SOME ASYMPTOTIC FORMULAE INVOLVING POWERS OF ARITHMETIC FUNCTIONS

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§1. Introduction. S. Ramanujan was probably the first mathematician to consider asymptotic formulae for sums of powers of certain arithmetic functions. For example, in 1916 in his paper [I0] he generalized the classical Dirichlet divisor problem and gave estimates, without proof, for \qquad $\tau^-(\mathfrak{n})$, where $\tau(\mathfrak{n})$ denotes the n<x number of divisors of n_1 . He also gave estimates for \qquad \q n<x σ (n) σ ,(n), again without proof, where σ (n) denotes the sum of the a-th $n \leq x$ powers of the divisors of n , with $\sigma_1(n) = \sigma(n)$. Another remarkable sum that he considered was $\left\langle \right\rangle$ r²(n) where r(n) denotes the number of representations of $\left\| n\right\rangle$ n<x as sum of two integral squares.

Ramanujan's results were proved, and in many cases, improved by B.M. Wilson [24] among others. However, Ramanujan did not give asymptotic formula for $\varphi^-(n)$ or for such related sums. Here $\varphi(n)$ is the Euler totient. n<x

Evidently inspired by the work of Ramanujan, S. Chowla in 1930 [3] obtained an asymptotic formula for $\sum_{i=1}^{\infty}$, where k is a fixed integer, with error term m<x $U(\log x)$. Among other things, in this paper we improve this U -term to $\mathcal{O}(\lambda(x)(\log x)^{k})$, where $\lambda(x) = (\log x)^{k/3}(\log \log x)$ if $x > 3$, and $x = 1$ for $0 \lt x \lt 3$ (see (2.7)). We also establish an asymptotic formula for $\lambda_{m}(\frac{m-1}{m})$, where ψ is Dedekind's ψ -function given by $\psi(n) = n_{n+1}(1+\frac{1}{n})$, p rime, with an error term $U((\log x)^{\sqrt{3k-1}})^3$. In fact, we establish asymptotic formulae for the sums $\sum_{k=1}^{\infty}$ ($\sum_{k=1}^{\infty}$) and $\sum_{k=1}^{\infty}$ where r is a positive integer with

rlm rim uniform ~-estimates of the error term (see Theorems 3.1 and 3.3). In Section 4, we consider the above sums with the restriction that $(m,n) = 1$. We also estimate the sum $\sum_{m\leq x}\left(\frac{\gamma_{k,m\ell}}{\psi(m)}\right)^{m}$, (see Theorem 3.4). The special case $k = 1$ of this sum was considered earlier by D. Suryanarayana ([18], Theorem 5) in 1982 who improved earlier estimates of S. Wigert ([22], [23]). Our estimate of the error term for this sum is superior to that of Suryanarayana.

We also consider (see Theorems 3.5 and 3.6 and Corollaries 3.4 and 3.5) asymptotic estimates for $\sum_{n\geq 0}$ (and $\sum_{n\geq 0}$ for positive non-integral values of t. The case when O < t < i was considered in 1969 by I.I. Iljasov [6].

In section 5, we estimate the sums $_{n \times n}$ ($\frac{(1+\frac{1}{n})}{n}$ and $_{n \times n}$ ($\frac{(1+\frac{1}{n})}{n}$), where ϕ^* and σ^* are the unitary versions of ϕ and σ , k being any positive

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integer. The case $k = 1$ was considered earlier in 1973 by Suryanarayana and Sitaramachandrarao [19].

In Section 2, we develope the necessary background by establishing several lemmas. Among other results, we need to utilize the deep result of Walfisz [19] that

$$
\sum_{m \leq x} \sigma(m)/m = \zeta(2)x + \mathcal{O}(\log^{2/3} x),
$$

where throughout the paper $\zeta(s)$ denotes, as usual, the Riemann zeta function. We may mention here that Ramanujan [i0] gave without proof the result:

$$
\sum_{m \leq x} \sigma^{2}(m) = (5/6)\zeta(3)x^{3} + E(x),
$$

where $E(x) = \mathcal{O}(x^2 \log^2 x)$, $E(x) \neq o(x^2 \log x)$.

R.A. Smith [16] improved the error term to $\theta(x^2 \log^{5/3} x)$. In [15], these results of Walfisz and Smith have been extended by Sitaramaiah and Suryanarayana to the general sum $\sum_{m\leq x} \sigma^{r}(m)$ in a remarkable manner.

tjx Regarding the asymptotic estimate for the summatory function for $\varphi(n)$, the well known elementary result

$$
\sum_{n \le x} \frac{\varphi(n)}{n} = \frac{6}{\pi^2} x + \mathcal{O}(\log x)
$$

was vastly improved by A. Walfisz ([21], Chapter 4) who used some deep estimates of exponential sums to establish the result:

$$
\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + \mathcal{O}(x(\log x)^{2/3} (\log \log x)^{4/3}).
$$

it is not generally noticed that this result was further improved by A.I. Saltykov in 1960 [ii] who showed that

$$
\sum_{n \le x} \varphi(n) = \frac{3x^2}{\pi^2} + \mathcal{O}(x(\log x)^{2/3}(\log \log x)^{1+\epsilon})
$$

for every $\varepsilon > 0$.

In obtaining our asymptotic results with error term for $\sum_{m\leq x}$ $\phi^{k}(m)$ and related sums, we need to establish several preliminary estimates. In doing so, to simplify our arguments, we utilize certain estimates of Walfisz. Thus our estimate of the error term for $\sum_{m\leq x} \varphi^k(m)$ is a direct generalization of that of Walfisz for $k = 1$. By similar arguments, we could improve our estimates by using the result of Saltykov. However, we shall not go into that here.

§2. Preliminaries. Throughout this paper the letter p stands for a prime number. The Dedekind ψ -function is known (cf. [5], page 123) to possess the following arithmetic form:

$$
\psi(m) = m \sum_{d \mid m} \frac{u^2(d)}{d} = m \left(1 + \frac{1}{p}\right) , \qquad (2.1)
$$

where μ is the Mobius function. It is clear that $\psi(m) \geq m$ and $\frac{\psi(m)}{m} \leq \theta(m) \leq \tau(m)$, where $\theta(m) = 2^{\omega(m)}$, the number of square-free divisors of m, and $\omega(m)$ is the number of distinct prime factors of m. Also, $\psi(mn)$ =

 $\frac{\gamma(\omega,\gamma(\omega),\ldots)}{\psi((m,n))}$, where (m,n) is the greatest common divisor of m and n, so that $\psi(\mathfrak{m},n) \leq \psi(\mathfrak{m})\psi(n)$. We frequently make use of the estimates

$$
\sum_{\substack{\text{m}{\leq}x}} \frac{1}{m} = \frac{\mathcal{O}}{\log x}, \quad \text{for} \quad x \geq 2 \quad , \tag{2.2}
$$

and for $s > 1$ and $x > 0$

$$
\sum_{\substack{m \to x \\ m \geq x}} \frac{1}{n^s} = \left(\frac{1}{\sqrt{2}-1} \right) \quad . \tag{2.3}
$$

We may have an occasion to use (2.2) for $x > 0$ also. In that case, without further mention we mean that $\int_{m\leq x} \frac{1}{m} = \mathcal{O}(f(x))$, where $f(x) = 1$ if $0 \leq x \leq 2$ and $f(x) = \log x$ if $x \ge 2$. A similar remark applies to all the asymptotic formulae in this paper and they are all valid for $x > 0$.

