Asymptotic of a sums of powers of binomial coefficients * x^k

(*Václav Kotěšovec, 20.9.2012*)

Main result:

Asymptotic formula
for $p \geq 1,\ x > 0,\ n \to \infty$

$$\sum_{k=0}^{n} \binom{n}{k}^{p} x^{k} \sim \frac{\left(1 + x^{\frac{1}{p}}\right)^{pn+p-1}}{\sqrt{(2\pi n)^{p-1} * p * x^{1-\frac{1}{p}}}}$$

Some partial results are already know, for example case $x = 1$.
Main asymptotic term see [1], exercise 9.18 (for $2n \to n$), p. 490 and p.593 or [2], p. 263

$$\sum_{k=0}^{n} \binom{n}{k}^{p} \sim \frac{2^{pn}}{\sqrt{p}} \left(\frac{2}{\pi n}\right)^{\frac{p-1}{2}} * \left(1 - \frac{(p-1)^2}{4pn} + O\left(\frac{1}{n^2}\right)\right)$$

or case $p = 2$ (see [3], Proposition 7, $r \to n, d \to x$)

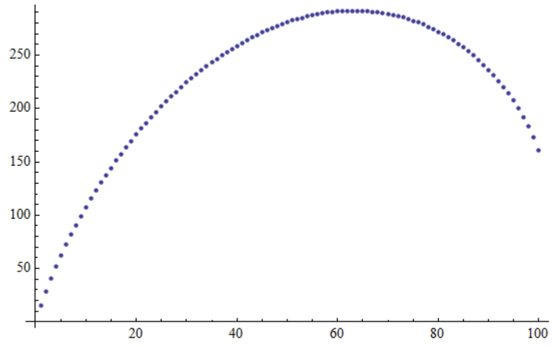
$$\sum_{k=0}^{n} \binom{n}{k}^{2} x^{k} \sim \frac{\left(1 + \sqrt{x}\right)^{2n+1}}{2x^{\frac{1}{4}} \sqrt{\pi n}}$$

For general case we find (with using of Stirling formula and same method as in [4]) maximal term in the sum

$$\sum_{k=0}^{n} \binom{n}{k}^{p} x^{k}$$

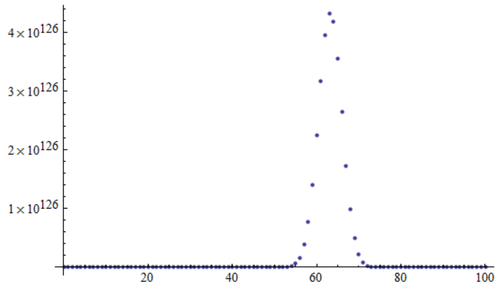
For x=1 is maximum in centre, but x>1 shift the maximum to the right. For first orientation see following graph (in logarithmical scale)

```
ListPlot[Table[Log[Binomial[n,k]^p*x^k]/.{n->100,p->3,x->5},{k,1,100}]]
```



In normal scale is the situation following:

```
ListPlot[
 Table[{k, Binomial[n, k]^p*x^k /. {n → 100, p → 3, x → 5}},
  {k, 0, 100}], PlotRange → All]
```



For finding of the maximum we must solve a equation

$$\frac{\mathrm{d}}{\mathrm{dk}} f(k) = \frac{\mathrm{d}}{\mathrm{dk}} \binom{n}{k}^p x^k = 0$$

