0 \le m \le q-1.$$

Show that this operation is isometric and free. The quotient $L(q,p) := \mathbf{S}^3/\mathbf{Z}_q$ is a so-called lens space. Compute its curvature and diameter.

Identify $\mathbf{R}^4$ with $\mathbf{C}^2$ by letting (x_1,x_2,x_3,x_4) correspond to (x_1+ix_2, x_3+ix_4). Let

$$\mathbf{S}^3 = \{(z_1,z_2) \in \mathbf{C}^2 \ : \ |z_1|^2 + |z_2|^2 = 1\},$$

and let $h : S^3 \to S^3$ be given by

$$h(z_1,z_2) = \left(e^{\frac{2\pi i}{q}} z_1, e^{\frac{2\pi i p}{q}} z_2\right), \quad (z_1,z_2) \in \mathbf{S}^3$$

where p and q are relatively prime integers and $q > 2$. For simplicity write $\alpha = 2\pi/q$ and $\beta = 2\pi p/q$.

We first show that the group action induced by h is isometric. Fix $k \in \{0,\dots,q-1\}$ and $u,v \in T_p\mathbf{S}^3 \subset \mathbf{C}^2$. Write

$$u = (u_1,u_2) = (u_1^1 + iu_1^2, u_2^1 + iu_2^2) \quad \text{and} \quad v = (v_1,v_2) = (v_1^1 + iv_1^2, v_2^1 + iv_2^2)$$

then

$$\begin{aligned}
(e^{ik\alpha}u_1, e^{ik\beta}u_2) &= ((\cos(k\alpha) + i\sin(k\alpha))(u_1^1 + iu_1^2), (\cos(k\beta) + i\sin(k\beta))(u_2^1 + iu_2^2)) \\
&= (u_1^1\cos(k\alpha) - u_1^2\sin(k\alpha)) + i(u_1^2\cos(k\alpha) + u_1^1\sin(k\alpha)), \\
&\qquad (u_2^1\cos(k\beta) - u_2^2\sin(k\beta)) + i(u_2^2\cos(k\beta) + u_2^1\sin(k\beta))) \\
&= (u_1^1\cos(k\alpha) - u_1^2\sin(k\alpha), u_1^2\cos(k\alpha) + u_1^1\sin(k\alpha), \\
&\qquad u_2^1\cos(k\beta) - u_2^2\sin(k\beta), u_2^2\cos(k\beta) + u_2^1\sin(k\beta)),
\end{aligned}$$

where the last equality follows since we identified $\mathbf{R}^4$ with $\mathbf{C}^2$; hence the inner product in question is that induced by $\mathbf{R}^4$, *not* the usual Hermitian inner product of $\mathbf{C}^2$. Similarly,

$$\begin{aligned}
(e^{ik\alpha}v_1, e^{ik\beta}v_2) = (&v_1^1\cos(k\alpha) - v_1^2\sin(k\alpha), v_1^1\cos(k\alpha) + v_1^1\sin(k\alpha), \\
&v_2^1\cos(k\beta) - v_2^2\sin(k\beta), v_2^2\cos(k\beta) + v_2^1\sin(k\beta)).
\end{aligned}$$

We compute

$$dh^k_{(z_1,z_2)} = \left(e^{ik\alpha}dz_1, e^{ik\beta}dz_2\right),$$

where $dz_i = dx_i + i dy_i$; therefore,

$$\begin{aligned}
\langle dh^k(u), dh^k(v)\rangle_{h^k(p)} &= \langle (e^{ik\alpha}u_1, e^{ik\beta}u_2), (e^{ik\alpha}v_1, e^{ik\beta}v_2)\rangle \\
&= (u_1^1\cos(k\alpha) - u_1^2\sin(k\alpha))((v_1^1\cos(k\alpha) - v_1^2\sin(k\alpha)) \\
&\quad + (u_1^2\cos(k\alpha) + u_1^1\sin(k\alpha))(v_1^2\cos(k\alpha) + v_1^1\sin(k\alpha)) \\
&\quad + (u_2^1\cos(k\beta) - u_2^2\sin(k\beta))(v_2^1\cos(k\beta) - v_2^2\sin(k\alpha)) \\
&\quad + (u_2^2\cos(k\beta) + u_2^1\sin(k\beta))(v_2^2\cos(k\beta) + v_2^1\sin(k\alpha)) \\
&= u_1^1v_1^1 + u_1^2v_1^2 + u_2^1v_2^1 + u_2^2v_2^2 \\
&= \langle (u_1^1, u_1^2, u_2^1, u_2^2), (v_1^1, v_1^2, v_2^1, v_2^2)\rangle \\
&= \langle (u_1^1 + iu_1^2, u_2^1 + iu_2^2), (v_1^1 + iv_1^2, v_2^1 + iv_2^2)\rangle \\
&= \langle (u_1, u_2), (v_1, v_2)\rangle \\
&= \langle u, v\rangle
\end{aligned}$$

and hence $h : S^3 \to S^3$ induces a discrete group of isometries given by $G = \{\mathrm{id}, h, h^2, h^3, \ldots, h^{q-1}\}$.

Now we show that the group action is free. It suffices to show that for all $1 \le k \le q-1$ that $h^k(z_1, z_2) \neq (z_1, z_2)$ for all $z_1, z_2 \in \mathbf{C}$. Since p and q are relatively prime, there exists $s, t \in \mathbf{Z}$ such that $sq + tp = 1$. Now if some $k \in \{1, \ldots, q-1\}$ satisfies

$$e^{ik\alpha} = 1 \quad \text{or} \quad e^{ik\beta} = 1$$

then $k = mq$ or $kp = mq$ for some $m \in \mathbf{Z}$. In the first case we immediately obtain a contradiction with the coprimality of p and q. In the second case we also obtain a contradiction since

$$k = skq + tkp = skq + tmq = (sk + tm)q.$$

So we have shown that the group action is free.

To ensure the the Lens space $\mathbf{S}^3/\mathbf{Z}_q$ is well defined we need to ensure that the group action is properly discontinuous. We need to demonstrate for all $p \in \mathbf{S}^3$ the existence of an open set U containing p such that $h^k(U) \cap U = \emptyset$ for all $k = 1, \ldots, q-1$. Fix $p = (z_1, z_2) \in \mathbf{S}^3$ and write $q_k = h^k(p)$ for all $k = 1, \ldots, q-1$. Since $\mathbf{S}^3$ is Hausdorff and since the group action is free there exists open sets V_k containing q_k, respectively, such that $U \cap V_k = \emptyset$. Since h is continuous, we may retract U such that $h^k(U) \subseteq V_k$ for all $k = 1, \ldots, q-1$. In particular, we see that such a U suffices, and so the group action induced by h is properly discontinuous. Now we deduce by the quotient manifold theorem that $\mathbf{S}^3/\mathbf{Z}_q$ is a smooth manifold endowed with a canonical smooth structure making the projection map a local diffeomorphism.

Now we study the geometric properties of the Lens space. Since the group action acts freely and isometrically on $\mathbf{S}^3$ we see that $\mathbf{S}^3/\mathbf{Z}_q$ inherits the Riemannian structure of constant curvature 1, and the projection $\mathbf{S}^3 \to \mathbf{S}^3/\mathbf{Z}_q$ is a local isometry. Since the projection is a local isometry we see that every geodesic on $L(q, p)$ lifts to a geodesic on $\mathbf{S}^3$; in particular, we deduce that all of the geodesics on $L(q, p)$ are closed.

Now we compute the diameter. From the Bonnet-Myers theorem we immediately deduce that $\mathrm{diam}(L(q, p)) \le \pi$, unfortunately this is too weak of a bound. Consider any orbit $\mathbf{Z}_q \cdot x$ of the action on $\mathbf{S}^3$, and the associated Voronoi tiling, that is: since $\mathbf{Z}_q \cdot x$ is a finite set of points in $\mathbf{S}^3$ we can partition S^3 into $n := |\mathbf{Z}_q \cdot x| = \{x_1, \ldots, x_n\}$ cells based on their distance; namely, for $i = 1, \ldots, n$ we let

$$V_i := \{y \in \mathbf{S}^3 \,:\, d(y, x_i) \le d(y, x_j) \text{ for all } j \neq i\}.$$

Note that the tile V_x, containing the point x, is the intersection of the hemispheres bounded by the bisectors of x and gx for all $g \in \mathbf{Z}_q \setminus \{1\}$. We have a similar result for all of the other tiles. In particular, we deduce that V_x is contained in a hemisphere centered at x, and so the distance from x to any point $y \in V_x$ is bounded above by $\pi/2$. It then follows that since the group action is isometric and since all of the point $\mathbf{Z}_q \cdot x$ are identified in $\mathbf{S}^3/\mathbf{Z}_q$

that the distance between $[x]$ and $[y]$ in $\mathbf{S}^3/\mathbf{Z}_q$ is at most $\pi/2$. Since the Voronoi cells cover $\mathbf{S}^3$, we deduce that $\operatorname{diam}(L(q,p)) \le \pi/2$. We claim that the diameter is exactly $\pi/2$, and so we need to exhibit a two points in $L(q,p)$ with distance exactly $\pi/2$. Let $\pi : \mathbf{S}^3 \to L(q,p)$ be the projection, and set $x = \pi(1,0,0,0)$ and $y = \pi(0,0,0,1)$. We claim that $d(x,y) = \pi/2$. Indeed, they are connected by a geodesic $\gamma(t) = \pi(\cos(t),0,0,\sin(t))$ for $t \in [0,\pi/2]$, and since $t \mapsto (\cos(t),0,0,\sin(t))$ is contained entirely in a single Voronoi cell we deduce that $\gamma(t)$ is globally minimizing, since it is the projection of a globally minimizing geodesic in S^3. In particular, we deduce that $L(\gamma) = \pi/2$. In particular, we have shown that $\operatorname{diam}(L(q,p)) = \pi/2$. ■

Exercise 5. Show that any compact odd-dimensional Riemannian manifold with positive sectional curvature is orientable.

Let M be a compact odd-dimensional Riemannian manifold with positive sectional curvature. Recall that from Exercise 3.4 that the parallel transport map, with respect to the Levi-Civita connection, induces a group homomorphism $\pi_1(M;p) \to \mathbf{O}(T_pM)$ for all $p \in M$. In particular, we see that for any closed curve $\gamma \in \pi_1(M;p)$ that $\mathscr{P}_\gamma : T_pM \to T_pM$ is an isometry; in particular $\det \mathscr{P}_\gamma = \pm 1$.

Assume, for the sake of contradiction, that M is not orientable. Then consider any non-orientable closed path, $\gamma^\times \in \pi_1(M;p)$. Then we see that $[\gamma^\times]$ is a nonzero homotopy class such that for any closed curve $\gamma : [0,1] \to M$ in $[\gamma^\times]$, $\det \mathscr{P}_\gamma = -1$. Now fix $\gamma \in [\gamma^\times]$ to be the representative which minimizes the length functional. Since $\mathscr{P}_\gamma(\dot\gamma(0)) = \dot\gamma(0)$, we see that

$$\det \mathscr{P}_\gamma|_{(\dot\gamma(0))^\perp} = -1,$$

where $(\dot\gamma(0))^\perp$ is the orthogonal complement in T_pM. Observe that

$$\dim(\dot\gamma(0))^\perp = \dim T_pM - 1 = \dim M - 1$$

is even. So in particular, there exists a unique fixed point $v \in (\dot\gamma(0))^\perp$ such that

$$\mathscr{P}_\gamma(v) = v.$$

Now let X be the parallel vector field along γ with $X(0) = v$. Now consider any variation $\overline{\gamma} : \mathbf{S}^1 \times (-\varepsilon,\varepsilon) : (t,s) \mapsto \overline{\gamma}(t,s)$ of γ with $\overline{\gamma}'(t,0) = X(t)$ for all $t \in \mathbf{S}^1$. Since $\overline{\gamma}$ is a geodesic variation, we see that

$$\frac{\partial}{\partial t} E(\gamma(t,\cdot))\bigg|_{t=0} = 0,$$

and since X is parallel and satisfies $X(0) = X(2\pi)$,

$$\begin{aligned}
\frac{\partial^2}{\partial s^2} E(\overline{\gamma}(\cdot,s))\bigg|_{s=0} &= \int_0^{2\pi} \left\langle \nabla_{\frac{\partial}{\partial t}} X(t), \nabla_{\frac{\partial}{\partial t}} X(t) \right\rangle \,\mathrm{d}t - \int_0^{2\pi} \langle R(\partial_s\gamma, X)X, \partial_s\gamma\rangle \,\mathrm{d}t \\
&= -\int_0^{2\pi} \langle R(\partial_s\gamma, X)X, \partial_s\gamma\rangle \,\mathrm{d}t \\
&< 0.
\end{aligned}$$

Hence, for sufficiently small $s > 0$,

$$E(\overline{\gamma}(\cdot,s)) < E(\gamma).$$

However, since the variation remains in the same homotopy class we have a contradiction with the fact that γ minimizes the length (and energy) of all closed curves in it's homotopy class. So we deduce that M is orientable. ■

Exercise 6. Show that the real projective space $\mathbf{RP}^n$ is orientable for odd n and non-orientable for even n.

