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On the Enumeration Function of Multiplicative Partitions

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Presented by V. Popov

Let $g(n)$ denote the enumeration function of multiplicative partitions of the natural number n . In this paper we give a simplified proof of the conjecture $g(n) \leq n$ and discuss a sum which is related to $g(n)$.

1. Introduction

Consider the set $T(n) = \{(m_1, m_2, \dots, m_s); n = m_1 m_2 \dots m_s, m_i > 1, 1 \leq i \leq s\}$ where n and $m_i, 1 \leq i \leq s$, all are natural numbers and identify those partitions which differ only by the order of factors. We define $g(n) = |T(n)|, n > 1$, and $g(1) = 1$. For example $g(12) = 4$, since $12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$ are the four multiplicative partitions.

In 1983 J. F. Hughes and J. O. Shallit [2] have proved that $g(n) \leq 2n^{\sqrt{2}}$ and made the conjecture $g(n) \leq n$. In 1987 Chen Xiao-Xia [1] has proved that $g(n) \leq n$. In this paper we shall prove

Theorem 1. $g(n) \leq n$.

The proof, we shall give to Theorem 1, is a simplified proof, which is much simpler than the proof in [1].

Then we shall prove that

Theorem 2. $\sum_{n \leq x} |\mu(n)| g(n) \ll x^{1 + \frac{1}{c}}$, where $\mu(n)$ is a Möbius function.

Theorem 2 gives a bound for the number of all multiplicative partitions of square-free integers $\leq x$.

2. Some lemmas

To prove the theorems, we need the following lemmas.

Lemma 1. Let $S(x) = 1 + \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{n_1, n_2, \dots, n_k \leq x \\ n_1, n_2, \dots, n_k > 1}} \frac{1}{\log n_1 \log n_2 \dots \log n_k}$ for $x \geq 1$,

where n_1, n_2, \dots, n_k are ordinary integers, then

$$S(x) = Cx + O(x \exp \{ -C_1(\log x)^3 (\log \log x)^2 \}),$$

where C and C_1 are constants $C > 1$, $C_1 > 0$.

Proof. See [3].

Lemma 2. If $x > 0$, then $\log x \leq e \sqrt{x}$.

Proof. Consider $f(x) = \frac{e \sqrt{x}}{\log x}$, we have

$$f'(x) = \frac{\frac{1}{e} x^{\frac{1}{e}-1} \log x - x^{\frac{1}{e}-1}}{\log^2 x} = \frac{\frac{1}{e} x^{\frac{1}{e}-1} (\log x - e)}{\log^2 x}.$$

Let $f'(x) = 0$, we get $x = e^e$. Obviously, when $0 < x < e^e$, $f'(x) < 0$; and when $x > e^e$, $f'(x) > 0$. Hence we get $f(x) \geq f(e^e) = 1$, $x > 0$. ■

In this paper $P(n)$ is the largest prime factor of n and $P_1(n)$ — the smallest.

Lemma 3. If $n > 1$, then $g(n) \leq \sum_{d|\frac{n}{P_1(n)}} g(d)$.

Proof. Let

$$n = \prod_{j=1}^r p_j^{a_j}, \quad p_1 < p_2 < \dots < p_r, \quad a_j \geq 1, \quad 1 \leq j \leq r.$$

Consider the sets:

$$T_{j_1 j_2 \dots j_r}(n) = \{(p_1 p_1^{a_1 - j_1} p_2^{a_2 - j_2} \dots p_r^{(a_r - 1) - j_r}, m_2, m_3, \dots, m_s);$$

$$n = p_1 p_1^{a_1 - j_1} p_2^{a_2 - j_2} \dots p_r^{(a_r - 1) - j_r} m_2 \dots m_s, \quad m_i > 1, \quad 2 \leq i \leq s),$$

$$0 \leq j_i \leq a_i, \quad 1 \leq i \leq r-1, \quad 0 \leq j_r \leq a_r - 1,$$

where we also identify those partitions which differ only by the order of factors.

We easily see

$$|T_{j_1 j_2 \dots j_r}(n)| = g(p_1^{j_1} p_2^{j_2} \dots p_r^{j_r})$$

and

$$T(n) = \bigcup_{j_1=0}^{a_1} \bigcup_{j_2=0}^{a_2} \dots \bigcup_{j_r=0}^{a_r-1} T_{j_1 j_2 \dots j_r}(n).$$

So we have

$$\begin{aligned} g(n) &= |T(n)| \leq \sum_{j_1=0}^{a_1} \sum_{j_2=0}^{a_2} \dots \sum_{j_r=0}^{a_r-1} |T_{j_1 j_2 \dots j_r}(n)| \\ &= \sum_{j_1=0}^{a_1} \sum_{j_2=0}^{a_2} \dots \sum_{j_r=0}^{a_r-1} g(p_1^{j_1} p_2^{j_2} \dots p_r^{j_r}) = \sum_{d|\frac{n}{P_1(n)}} g(d). \end{aligned}$$