With help of program Mathematica:

```
stirling[n_]:=n^n/E^n*Sqrt[2*Pi*n];
binom[n_,k_]:=stirling[n]/stirling[k]/stirling[n-k];
Simplify[D[Simplify[binom[n,k]^p*x^k],k]]
```

$$\frac{1}{k\,(k-n)}\, 2^{-1-\frac{p}{2}} \left(k^{-\frac{1}{2}-k}\, n^{\frac{1}{2}+n}\, (-k+n)^{-\frac{1}{2}+k-n}\right)^p \pi^{-p/2}\, x^k \left(-2\,k\,p + n\,p + 2\,k\,(-k+n)\,p\,\mathrm{Log}[k] + 2\,k\,(k-n)\,p\,\mathrm{Log}[-k+n] + 2\,k^2\,\mathrm{Log}[x] - 2\,k\,n\,\mathrm{Log}[x]\right)$$

```
Simplify[(-2k*p+n*p+2k(-k+n)p*Log[k]+2k(k-n)p*Log[-k+n]+2k^2*Log[x]-2k*n*Log[x])/.k->(q*n)]
```

$$n\left(p - 2\,p\,q - 2\,n\,p\,(-1+q)\,q\,\mathrm{Log}[n\,q] + 2\,n\,p\,(-1+q)\,q\,\mathrm{Log}[n-n\,q] - 2\,n\,q\,\mathrm{Log}[x] + 2\,n\,q^2\,\mathrm{Log}[x]\right)$$

Now we reduce terms which tends to zero if $n \to \infty$

```
Limit[(p-2p*q-2n*p(-1+q)q*Log[n*q]+2n*p(-1+q)q*Log[n-n*q]-2n*q*Log[x]+2n*q^2*Log[x])/n,n->Infinity]
```

$$2\,(-1+q)\,q\,(p\,\mathrm{Log}[1-q] - p\,\mathrm{Log}[q] + \mathrm{Log}[x])$$

```
Solve[p*Log[1-q]-p*Log[q]+Log[x]==0,q]
```

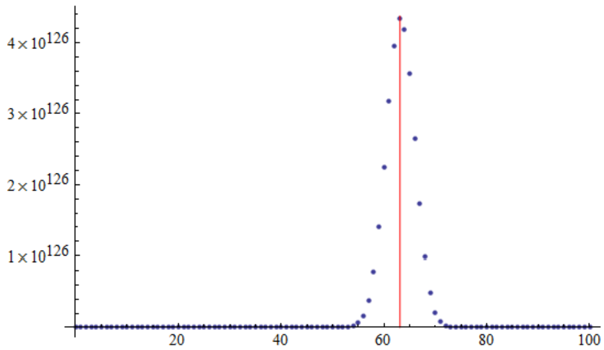
$$q \to \frac{x^{\frac{1}{p}}}{1 + x^{\frac{1}{p}}}$$

Maximum is in

$$t = k_{max} = q * n = \frac{x^{\frac{1}{p}}}{1 + x^{\frac{1}{p}}} * n$$

For example, in previous case $p = 3,\ x = 5,\ n = 100$, we have maximum in

$$t = \frac{5^{1/3}}{1 + 5^{1/3}} = \ 63.099303474397182 \ldots$$



Good estimate (lower bound) is now value in maximum:

$$f(k_{max}) = f(t) = \frac{(1+x^{1/p})^{(n+1)p}}{\sqrt{(2\pi n)^p * x}}$$

but right asymptotic has little bigger value. We will now analyze neighbourhood of maximum, $t \pm k$. From logarithmic form of the Stirling formula we obtain (with help of Mathematica)

```
lognfak[n_]:=n*Log[n]-n+1/2*Log[n]+1/2*Log[2*Pi];
t=x^(1/p)*n/(1+x^(1/p));
FullSimplify[p*(lognfak[n]-lognfak[t+k]-lognfak[n-t-k])+(t+k)*Log[x]]
```

$$\frac{1}{2\left(1+x^{\frac{1}{p}}\right)}\left((1+2n)\,p\left(1+x^{\frac{1}{p}}\right)\mathrm{Log}[n]+2\left(k+(k+n)\,x^{\frac{1}{p}}\right)\mathrm{Log}[x]-p\left(\left(1+x^{\frac{1}{p}}\right)\mathrm{Log}[2\pi]+\left(1+2k+(1+2k+2n)\,x^{\frac{1}{p}}\right)\mathrm{Log}\left[k+n-\frac{n}{1+x^{\frac{1}{p}}}\right]+\left(1+2n+x^{\frac{1}{p}}-2k\left(1+x^{\frac{1}{p}}\right)\right)\mathrm{Log}\left[-k+\frac{n}{1+x^{\frac{1}{p}}}\right]\right)\right)$$