Since $\mathbf{RP}^n$ is the quotient of $\mathbf{S}^n$ under the equivalence relation identifying oppositive points, we have a natural two-fold covering map $\mathbf{S}^n \to \mathbf{RP}^n$. Now recall that $\mathbf{S}^n$ is simply connected. Now it is a standard result from algebraic topology that if $\pi : \overline{X} \to X$ is a universal covering space, then $\pi_1(X; x_0) = \pi^{-1}(x_0)$, as sets. Hence, $\pi_1(\mathbf{RP}^n; x_0)$ has only two elements; now we deduce

$$\pi_1(\mathbf{RP}^n; x_0) = \mathbf{Z}_2,$$

since $\mathbf{Z}_2$ is the only two-element group up to isomorphism. Recall that the scalar curvature of $\mathbf{RP}^n$ is identically 1. Now by Synge's theorem we immediately see that $\mathbf{RP}^n$ is non-orientable for even n. On the other hand by Exercise 4.5 we see that for odd n that $\mathbf{RP}^n$ is orientable.

We can also obtain this result in a more direct way without using Synge's theorem. Let $\iota : \mathbf{S}^n \to \mathbf{R}^{n+1}$ be the inclusion map, $\widetilde{a} : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be the antipodal map $x \mapsto -x$, and $a : \mathbf{S}^n \to \mathbf{S}^n$ be the restriction of $\widetilde{a}$ to $\mathbf{S}^n$. We have the following commutative diagram:

$$\begin{tikzcd} \mathbf{S}^n \arrow[r, "a"] \arrow[d, "\iota"'] & \mathbf{S}^n \arrow[d, "\iota"] \\ \mathbf{R}^{n+1} \arrow[r, "\widetilde{a}"] & \mathbf{R}^{n+1} \end{tikzcd}$$

Now let $\omega = dx^1 \wedge \cdots \wedge dx^{n+1}$ be the standard volume form on $\mathbf{R}^{n+1}$, and consider the vector field

$$X = \sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i} \in \Gamma(\mathbf{R}^{n+1}; T\mathbf{R}^{n+1}).$$

Note that the function $f : \mathbf{R}^{n+1} \to \mathbf{R}$, given by $f(x) = \|x\|$, is constant on $\mathbf{S}^n$, and so df_p vanishes on $T_p\mathbf{S}^n \subseteq \mathbf{R}^{n+1}$. Now since

$$X(f) = \sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i}(f) = 2\sum_{i=1}^{n+1}(x^i)^2 = 2\|x\|^2,$$

$X(f)_p \neq 0$ for all $p \in \mathbf{S}^n$; hence $X_p \notin T_p\mathbf{S}^n$. Now since ω is a volume form on $\mathbf{R}^{n+1}$, it follows that the form

$$\alpha := \iota^*(i_X\omega) = (i_X\omega)|_{T\mathbf{S}^n}$$

is a volume form on $\mathbf{S}^n$. Since $\widetilde{a}^*\omega = (-1)^{n+1}\omega$ and $d\widetilde{a}(X) = X$ on $\mathbf{R}^{n+1}$,

$$a^*\alpha = a^*\iota^*(i_X\omega) = \iota^*\widetilde{a}^*(i_{d\widetilde{a}(X)}\omega) = \iota^*(i_X(\widetilde{a}^*\omega)) = (-1)^{n+1}\iota^*(i_X\omega) = (-1)^{n+1}\alpha.$$

So we see that $a : \mathbf{S}^n \to \mathbf{S}^n$ is orientation preserving if and only if $n+1$ is even. Now recall that if $\overline{M}$ is a smooth manifold and G is a group that acts on $\overline{M}$ smoothly and properly discontinuously that $M = \overline{M}/G$ is a smooth manifold, with smooth structure induced from that of $\pi : \overline{M} \to M$. In particular, we have that

$$\Omega^\bullet(\overline{M}/G) = \{\pi^*\alpha \ : \ \alpha \in \Omega^\bullet(M)\} = \{\overline{\alpha} \in \Omega^\bullet(\overline{M}) \ : \ g^*\overline{\alpha} = \overline{\alpha} \text{ for all } g \in G\}.$$

Now if n is odd, then $\alpha \in \Omega^n(\mathbf{S}^n/\mathbf{Z}_2) = \Omega^n(\mathbf{RP}^n)$ and defines an orientation on $\mathbf{RP}^n$; so $\mathbf{RP}^n$ is orientable if n is odd. On the other hand, if n is even, there exists no nonvanishing $\beta \in \Omega^n(\mathbf{S}^n/\mathbf{Z}_2) = \Omega^n(\mathbf{RP}^n)$ and so $\mathbf{RP}^n$ is not orientable in this case by the above. To see that there doesn't exists such a volume form, note that for $\beta \in \Omega^n(\mathbf{S}^n/\mathbf{Z}_2)$ that $\beta = f\alpha$ for some $f \in \mathscr{C}^\infty(\mathbf{S}^n)$. Hence,

$$f\alpha = \beta = a^*\beta = (f \circ a)a^*\alpha = -(f \circ a)\alpha,$$

which implies $f \circ a = -f$. Hence, f and β must vanish somewhere on $\mathbf{S}^n$. ∎

Exercise 7. Show that Synge's theorem does not hold in odd dimensions.

By the previous two exercises, we see that $\pi_1(\mathbf{RP}^n) \cong \mathbf{Z}_2$, $\mathbf{RP}^n$ has positive sectional curvature, is compact, and is oriented. Hence, this would produce a counter example to the following *false theorem*:

Theorem 5.1.2* (*False* odd-dimensional Synge)**.** *Any compact oriented odd-dimensional Riemannian manifold with positive sectional curvature is simply connected.*

∎

Exercise 8. Try to generalize the theory of Jacobi fields to other variational problems.

∎

Exercise 9 ***(A more difficult exercise).*** Compute the second variation of volume for a minimal submanifold of a Riemannian manifold.

Let $M \subseteq N$ be a minimal submanifold. Now consider a smooth variation $\widetilde{M} : M \times (-\varepsilon, \varepsilon) \to N$ for some $\varepsilon > 0$ that is compactly supported, i.e. there is a compact set $K \subseteq M$ such that for all $x \notin K$ $\widetilde{M}(x,t) = x$ for all $t \in (-\varepsilon, \varepsilon)$. By the implicit function theorem, by shrinking ε if need be, we can assume that $\Phi_t(\cdot) := \widetilde{M}(\cdot, t)$ is a diffeomorphism from $M \to M_t$ for all $t \in (-\varepsilon, \varepsilon)$. Since we are considering local variations we can assume that $\{x \in M : \widetilde{M}(x,t) \neq x\}$ is orientable. Now let $\{e_1, \ldots, e_m\}$ be a positively oriented orthonormal frame of TM, and note that $e_1 \wedge \cdots \wedge e_m$ has unit norm in $\bigwedge^m(TM)$ with the induced inner product.

Since Φ_t is a diffeomorphism we can write

$$\mathrm{vol}(M_t) = \int_{M_t} \mathrm{vol}_{M_t} = \int_M \Phi_t^* \mathrm{vol}_M = \int_M \langle \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m \rangle^{\frac{1}{2}} \ \mathrm{vol}_M \,.$$

Now we can compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(M_t) &= \int_M \frac{\mathrm{d}}{\mathrm{d}t} \langle \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m \rangle^{\frac{1}{2}} \ \mathrm{vol}_M \\
&= \frac{1}{2} \int_M \frac{\frac{\mathrm{d}}{\mathrm{d}t} \langle \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m \rangle}{\langle \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m \rangle^{\frac{1}{2}}} \ \mathrm{vol}_M \\
&= \sum_{j=1}^m \int_M \frac{\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial}{\partial t}\Phi_{t*}e_j \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m \rangle}{\|\Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\|} \ \mathrm{vol}_M \,.
\end{aligned}$$

We now explain the notation $\frac{\partial}{\partial t}\Phi_{t*}e_i$. Let $\gamma_i(s)$ be a smooth curve on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = d\gamma_i\left(\frac{\partial}{\partial s}\big|_{s=0}\right) = e_i$ and let $\gamma_i(s,t) = \Phi_t(\gamma_i(s))$. We then have by the chain rule

$$\Phi_{t*}e_i = d\Phi_t(e_i) = d\Phi_t\left(d\gamma_i\left(\frac{\partial}{\partial s}\bigg|_{s=0}\right)\right) = d(\Phi_t \circ \gamma_i)\left(\frac{\partial}{\partial s}\bigg|_{s=0}\right) = \frac{\partial}{\partial s}\gamma_i(s,t)\bigg|_{s=0} .$$

Now by $\frac{\partial}{\partial t}\Phi_{t*}e_i$ we mean the covariant derivative $\frac{\overline{\nabla}}{\partial t}\Phi_{t*}e_i$. In particular, we have

$$\frac{\partial}{\partial t}\Phi_{t*}e_i\bigg|_{t=0} = \frac{\overline{\nabla}}{\partial t}\frac{\partial}{\partial s}\gamma_i(s,t)\bigg|_{s=t=0} = \frac{\overline{\nabla}}{\partial s}\frac{\partial}{\partial t}\gamma_i(s,t)\bigg|_{s=t=0} = \overline{\nabla}_{\frac{\partial}{\partial s}}X\bigg|_{s=0} = \overline{\nabla}_{e_i}X,$$

where $X := \frac{\partial}{\partial t}\Phi_t\big|_{t=0}$.

Since a tangential variation of M does not affect its image in N, without loss of generality we can assume that $X \in \Gamma(TM^\perp)$. That is to say we only consider normal variations of M in N. *I'm still not sure how to make this entirely rigorous!*

We now compute the second variation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\operatorname{vol}(M_t) = \sum_{i=1}^m \int_M \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial}{\partial t}\Phi_{t*}e_i \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\rangle}{\|\Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\|}\right)\operatorname{vol}_M$$

$$\begin{aligned}
= \int_M \Bigg(& 2\sum_{i<j}^m \frac{\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial}{\partial t}\Phi_{t*}e_i \wedge \cdots \wedge \frac{\partial}{\partial t}\Phi_{t*}e_j \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\rangle}{\|\Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\|} \\
&+ \sum_{i,j=1}^m \frac{\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial}{\partial t}\Phi_{t*}e_i \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial}{\partial t}\Phi_{t*}e_j \wedge \cdots \wedge \Phi_{t*}e_m\rangle}{\|\Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\|} \\
&- \sum_{i,j=1}^m \frac{\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial}{\partial t}\Phi_{t*}e_i \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\rangle\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial}{\partial t}\Phi_{t*}e_j \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\rangle}{\|\Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\|^3} \\
&+ \sum_{i=1}^m \frac{\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial^2}{\partial t^2}\Phi_{t*}e_i \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\rangle}{\|\Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\|}\Bigg)\operatorname{vol}_M
\end{aligned}$$

The only new thing we need to compute is $\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial^2}{\partial t^2}\Phi_{t*}e_i \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\rangle$ at $t = 0$. Since $\Phi_{t*}e_i|_{t=0} = e_i$ form an orthonormal basis of T_pM we deduce that

$$\left\langle \Phi_{t*}e_1 \wedge \cdots \wedge \frac{\partial^2}{\partial t^2}\Phi_{t*}e_i \wedge \cdots \wedge \Phi_{t*}e_m, \Phi_{t*}e_1 \wedge \cdots \wedge \Phi_{t*}e_m\right\rangle\Bigg|_{t=0} = \left\langle \frac{\partial^2}{\partial t^2}\Phi_{t*}e_i, \Phi_{t*}e_i\right\rangle\Bigg|_{t=0}.$$

Again, let $\gamma_i(s)$ be a smooth curve on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = e_i$ and write $\gamma_i(s,t) = \Phi_t(\gamma_i(s))$. Then

