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On the Enumeration Function of Multiplicative Partitions

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Let $g(n)$ denote the enumeration function of multiplicative partitions of the natural number n. In this paper we give a simplified proof of the conjecture $g(n) \leqq n$ and discuss a sum which is related to $g(n)$.

1. Introduction

Consider the set $T(n)=\{(m_1, m_2, \ldots, m_s);\ n=m_1 m_2 \ldots m_s, m_i>1,\ 1 \leqq i \leqq s\}$ where n and m_i, $1 \leqq i \leqq s$, all are natural numbers and identify those partitions which differ only by the order of factors. We define $g(n)=|T(n)|$, $n>1$, and $g(1)=1$. For example $g(12)=4$, since $12=6 \cdot 2=4 \cdot 3=3 \cdot 2 \cdot 2$ are the four multiplicative partitions.

In 1983 J. F. Hughes and J. O. Shallit [2] have proved that $g(n) \leqq 2n^{\sqrt{2}}$ and made the conjecture $g(n) \leqq n$. In 1987 Chen Xiao-Xia [1] has proved that $g(n) \leqq n$. In this paper we shall prove

Theorem 1. $g(n) \leqq n$.

The proof, we shall give to Theorem 1, is a simplified proof, which is much simpler than the proof in [1].

Then we shall prove that

Theorem 2. $\Sigma_{n \leqq x} |\mu(n)| g(n) \ll x^{1+\frac{1}{e}}$, *where $\mu(n)$ is a Möbius function.*

Theorem 2 gives a bound for the number of all multiplicative partitions of square-free integers $\leqq x$.

2. Some lemmas

To prove the theorems, we need the following lemmas.

Lemma 1. *Let* $S(x)=1+\sum_{k \geqq 1} \frac{1}{k!} \sum_{\substack{n_1, n_2 \ldots n_k \leqq x \\ n_1, n_2, \ldots, n_k > 1}} \frac{1}{\log n_1 \log n_2 \ldots \log n_k}$ *for* $x \geqq 1$, *where $n_1, n_2, \ldots, n_k$ are ordinary integers, then*

$$S(x)=Cx+0(x\exp\{-C_1(\log x)^{\frac{3}{5}}(\log\log\log x)^{\frac{2}{5}}\}),$$

where C and C_1 are constants $C>1$, $C_1>0$.

Proof. See [3].

Lemma 2. *If $x>0$, then $\log x \leqq \sqrt[e]{x}$.*

Proof. Consider $f(x)=\dfrac{\sqrt[e]{x}}{\log x}$, we have

$$f'(x)=\frac{\frac{1}{e}x^{\frac{1}{e}-1}\log x - x^{\frac{1}{e}-1}}{\log^2 x}=\frac{\frac{1}{e}x^{\frac{1}{e}-1}(\log x - e)}{\log^2 x}.$$

Let $f'(x)=0$, we get $x=e^e$. Obviously, when $0<x<e^e$, $f'(x)<0$; and when $x>e^e$, $f'(x)>0$. Hence we get $f(x)\geqq f(e^e)=1$, $x>0$. ■

In this paper $P(n)$ is the largest prime factor of n and $P_1(n)$ — the smallest.

Lemma 3. *If $n>1$, then $g(n) \leqq \sum_{d\mid\frac{n}{p(n)}} g(d)$.*

Proof. Let

$$n=\prod_{j=1}^{r} p_j^{\alpha_j},\ p_1<p_2<\ldots<p_r,\ a_j\geqq 1,\ 1\leqq j\leqq r.$$

Consider the sets:

$$T_{j_1 j_2\ldots j_r}(n)=\{(p_r p_1^{\alpha_1-j_1}p_2^{\alpha_2-j_2}\ldots p_r^{(\alpha_r-1)-j_r},\ m_2, m_3,\ldots,m_s);$$
$$n=p_r p_1^{\alpha_1-j_1}p_2^{\alpha_2-j_2}\ldots p_r^{(\alpha_r-1)-j_r}m_2\ldots m_s,\ m_i>1,\ 2\leqq i\leqq s\},$$
$$0\leqq j_i\leqq \alpha_i,\ 1\leqq i\leqq r-1,\ 0\leqq j_r\leqq \alpha_r-1,$$

where we also identify those partitions which differ only by the order of factors.