We now apply the first two terms from Taylor series (near 0)

$$\log(1+\mathrm{z}) = z - \frac{z^2}{2} + \cdots$$

and approximate

```
slog[k_,n_]:=Log[n]+k/n-1/2*(k/n)^2;
```

```
Assuming[x ≥ 1 && p ≥ 1 && n ≥ 1 && k ≥ 0,
 FullSimplify[
```

$$\frac{1}{2\left(1+x^{\frac{1}{p}}\right)}\left((1+2n)\,p\left(1+x^{\frac{1}{p}}\right)\mathrm{Log}[n]+2\left(k+(k+n)\,x^{\frac{1}{p}}\right)\mathrm{Log}[x]-p\left(\left(1+x^{\frac{1}{p}}\right)\mathrm{Log}[2\pi]+\left(1+2k+(1+2k+2n)\,x^{\frac{1}{p}}\right)*\mathrm{slog}\left[k,\,n-\frac{n}{1+x^{\frac{1}{p}}}\right]+\left(1+2n+x^{\frac{1}{p}}-2k\left(1+x^{\frac{1}{p}}\right)\right)*\mathrm{slog}\left[-k,\,\frac{n}{1+x^{\frac{1}{p}}}\right]\right)\right)\Big]\Big]$$

$$\frac{1}{4n^2}x^{-2/p}\left(k\,p\left(-2k^2\left(-1+x^{\frac{1}{p}}\right)\left(1+x^{\frac{1}{p}}\right)^3+2n\,x^{\frac{1}{p}}\left(-1+x^{2/p}\right)+k\left(1+x^{\frac{1}{p}}\right)^2\left(1-2n\,x^{\frac{1}{p}}+x^{2/p}\right)\right)+2n^2x^{2/p}\left(-p\,\mathrm{Log}[2n\pi]-\mathrm{Log}[x]+2(1+n)\,p\,\mathrm{Log}\left[1+x^{\frac{1}{p}}\right]\right)\right)$$

Result has **two parts**, each we must return back to exponential form.

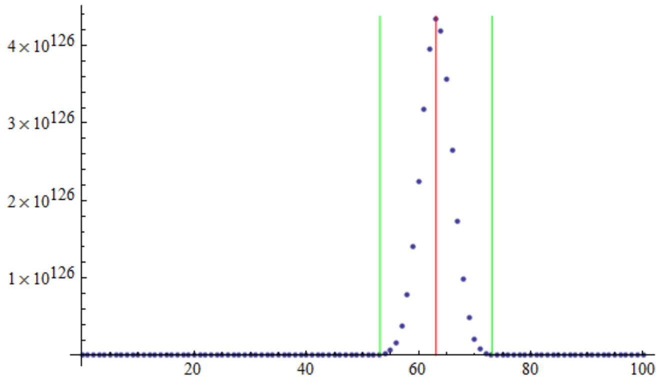
Second part is equal to value in maximum:

$$\mathrm{Factor}\left[\mathrm{Exp}\left[\left(-p\,\mathrm{Log}[2n\pi]-\mathrm{Log}[x]+2(1+n)\,p\,\mathrm{Log}\left[1+x^{\frac{1}{p}}\right]\right)\Big/2\right]\right]$$

$$\frac{n^{-p/2}\,(2\pi)^{-p/2}\left(1+x^{\frac{1}{p}}\right)^{(1+n)\,p}}{\sqrt{x}}$$

First part is interesting. Here are 3 cases. If we compare $k = n^{\varepsilon}$, then

if	limit	Exp(limit)
$\varepsilon < 1/2$	0	1
$\varepsilon = 1/2$	$\neq 0$	>1
$\varepsilon > 1/2$	$-\infty$	0