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\Phi_{t*}e_i\Bigg|_{t=0} &= \frac{\overline\nabla}{\partial t}\frac{\overline\nabla}{\partial t}\frac{\partial}{\partial s}\gamma_i(s,t)\Bigg|_{s=t=0} \\
&= \frac{\overline\nabla}{\partial t}\frac{\overline\nabla}{\partial s}\frac{\partial}{\partial t}\gamma_i(s,t)\Bigg|_{s=t=0} \\
&= \frac{\overline\nabla}{\partial s}\frac{\overline\nabla}{\partial t}\frac{\partial\gamma_i}{\partial t} + R\left(\frac{\partial\gamma_i}{\partial s}, \frac{\partial\gamma_i}{\partial t}\right)\frac{\partial\gamma_i}{\partial t}\Bigg|_{s=t=0} \\
&= R(e_i, X)X,
\end{aligned}$$

where R is the $(3,1)$-type Riemann curvature tensor field on N. Note that the first term dropped out since $\gamma_i(s,0)$ is a geodesic and the geodesic equations are simply $\frac{\nabla}{\partial t}\frac{\partial\gamma}{\partial t} = 0$. So we see that

$$\left\langle \frac{\partial^2}{\partial t^2}\Phi_{t*}e_i, \Phi_{t*}e_i\right\rangle\Bigg|_{t=0} = \langle R(e_i, X)X, e_i\rangle.$$

Plugging this into out equation for the second variation of volume gives us

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\operatorname{vol}(M_t)\bigg|_{t=0} &= \int_M \Bigg(2\sum_{i<j}\langle e_1\wedge\cdots\wedge\overline{\nabla}_{e_i}X\wedge\cdots\wedge\overline{\nabla}_{e_j}X\wedge\cdots\wedge e_m, e_1\wedge\cdots\wedge e_m\rangle \\
&\qquad + \sum_{i,j=1}^m\langle e_1\wedge\cdots\wedge\overline{\nabla}_{e_i}X\wedge\cdots\wedge e_m, e_1\wedge\cdots\wedge\overline{\nabla}_{e_j}X\wedge\cdots\wedge e_m\rangle \\
&\qquad - \sum_{i,j=1}^m\langle e_1\wedge\cdots\wedge\overline{\nabla}_{e_i}X\wedge\cdots\wedge e_m, e_1\wedge\cdots\wedge e_m\rangle\langle e_1\wedge\cdots\wedge\overline{\nabla}_{e_j}X\wedge\cdots\wedge e_m, e_1\wedge\cdots\wedge e_m\rangle \\
&\qquad + \sum_{i=1}^m\langle e_1\wedge\cdots\wedge R(e_i,X)X\wedge\cdots\wedge e_m, e_1\wedge\cdots\wedge e_m\rangle\Bigg)\operatorname{vol}_M \\
&= \int_M\Bigg(2\sum_{i<j}\langle\overline{\nabla}_{e_i}X\wedge\overline{\nabla}_{e_j}X, e_i\wedge e_j\rangle + \sum_{i,j=1}^m\langle e_1\wedge\cdots\wedge\overline{\nabla}_{e_i}X\wedge\cdots\wedge e_m, e_1\wedge\cdots\wedge\overline{\nabla}_{e_j}X\wedge\cdots\wedge e_m\rangle \\
&\qquad - \sum_{i,j=1}^m\langle\overline{\nabla}_{e_i}X, e_i\rangle\langle\overline{\nabla}_{e_j}X, e_j\rangle + \sum_{i=1}^m\langle R(e_i,X)X, e_i\rangle\Bigg)\operatorname{vol}_M
\end{aligned}$$

where Ric is the Ricci curvature tensor.

Let $\nu\in T_pM^\perp$ and consider the second fundamental tensor $S_\nu : T_pM\to T_pM$, defined as $S_\nu(X) = (\overline{\nabla}_X\nu)^\top$. Now consider the *normal connection* $\nabla^\perp$ on $TM^\perp$ given as follows: if $\nu\in\Gamma(TM^\perp)$ and $X\in\Gamma(TM)$ then

$$\nabla^\perp_X\nu = \Pi^\perp\overline{\nabla}_X\nu,$$

where $\Pi^\perp(p)$ is the orthogonal projection of T_pN onto $TM^\perp$. Now we see that $\nabla^\perp_X\nu = \overline{\nabla}_X\nu - S_\nu(X)$. For all $i=1,\dots,m$ write S_X in the $\{e_1,\dots,e_m\}$ basis as

$$S_Xe_i = s_X^{i\ell}e_\ell.$$

Now since we are only considering normal variations we see that $X^\top = 0$. We simplify the first sum in the integrand as

$$\sum_{i<j}\langle\overline{\nabla}_{e_i}X\wedge\overline{\nabla}_{e_j}X, e_i\wedge e_j\rangle = \sum_{i<j}\langle S_X(e_i)\wedge S_Xe_j, e_i\wedge e_j\rangle = \sum_{i<j}\sum_{\ell,k=1}^m s_X^{i\ell}s_X^{ik}\langle e_\ell\wedge e_k, e_i\wedge e_j\rangle = \sum_{i<j}\left(s_X^{ii}s_X^{jj} - s_X^{ij}s_X^{ji}\right).$$

We can write this last expression as $\operatorname{tr}\left(\bigwedge^2 S_X\right)$, where we use the functoriality of $\bigwedge^2$ to obtain the map $\bigwedge^2 S_X : \bigwedge^2 TM\to\bigwedge^2 TM$.

Similarly, we deduce that the second sum in the integrand simplifies to

$$\sum_{i,j=1}^m\langle e_1\wedge\cdots\wedge\overline{\nabla}_{e_i}X\wedge\cdots\wedge e_m, e_1\wedge\cdots\wedge\overline{\nabla}_{e_j}X\wedge\cdots\wedge e_m\rangle = \sum_{i,j=1}^m s_X^{ii}s_X^{jj} + \langle\nabla^\perp_{e_i}X, \nabla^\perp_{e_j}X\rangle.$$

Finally, the third sum in the integrand simplifies to

$$\sum_{i,j=1}^m\langle\overline{\nabla}_{e_i}X, e_i\rangle\langle\overline{\nabla}_{e_j}X, e_j\rangle = \sum_{i,j=1}^m s_X^{ii}s_X^{jj}.$$

Now we see that our second variation of volume formula becomes

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\operatorname{vol}(M_t)\bigg|_{t=0} &= \int_M \left(2\operatorname{tr}\left(\bigwedge^2 S_X\right) + \|\nabla^\perp X\|^2 + \operatorname{Ric}(X,X)\right)\operatorname{vol}_M \\ &= \int_M \left((\operatorname{tr} S_X)^2 - \operatorname{tr}(S_X^* S_X) + \|\nabla^\perp X\|^2 + \operatorname{Ric}(X,X)\right)\operatorname{vol}_M \\ &= \int_M \left(m^2 H_X - \operatorname{tr}(S_X^* S_X) + \|\nabla^\perp X\|^2 + \operatorname{Ric}(X,X)\right)\operatorname{vol}_M,\end{aligned}$$

where H_X is the mean curvature of M in the direction of X in N. Now we see that if $M = M_0$ is a minimal submanifold that the mean curvature vanishes identically and we have that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\operatorname{vol}(M_t)\bigg|_{t=0} = \int_M \left(\|\nabla^\perp X\|^2 - \operatorname{tr}(S_X^* S_X) + \operatorname{Ric}(X,X)\right)\operatorname{vol}_M .$$

■

Exercise 10. Give examples to show that the curve $\exp_p tv$ as in Corollary 4.2.4 need not be the shortest connection of its endpoints.

Let M be a the flat torus $\mathbf{T}^2 \subseteq \mathbf{R}^4$ parameterized by $(s,t) \mapsto (e^{is}, e^{it})$, where we make the natural identification of $\mathbf{C}^2 \times \mathbf{C}^2 \cong \mathbf{R}^4$. We endow $\mathbf{T}^2$ with the induced Riemannian metric from $\mathbf{R}^4$. Since $\mathbf{T}^2$ is compact we have that $\mathbf{T}^2$ is geodesically complete and so the domain of $\exp_p$ is all of $T_p\mathbf{T}^2$ for all $p \in \mathbf{T}^2$. Now consider the points $p = (e^{7i\pi/4}, e^{i\pi})$ and $q = (e^{i\pi/4}, e^{i\pi})$, the vector $v \in T_p\mathbf{T}^2$ be given by $v = (-3i\pi/2, 0) \in \mathbf{R}^4$ and $w = (i\pi/2, 0)$, and the curves $\gamma_v : \mathbf{R} \to \mathbf{T}^2$ defined by

$$\gamma_v(t) = \exp_p(tv) = \left(e^{i\left(\frac{7\pi}{4} - \frac{3\pi}{2}t\right)}, e^{i\pi}\right),$$

and $\gamma_w : \mathbf{R} \to \mathbf{T}^2$ defined by

$$\gamma_w(t) = \exp_p(tw) = \left(e^{i\left(\frac{7\pi}{4} + \frac{\pi}{2}t\right)}, e^{i\pi}\right).$$

A direct computation shows that

$$L((\gamma_v)|_{[0,1]}) = \int_0^1 \|\dot{\gamma_v}(t)\|\,\mathrm{d}t = \int_0^1 \frac{3\pi}{2}\,\mathrm{d}t = \frac{3\pi}{2},$$

but

$$L((\gamma_w)|_{[0,1]}) = \int_0^1 \|\dot{\gamma_w}(t)\|\,\mathrm{d}t = \int_0^1 \frac{\pi}{2}\,\mathrm{d}t = \frac{\pi}{2}.$$

We deduce that the curve $\gamma_v(t) = \exp_p(tw)$ is not globally minimizing.

Another way to show that the exponential map need not be globally minimizing is as follows. Again, consider $\mathbf{T}^2$. Consider the dense geodesic given by

$$\gamma(t) = \left(e^{it}, e^{i\sqrt{2}t}\right).$$

This curve extends for all time and never intersects itself. Since the diameter of $\mathbf{T}^2$ is finite we see that there exists some time $t > 0$ such that $L(\gamma|_{[0,t]}) > \operatorname{diam}(\mathbf{T}^2)$. Now take $p = \gamma(0)$ and $q = \gamma(t)$, and clearly $\gamma|_{[0,t]}$ is not the shortest connection of its endpoints. ■

Exercise 11. Let $c : [0,\infty) \to \mathbf{S}^n$ be a geodesic parameterized by arc length. For $t > 0$, compute the dimension of the space J_c^t of Jacobi fields X along c with $X(0) = 0 = X(t)$. Use the Morse index theorem 4.3.2 to compute the indices and nullities of geodesics on $\mathbf{S}^n$.

Let $c : [0,\infty) \to \mathbf{S}^n$ be a unit speed geodesic. Let J be any Jacobi field. Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J, \dot{c}\rangle = \langle \ddot{J}, \dot{c}\rangle = -\langle R(J,\dot{c})\dot{c}, \dot{c}\rangle = 0,$$

and so $\langle J, \dot{c}\rangle$ is a linear function of t. We compute $(\langle J, \dot{c}\rangle)^{\cdot} = \langle \dot{J}, \dot{c}\rangle$, and so we deduce that J, and $\dot{J}$ are orthogonal to $\dot{\gamma}$ if and only if $\langle J, \dot{c}\rangle = \langle \dot{J}, \dot{c}\rangle = 0$, which is clearly equivalent to $\langle J, \dot{c}\rangle = 0$ for all $t \geq 0$. Note that in particular, if $J(t_1) = J(t_2) = 0$ for two distinct times $t_1 \neq t_2$ then $\langle J, \dot{c}\rangle = 0$ for all t since the map is linear.

From this we deduce that all Jacobi fields J along c with $J(0) = 0 = J(t)$ for any $t > 0$ must be normal to $\dot{c}$ for all $t > 0$.

First we characterize all of the normal Jacobi fields, J, along c with $J(0) = 0$. We claim that they are all of the form $J(t) = s_1(t)E(t) = \sin(t)E(t)$ where E is any parallel normal vector field along c. Recall that the Riemann curvature tensor of $\mathbf{S}^n$ is given by $R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$. Plugging this into the Jacobi equation, and using the facts $\|\dot{c}\| = 1$ and $\langle J, \dot{c}\rangle = 0$, gives us

$$\ddot{J} + \langle \dot{c}, \dot{c}\rangle J - \langle J, \dot{c}\rangle \dot{c} = \ddot{J} + J = 0.$$

Now we see that $\ddot{J} = -J$, and so it is reasonable to look for J of the form $J(t) = \alpha(t)E(t)$ where E is a fixed parallel normal vector field to $\dot{c}$. We compute $\dot{J}(t) = \dot{\alpha}(t)E(t) + \alpha(t)\dot{E}(t)$, $\ddot{J}(t) = \ddot{\alpha}(t)E(t) + 2\dot{\alpha}(t)\dot{E}(t) + \alpha(t)\ddot{E}(t)$, and so we find that the Jacobi equation reduces to

$$(\ddot{\alpha}(t) + \alpha(t))E(t) = 0,$$

where we used the fact that $E(t)$ is parallel to c to deduce that $\dot{E} = \ddot{E} = 0$ identically. By standard uniqueness of ordinary differential equations we find that all solutions are constant multiples of $\alpha(t) = s_1(t) = \sin(t)$. We find that these are all possible solutions since the dimension of the space of all such solutions is $n-1$, and the dimension of the space of all normal Jacobi fields with $J(0) = 0$ is also $n-1$.