We easily see

$$|T_{j_1 j_2\ldots j_r}(n)|=g(p_1^{j_1}p_2^{j_2}\ldots p_r^{j_r})$$

and

$$T(n)=\bigcup_{j_1=0}^{\alpha_1}\bigcup_{j_2=0}^{\alpha_2}\ldots\bigcup_{j_r=0}^{\alpha_r-1}T_{j_1 j_2\ldots j_r}(n).$$

So we have

$$g(n)=|T(n)|\leqq \sum_{j_1=0}^{\alpha_1}\sum_{j_2=0}^{\alpha_2}\ldots\sum_{j_r=0}^{\alpha_r-1}|T_{j_1 j_2\ldots j_r}(n)|$$
$$=\sum_{j_1=0}^{\alpha_1}\sum_{j_2=0}^{\alpha_2}\ldots\sum_{j_r=0}^{\alpha_r-1}g(p_1^{j_1}p_2^{j_2}\ldots p_r^{j_r})=\sum_{d\mid\frac{n}{p(n)}}g(d).$$

Lemma 4. *If $n>1$, then*

$$\sum_{d\mid\frac{n}{p(n)}} d \leqq \frac{1}{p_1(n)-1} n.$$

Proof. Let $n=\prod_{i=1}^{r} p_i^{\alpha_i}$, $p_1<p_2<\dots<p_r$; $\alpha_i \geqq 1$, $1\leqq i\leqq r$,

we either have

$$\sum_{d\mid\frac{n}{p(n)}} d=\frac{p_1^{\alpha_1}-1}{p_1-1}\leqq\frac{1}{p_1(n)-1}n \quad (r=1)$$

or

$$\sum_{d\mid\frac{n}{p(n)}} d=\frac{p_r^{\alpha_r}-1}{p_r-1}\prod_{i=1}^{r-1}\frac{p_i^{\alpha_i+1}-1}{p_i-1}=\frac{p_r^{\alpha_r}-1}{p_1-1}\prod_{i=1}^{r-1}\frac{p_i^{\alpha_i+1}-1}{p_{i+1}-1}$$

$$\leqq\frac{p_r^{\alpha_r}}{p_1-1}\prod_{i=1}^{r-1}\frac{p_i^{\alpha_i+1}-1}{p_i}\leqq\frac{1}{p_1(n)-1}n (r\geqq 2). \blacksquare$$

3. Proof of theorems

Proof of Theorem 1. We use induction on n.
When $n=1$, we have $g(1)=1$.
Suppose $g(d)\leqq d$, $d\leqq n-1$, $n-1\geqq 1$. We shall prove that $g(n)\leqq n$.
By Lemma 3 and Lemma 4 we have

$$g(n)\leqq\sum_{d\mid\frac{n}{p(n)}} g(d)\leqq\sum_{d\mid\frac{n}{p(n)}} d\leqq\frac{1}{p_1(n)-1}n\leqq n. \blacksquare$$

Proof of Theorem 2. By Lemma 1 and Lemma 2, we have

$$\sum_{n\leqq x}|\mu(n)|\frac{g(n)}{\sqrt[c]{n}}=\sum_{n\leqq x}\sum_{k\geqq 1}\frac{1}{k!}\sum_{\substack{n_1,n_2,\dots n_k=n\\ n_1,n_2,\dots,n_k>1}}\frac{|\mu(n)|}{\sqrt[c]{n_1 n_2,\dots,n_k}}$$

$$(1)\qquad \leqq\sum_{n\leqq x}\sum_{k\geqq 1}\frac{1}{k!}\sum_{\substack{n_1 n_2\dots n_k=n\\ n_1,n_2,\dots,n_k>1}}\frac{1}{\log n_1\log n_2\dots\log n_k}$$

$$=\sum_{k\geqq 1}\frac{1}{k!}\sum_{\substack{n_1 n_2\dots n_k\leqq x\\ n_1,n_2,\dots,n_k>1}}\frac{1}{\log n_1\log n_2\dots\log n_k}\ll S(x)\ll x.$$

Let $S_1(t)=\sum_{n\leqq t}|\mu(n)|\frac{g(n)}{\sqrt[c]{n}}$, by Abel's summation formula, we get

$$(2)\quad \sum_{n\leqq x}|\mu(n)|g(n)=\sum_{n\leqq x}|\mu(n)|\frac{g(n)}{\sqrt[c]{n}}\sqrt[c]{n}=S_1(x)\sqrt[c]{x}=\int_1^x S_1(t)(\sqrt[c]{t})'dt=S_1-S_2.$$

By (1) we get

$$S_1 \ll x^{1+\frac{1}{\epsilon}}, \tag{3}$$

$$S_2 \ll \int_1^x t(\sqrt[\epsilon]{t})' dt \leq x \int_1^x (\sqrt[\epsilon]{t})' dt \ll x^{1+\frac{1}{\epsilon}}. \tag{4}$$

By (2), (3) and (4), we get

$$\sum_{n \leq x} |\mu(n)| g(n) \ll x^{1+\frac{1}{\epsilon}}. \blacksquare$$

Remark. By $\sum_{n \leq x} |\mu(n)| = \frac{6}{\pi^2} x + 0(\sqrt{x})$ and $g(n) \leq n$, we only can obtain

$$\sum_{n \leq x} |\mu(n)| g(n) \leq \sum_{n \leq x} |\mu(n)| n = \frac{3}{\pi^2} x^2 + 0(x^{\frac{3}{2}}) \ll x^2.$$

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