Lemma 4. *If $n > 1$, then*

$$\sum_{d|\frac{n}{p(n)}} d \leq \frac{1}{p_1(n)-1} n.$$

Proof. Let $n = \prod_{i=1}^r p_i^{\alpha_i}$, $p_1 < p_2 < \dots < p_r$; $\alpha_i \geq 1$, $1 \leq i \leq r$,

we either have

$$\sum_{d|\frac{n}{p(n)}} d = \frac{p_1^{\alpha_1} - 1}{p_1 - 1} \leq \frac{1}{p_1(n) - 1} n \quad (r=1)$$

or

$$\begin{aligned} \sum_{d|\frac{n}{p(n)}} d &= \frac{p_r^{\alpha_r} - 1}{p_r - 1} \prod_{i=1}^{r-1} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} = \frac{p_r^{\alpha_r} - 1}{p_1 - 1} \prod_{i=1}^{r-1} \frac{p_i^{\alpha_i+1} - 1}{p_{i+1} - 1} \\ &\leq \frac{p_r^{\alpha_r}}{p_1 - 1} \prod_{i=1}^{r-1} \frac{p_i^{\alpha_i+1} - 1}{p_i} \leq \frac{1}{p_1(n) - 1} n \quad (r \geq 2). \blacksquare \end{aligned}$$

3. Proof of theorems

Proof of Theorem 1. We use induction on n .

When $n=1$, we have $g(1)=1$.

Suppose $g(d) \leq d$, $d \leq n-1$, $n-1 \geq 1$. We shall prove that $g(n) \leq n$.

By Lemma 3 and Lemma 4 we have

$$g(n) \leq \sum_{d|\frac{n}{p(n)}} g(d) \leq \sum_{d|\frac{n}{p(n)}} d \leq \frac{1}{p_1(n)-1} n \leq n. \blacksquare$$

Proof of Theorem 2. By Lemma 1 and Lemma 2, we have

$$\begin{aligned} \sum_{n \leq x} |\mu(n)| \frac{g(n)}{\sqrt[n]{n}} &= \sum_{n \leq x} \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{n_1 n_2 \dots n_k = n \\ n_1, n_2, \dots, n_k > 1}} \frac{|\mu(n)|}{\sqrt[n_1 n_2 \dots n_k]{n_1 n_2 \dots n_k}} \\ (1) \quad &\leq \sum_{n \leq x} \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{n_1 n_2 \dots n_k = n \\ n_1, n_2, \dots, n_k > 1}} \frac{1}{\log n_1 \log n_2 \dots \log n_k} \\ &= \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{n_1 n_2 \dots n_k \leq x \\ n_1, n_2, \dots, n_k > 1}} \frac{1}{\log n_1 \log n_2 \dots \log n_k} \ll S(x) \ll x. \end{aligned}$$

Let $S_1(t) = \sum_{n \leq t} |\mu(n)| \frac{g(n)}{\sqrt[n]{n}}$, by Abel's summation formula, we get

$$(2) \quad \sum_{n \leq x} |\mu(n)| g(n) = \sum_{n \leq x} |\mu(n)| \frac{g(n)}{\sqrt[n]{n}} \sqrt[n]{n} = S_1(x) \sqrt{x} = \int_1^x S_1(t) (\sqrt[t]{t})' dt = S_1 - S_2.$$

By (1) we get

$$(3) \quad S_1 \ll x^{1+\frac{1}{c}},$$

$$(4) \quad S_2 \ll \int_1^x t(\sqrt[c]{t})' dt \leq x \int_1^x (\sqrt[c]{t})' dt \ll x^{1+\frac{1}{c}}.$$

By (2), (3) and (4), we get

$$\sum_{n \leq x} |\mu(n)|g(n) \ll x^{1+\frac{1}{c}}. \quad \blacksquare$$

Remark. By $\sum_{n \leq x} |\mu(n)| = \frac{6}{\pi^2}x + O(\sqrt{x})$ and $g(n) \leq n$, we only can obtain

$$\sum_{n \leq x} |\mu(n)|g(n) \leq \sum_{n \leq x} |\mu(n)|n = \frac{3}{\pi^2}x^2 + O(x^{\frac{3}{2}}) \ll x^2.$$

References

1. Chen Xiao-Xia. On multiplicative partitions of natural number. *Acta Math. Sinica*, **30**, 1987, 268-271.
2. John F. Hughes, J. O. Shallit. On the number of Multiplicative partitions. *Amer. Math. Monthly*, **90**, 1983, 468-471.
3. Wen-Bin Zhang. On a number theoretic series. *J. Number Theory*, **30**, 1988, 109-119.

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