Since $\sin(t) = 0$ only when $t \in \mathbf{N}\pi$ we deduce that if $t \in \mathbf{N}\pi$ that $\dim J_c^t = n-1$. On the other hand, for all other t we have that $\dim J_c^t = 0$. This follows since the Levi-Civita connection is flat, and so the parallel transport map is an isomerty of the tangent spaces; i.e. if $E(0) \neq 0$ then $E(t) \neq 0$ and since $\sin(t) \neq 0$ we see that $J(t) \neq 0$ as well. Hence, $J_c^t = \{0\}$ for all $t \notin \mathbf{N}\pi$. Summarizing, we have

$$\dim J_c^t = \begin{cases} n-1 & \text{if } t = n\pi \text{ for some } n \in \mathbf{N}, \\ 0 & \text{else.} \end{cases}$$

Note that every geodesic $c : [0,\infty) \to \mathbf{S}^n$ has a period of 2π, so we only consider $c : [0,2\pi] \to \mathbf{S}^n$. By the Morse-Index theorem we have

$$\mathrm{Ind}(c) = \sum_{t\in(0,2\pi)} \dim J_c^t = n-1,$$

$$\mathrm{Ind}_0(c) = \sum_{t\in(0,2\pi]} \dim J_c^t = 2n-2.$$

So we deduce that the nullity of c is

$$N(c) = \mathrm{Ind}_0(c) - \mathrm{Ind}(c) = n-1.$$

We could have also used Lemma 4.3.3 to immediately deduce that $N(c) = n-1$. ∎

Exercise 12. Show that if under the assumptions of Theorem 4.5.1 we have equality in (4.5.6) for some t with $0 < t \le \tau$, then the sectional curvature of the plane spanned by $\dot{c}(s)$ and $J(s)$ is equal to μ for all s with $0 \le s \le t$.

We first recall the theorem:

Theorem 4.5.1. *Suppose $K \le \mu$, and as always, $\|\dot{c}\| \equiv 1$. Assume either $\mu \ge 0$ or $J^{\tan} \equiv 0$. Let $f_\mu := |J(0)|c_\mu + |J|^{\cdot}(0)s_\mu$ solve*

$$\ddot{f} + \mu f = 0$$

with $f(0) = |J(0)|$, $\dot{f}(0) = |J|^{\cdot}(0)$, i.e. $f_\mu = |J(0)|c_\mu + |J|^{\cdot}(0)s_\mu$.

If

$$f_\mu(t) > 0 \qquad \text{for } 0 < t < \tau, \tag{4.5.3}$$

then

$$\langle J, \dot{J}\rangle f_\mu \ge \langle J, J\rangle \dot{f}_\mu \qquad \text{on } [0,\tau], \tag{4.5.4}$$

$$1 \le \frac{|J(t_1)|}{f_\mu(t_1)} \le \frac{|J(t_2)|}{f_\mu(t_2)}, \qquad \text{if } 0 < t_1 \le t_2 < \tau, \tag{4.5.5}$$

$$|J(0)|c_\mu(t) + |J|^{\cdot}(0)s_\mu(t) \le |J(t)| \qquad \text{for } 0 \le t \le \tau. \tag{4.5.6}$$

We now solve the exercise.

Suppose that there exists some t with $0 < t < \tau$ such that

$$|J(0)|c_\mu(t) + |J|^{\cdot}(0)s_\mu(t) = |J(t)|.$$

We will consider the function $\Psi(s) = |J|^{\cdot}(s)f_\mu(s) - |J(s)|\dot{f}_\mu(s)$ for $0 < s < \tau$. Note that

$$\Psi(0) = |J|^{\cdot}(0)f_\mu(0) - |J(0)|\dot{f}_\mu(0) = |J|^{\cdot}(0)|J(0)| - |J(0)||J|^{\cdot}(0) = 0.$$

Furthermore, we have that $\dot{\Psi}(s) \ge 0$ since $\ddot{f}_\mu + \mu f_\mu = 0$. Note that this implies

$$\left(\frac{|J|}{f_\mu}\right)^{\cdot} = \frac{1}{f_\mu^2}\left(|J|^{\cdot} f_\mu - |J|\dot{f}_\mu\right) \ge 0.$$

In particular, since we have equality in (4.5.6) at time t we also have that $\left(\frac{|J|}{f_\mu}\right)^{\cdot} = 0$. Hence,

$$\Psi(t) = |J|^{\cdot}(t)f_\mu(t) - |J|(0)\dot{f}_\mu(t) = |J|^{\cdot}(t)|J(t)| - |J(t)||J|^{\cdot}(t) = 0.$$

So we have shown that $\Psi(s) = 0$ for all $0 \le s \le t$, and so $|J(s)| = f_\mu(s)$ for all $0 \le s \le t$. Since $|J|^{\cdot\cdot} + \mu|J| = 0$ for all $0 \le s \le t$, we have

$$\begin{aligned}
|J|^{\cdot\cdot} + \mu|J| &= \frac{1}{|J|}\left(\mu\langle J, J\rangle - \langle R(J,\dot{c})\dot{c}, J\rangle\right) + \frac{1}{|J|^3}\left(|\dot{J}|^2|J|^2 - \langle J, \dot{J}\rangle^2\right) \\
&= \frac{1}{|J|}\left(\mu\langle J, J\rangle - K(J,\dot{c})|J|^2\right) + \frac{1}{|J|^3}\left(|\dot{J}|^2|J|^2 - \langle J, \dot{J}\rangle^2\right) \\
&= 0.
\end{aligned}$$

By the Cauchy-Schwartz inequality we have that $\langle J, \dot{J}\rangle \le |J||\dot{J}|$, and so from the above equality we deduce that J and $\dot{J}$ are linearly dependent for all $0 \le s \le t$. In light of this, the above chain of equalities reduces to

$$\frac{1}{|J|}\left(\mu|J|^2 - K(J,\dot{c})|J|^2\right) = \mu|J| - K(J,\dot{c})|J| = 0.$$

Since $|J| \ne 0$ we deduce that $\mu = K(J,\dot{c})$ for all $0 \le s \le t$, as desired. ∎

Exercise 13. Let $p \in M$, $n = \dim M$, $r(x) = d(x,p)$,

$$w(x,t) := \frac{1}{t^{n/2}} \exp\left(-\frac{r^2(x)}{4t}\right).$$

In the Euclidean case, $w(x,t)$ is the fundamental solution of the heat operator, i.e. for $(x,t) \neq (p,0)$

$$\left(\frac{\partial}{\partial t} + \Delta\right) w(x,t) = 0.$$

Under the assumptions of Lemma 4.7.1, derive the estimate

$$\left|\left(\frac{\partial}{\partial t} + \Delta\right) w(x,t)\right| \leq 2\Lambda^2 \frac{r^2(x)}{4t} w(x,t)$$

for $(x,t) \neq (p,0)$.

Lemma 4.7.1. *Suppose* $\exp_p : T_pM \to M$ *is a diffeomorphism on the ball* $\{v \in T_pM \ : \ \|v\| \leq \rho\}$*, and suppose that the sectional curvature in* $B(p,\rho)$ *satisfies*

$$\lambda \leq K \leq \mu \qquad \text{with } \lambda \leq 0,\ \mu \geq 0,$$

put $\Lambda := \max(-\lambda, \mu)$*, and assume*

$$\rho < \frac{\pi}{\sqrt{\mu}} \qquad \text{in case } \mu > 0.$$

Then, with $r(x) = d(x,p)$ *for* $x \neq p$

$$|\Delta \log r(x)| \leq 2\Lambda \qquad \text{if } n = \dim M = 2, \tag{4.7.1}$$

$$|\Delta(r(x)^{2-n})| \leq \frac{n-2}{2}\Lambda r^{2-n}(x) \qquad \text{if } n = \dim M \geq 3. \tag{4.7.2}$$

We begin by computing

$$\frac{\partial}{\partial t} w(x,t) = -\frac{n \exp\left(-\frac{r^2(x)}{4t}\right)}{2t^{1+\frac{n}{2}}} + \frac{r^2(x)\exp\left(-\frac{r^2(x)}{4t}\right)}{4t^{2+\frac{n}{2}}} = \left(\frac{r^2(x) - 2nt}{4t^2}\right) w(x).$$

Computing the Laplacian of w is a lot more involved since we need to work out some sort of chain rule for the Laplace-Beltrami operator on M.

Let $h : M \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$. Then $\operatorname{grad}(f \circ h)(x) = f'(h(x)) \operatorname{grad} h(x)$. Recall that divergence can be defined as the trace of the covariant derivative, from which we can easily deduce that $\operatorname{div}(FX) = F \operatorname{div}(X) + X(F)$ for any $F \in \mathscr{C}^\infty(M)$ and $X \in \Gamma(TM)$. We deduce

$$\begin{aligned}
\Delta(f \circ h) &= -\operatorname{div}(\operatorname{grad}(f \circ h)) \\
&= -\operatorname{div}((f' \circ h) \operatorname{grad} h) \\
&= -(f' \circ h) \operatorname{div}(\operatorname{grad}(h)) - \operatorname{grad}(h)\left(f' \circ h\right) \\
&= (f' \circ h)\Delta h - \operatorname{grad}(h)\left(f' \circ h\right) \\
&= (f' \circ h)\Delta h - d(f' \circ h)(\operatorname{grad}(h)) \\
&= (f' \circ h)\Delta h - \left(df'_{h(\cdot)} \circ dh_{(\cdot)}\right)(\operatorname{grad}(h)) \\
&= (f' \circ h)\Delta h - (f'' \circ h) dh(\operatorname{grad}(h)) \\
&= (f' \circ h)\Delta h - (f'' \circ h)\|\operatorname{grad}(h)\|^2.
\end{aligned}$$

We now can try and apply this to $w(x,t)$ where $f(s) = \frac{1}{t^{n/2}} \exp\left(-\frac{s}{4t}\right)$ and $h(x) = r^2(x)$. We find that

$$\begin{aligned}\Delta w(x,t) &= -\frac{1}{4t^{1+\frac{n}{2}}} \exp\left(-\frac{r^2(x)}{4t}\right) \Delta\left(r^2(x)\right) - \frac{1}{16t^{2+\frac{n}{2}}} \exp\left(-\frac{r^2(x)}{4t}\right) \left\|\mathrm{grad}\left(r^2(x)\right)\right\|^2 \\ &= -\frac{1}{4t^{1+\frac{n}{2}}} \exp\left(-\frac{r^2(x)}{4t}\right) \Delta\left(r^2(x)\right) - \frac{1}{4t^{2+\frac{n}{2}}} \exp\left(-\frac{r^2(x)}{4t}\right) r^2(x) \\ &= -\left(\frac{r^2(x) + t\Delta\left(r^2(x)\right)}{4t^2}\right) w(x,t).\end{aligned}$$

So we have

$$\left|\left(\frac{\partial}{\partial t} + \Delta\right) w(x,t)\right| = \left|\frac{-t\Delta\left(r^2(x)\right) - 2nt}{4t^2}\right| w(x,t)$$

Given our manifolds sectional curvature bounds we deduce that

$$2n(1 - \Lambda r^2(x)) \leq -\Delta r^2(x) \leq 2n(1 + \Lambda r^2(x)).$$

Using this we have

$$\frac{-t\Delta\left(r^2(x)\right) - 2nt}{4t^2} \leq \frac{2nt(1 + \Lambda r^2(x)) - 2nt}{4t^2} = \frac{2\Lambda r^2(x)}{4t^2},$$

and

$$\frac{-t\Delta\left(r^2(x)\right) - 2nt}{4t^2} \geq \frac{2nt(1 - \Lambda r^2(x)) - 2nt}{4t^2} = -\frac{2\Lambda r^2(x)}{4t}.$$

Hence,

$$\left|\frac{-t\Delta\left(r^2(x)\right) - 2nt}{4t^2}\right| \leq \frac{2\Lambda r^2(x)}{4t},$$

and we conclude

$$\left|\left(\frac{\partial}{\partial t} + \Delta\right) w(x,t)\right| \leq 2\Lambda \frac{r^2(x)}{4t} w(x,t).$$

I think there is a typo in the question since the bound I have does not have the Λ^2 term, only Λ.