We prove

LEMMA 2.1. For any positive integer k,

$$
\sum_{m\leq x} \left(\frac{\psi(m)}{m}\right)^k = \mathcal{O}(x) \quad , \tag{2.4}
$$

where the ℓ -constant depends only on k.

PROOF: By (2.1) , we have

$$
\sum_{\substack{m\leq x \\ m\leq x}} \frac{\psi(m)}{m} = \sum_{d\leq x} \frac{\mu^2(d)}{d} \sum_{\delta \leq x \mid d} 1 = \mathcal{O}(x) \sum_{\substack{d\leq x \\ d\leq x} d^2} = \mathcal{O}(x).
$$

Hence (2.4) is true for $k = 1$. We now assume (2.4) for some $k \ge 1$. We have by (2.1) and the induction hypothesis,

$$
\sum_{m \leq x} \left(\frac{\psi(m)}{m}\right)^{k+1} = \sum_{m \leq x} \left(\frac{\psi(m)}{m}\right)^{k} d\delta = m \frac{\nu^{2}(d)}{d}
$$
\n
$$
\leq \sum_{d\delta = m} \frac{(\psi(d))^{k}}{d^{k+1}} \left(\frac{\psi(\delta)}{\delta}\right)^{k}
$$
\n
$$
= \sum_{d \leq x} \frac{(\psi(d))^{k}}{d^{k+1}} \sum_{\delta \leq x \mid d} \left(\frac{\psi(\delta)}{\delta}\right)^{k}
$$
\n
$$
= \hat{U} \times \sum_{d \leq x} \frac{(\psi(d))^{k}}{d^{k+2}} = \hat{U}(x),
$$

where we used the result that

$$
\sum_{\substack{d \leq x \\ d \leq x}} \frac{\left(\psi(d)\right)^k}{d^{k+2}} \leq \sum_{\substack{d \leq x \\ d \leq x}} \frac{\left(\tau(d)\right)^k}{d^2} = O(1) ,
$$

since $\tau(d) = \hat{U}(d^{\epsilon})$, for every $\epsilon > 0$. Hence the induction is complete and Lemma 2. i follows.

LEMMA 2.2. For $t > 0$, we have

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{m}\leq \mathbf{x}}} \frac{\sigma^{\mathbf{x}}_{-\mathbf{t}}(\mathbf{m})}{\mathbf{m}} = \theta_{(\log \mathbf{x})},
$$

where $\sigma^{\star}(\mathbb{m})$ is the sum of the s-th powers of the square-free divisors of \mathbb{m} . PROOF. By (2.2) , we have

$$
\sum_{m \leq x} \frac{\sigma_{\pm}^{\star}(m)}{m} = \sum_{m \leq x} \frac{1}{m} \sum_{d\delta = m} \mu^{2}(d)d^{-t}
$$

$$
\leq \sum_{d \leq x} \frac{1}{d^{t+1}} \sum_{\delta \leq x/d} \frac{1}{\delta}
$$

$$
= \mathcal{O}\left(\frac{\log x}{\log x}\right) = \mathcal{O}\left(\log x\right).
$$

LEMMA 2.3. For $t > 0$ and $k \ge 1$

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{m}\leq \mathbf{x}}} \frac{\sigma_{\mathbf{t}}^{\mathbf{x}}(\mathbf{m})\psi^{k-1}(\mathbf{m})}{\mathbf{m}^{k}} = \hat{\mathcal{O}}(\log \mathbf{x}). \tag{2.5}
$$

PROOF. For $k = 1$, (2.5) is true by Lemma 2.2. We assume (2.5) for some integer

 $k \geq 1$. We have by (2.1) ,

$$
\sum_{m\leq x} \frac{\sigma_{\pm}^{*}(m)\psi^{k}(m)}{m^{k+1}} = \sum_{m\leq x} \frac{\sigma_{\pm}^{*}(m)\psi^{k-1}(m)}{m^{k}} \sum_{d\delta=m} \frac{\mu^{2}(d)}{d}
$$

$$
\leq \sum_{d\delta\leq x} \frac{\sigma_{\pm}^{*}(d)\sigma_{\pm}^{*}(d)\psi^{k-1}(d)\psi^{k-1}(\delta)}{d^{k+1}\delta^{k}}
$$

$$
= \sum_{d\leq x} \frac{\sigma_{\pm}^{*}(d)\psi^{k-1}(d)}{d^{k+1}} \sum_{\delta\leq x|d} \frac{\sigma_{\pm}^{*}(d)\psi^{k-1}(d)}{\delta^{k}}
$$

$$
= \mathcal{O}((\log x) \cdot \sum_{d\leq x} \frac{\sigma_{\pm}^{*}(d)\psi^{k-1}(d)}{d^{k+1}}),
$$

where we used the result that $\sigma_{-t}^{\star}(d\delta) \leq \sigma_{-t}^{\star}(d)\sigma_{-t}^{\star}(\delta)$ and the induction hypothesis.

 $\ddot{}$

We have

$$
\sigma_{-r}^{\star}(m) \leq \theta(m) \leq \tau(m),
$$

so that

$$
\sum_{d \leq x} \frac{\sigma_{-k}^{\star}(d)\psi^{k-1}(d)}{d^{k+1}} \leq \sum_{d \leq x} \frac{(\tau(d))^k}{d^2} = \mathcal{O}(1)
$$

Hence

$$
\sum_{\substack{\underline{\gamma} \\ \underline{m} \leq x}} \frac{\sigma_{-\underline{t}}^*(\underline{m}) \psi^{K}(\underline{m})}{\underline{m}^{k+1}} = \hat{U}(\log x).
$$

The induction is complete.