$\varepsilon > \frac{1}{2}$ $\varepsilon = \frac{1}{2}$ $\varepsilon < \frac{1}{2}$ $\varepsilon = \frac{1}{2}$ $\varepsilon > \frac{1}{2}$

Green lines are $k = t - c * n^{\frac{1}{2}}$ and $k = t + c * n^{\frac{1}{2}}$, where t is maximum and c is some constant. Therefore only case $\varepsilon \leq 1/2$ has some asymptotic weight (only between green lines are dominant terms). Values out of these bounds tends (in comparing with value in the maximum) asymptotically to zero.

Now we compute contributions of other terms (near maximum) yet

$$\text{Limit}\left[\frac{1}{4\,n^2}\,x^{-2/p}\left(k\,p\left(-2\,k^2\left(-1+x^{\frac{1}{p}}\right)\left(1+x^{\frac{1}{p}}\right)^3+2\,n\,x^{\frac{1}{p}}\left(-1+x^{2/p}\right)+k\left(1+x^{\frac{1}{p}}\right)^2\left(1-2\,n\,x^{\frac{1}{p}}+x^{2/p}\right)\right)\right)\;/.\;k\to(c*\text{Sqrt}[n]),\;n\to\text{Infinity}\right]$$

$$-\frac{1}{2}\,c^2\,p\,x^{-1/p}\left(1+x^{\frac{1}{p}}\right)^2$$

$$c^2 = \frac{k^2}{n}$$

By merging of the both parts we obtain

$$\sum_{k=0}^{n}\binom{n}{k}^p x^k \sim \frac{\left(1+x^{1/p}\right)^{(n+1)p}}{\sqrt{(2\pi n)^p * x}} * \sum_k exp\left(-\frac{1}{2} * \frac{k^2}{n} * \frac{p}{x^{\frac{1}{p}}} * \left(1+x^{\frac{1}{p}}\right)^2\right)$$

But

$$\sum_k e^{-\frac{k^2}{N}} \sim \sqrt{\pi N}$$

(for proof see [1], pp.482-485). Here is

$$N = \frac{2n * x^{\frac{1}{p}}}{p * \left(1+x^{\frac{1}{p}}\right)^2}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}^p x^k \sim \frac{\left(1+x^{1/p}\right)^{(n+1)p}}{\sqrt{(2\pi n)^p * x}} * \sqrt{\frac{2\pi n * x^{\frac{1}{p}}}{p * \left(1+x^{\frac{1}{p}}\right)^2}} = \frac{\left(1+x^{\frac{1}{p}}\right)^{pn+p-1}}{\sqrt{(2\pi n)^{p-1} * p * x^{1-\frac{1}{p}}}}$$

More detailed asymptotic is then

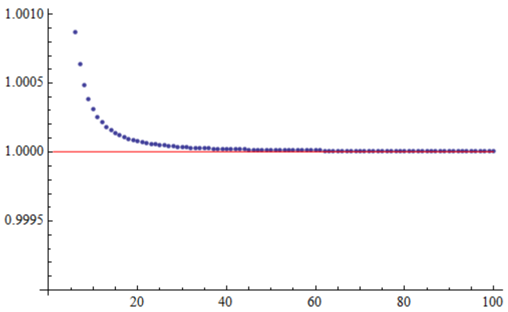
$$\sum_{k=0}^{n} \binom{n}{k}^p x^k \sim \frac{\left(1+x^{\frac{1}{p}}\right)^{pn+p-1}}{\sqrt{(2\pi n)^{p-1} * p * x^{1-\frac{1}{p}}}} * \left(1 - \frac{(p-1)(2p-1)}{6pn} + \frac{(p-1)(p+1)\left(x^{-1/p}+x^{1/p}\right)}{24pn} + O\left(\frac{1}{n^2}\right)\right)$$