■

Exercise 1. Show that the real projective space $\mathbf{RP}^n$ can be obtained as the space of all (real) lines in $\mathbf{R}^{n+1}$. Show that $\mathbf{RP}^1$ is diffeomorphic to $\mathbf{S}^1$. Compute the cohomology of $\mathbf{RP}^n$. Show that $\mathbf{RP}^n$ carries the structure of a symmetric space.

I already showed the fact that $\mathbf{RP}^n$ can be obtained as the space of all real lines in $\mathbf{R}^{n+1}$ in Chapter 1, Exercises 1 and 3.

Since $\mathbf{RP}^1$ and $\mathbf{S}^1$ are both 1-dimensional smooth manifolds, to show that they are diffeomorphic it suffices to show that they are homeomorphic. Note that $\mathbf{RP}^1 \cong \mathbf{S}^1/\tau$, where $\tau(x) = -x$ is the antipodal involution of $\mathbf{S}^1$. Now we just need to show that $\mathbf{S}^1/\tau$ is homeomorphic to $\mathbf{S}^1$. Write $\mathbf{S}^1 = \{z \in \mathbf{C} : |z| = 1\}$ and consider the map $f : \mathbf{S}^1 \to \mathbf{S}^1$ given by $f(z) = z^2$. Note that f is clearly a continuous surjective function with $f(z) = f(-z)$. Hence, f factors through the quotient map under the equivalence determined by τ. Now by the universal property of the quotient topology we find that there exists a unique homeomorphism $\varphi : \mathbf{S}^1/\!\sim\, \to \mathbf{S}^1$. Hence, $\mathbf{RP}^1$ is homeomorphic (and therefore diffeomorphic) to $\mathbf{S}^1$.

Note that the involution $\tau(x) = -x$ of $\mathbf{S}^n$ induces a splitting $\Omega^p(\mathbf{S}^n) = \Omega^p_+(\mathbf{S}^n) \oplus \Omega^p_-(\mathbf{S}^n)$, where $\Omega^p_\pm(\mathbf{S}^n)$ only consists of the differential forms with $\tau^*\omega = \pm\omega$. So we have a splitting of the de Rham cohomology groups $H^*(\mathbf{S}^n) = H^*_+(\mathbf{S}^n) \oplus H^*_-(\mathbf{S}^n) \cong \mathbf{R}$. Now the quotient map $\varphi : \mathbf{S}^n \to \mathbf{RP}^n$ induces an isomorphism $\varphi^* : \Omega^*(\mathbf{RP}^n) \xrightarrow{\cong} \Omega^*_+(\mathbf{S}^n)$, and hence an isomorphism $\varphi^* : H^*(\mathbf{RP}^n) \xrightarrow{\cong} H^*_+(\mathbf{S}^n)$. Finally, note that the canonical generator $[\omega_0] \in H^n(\mathbf{S}^n) = \mathbf{R}[\omega_0]$ is in $H^n_+(\mathbf{S}^n)$ or $H^n_-(\mathbf{S}^n)$ depending on whether n is even or odd (this follows since $\tau^*(\omega_0) = (-1)^{n+1}\omega_0$. So we have computed

$$H^n(\mathbf{RP}^n) = \begin{cases} \mathbf{R} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Now since $\mathbf{RP}^n$ is connected we have that $H^0(\mathbf{RP}^n) = \mathbf{R}$. Now since $H^p(\mathbf{S}^n) = 0$ for all $0 < p < n$ we deduce that

$$H^p(\mathbf{RP}^n) = \begin{cases} \mathbf{R} & \text{if } p = 0, \text{ or } p = n \text{ and } n \text{ is odd,} \\ 0 & \text{else.} \end{cases}$$

Consider the symmetry $\sigma_N : \mathbf{S}^n \to \mathbf{S}^n$ on $\mathbf{S}^n$ around the north pole $N = (0, \ldots, 0, 1) \in \mathbf{S}^n$ given by

$$\sigma_N(x) = 2(N \cdot x)N - x,$$

where we consider $\mathbf{S}^n \subseteq \mathbf{R}^{n+1}$. Note that $\sigma_N \circ \tau = \tau \circ \sigma_N$, and so σ_N descends to an isometry of the quotient space $\mathbf{RP}^n = \mathbf{S}^n/\tau$. We compute

$$\sigma_N(N) = 2(N \cdot N)N - N = 2\|N\|N - N = N$$

and since we can write $\sigma_N(x) = (2NN^\top - \mathbf{1})x$ we have

$$D\sigma_N(x) = (2NN^\top - \mathbf{1}) = \left(\begin{array}{c|c} -\mathbf{1}_{n\times n} & 0 \\ \hline 0 & 1 \end{array}\right)$$

Since $T_N\mathbf{RP}^n = T_N\mathbf{S}^n = \mathbf{R}^n \times \{0\}$ we see that the restriction $D\sigma_N(N) : T_N\mathbf{RP}^n \to T_N\mathbf{RP}^n$ is given by $D\sigma_N(N) = -\mathrm{id}$, as desired. So we have shown that $\mathbf{RP}^n$ is a symmetric space. ∎

Exercise 2. Similarly, define and discuss the quaternionic projective space $\mathbf{HP}^n$ as the space of all quaternionic lines in quaternionic space $\mathbf{H}^{n+1}$. In particular, show that it is a symmetric space.

We define $\mathbf{HP}^n$ as the quotient of $\mathbf{H}^{n+1}\setminus\{0\}$ by the equivalence relation $Z\sim\lambda Z$ whenever $Z=(Z_0,\dots,Z_n)\in\mathbf{H}^{n+1}\setminus\{0\}$ and $0\neq\lambda\in\mathbf{H}$. Now consider the unit sphere $\mathbf{S}^{4n+3}\subseteq\mathbf{H}^{n+1}$ as the set of points such that $Z\cdot\overline{Z}=1$, where $\cdot$ is the usual dot product and the $\bar{\cdot}$ denotes the quaternionic conjugation. Any quaternionic line in $\mathbf{H}^{n+1}$ must intersect the unit sphere. Furthermore, if $Z\in\mathbf{S}^{4n+3}$, then $\lambda Z\in\mathbf{S}^{4n+3}$ if and only if $|\lambda|=1$. Now by identifying $\mathbf{Sp}(1)$ with the group of unital quaternions we see that $\mathbf{HP}^{n+1}=\mathbf{S}^{4n+3}/\mathbf{Sp}(1)$ (as sets). Since $\mathbf{Sp}(1)$ is a compact Lie group acting effectively on $\mathbf{S}^{4n+3}$ we deduce by the quotient manifold theorem that $\mathbf{HP}^{n+1}$ is a smooth manifold. Note that $\mathbf{Sp}(n+1)$, the group of $(n+1)\times(n+1)$ quaternionic matrices which preserve the natural quaternionic Hermitian form $X\cdot\overline{Y}$, acts transitively on the unit sphere in $\mathbf{H}^{n+1}$.

Let $[N]$ be the coset of the north pole $N\in\mathbf{S}^{4n+3}$. Let L be the quaternionic line through $[N]$, and define a linear operator $\sigma:\mathbf{H}^{n+1}\to\mathbf{H}^{n+1}$ such that $\sigma|_L=\mathrm{id}$ and $\sigma|_{L^\perp}=-\mathrm{id}$, i.e. $\sigma(Z_0,Z_1,\dots,Z_n)=(Z_0,-Z_1,\dots,-Z_n)$. Since σ is linear, σ factors through the quotient to $\mathbf{HP}^n$ to define the involution $\sigma_{[N]}:\mathbf{HP}^n\to\mathbf{HP}^n$. It is straighforward to see that $\sigma_{[N]}([N])=[N]$ and $D\sigma_{[N]}([N])=-\mathrm{id}$. Since the isometry group acts isometrically on $\mathbf{HP}^n$ we deduce that this involution extends to all points in the quaternionic projective space. Hence, $\mathbf{HP}^n$ is a symmetric space.

■

Exercise 3. Determine all Killing fields on $\mathbf{S}^n$.

We claim the the following vector fields form a basis of the vector space of Killing fields on $\mathbf{S}^n$: Consider $\mathbf{S}^n\subseteq\mathbf{R}^{n+1}$ and the vector fields

$$K_{ij}=x^i\frac{\partial}{\partial x^j}-x^j\frac{\partial}{\partial x^i}$$

for any $1\le i,j\le n+1$. Note that the flow of K_{ij} is given by $\Phi_{ij}(x,t)=\exp\big(tA^{(ij)}\big)x$ where $(A^{(ij)})_{ij}=-(A^{ij})_{ji}=1$ and all other entries are zero. Since for any t the flow map Φ_{ij} is an isometry of $\mathbf{R}^{n+1}$ that preserves the sphere, we see that $\Phi_{ij}(x,\cdot)$ is a local group of isometries of $\mathbf{S}^n$ for any $x\in\mathbf{S}^n$, and so we deduce that K_{ij} is a Killing field.

Note that

$$\dim\big(\mathrm{span}\{K_{ij}\ :\ 1\le i,j\le n+1\}\big)=\frac{n(n+1)}{2}.$$

Now since the isometry group of $\mathbf{S}^n$ is $\mathbf{SO}(n+1)$ and since $\dim\mathbf{SO}(n+1)=\frac{n(n+1)}{2}$, we deduce that $\mathrm{span}\{K_{ij}\ :\ 1\le i,j\le n\}$ is the set of all Killing fields on $\mathbf{S}^n$.

■

Exercise 4. Determine the Killing forms of the groups $\mathbf{SL}(n,\mathbf{C})$, $\mathbf{Sp}(n,\mathbf{R})$, $\mathbf{SU}(n)$, $\mathbf{U}(n)$.

Note that if we have any basis $\{e_1,\dots,e_n\}$ of a Lie algebra $\mathfrak{g}$ then we can find coefficients c_k^{ij} such that

$$[e^i,e^j]=c_k^{ij}e^k.$$

Then we find that

$$(\mathrm{ad}(e^i)\circ\mathrm{ad}(e^j))(e^k)=[e^i,[e^j,e^k]]=[e^i,c_\ell^{jk}e^\ell]=c_m^{i\ell}c_\ell^{jk}e^m.$$

Since the Lie bracket is anticommutative we deduce that $c_k^{ij}=-c_{jk}^k$ and since the Lie bracket satisfies the Jacobi identity we have

$$c_\ell^{ij}c_j^{km}+c_\ell^{mj}c_j^{ik}+c_\ell^{kj}c_j^{mi}=0.$$

Now we can easily check that

$$B_{\mathfrak{g}}(e_i,e_j)=c_k^{i\ell}c_\ell^{jk}.$$

- **SL**$(n, \mathbf{C})$*:* Let $E^{(ij)}$ be the matrix with $(E^{(ij)})_{i,j} = 1$ and all other entries zero. Define $D_{ij} = E_{ii} - E_{jj}$ and let $h_i = D_{i,i+1}$. Note that $h_1, \ldots, h_{n-1}$ and the $E^{(ij)}$ for $i \neq j$ form a basis of $\mathfrak{sl}(n, \mathbf{C})$. Since $\mathfrak{sl}(n, \mathbf{C})$ is a simple Lie algebra over $\mathbf{C}$, we know that any two symmetric invariant bilinear forms are scalar multiples of each other. So we deduce that
$$B_{\mathfrak{sl}(n,\mathbf{C})}(X, Y) = \lambda \operatorname{tr}(XY)$$
for all $X, Y \in \mathfrak{sl}(n, \mathbf{C})$, where λ is a constant. If Z is a diagonal matrix with $\operatorname{tr} Z = 0$ then
$$[Z, E^{(ij)}] = (Z_{ii} - Z_{jj})E^{(ij)},$$
and
$$\operatorname{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_Z) = \sum_{i,j} \left(Z_{ii} - Z_{ij}\right)^2 = 2n \operatorname{tr}(Z^2).$$
In particular, $B_{\mathfrak{sl}(n,\mathbf{C})}(h_1, h_1) = 2n \operatorname{tr}(h_1^2) = 4n$. This shows that $\lambda = 2n$, and so
$$B_{\mathfrak{sl}(n,\mathbf{C})}(X, Y) = 2n \operatorname{tr}(XY).$$
Alternatively, we can just use determine the structure constants of the Lie bracket in $\mathfrak{sl}(n, \mathscr{C})$ to determine the result in the same way.