LEMMA 2.4. For any integer $t \geq 1$,

$$
\sum_{m \le x} \left(\frac{\psi(m)}{m}\right)^t A_1(m) = \mathcal{O}(x) ,
$$
\n
$$
A_1(m) = \sum_{n \text{ odd}} \frac{\varphi(q)}{n^2} .
$$
\n(2.6)

 \ddot{r}

where

PROOF: We have
$$
A_1(m) \leq \sum_{q|m} \frac{1}{q}
$$
, so that

$$
\sum_{m \le x} \left(\frac{\psi(m)}{m}\right)^t A_1(m) \le \sum_{d\delta \le x} \frac{\left(\psi(d)\right)^t}{d^{t+1}} \left(\frac{\psi(\delta)}{\delta}\right)^t
$$
\n
$$
= \sum_{\substack{d \le x} \le x} \frac{\left(\psi(d)\right)^t}{d^{t+1}} \sum_{\substack{\delta \le x \mid d}} \left(\frac{\psi(\delta)}{\delta}\right)^t
$$

$$
= \mathcal{O}\left(\begin{array}{cc} x & \frac{\left(\sqrt{d}\right)^{t}}{t+2} \\ \frac{d}{x} & d \end{array}\right)
$$

 $=$ $\mathcal{O}(x)$,

where we used Lemma 2.1 and the fact

$$
\sum_{\substack{d \leq x \\ d \leq x}} \frac{(\psi(d))^{\mathsf{t}}}{d^{\mathsf{t}+2}} \leq \sum_{\substack{d \leq x \\ d \leq x}} \frac{(\tau(d))^{\mathsf{t}}}{d^2} = 0 \quad (1).
$$

LEMMA 2.5. We have

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{m}\leq \mathbf{x}}} \frac{\tau(\mathbf{m})}{\varphi(\mathbf{m})} = \mathcal{O}(\log^2 \mathbf{x}).
$$

PROOF. Since $\frac{m}{\mu(A)} = \sum_{d|n} \frac{\mu^2(d)}{\mu(d)}$ $_{\infty}$ (m) $^{-}$ $_{d}$ lm $_{\infty}$ (d) $'$ we have

$$
\sum_{m \leq x} \frac{\tau(m)}{\varphi(m)} \leq \sum_{d \leq x} \frac{\tau(d)}{d\varphi(d)} \sum_{\delta \leq x \mid d} \frac{\tau(\delta)}{\delta}
$$

$$
= \mathcal{O}\left(\left(\log x\right)^2 \cdot \sum_{d \leq x} \frac{\tau(d)}{d\varphi(d)}\right) = \mathcal{O}\left(\log^2 x\right) ,
$$

where we used the result that $\sum_{m\leq x} \frac{\tau(m)}{m} = 0$ (log²x) and the fact that

$$
\sum_{\substack{d \leq x \\ d \leq x}} \frac{\tau(d)}{d\varphi(d)} \leq \sum_{\substack{d \leq x \\ d \leq x}} \frac{\left(\tau(d)\right)^2}{d^2} = 0 \quad (1)
$$

since $\frac{d}{\varphi(d)} \leq \theta(d) \leq \tau(d)$. Hence Lemma 2.5 follows.

LEMMA 2.6. (cf. [14], Lemma 2.2). For any positive integer $\,$ n and $\,$ x $>$ 0, we $\,$ have

$$
\sum_{\substack{m\leq x\\(m,\,n)=1}}\frac{\mu(m)}{n}\,\rho\left(\frac{x}{m}\right)=\mathcal{O}\left(\sigma_{-1}^{\star}+\varepsilon\left(m\right)\lambda\left(x\right)\right),
$$

for every $\varepsilon > 0$, where $\rho(x) = x - [x] - 1/2$ and, as stated earlier,

$$
\lambda(x) = \begin{cases} (\log x)^{2/3} (\log \log x)^{4/3}, & \text{if } x \ge 3 \\ 1, & \text{if } 0 \le x \le 3. \end{cases}
$$
 (2.7)

Also, the \hat{U} -constant depends only on ε .

REMARK 2.1: It is clear that $\lambda(x)$ is increasing for be shown that $\lambda(x) \leq \frac{1}{\sqrt{3}} \lambda(y)$ whenever $0 \leq x \leq y$. $\lambda(x) = U(\lambda(y))$, for $0 \le x \le y$. x > 3. Using this it can In particular,

LEMMA $2.7.$ (cf. $[4]$, page 10). We have

$$
\sum_{\substack{\underline{m\leq x}\\(m,n)=1}}\frac{\mu(m)}{m}=0\ (1),
$$

where the \hat{U} -estimate is uniform in x and n.

LEMMA 2.8. (A. Walfisz [21]). For $x > 2$, we have

$$
\sum_{\substack{\text{m}\leq x}}\frac{\sigma(\text{m})}{\text{m}} = x\zeta(2) + \mathcal{O}(\log^{2/3} x).
$$

We now prove

LEMMA 2.9. Let $\sigma'(\mathfrak{m};n)$ denote the sum of the reciprocals of the divisors of m which are prime to n, that is, $\sigma'(\mathbf{m}; \mathbf{n}) = \int_{a}^{a} \frac{1}{a}$. Then we have $(a, n) = 1$

$$
\sum_{\substack{n \leq x \\ n \leq x}} \sigma'(\mathfrak{m}; \mathfrak{n}) = \frac{\zeta(2) J_2(n) x}{n^2} + \mathcal{O}\left(\frac{\psi(n)}{n} \cdot \log^{2/3} x\right),
$$
\n
$$
J_2(n) = n^2 \frac{1}{p} \left(n^2 + \frac{1}{p^2}\right),
$$
\n(2.8)

where

the θ -estimate being uniform in x and n.

PROOF. Since $\int_{d \ln} \mu(d) = 1$ or 0 according as $m = 1$ or $m > 1$, we have by i Lemma 2.8 ,

$$
\sum_{m \leq x} \sigma'(m; n) = \sum_{\substack{ab \leq x \\ (a, n) = 1}} \frac{1}{a} = \sum_{d \mid n} \frac{\mu(d)}{d} \sum_{\substack{b \leq x/d \\ (a, n) = 1}} \frac{1}{6}
$$

$$
= \sum_{\substack{d \mid n \\ d \mid n}} \frac{\mu(d)}{d} \sum_{\substack{m \leq x/d \\ (a, n) = 1}} \frac{\sigma(m)}{m}
$$

$$
= \sum_{\substack{d \mid n \\ d \mid n}} \frac{\mu(d)}{d} \left\{ \frac{x \zeta(2)}{d} + \mathcal{O}(\log^{2/3} x) \right\}
$$

$$
= x \zeta(2) \sum_{\substack{d \mid n \\ d \mid n}} \frac{\mu(d)}{d} + \mathcal{O}((\log^{2/3} x) \cdot \sum_{\substack{d \mid n \\ d \mid n}} \frac{\mu^2(d)}{d})
$$

$$
= \frac{x \zeta(2) J_2(n)}{n^2} + \mathcal{O}(\frac{\psi(n)}{n} \cdot \log^{2/3} x).
$$

LEMMA 2.10. We have for any positive integer r,

$$
\sum_{\substack{\underline{n} \le x \\ \underline{r} \mid \underline{n}}} \frac{\sigma(\underline{n})}{\underline{n}} = \frac{x\zeta(2)A_1(r)}{r} + \mathcal{O}(s(r)\log^{2/3}x) ,
$$

where

and

$$
A_1(r) = \sum_{q \mid r} \frac{\varphi(q)}{q^2}
$$
 (2.9)

$$
S(r) = \sum_{d \mid r} \frac{\mu^2(d)\theta(d)}{\varphi(d)} = \frac{(1 + \frac{2}{p-1})}{p \mid r} \left(\frac{1 + \frac{2}{p-1}}{1 + \frac{2}{p-1}} \right) = \frac{\psi(r)}{\varphi(r)}
$$
(2.10)

the θ -estimate being uniform in x and r.