For x=1 this result according to

$$\frac{\sum_{i=0}^{n}\binom{n}{i}^p}{\binom{n}{n/2}^p} \sim \sqrt{\frac{\pi n}{2p}}$$

Numerical verification, example for $p = 3,\ x = 4,\ n = 100$

```
Show[ListPlot[Table[(Sum[Binomial[n,k]^p*x^k,{k,0,n}])/
((1+x^(1/p))^(p*n+p-1)/Sqrt[(2*Pi*n)^(p-1)*p*x^(1-1/p)]*
(1-(p-1)*(2p-1)/(6*p*n)+(p-1)*(p+1)*(x^(1/p)+1/x^(1/p))/(24*p*n)))/.{p->3,x->4},{n,1,100}]],
Plot[1,{n,1,100},PlotStyle->Red],PlotRange->{0.999,1.001},AxesOrigin->{0,0.999}]
```



Sequences in OEIS

	x=1	x=2	x=3	x=4	x=5	x=6	
p=1	A000079	3^n	4^n	5^n	6^n	7^n	*(binomial theorem)*
p=2	A000984	A001850	A069835	A084771	A084772	A098659	
p=3	A000172	A206178	A206180	A216483	A216636	A216698	
p=4	A005260	A216696	A216795				
p=5	A005261						
p=6	A069865						
p=7	A182421						
p=8	A182422						
p=9	A182446						
p=10	A182447						

Recurrences

	x=1	general x
p=2	A000984	H. A. Verrill, 1.2.2008 [9]
p=3	J. Franel, 1894 [6], A000172	V. Kotěšovec, 15.9.2012 [5]
p=4	J. Franel, 1895 [6], A005260	V. Kotěšovec, 16.9.2012 [5]
p=5	M. A. Perlstadt 1987 [7], [8], A005261	
p=6	M. A. Perlstadt 1987 [7], [8], A069865	
p=7	V. Kotěšovec, 28.4.2012 [5], A182421	
p=8	V. Kotěšovec, 28.4.2012 [5], A182422	
p=9	V. Kotěšovec, 29.4.2012 [5], A182446	
p=10	V. Kotěšovec, 29.4.2012 [5], A182447	

General recurrences

$p = 2$

$$(n+2)*a(n+2)-(x+1)*(2n+3)*a(n+1)+(x-1)^2*(n+1)*a(n)=0$$

$p = 3$

$$(n+3)^2*(3n+4)*a(n+3)-(9n^3+57n^2+116n+74)*(x+1)*a(n+2)$$
$$+(3n+5)*(3n^2*(x^2-7x+1)+11n*(x^2-7x+1)+9x^2-66x+9)*a(n+1)$$
$$-(n+1)^2*(3n+7)*(x+1)^3*a(n)= 0$$