- **Sp**$(n, \mathbf{R})$*:* We just use the structure constants of the Lie bracket of $\mathfrak{sp}(n, \mathbf{R})$ to find that
$$B_{\mathfrak{sp}(n,\mathbf{R})}(X, Y) = (2n + 2) \operatorname{tr}(XY).$$

- **SU**(n)*:* We compute the structure constants of $\mathfrak{su}(n)$. First note that every Hermitian $n \times n$-matrix can be written as a real linear combination of the following matrices:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -i & \cdots & 0 & 0 \\ i & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -i \\ 0 & 0 & \cdots & i & 0 \end{pmatrix} \tag{14}$$

Note that the matrices in (14) are traceless. Now if we replace the matrices in (12) by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \tag{14'}$$

then we obtain a basis for all Hermitian matrices with zero trace. Now let these basis matrices in (14') and (14) be denoted as λ^i. Then the basis of $\mathfrak{su}(n)$ is given by

$$e^i = -\frac{1}{2} i\lambda_i.$$

Now we can write

$$[e^i, e^j] = c_\ell^{ij} e^\ell,$$

and

$$[\lambda^i, \lambda^j] = 2ic_\ell^{ij}\lambda^\ell.$$

It is easy to compute that $\operatorname{tr}(\lambda^i\lambda^j) = 2\delta_{ij}$ for λ^i, λ^j in (14). Now we find that

$$\operatorname{tr}([\lambda^i, \lambda^j], \lambda^k) = 2ic_\ell^{ij} \operatorname{tr}(\lambda^\ell \lambda^k) = 4ic_k^{ij}.$$

So we can compute the structure coefficients as

$$c_k^{ij} = \frac{1}{4i} \operatorname{tr}([\lambda^i, \lambda^j], \lambda^k).$$

Since the trace is invariant under cyclic permutations of its operator we see that

$$4ic_k^{ij} = -4ic_j^{ik}.$$

Similarly, we find that c_k^{ij} is antisymmetric in all indices. Now we deduce that

$$B_{\mathfrak{su}(n)}(e^i, e^j) = \sum_{\ell,k=1}^{n} -c_\ell^{ik} c_\ell^{jk}.$$

In particular, we find that

$$B_{\mathfrak{su}(n)}(X, Y) = 2n \operatorname{tr}(XY).$$

- $\mathbf{U}(n)$*:* Since $\mathfrak{su}(n)$ is an ideal in $\mathfrak{u}(n)$ we deduce that they have the same Killing form. That is to say

$$B_{\mathfrak{u}(n)}(X, Y) = 2n \operatorname{tr}(XY).$$

■

Exercise 5. Discuss the geometry of $\mathbf{S}^n$ by viewing it as the symmetric space $\mathbf{SO}(n+1)/\mathbf{SO}(n)$.

We prove that $\mathbf{S}^n \cong \mathbf{SO}(n+1)/\mathbf{SO}(n)$, and discuss the geometric interpretations along the way. This is essentially just the orbit stabilizer theorem.

Intuitively, note that $\mathbf{SO}(n+1)$ acts by rotations on $\mathbf{R}^{n+1}$. This action restricts to a transitive action on $\mathbf{S}^n$. Now fix a vector $e_1 = (1, 0, \dots, 0) \in \mathbf{S}^n$. Note that we have a continuous map $\mathbf{SO}(n+1) \to \mathbf{S}^n$ given by $A \mapsto Ae_1$. The subgroup of $\mathbf{SO}(n+1)$ which stabilizes e_1 is the kernel of this map. It is the block diagonal subgroup $H = \{1\} \times \mathbf{SO}(n)$. It follows that the quotient $\mathbf{SO}(n+1)/H \cong \mathbf{SO}(n+1)/\mathbf{SO}(n)$ is in continuous bijection with $\mathbf{S}^n$. Since both spaces are compact Hausdorff spaces, we deduce that the map is a homeomorphism. ■

Exercise 6. Show that $\mathbf{CP}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(n))$. Compute the rank of $\mathbf{CP}^n$ as a symmetric space.

We will use the Hopf fibration $\mathbf{S}^{2n+1} \to \mathbf{CP}^n$. Note that $\mathbf{U}(n+1)$ is the isometry group of $\mathbf{C}^{n+1}$ which preserve the complex structure. This $U(n+1)$ acts transitively on $\mathbf{S}^{2n+1}$, and the action descends to a transitive action on $\mathbf{CP}^n$. Now if $x \in \mathbf{U}(n+1)$ fixes a point $p \in \mathbf{S}^{2n+1}$, then x also stabilizes the orthogonal complement of x, and so the stabilizer of a point is $\mathbf{U}(n)$, where we view $\mathbf{U}(n)$ is embedded in $\mathbf{U}(n+1)$ as $\mathbf{U}(1) \times \mathbf{U}(n) \subset \mathbf{U}(n+1)$. In particular, we deduce that

$$\mathbf{CP}^n = \mathbf{U}(n+1)/(\mathbf{U}(1) \times \mathbf{U}(n)) = \mathbf{U}(n+1)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(n)).$$

The rank of $\mathbf{CP}^n$ as a symmetric space is 1. ■

Exercise 7. Determine the closed geodesics and compute the injectivity radius of the symmetric space $\mathbf{RP}^n$.

First note that the geodesics of $\mathbf{RP}^n$ are simply the projections of the geodesics of $\mathbf{S}^n$. This follows since the map $\mathbf{S}^n \to \mathbf{RP}^n$ is a local isometry, and hence preserves the geodesics. Using this, we immediately find that a geodesic $c : \mathbf{R} \to \mathbf{RP}^n$ satisfying $c(0) = p = \pi(x)$, and $\dot{c}(0) = X = \pi(x, u)$, with $\|X\| = 1$ is written as

$$c(t) = \pi(x\cos(t) + u\sin(t)) = \exp_p(tX).$$

From this we immediately deduce that all of the geodesics in $\mathbf{RP}^n$ are closed. Note that $L(c) = \pi/2$, and so $\mathrm{diam}(\mathbf{RP}^n) = \pi/2$. Now since we have an explicit expression of the exponential map, and since $\pi : \mathbf{S}^n \to \mathbf{RP}^n$ is non-singular that the injectivity radius of $\mathbf{RP}^n$ is exactly $\pi/2$. ■

Exercise 1. Show that if f is a Morse function on the compact manifold X, $a < b$, and if f has no critical point p with $a \leq f(p) \leq b$, then the sublevel set $\{x \in X : f(x) \leq a\}$ is diffeomorphic to $\{x \in X : f(x) \leq b\}$.

Let $\mathrm{Crit}(f)$ denote the set of critical points of f in M. Note that since M is compact we have that the sublevel sets are compact, and that the set $\mathrm{Crit}(f)$ is closed. So we can find some $\varepsilon > 0$ such that

$$\{a - \varepsilon < f < b + \varepsilon\} \subset M \setminus \mathrm{Crit}(f).$$

Now consider a smooth function $\varphi : M \to \mathbf{R}^+$ satisfying

$$\varphi(x) = \begin{cases} |\operatorname{grad}(f)f|^{-1} & \text{if } f(x) \in [a, b], \\ 0 & \text{if } f(x) \notin (a - \varepsilon, b + \varepsilon). \end{cases}$$

Now consider the vector field $X = -\varphi \operatorname{grad}(f)$ on M and the associated flow map $\Phi : \mathbf{R} \times M \to M$. Now consider an integral curve, γ, of X. Now we consider the derivative of f along γ to find that in the region $\{a \leq f \leq b\}$ that

$$\frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t} = Xf = -\frac{\operatorname{grad}(f)f}{\operatorname{grad}(f)f} = -1.$$

So we see that in $\{a \leq f \leq b\}$ the value of f decreases at a rate of one unit per second. This implies, by the fundamental theorem of calculus, that

$$\Phi_{b-a}(\{x \in X : f(x) \leq b\}) = \{x \in X : f(x) \leq a\}$$

and

$$\Phi_{a-b}(\{x \in X : f(x) \leq a\}) = \{x \in X : f(x) \leq b\}.$$

Hence, Φ_{a-b} is a diffeomorphism between $\{x \in X : f(x) \leq a\}$ and $\{x \in X : f(x) \leq b\}$.

Note that we don't need f to be a Morse function for this proof to go through. Similarly, we don't need M to be compact if we assume that all of sublevel sets are compact.

■

Exercise 2. Compute the Euler characteristic of a torus by constructing a suitable Morse function.

Consider the n-torus $\mathbf{T}^n = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1 \subseteq \mathbf{C}^n$. Now consider the function $f : \mathbf{T}^n \to \mathbf{R}$ given by

$$f\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) = \sum_{j=1}^{n} \cos\theta_j.$$

It is easy to compute

$$\operatorname{grad}(f) = -(\sin\theta_1, \ldots, \sin\theta_n),$$

which vanishes if and only if $e^{i\theta_j} = \pm 1$ for all $1 \leq j \leq n$. In particular,

$$\mathrm{Crit}(f) = \{(b_1, \ldots, b_n) : b_j = \pm 1\},$$

with $|\operatorname{Crit}(f)| = 2^n$. We can partition $\operatorname{Crit}(f) = \bigcup \operatorname{Crit}_k(f)$, where $\operatorname{Crit}_k(f)$ is the set of critical points of f with k indices equal to $+1$ and $n-k$ indices of f equal to -1. For any $p \in \operatorname{Crit}_k(f)$ we see that k of the second derivatives of each term, $\cos\theta_j$, are equal to -1 and $n-k$ are equal to 1, with no other contributions to the second-order Taylor series expansion of f. So we deduce that all critical values of f are non-degenerate with index k; hence f is a Morse function. Now we compute the Euler characteristic

$$\chi(\mathbf{T}^n) = \sum_{k=0}^{n} (-1)^k |\operatorname{Crit}_k(f)| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

■

Exercise 3. Show that the Euler characteristic of any compact odd-dimensional differentiable manifold is zero.

Let M be a smooth n-manifold with $n = \dim M$ odd. Consider a Morse function $f : M \to \mathbf{R}$. Since $|\operatorname{Crit}_k(f)| = |\operatorname{Crit}_{n-k}(-f)|$ we have

$$\begin{aligned}
\chi(M) &= \sum_{k=0}^{n} (-1)^k |\operatorname{Crit}_k(f)| \\
&= \sum_{k=0}^{n} (-1)^k |\operatorname{Crit}_{n-k}(-f)| \\
&= (-1)^n \sum_{k=0}^{n} (-1)^{n-k} |\operatorname{Crit}_{n-k}(-f)| \\
&= (-1)^n \sum_{k=0}^{n} (-1)^k |\operatorname{Crit}_k(-f)| && \text{(by reindexing)} \\
&= (-1)^n \chi(M).
\end{aligned}$$

Since n is odd we deduce that $\chi(M) = 0$. ■

Exercise 4. Show that any smooth function $f : \mathbf{S}^n \to \mathbf{R}$ always has an even number of critical points, provided all of them are nondegenerate.

By using the standard height function $h : \mathbf{S}^n \to \mathbf{R}$ given by $\boldsymbol{x} \mapsto x_{n+1}$ we find that

$$\chi(\mathbf{S}^n) = 1 + (-1)^n,$$

which is even. Now consider any Morse function $f : \mathbf{S}^n \to \mathbf{R}$ and note that

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \mu_k(f) = 1 + (-1)^n.$$

Now we compute

$$|\operatorname{Crit}(f)| = \sum_{k=0}^{n} \mu_k(f) = \sum_{k=0}^{n} (-1)^k \mu_k(f) + 2 \sum_{k \text{ is odd},\ 0 \le k \le n} \mu_k(n) = (1 + (-1)^n) + 2 \sum_{k \text{ is odd},\ 0 \le k \le n} \mu_k(n),$$

which is clearly even. ■

Exercise 5. Prove the following

Theorem (Reeb)**.** *Let M be a compact differentiable manifold, and let $f \in \mathscr{C}^3(M, \mathbf{R})$ have precisely two critical point, both of them nondegenerate. Then M is homeomorphic to the sphere $\mathbf{S}^n$ ($n = \dim M$).*

Let $f \in \mathscr{C}^3(M, \mathbf{R})$ have precisely two critical point. Since M is compact we see that f attains its maximum at some point $p_{\max} \in M$ and a minimum at some point $p_{\min} \in M$. In particular, these are the two nondegenerate critical points of f. By rescaling if need be, we may assume that $f(M) = [0,1]$, i.e. we define $\widetilde{f}(x) = \frac{f(x)-f(p_{\min})}{f(p_{\max})-f(p_{\min})}$. Note that this transformation is well defined since f has two nondegenerate critical points and so $f(p_{\max}) \neq f(p_{\min})$. Now we can use the Morse-Palais lemma to find some $\varepsilon > 0$ such that $f^{-1}([0,\varepsilon])$ and $f^{-1}([1-\varepsilon, 1])$ are diffeomorphic to closed balls in $\mathbf{R}^n$. By Exercise 6.1 we know that the sublevel set $M_\varepsilon := \{x \in M \,:\, f(x) \le \varepsilon\}$ and $M_{1-\varepsilon} := \{x \in M \,:\, f(x) \le 1-\varepsilon\}$ are diffeomorphic. So we deduce that $M_{1-\varepsilon}$ is also diffeomorphic to a closed ball in $\mathbf{R}^n$. In particular, we see that M is the union of two disks glued along their common boundaries.