PROOF: We have by Lemma 2.9,

$$
\sum_{m \leq x} \frac{\sigma(m)}{m} = \sum_{d \leq x} \frac{1}{\delta} = \sum_{d \leq x} \frac{1}{\delta} = \sum_{r(\delta, r)} \frac{1}{\delta}
$$
\n
$$
r|m = r|d\delta = \frac{r}{(\delta, r)}|d
$$
\n
$$
= \sum_{q \mid r} \frac{1}{q} - \sum_{b \leq x} \frac{1}{\delta} = \sum_{q \mid r} \frac{1}{q} - \sum_{\substack{a,b \leq x/r \\ (a, r) = q}} \frac{1}{a} - \sum_{\substack{a,b \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a,b \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack{a \leq x/r \\ (a, r) = 1}} \frac{1}{a} - \sum_{\substack
$$

It is not difficult to show that

$$
\sqrt{1 + \frac{1}{q} \frac{J_2(r/q)}{(r/q)^2}} = \sqrt{1 + \frac{q(q)}{q}} = A_1(r),
$$
\n
$$
\sqrt{1 + \frac{1}{q} \frac{\psi(r/q)}{(r/q)}} = \sqrt{1 + \frac{q(q)}{q}}.
$$

and

As observed in [15], p. 1194, we have

$$
\sum_{q \mid r} \frac{\theta(q)}{q} \leq S(r) ,
$$

from which we obtain Lemma 2.10.

COROLLARY 2.1. We have for $x \ge 2$ and $r \ge 1$,

$$
\sum_{\substack{m\leq x\\ m\leq x}} \sigma(m) = \frac{\pi^{2} x^{2} A_{1}(r)}{12 r} + \theta(s(r)x \log^{2/3} x),
$$

the O-estimate being uniform in x and r.

PROOF. Follows from Lemma 2.10 and partial summation.

REMARK 2.1: Corollary 2.1 is due to V. Sitaramaiah and D. Suryanarayana (cf. [15], Lemma 2.3).

LEMMA 2.11. For any positive integer n, we have

$$
\sum_{\substack{n\leq x\\(m,n)=1\\(m,n)=1}}\frac{\sigma(m)}{m}=\frac{x\zeta(2)\varphi(n)J_{2}(n)}{n^{3}}+\mathcal{O}(\frac{\theta(n)n}{\varphi(n)}\log^{2/3}x),
$$

uniformly in x and n.

PROOF. We have by Lemma 2.10,

$$
\sum_{\substack{m \leq x \\ (m,n)=1}} \frac{\sigma(m)}{m} = \sum_{d \mid n} \mu(d) \sum_{\substack{m \leq x \\ d \mid m}} \frac{\sigma(m)}{m} = \sum_{d \mid n} \mu(d) \left\{ \frac{x\zeta(2)A_1(d)}{d} + \theta(s(d) \cdot \log^{2/3} x) \right\}
$$

$$
= x\zeta(2) \sum_{d \mid n} \frac{\mu(d)A_1(d)}{d} + \theta(\log^{2/3} x) \cdot \sum_{d \mid n} \mu^2(d)S(d))
$$

$$
= \frac{x\zeta(2)\varphi(n)J_2(n)}{n^3} + \theta(\frac{n\theta(n)}{\varphi(n)} \cdot \log^{2/3} x) ,
$$

since by (2.9) and (2.10),

$$
\int_{d|n} \frac{\mu(d)A_1(d)}{d} = \frac{\varphi(n)D_2(n)}{n^3}, \quad \int_{d|n} \mu^2(d)S(d) = \frac{\theta(n)n}{\varphi(n)}.
$$

Hence Lema 2.11 follows.

for every e > 0, where _{ო(m)} አዋ\^{L/D}1 m.<x "r^{-B}.(r) rlm $-\frac{1}{2} + U(\sigma^*, (r)_{\phi}(r)r^{-1}(\chi))$ $B_1 = H B_1(P)$, $B_1(P) = H B_1(P)$, P Plr and $B_1(p) = 1 - \frac{1}{2}$. P PROOF: Since $\frac{r}{m} = \lambda_{d\delta=m} \frac{r}{d}$, we have $\frac{L}{m}$ $\frac{L}{m}$ $\frac{L}{dS}$ $\frac{d}{d}$ $\frac{L}{dr(d,r)}$ $\frac{L}{d}$ $\frac{1}{d}$ rim rid6 $=\begin{array}{cc} \n\lambda & \lambda & \lambda \\ \nq & r & (d|q) b \lambda x / r & d\n\end{array}$ $\begin{array}{cc} \n\lambda & \lambda & \lambda \\ \nq & q & r \end{array}$ $(d,r) = q$ (a,r/q)=1 **= [p(q) [~(a)** $q \mid r$ ab $\langle x \rangle r$ $(a, r/q)$ =1 $(a, a)=1$ $-$, since $\mu(aq) = 0$ if $(a,q) > 1$ **= [~(q) [u(a~ = ~(r) [p(a)** q \vert r ab $\langle x/r \rangle$ r a $\langle x/r \rangle$ r a $\langle x/r \rangle$ $(a,r)=1$ $(a,r)=1$ b<x/ar **r a<xlr** (a, r)=l **= ~(r) [** $r = a \langle x/r$ (a,r)=l $\mu(a)$ (X $(X \setminus -1)$ **a** $\{$ ar $\{$ $\}$ $\{$ $2\}$ **_ xφ(r)** τ μ(a) φ(r) r^2 $a\overline{\langle x/r\rangle}$ a^2 r $a\overline{\langle x/r\rangle}$ (a,r)=l (a,r)=l μ(a) _{.(}x _ι φ(r) . a $\arctan 2r$ $ab \lt x/r$ (a,r)=l **~(a)**

a

LEMMA 2.12. For any positive integer r, we have

$$
= \frac{x \varphi(r)}{r^2} \sum_{\substack{a=1 \ a^2}} \frac{\mu(a)}{2} + \frac{\ell(x \varphi(r)}{r^2} \cdot \frac{r}{x})
$$

\n
$$
(a, r)=1 + \frac{\ell(\varphi(r)}{r} \sigma_{-1+\varepsilon}^*(r) \lambda(x)) + (\frac{\varphi(r)}{r}) , \qquad (2.11)
$$

where we used the result that

$$
\sum_{\substack{a \ge x \\ (a, r) = 1}} \frac{\mu(a)}{2} = \mathcal{O}\left(\sum_{\substack{a \ge x \\ a \ge x}} \frac{1}{2}\right) = \mathcal{O}\left(\frac{1}{x}\right),
$$

and the Lemmas 2.6 and 2.7. Also it is clear that

$$
\sum_{a=1}^{\infty} \frac{\mu(a)}{a^2} = \frac{B_1}{B_1(r)}.
$$

(a, r)=1

On combining the \hat{U} -terms in (2.11) we obtain Lemma 2.12.