$p = 4$

-(x-1)^4*(n+1)^3*(n+2)*(16*(4*x^2 + 17*x + 4)*n^4 + 184*(4*x^2 + 17*x + 4)*n^3 + (3137*x^2 + 13351*x + 3137)*n^2 + (5867*x^2 + 25051*x + 5867)*n + 4061*x^2 + 17438*x + 4061)***a(n)**
+ (n+2)*(64*(4*x^5 + 141*x^4 + 655*x^3 + 655*x^2 + 141*x + 4)*n^7 + 1024*(4*x^5 + 141*x^4 + 655*x^3 + 655*x^2 + 141*x + 4)*n^6 + 4*(6857*x^5 + 242368*x^4 + 1126775*x^3 + 1126775*x^2 + 242368*x + 6857)*n^5 + 8*(12439*x^5 + 442336*x^4 + 2059985*x^3 + 2059985*x^2 + 442336*x + 12439)*n^4 + (211031*x^5 + 7579744*x^4 + 35400065*x^3 + 35400065*x^2 + 7579744*x + 211031)*n^3 + (261344*x^5 + 9524206*x^4 + 44667470*x^3 + 44667470*x^2 + 9524206*x + 261344)*n^2 + (174888*x^5 + 6498997*x^4 + 30655175*x^3 + 30655175*x^2 + 6498997*x + 174888)*n + 15*(3251*x^5 + 123835*x^4 + 588594*x^3 + 588594*x^2 + 123835*x + 3251))***a(n+1)**
-(32*(12*x^4-197*x^3-1030*x^2-197*x + 12)*n^8 + 624*(12*x^4-197*x^3-1030*x^2-197*x + 12)*n^7 + 10*(6295*x^4-103673*x^3-542004*x^2-103673*x + 6295)*n^6 + 4*(74418*x^4-1233343*x^3-6448430*x^2-1233343*x + 74418)*n^5 + (864893*x^4-14467663*x^3-75685660*x^2-14467663*x + 864893)*n^4 + 20*(78938*x^4-1336491*x^3-7002327*x^2-1336491*x + 78938)*n^3 + (1764932*x^4-30321697*x^3-159367410*x^2-30321697*x + 1764932)*n^2 + (1102551*x^4-19262286*x^3-101826010*x^2-19262286*x + 1102551)*n + 10*(29405*x^4-523232*x^3-2793306*x^2-523232*x + 29405))***a(n+2)**
+ (n+3)*(64*(4*x^3 + 21*x^2 + 21*x + 4)*n^7 + 1152*(4*x^3 + 21*x^2 + 21*x + 4)*n^6 + 12*(2899*x^3 + 15226*x^2 + 15226*x + 2899)*n^5 + 4*(35609*x^3 + 187226*x^2 + 187226*x + 35609)*n^4 + (340693*x^3 + 1795162*x^2 + 1795162*x + 340693)*n^3 + (474743*x^3 + 2511132*x^2 + 2511132*x + 474743)*n^2 + (355831*x^3 + 1894439*x^2 + 1894439*x + 355831)*n + 10*(11039*x^3 + 59401*x^2 + 59401*x + 11039))***a(n+3)**
-(n+3)*(n+4)^3*(16*(4*x^2 + 17*x + 4)*n^4 + 120*(4*x^2 + 17*x + 4)*n^3 + (1313*x^2 + 5599*x + 1313)*n^2 + 15*(103*x^2 + 443*x + 103)*n + 659*x^2 + 2882*x + 659)***a(n+4)** = 0

Recurrence order is p.

Conjecture (V. Kotěšovec, 19.9.2012): for $p > 1$ is maximal degree of polynomials in *n*

$$\sum_{k=1}^{p-1}\left(k^2-\left\lfloor\frac{k^2}{2}\right\rfloor\right)=\frac{4p^3-6p^2+8p-3+3*(-1)^p}{24}$$

(=A131941($p-1$), holds for $p \leq 19$)

References:

[1] Graham R. L., Knuth D. E. and Patashnik O., Concrete Mathematics: A Foundation for Computer Science, 2nd edition, 1994.
[2] Kotěšovec V., Non-attacking chess pieces, 5th edition, 9.1.2012
[3] Noble R., Asymptotics of a family of binomial sums, J. Number Theory 130 (2010), no. 11, 2561-2585
[4] Kotěšovec V., Asymptotic formula for number of fat trees on n labeled vertices, website 25.8.2012
[5] OEIS - The On-Line Encyclopedia of Integer Sequences
[6] Franel J., Intermediaire des Mathematiciens, 1894, p.45-47, 1895, p.33-35
[7] Perlstadt M. A., Some Recurrences for Sums of Powers of Binomial Coefficients, Journal of Number Theory 27 (1987), pp. 304-309
[8] Elsner C., On recurrence formulae for sums involving binomial coefficients, 2002-2003
[9] Verrill H. A., Sums of squares of binomial coefficients, with applications to Picard-Fuchs equations, 1.2.2008

This article was published on website http://web.telecom.cz/vaclav.kotesovec/math.htm, 20.9.2012