Now we construct an explict map $\eta : \mathbf{S}^n \to M$ as follows. Write $\mathbf{S}^n = \mathbf{S}^n_+ \cup \mathbf{S}^n_-$ where $\mathbf{S}^n_\pm = \mathbf{S}^n \cap \overline{\{\operatorname{sign}(x_{n+1}) = \pm 1\}}$. We immediately have a diffeomorphism $\varphi : \mathbf{S}^n_- \to M_{1-\varepsilon}$ from the work in the previous paragraph. Let ψ_0 be the restriction of φ to $\partial \mathbf{S}^n_+ \cong \partial f^{-1}([1-\varepsilon, 1]) \cong \mathbf{S}^{n-1}$, and extend ψ_0 radially to a homeomorphism $\psi : f^{-1}([1-\varepsilon, \varepsilon]) \to \mathbf{S}^n_+$. Explicitly, we can write

$$\psi(x) = \begin{cases} \|x\| \psi_0\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Now by considering the map

$$\eta(x) = \begin{cases} \varphi(x) & \text{if } x \in \mathbf{S}^n_-, \\ \psi(x) & \text{if } x \in \mathbf{S}^n_+. \end{cases}$$

This map is clearly a homeomorphism between $\mathbf{S}^n$ and M. ■

Exercise 6. Is it possible, for any compact differentiable manifold M, to find a smooth function $f : M \to \mathbf{R}$ with only nondegenerate critical points, and with $\mu_j = b_j$ for all j (notations of Theorem 6.10.2)?

We will show that this is not always possible by considering $M = \mathbf{RP}^3$. Note that by Poincaré duality and Bochner's theorem (alternatively from Exercise 5.1) we obtain $b_0(\mathbf{RP}^3) = b_3(\mathbf{RP}^3) = 1$ and $b_1(\mathbf{RP}^3) = b_2(\mathbf{RP}^3) = 0$. Assume, for the sake of contradiction, that there exists a smooth function $f : \mathbf{RP}^3 \to \mathbf{R}$ with only nondegenerate critical points satisfying $\mu_j(f) = b_j(\mathbf{RP}^3)$ for all $j = 0, \ldots, 3$. That is to say that

$$|\operatorname{Crit}(f)| = \sum_{i=0}^{3} \mu_j(f) = \sum_{i=0}^{3} b_j(\mathbf{RP}^3) = 2.$$

Now by Reeb's theorem (Exercise 6.5) we deduce that $\mathbf{RP}^3$ is homeomorphic to $\mathbf{S}^3$. This is clearly impossible since $\mathbf{RP}^3$ is not simply connected, while $\mathbf{S}^3$ is. So we deduce that such an f cannot exist. ■

Exercise 7. State conditions for a complete, but noncompact, Riemannian manifold to contain a nontrivial closed geodesic. (Note that such conditions will depend not only on the topology, but also on the metric as is already seen for surfaces of revolution in $\mathbf{R}^3$)

■

Exercise 8. Let M be a compact Riemannian manifold, $p, q \in M$, $p \neq q$. Show that there exist at least two geodesic arcs with endpoints p and q.

■

Exercise 9. In (6.2.1), assume that f has two relative minima, not necessarily strict anymore. Show that again there exists another critical point x_3 of f with $f(x_3) \geq \max\{f(x_1), f(x_2)\}$. Furthermore, if $\kappa = \inf_{\gamma\in\Gamma} \max_{x\in\gamma} f(x) = f(x_1) = f(x_2)$, show that f has infinitely many critical points.

■

Exercise 10. Prove the following statement:
Let γ be a smooth convex closed Jordan curve in the plane $\mathbf{R}^2$. Show that there exists a straight line ℓ in $\mathbf{R}^2$ (not necessarily through the origin, i.e. $\ell = \{ax^1 + bx^2 + c = 0\}$ with fixed coefficients a, b, c) intersecting γ orthogonally in two points.

By the Jordan curve theorem we know that γ bounds a compact set $\Omega \subseteq \mathbf{R}^2$. Now let $\mathcal{L}$ denote the set of all line segments ℓ in Ω with $\partial\ell \subset \gamma$, and let $\mathcal{P}$ denote the set of all points on γ. In $\mathcal{L}$ we will admit trivial curves, i.e. a single point in $\mathcal{P}$ is a line in Ω. This will allow our space to be closed with respect to the following notion of convergence. We now say that a sequence of lines $\{\ell_n\}$ converges to $\ell \in \mathcal{L}$ if the endpoints converge as points.

Now consider any continuous map $v : [0,1] \to \mathcal{L}$ satisfying $v(0) = v(1)$. Let $\mathcal{V}$ denote the set of all such v. For every $t \in [0,1]$ we can partition Ω across $v(t)$ into $\Omega_t^1 \cup \Omega_t^2$. We pick an arbitrary choice for $t = 0$ and determine Ω_t^1 and Ω_t^2 for $t > 0$ by continuity. Note that this implies $\Omega_0^1 = \Omega_1^2$. Now define

$$\kappa := \inf_{v\in\mathcal{V}} \sup_{t\in[0,1]} L(v(t)).$$

We claim that $\kappa > 0$. Let

$$r := \sup\{\rho > 0 : \text{for some } x_0 \in \Omega,\ B(x_0, \rho) \subset \Omega\},$$

and let x_0 be the corresponding center of the ball. Note that

$$\kappa \geq \widetilde{\kappa} := \inf_{v\in\mathcal{V}} \sup_{t\in[0,1]} L(v(t) \cap B(x, r)).$$

Set $\widetilde{\Omega}_t^i = \Omega_t^i \cap B(x_0, r)$. Note that $\widetilde{\Omega}_t^i$ varies continuously as well, and since $\widetilde{\Omega}_0^1 = \widetilde{\Omega}_0^2$ there exists some $t_0 \in [0,1]$ such that

$$\mathcal{L}^2(\widetilde{\Omega}_{t_0}^1) = \mathcal{L}^2(\widetilde{\Omega}_{t_0}^2),$$

where $\mathcal{L}^2$ denotes the Lebesgue measure on $\mathbf{R}^2$. Since $v(t_0)$ divides $B(x_0, r)$ into two subregions of equal area, we deduce that $L(v(t_0) \cap B(x_0, r)) = 2r$. Hence,

$$\kappa \geq \widetilde{\kappa} = 2r > 0,$$

and so our claim that $\kappa > 0$ is shown.

We want to show that κ is realized by a critical point of L among all lines $\ell \in \mathcal{L}$. Let $\{v_n\} \subset \mathscr{C}([0,1]; \mathcal{L})$ be an infimizing sequence of paths in $\mathcal{L}$, i.e.

$$\inf_{v\in\mathcal{V}} \sup_{t\in[0,1]} L(v(t)) = \lim_{n\to\infty} \sup_{t\in[0,1]} L(v_n(t)).$$

Now find a subsequence $\{v_{n_k}\}_{k\in\mathbf{N}}$ such that for every $t \in [0,1]$ we have that $v_{n_k}(t) \to v^*(t) \in \mathcal{L}$. We can pick such a subsequence dues to the continuity of v and the obvious compactness of $\gamma \subset \mathbf{R}^2$. Furthermore, we deduce that $v^* \in \mathcal{V}$ since the convergence of the lines is clearly uniform. Now since L is sequentially lower semicontinuous we deduce that

$$\inf_{v\in\mathcal{V}} \sup_{t\in[0,1]} L(v(t)) = \sup_{t\in[0,1]} v^*(t).$$

Since v^* is continuous we deduce the existence of some $t^* \in [0,1]$ such that

$$\inf_{v\in\mathcal{V}} \sup_{t\in[0,1]} L(v(t)) = \sup_{t\in[0,1]} v^*(t) = v^*(t^*).$$

Now it is clear that $v^*(t^*)$ is a critical point for L among all smooth curves with endpoints on γ. Now by Exercise 4.1 we deduce that $v^*(t^*)$ must intersect γ orthogonally at both of it's endpoints. ∎

Exercise 11. Generalize the result of Exercise 10 as follows:
Let M be diffeomorphic to $\mathbf{S}^2$, γ a smooth closed Jordan curve in M. Show that there exists a nontrivial geodesic arc in M meeting γ orthogonally at both endpoints.

∎

Exercise 12. If you know some algebraic topology (relative homotopy groups and a suitable extension of Lemma 6.11.3, see E. Spanier, Algebraic topology, McGraw Hill (1966)), you should be able to show the following generalization of 11:
Let M_0 be a compact (differentiable) submanifold of the compact Riemannian manifold M. Show that there exists a nontrivial geodesic arc in M meeting M_0 orthogonally at both endpoints.

∎

Exercise 13. For $p > 1$ and a smooth curve $c(t)$ in M, define

$$E_p(c) := \frac{1}{p}\int \|\dot{c}\|^p \,\mathrm{d}t.$$

Define more generally a space $H^{1,p}(M)$ of curves with finite value of E_p. What are the critical points of E_p (derive the Euler-Lagrange equations)? If M is compact, does E_p satisfy the Palais-Smale condition?

Let $c : [a,b] \to M$ be a smooth curve.

A direct computation shows that if $c : [a,b] \times (-\varepsilon, \varepsilon) \to M$ is a variation of c then

$$\frac{\mathrm{d}}{\mathrm{d}s}E(c_s) = \frac{1}{2}\int_a^b \left\langle \frac{\partial c}{\partial t}(t,s), \frac{\partial c}{\partial t}(t,s)\right\rangle^{\frac{p-2}{2}} \frac{\partial}{\partial s}\left\langle \frac{\partial c}{\partial t}(t,s), \frac{\partial c}{\partial t}(t,s)\right\rangle \mathrm{d}t.$$

It is now identical to the computation of the first variation of the energy. We deduce that

$$\frac{\mathrm{d}}{\mathrm{d}s}E(c_s)\bigg|_{s=0} = \int_a^b \langle \dot{c}, \dot{c}\rangle^{\frac{p-2}{2}} \left(\frac{\partial}{\partial t}\langle c', \dot{c}\rangle - \langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c}\rangle\right) \mathrm{d}t.$$

So we see that c is critical for E_p if and only if

$$\langle \dot{c}, \dot{c} \rangle^{\frac{p-2}{2}} \left(\frac{\partial}{\partial t} \langle X, \dot{c} \rangle - \langle X, \nabla_{\frac{\partial}{\partial t}} \dot{c} \rangle \right) = 0$$

for all $t \in [a, b]$ for all $X \in \Gamma(c^* TM)$. Since $\langle \dot{c}, \dot{c} \rangle \neq 0$ we deduce that c is critical for E_p if and only if

$$\left(\frac{\partial}{\partial t} \langle X, \dot{c} \rangle - \langle X, \nabla_{\frac{\partial}{\partial t}} \dot{c} \rangle \right) = 0$$

for all $X \in \Gamma(c^* TM)$, but this is the same condition for c to be critical for E_2 and L. Hence, c is critical for E_p if and only if c is a geodesic, and the Euler-Lagrange equations of E_p are just the geodesic equations.

Now we define the Sobolev space $H^{1,p}(M)$ to be the space of curves with finite E_p-energy (really the closure of $\mathscr{C}^\infty(M)$ with respect to the E_p norm). Now we say that $c_n \to c$ in $H^{1,p}$ as $n \to \infty$ if c_n converges uniformly to c and if $E_p(c_n) \to E_p(c)$ as $n \to \infty$.

Now we claim that E_p satisfies the Palais-Smale condition. Consider a sequence $\{c_n\}_{n \in \mathbf{N}} \subset H^{1,p}(M)$ satisfying

$$E(c_n) \leq C_1, \qquad \|DE_p(c_n)\| \to 0 \text{ as } n \to \infty.$$

Now let $\gamma \in H^{1,p}(M)$ and $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, and compute

$$\begin{aligned} d(\gamma(t_1), \gamma(t_2)) &\leq \int_{t_1}^{t_2} \left((g_{ij} \circ \gamma) \dot{\gamma}^i \dot{\gamma}^j \right)^{1/2} \, \mathrm{d}t \\ &\leq (t_2 - t_1)^{\frac{p}{p-1}} \left(\int_{t_1}^{t_2} \left((g_{ij} \circ \gamma) \dot{\gamma}^i \dot{\gamma}^j \right)^{p/2} \mathrm{d}t \right)^{1/p} \\ &\leq \sqrt[p]{p} |t_2 - t_1|^{\frac{p}{p-1}} \sqrt[p]{E_p(\gamma)}. \end{aligned}$$

So we see that $H^{1,p}(M) \subseteq \mathscr{C}^{0,1/2}([a, b]; M)$. Note that this is just an easy case of the Sobolev embeddings where $n = 1$ and the target space is M. Now since $\{E_p(c_n)\}_{n \in \mathbf{N}}$ is bounded by C_1 we can use the Arzela-Ascoli theorem to find a uniformly convergent (in $\mathscr{C}([a, b]; M)$) subsequence. Let us call the limit c. Now we need to show that $c \in H^{1,p}(M)$. Note that the $H^{1,p}$-norm in local coordiantes is lower semicontinuous with respect to L^p-convergence (which we have since M is compact and the convergence $c_n \to c$ is uniform), and so the fact that $E_p(c) < +\infty$ follows immediately.