COROLLARY 2.2. We have

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{r} \mid \mathbf{m}}} \varphi(\mathbf{m}) = \frac{\mathbf{x}^2 \varphi(\mathbf{r}) \mathbf{B}_1}{2 \mathbf{r}^2 \mathbf{B}_1(\mathbf{r})} + \mathcal{O} \left(\sigma_{-1+\varepsilon}^{\star}(\mathbf{r}) \varphi(\mathbf{r}) \mathbf{r}^{-1} \mathbf{x} \lambda(\mathbf{x}) \right), \tag{2.12}
$$

for every $\epsilon > 0$, where the U -constant depends only on ϵ .

 $PROOF:$ Follows from Lemma 2.12 and partial summation.

LEMMA 2.13. We have

$$
\psi(m) = \sum_{a^2 b = m} \mu(a) \sigma(b).
$$

We omit the proof which is easy.

LEMMA 2.14. For square-free r, we have

$$
\sum_{\substack{m\leq x\\r\mid m}}\frac{\psi(m)}{m}=\frac{x\psi(r)c_1}{r^2c_1(r)}+\mathcal{O}\left(s(r)+\log^{2/3}x\right),
$$

S(r) is as given in (2.10), the \hat{U} -estimate being uniform in x and r. PROOF: By Lemma 2.13 we have

$$
\sum_{\substack{m \le x \\ r | m}} \frac{\psi(m)}{m} = \sum_{\substack{a^2 \le x \\ a^2b \le x}} \frac{\mu(a)\sigma(b)}{a} = \sum_{\substack{a \le y \\ a \le y \\ a \le x}} \frac{\mu(a)}{a} \sum_{\substack{b \le x \\ a^2b}} \frac{\sigma(b)}{b}
$$

For square-free $r,(a^2,r) = (a,r)$. Hence by Lemma 2.10, we have

$$
\sum_{m \le x} \frac{\psi(m)}{m} = \sum_{a \le x} \frac{\mu(a)}{a} \sum_{b \le x/a} \frac{\sigma(b)}{b}
$$
\n
$$
r|m
$$
\n
$$
= \sum_{\substack{a \le x \\ a \le x/a}} \frac{\mu(a)}{a} \left\{ \frac{x \zeta(2)(a, r) A_1(r/(a, r))}{a^2 r} + \mathcal{O}\left(s\left(\frac{r}{(a, r)}\right) \log^{2/3} x\right) \right\}
$$
\n
$$
= \frac{x \zeta(2)}{r} \sum_{\substack{a \le x/a \\ a \le x/a}} \frac{\mu(a)(a, r) A_1(r/(a, r))}{a} + \mathcal{O}\left(\sum_{\substack{a \le x/a \\ a \le x/a}} \frac{1}{a^2} s\left(\frac{r}{(a, r)}\right) \log^{2/3} x\right)
$$
\n
$$
= \frac{x \zeta(2)}{r} \sum_{\substack{a \le x/a \\ a \le x/a}} \frac{\mu(a)(a, r) A_1(r/(a, r))}{a} + \mathcal{O}\left(\sum_{\substack{a \le x/a \\ a \le x/a}} \frac{1}{a^2} s\left(\frac{r}{(a, r)}\right) \log^{2/3} x\right)
$$
\n
$$
= \frac{x \zeta(2)}{r} \sum_{\substack{a \le x/a \\ a \le x/a}} \frac{\mu(a)(a, r) A_1(r/(a, r))}{a} + \mathcal{O}\left(\sum_{\substack{a \le x/a \\ a \le x/a}} \frac{1}{a^2} s\left(\frac{r}{(a, r)}\right) \log^{2/3} x\right)
$$
\nhave

\n(2.13)

We

$$
\sum_{1}^{n} = \sum_{q \mid r} q A_1(r/q) \sum_{\substack{a \leq \sqrt{x} \\ (a, r) = q}} \frac{\mu(a)}{a}
$$

$$
= \sum_{q \mid r} qA_1(r/q) \sum_{\substack{b \le r \le q \\ b \le \sqrt{x}/q}} \frac{\mu(bq)}{b^4 q} = \sum_{q \mid r} \frac{\mu(q)A_1(r/q)}{q} \sum_{\substack{b \le r \le q \\ b \le r \ge 1}} \frac{\mu(b)}{q}
$$

$$
\sum_{\substack{b \le x \\ (b, r) = 1}} \frac{\mu(b)}{b} = \sum_{\substack{b = 1 \\ (b, r) = 1}}^{\infty} \frac{\mu(b)}{b} + \mathcal{O}\left(\sum_{\substack{b \le x \\ b \ge x}} \frac{1}{b}\right)
$$

$$
= \prod_{\substack{p \mid r \\ p \nmid r}} \left(1 - \frac{1}{\mu}\right) + \mathcal{O}\left(\frac{1}{3}\right) ,
$$

so that

$$
\sum_{i=1}^{r} = \prod_{p \nmid r} (1 - \frac{1}{p}) \prod_{q \mid r} \frac{\mu(q) A_1(r/q)}{q^3} + \mathcal{O}\left(\frac{1}{x^{3/2}} \cdot \sum_{q \mid r} A_1(r/q)\right).
$$

By (2.9) , we have

$$
\sum_{q \mid r} \frac{\mu(q)A_1(r/q)}{q^3} = \prod_{p \mid r} (A_1(p) - \frac{1}{p^3}) = \prod_{p \mid r} (1 + \frac{p-1}{p^2} - \frac{1}{p^3})
$$

$$
= \prod_{p \mid r} (1 + \frac{1}{p})(1 - \frac{1}{p^2}) = \frac{\psi(r)}{r} \prod_{p \mid r} (1 - \frac{1}{p^2}).
$$

Hence, we have

$$
\prod_{p \nmid r} \left(1 - \frac{1}{p^4}\right) \int_{q \mid r} \frac{\mu(q)A_1(r/q)}{q^3} = \frac{\psi(r)}{\zeta(4)rc_1(r)}.
$$

A1 so

$$
\sum_{\substack{q \mid r}} A_1(r/q) = \sum_{\substack{p \mid r}} A_1(q) = \frac{(1+A_1(p))}{p \mid r} \leq \frac{\prod_{p \mid r} (1+\frac{1}{p}) = \frac{\psi(r)}{r}.
$$

Therefore,

$$
\sum_{i=1}^{n} t = \frac{\psi(r)}{\zeta(4)rc_{1}(r)} + \mathcal{O}\left(\frac{\psi(r)}{rx^{3/2}}\right) .
$$
 (2.14)

For d|r, S(r/d) \leq S(r). Hence if $\frac{1}{2}$ is as given in (2.13), we have

$$
\sum_{\substack{\tau \\ 2}} \mathbf{r} = \mathcal{O}(\mathbf{S}(\mathbf{r}) \cdot \sum_{\substack{\mathbf{a} \\ \mathbf{a} \\ \mathbf{a} \leq \mathbf{a}}} \frac{1}{\mathbf{a}}) = \mathcal{O}(\mathbf{S}(\mathbf{r})). \tag{2.15}
$$

Using (2.14) and (2.15) in (2.13), we obtain,

$$
\sum_{\substack{m\leq x \\ r|m}} \frac{\psi(m)}{m} = \frac{x\zeta(2)\psi(r)}{\zeta(4)r^2c_1(r)} + (\frac{\psi(r)}{r^2} \cdot \frac{1}{\sqrt{x}}) + \mathcal{O}(S(r) \cdot \log^{2/3} x).
$$

On noting that $c_1 = \frac{2.52}{\zeta(4)}$, we obtain Lemma 2.14.

LEMMA 2.15. For any positive integer r, we have

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{m} \mid \mathbf{m}}} \frac{\psi(\mathbf{m})}{\mathbf{m}} = \frac{\mathbf{x}\zeta(2)\mathbf{D}_1(\mathbf{r})}{\mathbf{r}} + \mathcal{O}(\mathbf{S}(\mathbf{r})\log^{2/3}\mathbf{x}),
$$

where

$$
D_1(r) = \sum_{a=1}^{\infty} \frac{\mu(a)A_1(r/(a^2,r))(a^2,r)}{a^4}
$$
 (2.16)

A1 so,

$$
D_1(r) = \hat{U}(A_1(r)) . \qquad (2.17)
$$

PROOF: By Lemma 2.13 and 2.10, we have

$$
\sum_{m \leq x} \frac{\psi(m)}{m} = \sum_{a \leq \sqrt{x}} \frac{\psi(a)}{a^2} \sum_{b \leq x/a^2} \frac{\sigma(b)}{b}
$$
\n
$$
r|m
$$
\n
$$
(\frac{r}{a^2,r})|b
$$
\n
$$
= \frac{x\zeta(2)}{r} \sum_{a \leq \sqrt{x}} \frac{\psi(a)A_1(r/(a^2,r))(a^2,r)}{a^4} + \mathcal{O}(S(r) \cdot \log^{2/3} x) .
$$

Since $A_1(r/(a^2, r)) \leq A_1(r)$ and $(a^2, r) \leq r$, we have

$$
\sum_{a \geq \sqrt{x}} \frac{\mu(a) A_1(r/(a^2, r))(a^2, r)}{a} = \mathcal{O} \left(r A_1(r) \sum_{\substack{a \leq \sqrt{x} \\ a^2 \geq 0}} \frac{1}{a} \right)
$$

$$
= o(r A_1(r) \frac{1}{x^{3/2}}) .
$$

Hence we have

$$
\sum_{\substack{m \le x \\ r|m}} \frac{\psi(m)}{m} = \frac{x \zeta(2)}{r} D_1(r) + \sqrt{(2 \pi \zeta(2))} + \sqrt[2]{(x \zeta(2))} D_2^{2/3} x.
$$

By noting that

$$
A_1(r) \leq \sum_{q \mid r} \frac{1}{q} = \prod_{p \alpha_{\parallel r}} (1 + \frac{1}{p} + \dots + \frac{1}{p^{\alpha}})
$$

$$
\leq \prod_{p \mid r} \left(1 + \frac{1}{p} + \frac{1}{p^2} \cdots \right)
$$
\n
$$
= \prod_{p \mid r} \left(1 + \frac{1}{p-1} \right) \leq \prod_{p \mid r} \left(1 + \frac{2}{p-1} \right) = S(r),
$$

we obtain Lemma 2.15. (2.16) follows by using $(a^-,r) < a^-$ and $A,(r/(a^-,r))$ $\leq A_1(r)$.

Lemma 2.16. For any positive integer r, we have

$$
\sum_{\substack{m \leq x \\ r \mid m}} \left(\frac{\psi(m)}{m}\right)^2 = \frac{x \zeta(2) \psi(r) D_2(r)}{r^2} + \mathcal{O}(s(r) \cdot \log^{5/3} x),
$$

where

$$
D_2(r) = \sum_{\substack{a=1 \ a, r \ge 1}}^{\infty} \frac{\mu^2(a)D_1(ar)}{a^2}
$$
 (2.18)

Also,

$$
D_2(r) = \mathcal{O}(A_1(r)) . \qquad (2.19)
$$

PROOF. By (2.1), we have

$$
\sum_{\substack{m\leq x\\r|m}}\left(\frac{\psi(m)}{m}\right)^2 = \sum_{\substack{d\leq x\\d\leq x}}\frac{\mu^2(d)}{d}\sum_{\substack{m\leq x\\(d,r)\mid m}}\frac{\psi(m)}{m},
$$

where $\{d, r\}$ denotes the least common multiple of d and r. Now, by Lemma 2.5, since $\{d,r\} = rd/(r,d)$, we obtain

$$
\sum_{\substack{m \leq x \\ r \mid m}} \left(\frac{\psi(m)}{m} \right)^2 = \sum_{\substack{d \leq x \\ r \text{ odd}}} \frac{\mu^2(d)}{d} \left\{ \frac{x \zeta(2) D_1(\text{rd}/(\text{r}, d))}{(\text{rd}/(\text{r}, d))} \right\} + \mathcal{O}\left(\frac{s \frac{d}{\zeta(1 + d)} \cdot \log^{2/3} x\right)
$$
\n
$$
= \frac{x \zeta(2)}{r} \sum_{\substack{d \leq x \\ r \text{ odd}}} \frac{\mu^2(d) D_1(\text{rd}/(\text{r}, d))(\text{r}, d)}{d^2} + \left(\log^{2/3} x \cdot \sum_{\substack{d \leq x \\ r \text{ odd}}} \frac{s \left(\frac{\text{rd}}{(\text{r}, d)} \right) \cdot \frac{1}{d}}{d^2} \right)
$$
\n
$$
= \frac{x \zeta(2)}{r} \sum_{\substack{d \leq x \\ r \text{ odd}}} \frac{\mu^2(d) D_1(\text{rd}/(\text{r}, d))(\text{r}, d)}{d^2} \qquad (2.20)
$$

say. We have

$$
\sum_{4}^{r} = \sum_{\substack{d \leq x}} S\left(\frac{rd}{(r,d)}\right) \cdot \frac{1}{d} \leq S(r) \sum_{\substack{d \leq x}} \frac{S(d)}{d} = \hat{U}(S(r) \cdot \log x),
$$

since by (2.10) ,

$$
\sum_{m \leq x} \frac{S(m)}{m} = \sum_{d \leq x} \frac{\mu^2(d) \theta(d)}{d \delta \varphi(d)}
$$

$$
\leq \sum_{d \leq x} \frac{I(d)}{d \varphi(d)} \sum_{\delta \leq \frac{x}{d}} \frac{1}{\delta}
$$

$$
= \mathcal{O} \left(\log x \cdot \sum_{\substack{d \leq x \\ d \leq x}} \frac{I(d)}{d \varphi(d)} \right) = \mathcal{O} \left(\log x \right).
$$