By the uniform convergence $c_n \to c$ we can find coordiante charts $f_\mu : U_\mu \to \mathbf{R}^n$ for $\mu = 1, \ldots, m$ and a covering $[a, b] = V_1 \cup \cdots \cup V_m$ by open sets such that for sufficiently large n,

$$c_n(V_\mu) \subset U_\mu \text{ and } c(V_\mu) \subset U_\mu \text{ for } \mu = 1, \ldots, m.$$

Now if $\xi \in \mathscr{C}_0^\infty(V_\mu, \mathbf{R}^n)$ for some μ then for sufficiently small $|\varepsilon|$ we have that

$$f_\mu(c(t) + \varepsilon \varphi(t)) \subset f_\mu(U_\mu) \text{ for all } t \in V_\mu.$$

That is we can compute local variations without leaving the coordinate chart. This is what we mean in the following when we write $c + \varepsilon \varphi$ (or things of this form). Now we can view the first variation of DE_p as a linear functional from $H_0^{1,p}(M) \to \mathbf{R}$, and so we endow it with the dual norm:

$$\|DE_p(\gamma)\| = \sup \left\{ \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E_p(\gamma + \varepsilon \xi) \; : \; \xi \in H_0^{1,p}([a, b]; M), \; \int \|\dot{\xi}\|_g \, \mathrm{d}t \leq 1 \right\}.$$

Now we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}E_p(\gamma+\varepsilon\xi) &= \frac{1}{p}\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int\left((g_{ij}\circ(\gamma+\varepsilon\xi))(\dot\gamma^i+\varepsilon\dot\xi^i)(\dot\gamma^j+\varepsilon\dot\xi^j)\right)^{p/2}\ \mathrm{d}t\right)\\ &= \int\left((g_{ij}\circ(\gamma+\varepsilon\xi))(\dot\gamma^i+\varepsilon\dot\xi^i)(\dot\gamma^j+\varepsilon\dot\xi^j)\right)^{\frac{p-2}{2}}\left((g_{ij}\circ(\gamma+\varepsilon\xi))\dot\gamma^i\dot\xi^j+\frac{1}{2}(g_{ij,k}\circ(\gamma+\varepsilon\xi))(\dot\gamma^i+\varepsilon\dot\xi^i)(\dot\gamma^j+\varepsilon\dot\xi^j)\xi^k\right)\mathrm{d}t.\end{aligned}$$

Now at $\varepsilon=0$ we see that this simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}E_p(\gamma+\varepsilon\xi)\bigg|_{\varepsilon=0}=\int\left((g_{ij}\circ\gamma)\dot\gamma^i\dot\gamma^j\right)^{\frac{p-2}{2}}\left((g_{ij}\circ\gamma)\dot\gamma^i\dot\xi^j+\frac{1}{2}(g_{ij,k}\circ\gamma)\dot\gamma^i\dot\gamma^j\xi^k\right)\mathrm{d}t.$$

Now let $\{\eta_\mu\}$ be a partition of unity subordinated to the V_μ. We want to show that $E_p(c_n)\to E_p(c)$. So in the above, we take $\xi^j=\eta_\mu(c_n^j-c^j)$ and compute the second term in the integrand

$$\int\left((g_{ij}\circ c_n)\dot c_n^i\dot c_n^j\right)^{\frac{p-2}{2}}(g_{ij,k}\circ c_n)\dot c_n^i\dot c_n^j\eta_\mu(c_n^k-c^k)\ \mathrm{d}t\le C_2\sup_{t\in V_\mu}d(c_n(t),c(t))E_p(c_n)\to 0$$

where the limit follows by the uniform convergence of $c_n\to c$ and since $E_p(c_n)\le C_1$. Since $\|DE_p(c_n)\|\to 0$ as $n\to\infty$ we deduce that the first term in the integrand also goes to zero. So we see that

$$E_p(c_n)-E_p(c)=\int((g_{ij}\circ c_n)\dot c_n^i\dot c_n^j-(g_{ij}\circ c)\dot c^i\dot c^j)\eta_\mu\ \mathrm{d}t\to 0.$$

That is to say that $E_p(c_n)\to E_p(c)$ as $n\to\infty$. Again, by the lower semicontinuity of E_p with respect to convergence in L^p we deduce that $DE_p(c)=0$. So we have shown that E_p does indeed satisfy the Palais-Smale condition on $H^{1,p}(M)$.

■

Exercise 1. Determine all harmonic maps between tori.

■

Exercise 2.

(a) We call a closed subset A of a Riemannian manifold N convex if any two points in A can be connected by a geodesic arc in A. We call A strictly convex if this geodesic arc is contained in the interior of A with the possible exception of its endpoints. We call A strongly convex, if its boundary ∂A is a smooth submanifold (of codimension 1) in N and if all its principal curvatures with respect to the normal vector pointing to the interior of A are positive. Show that a strongly convex set is strictly convex.

(b) Show that a strongly convex subset A of a complete Riemannian manifold N has a neighborhood whose closure B_1 and $B_0 := A$ satisfy the conclusions of Lemma 8.2.2.

(c) Show that Theorem 8.2.1 continues to hold if N is only complete, but not necessarily compact, again with $\pi_2(N) = 0$, provided $\varphi(\Sigma)$ is contained in a compact, strongly convex subset A of N. In that case, the harmonic $f : \Sigma \to N$ also satisfies $f(\Sigma) \subset A$.

■

Exercise 3. In this exercise, still another definition of the Sobolev space $H^{1,2}(M,N)$ will be given. The embedding theorem of Nash implies that there exists an isometric embedding

$$i : N \to \mathbf{R}^k$$

into some Euclidean space.
We then define

$$H_i^{1,2}(M,N) := \{f \in H^{1,2}(M,\mathbf{R}^k) \,:\, f(x) \in i(N) \text{ for almost all } x \in M\}.$$

Show that

$$H^{1,2}(M,N) = H_i^{1,2}(M,N).$$

■

Exercise 4.

(a) For $1 < p < \infty$ and $f \in L^p(M,N)$, we define

$$E_{p,\varepsilon}(f) := \frac{1}{\omega_m \varepsilon^{m+p}} \int_M \int_{B(x,\varepsilon)} d^p(f(x), f(y))\, d\,\mathrm{vol}(y) d\,\mathrm{vol}(x),$$

and

$$E_p(f) := \lim_{\varepsilon \to 0^+} E_{p,\varepsilon}(f) \in \mathbf{R} \cup \{\infty\}$$

(show that this limit exists). We say that $f \in L^p(M,N)$ belongs to the Sobolev space $H^{1,p}(M,N)$ if $E_p(f) < \infty$. Characterize the localizable maps belonging to $H^{1,p}(M,N)$.

(b) Show lower semicontinuity of E_p with respect to L^p-convergence, i.e. if $(f_\nu)_{\nu \in \mathbf{N}}$ converges to $f \in L^p(M,N)$, then

$$E_p(f) \le \liminf_{\nu \to \infty} E_p(f_\nu).$$

(c) Derive the Euler-Lagrange equations for critical points of E_p. (The smooth critical points are called p-harmonic maps. The regularity theory for p-harmonic maps, however, is not as good as the one for harmonic maps. In general, one only obtains weakly p-harmonic maps of regularity $\mathscr{C}^{1,\alpha}$ for some $\alpha > 0$.)

(d) Show the existence of a continuous weakly p-harmonic map (minimizing E_p) under the assumptions of Theorem 8.2.1.

(e) Extend the existence theory of §7.5 to E_p.

■

Exercise 5. Derive formula (7.2.13) in an invariant fashion, i.e. without using local coordinates.

■

Exercise 6. Prove the following result that is analogous to Corollary 7.2.4. A smooth map $f : M \to N$ between Riemannian manifolds is totally geodesic if and only if whenever V is open in N, $U = f^{-1}(V)$, $h : V \to \mathbf{R}$ is convex, then $h \circ f : U \to \mathbf{R}$ is convex.

■

Exercise 7. Let M be a compact Riemannian manifold with boundary, N a Riemannian manifold, $f : M \to N$ harmonic with $f(\partial M) = p$ for some point p in N. Show that if there exists a strictly convex function h on $f(M)$ with a minimum at p, then f is constant itself.

■

Exercise 8. State and prove a version of the uniqueness theorem 7.7.2 for minimizers of the functionals E_ε. Show that, as for the energy functional E, any critical point of E_ε (with values in a space of non-positive sectional curvature, as always) is a minimizer.

■

Exercise 1. Show that every two-dimensional torus carries the structure of a Riemann surface.

■

Exercise 2. Determine all holomorphic quadratic differentials on a two-dimensional torus, and all holomorphic quadratic differentials on an annular region $\{z \in \mathbf{C} \,:\, r_1 \leq |z| \leq r_2\}$ $(0 < r_1 < r_2)$ that are real on the boundary.

■

Exercise 3. Show that the conclusions of the Hartman-Wintner-Lemma 8.1.7 continue to hold if (8.1.17) is replaced by

$$|u_{z\bar{z}}| \leq K(|u_z| + |u|).$$

■

Exercise 4. We let Σ be a Riemann surface and $H : \mathbf{R}^3 \to \mathbf{R}$ be a smooth function. For a map $f : \Sigma \to \mathbf{R}^3$ we consider the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 2H(f(z)) f_x \wedge f_y$$

where $z = \kappa + iy$ is a conformal parameter on Σ and $\wedge$ denotes the standard vector product in $\mathbf{R}^3$.

(a) Show that, if f is conformal, $H(f(z))$ is the mean curvature of the surface $f(\Sigma)$ at the point $f(z)$.

(b) If $\Sigma = \mathbf{S}^2$, show that every solution is conformal.

(c) If Σ is the unit disk $\mathbf{D}$ and f is a solution which is constant on $\partial \mathbf{D}$, show that it is constant on all of $\mathbf{D}$.

(d) Show that for a nonconstant solution, f_x and f_y have only isolated zeros.

(e) At those points where f_x and f_y do not vanish, we define

$$L := \frac{\langle f_{xx}, f_x \wedge f_y \rangle}{|f_x \wedge f_y|},$$
$$M := \frac{\langle f_{xy}, f_x \wedge f_y \rangle}{|f_x \wedge f_y|},$$
$$N := \frac{\langle f_{yy}, f_x \wedge f_y \rangle}{|f_x \wedge f_y|}$$

(using the Euclidean metric of $\mathbf{R}^3$).

Show that for a solution with $H \equiv \text{const}$, $\varphi \; dz^2 := (L - N - 2iM) \; dz^2$ is a holomorphic quadratic differential.

Conclude that φ, since holomorphic and bounded, extends to all of Σ as a holomorphic quadratic differential.

(f) If $H \equiv \text{const}$ and $\Sigma = \mathbf{S}^2$, show that every solution $f(\Sigma)$ has constant and equal principal curvatures at each point. Conclude that it is a standard sphere of radius $\frac{1}{\sqrt{H}}$, i.e. $f(\Sigma) = \{x \in \mathbf{R}^3 \; : \; |x - x_0|^2 = \frac{1}{H}\}$ for some x_0.

Remark 10. *By the uniformization theorem, every two dimensional Riemannian manifold M diffeomorphic to $\mathbf{S}^2$ admits the structure of a Riemann surface and a conformal diffeomorphism $K : \mathbf{S}^2 \to M$. It thus is conformally equivalent to $\mathbf{S}^2$. The exercise then implies that every surface diffeomorphic to $\mathbf{S}^2$ and immersed into $\mathbf{R}^3$ with constant mean curvature is a standard "round" sphere. This result, as well as the method of proof presented here, were discovered by H. Hopf.* ◇

■

Exercise 5. Prove theorem 8.2.3, assuming only that N is complete but not necessarily compact.

■

Exercise 1. Show by a direct computation that (9.1.28), (9.1.29) imply (9.1.6), (9.1.7).

∎

Exercise 2. Derive the Euler-Lagrange equations for the functional defined in (9.2.16).

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