Hence

 $[$ ' = U (S(r).log x). 4 **(2.2t)**

Also,

$$
\sum_{3} \cdot = \sum_{q \mid r} \frac{\mu^{2}(q)}{q} \sum_{\substack{a \leq x/q \\ (a, r) = 1}} \frac{\nu^{2}(a)D_{1}(ar)}{a^{2}}.
$$

By (2.17) for $(a,r) = 1$, we have $D_1(ar) = 0$ $(A_1(ar)) = 0$ $(A_1(a) \cdot A_1(r))$. Also since

$$
A_1(m) \leq \sum_{q \mid m} \frac{1}{q}, \sum_{m \leq x} A_1(m) \leq \sum_{d \delta \leq x} \frac{1}{d} = \mathcal{O}\left(x, \sum_{d \leq x} \frac{1}{d^2}\right) = \mathcal{O}(x).
$$

Hence

$$
\sum_{\substack{a \ge x/q \\ (a,n)=1}} \frac{u^2(a)D_1(ar)}{a^2} = \mathcal{O}\left(A_1(r) - \sum_{a \ge x/q} \frac{A_1(a)}{a^2}\right) = \mathcal{O}\left(A_1(r) \cdot \frac{q}{x}\right) ,
$$

so that

$$
\sum_{3}^{t} = \sum_{\substack{q \mid r}} \frac{\mu^{2}(d)}{d} \left\{ \sum_{\substack{a=1 \ (a,r)=1}}^{\infty} \frac{\mu^{2}(a)D_{1}(ar)}{a^{2}} + \mathcal{O}(A_{1}(r) \cdot \frac{q}{x}) \right\}
$$
\n
$$
= \frac{\psi(r)}{r} D_{2}(r) + \mathcal{O}\left(\frac{A_{1}(r)\theta(r)}{x}\right) \quad . \tag{2.22}
$$

Substituting (2.22) and (2.21) into (2.20) , we obtain

$$
\sum_{\substack{m\leq x\\r|m}}\left(\frac{\psi(m)}{m}\right)^2=\frac{x\zeta(2)\psi(r)D_2(r)}{r^2}+\mathcal{O}\left(\frac{A_1(r)\theta(r)}{r}\right)+\mathcal{O}\left(s(r)\log^{5/3}x\right).
$$

Hence Lemma 2.16 follows

§3. Main Results. Throughout the following k stands for a fixed positive integer and $0 \leq \varepsilon \leq 1$. All the error terms in the asymptotic formulae given in this section depend at most on k and ε . First we prove

THEOREM 3.1. We have

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{m} \mid \mathbf{m}}} \left(\frac{\varphi(\mathbf{m})}{\mathbf{m}}\right)^k = \frac{\mathbf{x}(\varphi(\mathbf{r}))^k B_k}{\mathbf{r}^{k+1} B_k(\mathbf{r})} + \mathcal{O}(\sigma_{-1+\varepsilon}^{\star}(\mathbf{r})\left(\frac{\psi(\mathbf{r})}{\mathbf{r}}\right)^{k-1} \lambda(\mathbf{x}) (\log \mathbf{x})^{k-1}), \tag{3.1}
$$

where

$$
B_{k} = p_{k}(p) , B_{k}(r) = p|r \qquad B_{k}(p),
$$

$$
B_{k}(p) = 1 + \sum_{a=1}^{k} (-1)^{a} {k \choose a} \frac{1}{p^{a+1}} = 1 + \frac{1}{p} (1 - \frac{1}{p})^{k} - 1).
$$
 (3.2)

REMARK 3.1. Clearly all p , $p_{\mid r}$ $p_k(p)$ > depends only on k. \mathbf{B}_{μ} (p) is absolutely convergent. Since $0 \lt B_{\mu}$ (p) $\lt 1$, for $p B_k(p)$, so that $\frac{1}{B_k(r)} = U(1)$, where the U -constant

PROOF OF THEOREM 3.1: By Lemma 2.12, Theorem 3.1 is true for k = 1 and for all r. We assume (3.1) for some k > i and all r. We have, since $\varphi(m)$ m = λ_{a1} μ (d)d ,

$$
\sum_{\substack{\underline{n} \leq x \\ \underline{r} \mid \underline{m}}} \left(\frac{\varphi(\underline{n})}{m}\right)^{k+1} = \sum_{\substack{d \leq x \\ \underline{r} \leq x}} \frac{\mu(d)}{d} \sum_{\substack{\underline{n} \leq x \\ \underline{m} \leq x}} \left(\frac{\varphi(\underline{n})}{m}\right)^{k}.
$$

Hence by the induction hypothesis, we have

$$
\sum_{m \leq x} \left(\frac{\varphi(m)}{m} \right)^{k+1} = \frac{xB_k}{r^{k+1}} \sum_{d \leq x} \frac{\mu(d)\varphi^k (rd/(r,d))(r,d)^{k+1}}{d^{k+2}B_k (rd/(r,d))}
$$
\n
$$
r|m
$$
\n
$$
+ \mathcal{O}(\lambda(x)(\log x)^{k-1} \frac{1}{r^{k-1}} \sum_{d \leq x} \frac{\mu^2(d)\sigma_{-1+\varepsilon}^k (rd/(r,d))\psi^{k-1}(rd/(r,d))(r,d)^{k-1}}{d^k})
$$
\n
$$
= \frac{xB_k}{r^{k+1}} \sum_{j} + \mathcal{O}(\lambda(x)(\log x)^{k-1} \frac{1}{r^{k-1}} \sum_{d \leq x} \frac{1}{r^{k-1}} \frac{1}{\xi}) ,
$$
\n(3.3)

say. We have

$$
\sum_{5}^{1} = \frac{\mathfrak{g}^{k}(\mathbf{r})}{B_{k}(\mathbf{r})} \sum_{q \mid \mathbf{r}} \frac{\mu(q)}{q} \sum_{\substack{\mathbf{a} \leq x/q \\ (\mathbf{a}, \mathbf{r}) = 1}} \frac{\mu(\mathbf{a}) \mathfrak{g}^{k}(\mathbf{a})}{\mathfrak{g}^{k+2} B_{k}(\mathbf{a})} .
$$
 (3.4)

Since $\varphi(a) \le a$ and $\frac{1}{B_k(a)} = U(1)$, the series $\sum_{\substack{a=1 \ (a,r)=1}}^{\infty} \frac{\mu(a)\psi(a)}{a^{k+2}B_k(a)}$ is absolutely

convergent. Also, the general term of the series is multiplicative in a. Hence expanding the series as an infinite product of Euler-type, we obtain

$$
\sum_{\substack{a=1 \ a^{k+2} \ b^{k} \ (a) \ p|r}}^{\infty} \frac{\mu(a) \phi^{k}(a)}{a^{k+2} B_{k}(a)} = \frac{p^{(1 - \frac{(p-1)^{k}}{p^{k+2} B_{k}(p)})}}{p | r} \cdot \frac{(1 - \frac{(p-1)^{k}}{p^{k+2} B_{k}(p)})}{(1 - \frac{(p-1)^{k}}{p^{k+2} B_{k}(p)})}
$$
(3.5)

From (3.2) it is easily seen that

$$
B_{k+1}(p) = B_k(p) - \frac{(p-1)^k}{p^{k+2}}
$$
,

so that

$$
B_{k+1} = B_k \binom{p-1}{p}^{k+2} B_k(p)
$$

and

$$
B_{k+1}(r) = B_k(r) \left[1 - \frac{(p-1)^k}{p^{k+2} B_k(p)}\right]
$$
 (3.6)

Also,

$$
\sum_{\substack{a \ge x \\ (a, r) = 1}} \frac{\mu(a) \varphi^{k}(a)}{x^{k+2} B_{k}(a)} \approx \mathcal{O}\left(\sum_{\substack{a \ge x \\ (a, r) = 1}} \frac{1}{2}\right) \approx \mathcal{O}\left(\frac{1}{x}\right) .
$$
 (3.7)

From (3.5), (3.6), (3.7) and (3.4), we obtain

$$
\frac{xB_k}{r^{k+1}} = \frac{x\varphi^k(r)B_{k+1}}{r^{k+1}B_{k+1}(r)} \sum_{q \mid r} \frac{\mu(q)}{q} + \mathcal{O}\left(\frac{\varphi^k(r)}{r^{k+1}}\right) \sum_{q \mid r} \mu^2(q)\right)
$$

$$
= \frac{x\varphi^{k+1}(r)B_{k+1}}{r^{k+1}B_{k+1}(r)} + \mathcal{O}\left(\frac{\theta(r)}{r}\right) \tag{3.8}
$$

From (3.3),

$$
\sum_{6}^{\cdot} = \sigma_{-1+\epsilon}^{*}(r)\psi^{k-1}(r)\sum_{q|r}^{\cdot} \frac{\mu^{2}(q)}{q} \sum_{\substack{a \le x/q \\ (a,r)=1}}^{\cdot} \frac{\mu^{2}(a)\sigma_{-1+\epsilon}^{*}(a)\psi^{k-1}(a)}{a^{k}}
$$

$$
\le \sigma_{-1+\epsilon}^{*}(r)\psi^{k-1}(r)\sum_{q|r}^{\cdot} \frac{\mu^{2}(q)}{q} \sum_{\substack{a \le x/q \\ (a \le x/q)}}^{\cdot} \frac{\sigma_{-1+\epsilon}^{*}(a)\psi^{k-1}(a)}{a^{k}}
$$

$$
= \mathcal{O}(\sigma_{-1+\epsilon}^{*}(r)) \frac{\psi^{k}(r)}{r} \log x,
$$
 (3.9)

by Lemma 2.3.

From (3.8) , (3.9) and (3.3) we get

$$
\sum_{\substack{\underline{\mathfrak{m}} \leq x \\ \underline{\mathfrak{m}}} \binom{\underline{\varphi(\underline{\mathfrak{m}})}{m}^{k+1}}{m} = \frac{x(\varphi(\underline{r}))^{k+1}B_{k+1}}{r^{k+1}B_{k+1}(\underline{r})} + \mathcal{O}(\sigma_{-1+\underline{\mathfrak{m}}}^{k}(\underline{r})\left(\frac{\psi(\underline{r})}{r}\right)^k \lambda(x) (\log x)^k).
$$

Hence the induction is complete and Theorem 3.1 follows.

THEOREM 3.2 For any square-free r, we have

$$
\sum_{\substack{\underline{n}\leq x \\ \underline{n} \leq x}} \left(\frac{\psi(\underline{n})}{\underline{n}}\right)^k = \frac{x\psi^k(r)c_k}{r^{k+1}c_k(r)} + \mathcal{O}(s(r)(\log x)^{(3k-1)/3}), \quad (3.10)
$$

where

$$
c_k = \prod_p c_k(p) , c_k(r) = \prod_{p \mid r} c_k(p) ,
$$

and

$$
c_{k}(p) = 1 + \sum_{a=1}^{k} {k \choose a} \frac{1}{p^{a+1}} = 1 + \frac{1}{p} \left((1 + \frac{1}{p})^{k} - 1 \right) . \tag{3.11}
$$

PROOF. By Lemma 2.14, (3.10) is true for $k = 1$ and all square-free r. We assume (3.10) for all square-free r. By (2.1) we have

$$
\sum_{\substack{m\leq x\\r|m}} \left(\frac{\psi(m)}{m}\right)^{k+1} = \sum_{\substack{d\leq x\\d\leq x}} \frac{\psi(d)}{d} \sum_{\substack{m\leq x\\m\leq x}} \left(\frac{\psi(m)}{m}\right)^{k}.
$$

We can assume that d is square-free. Since r is also square-free $\{r,d\}$ is square-free. Therefore by the induction hypothesis, we obtain

$$
\sum_{\substack{m \leq x \\ m \leq x}} \left(\frac{\psi(m)}{m}\right)^{k+1} = \frac{x c_k}{r^{k+1}} \sum_{d \leq x} \frac{\psi^2(d)\psi^k (rd/(r,d))(r,d)^{k+1}}{d^{k+2}c_k (rd/(r,d))}
$$
\n
$$
r \mid m
$$
\n
$$
+ \mathcal{O}\left((\log x)^{\frac{3k-1}{3}} \cdot \sum_{d \leq x} \frac{1}{d} S\left(\frac{rd}{(r,d)}\right)\right)
$$
\n
$$
= \frac{x c_k}{r^{k+1}} \sum_{j} + \mathcal{O}\left(S(r) \cdot (\log x)^{\frac{3k-1}{3}} \cdot \log x\right), \qquad (3.12)
$$

say. We have

$$
\sum_{7} r = \frac{\psi^{k}(r)}{c_{k}(r)} \sum_{q \mid r} \frac{\mu^{2}(q)}{q} \sum_{\substack{a \leq x/q \\ (a, r) = 1}} \frac{\mu^{2}(a)\psi^{k}(a)}{a^{k+2}c_{k}(a)}
$$
(3.13)

By Lemma 2.1, $\sum_{m \leq x} \left(\frac{m(n-1)}{m} \right)^m = U(x)$. Hence by partial Summation,

$$
\sum_{a \ge x} \frac{\psi^{k}(a)a^{-k}}{a^{2}} = \mathcal{O}(\frac{1}{x}) ,
$$

so that

$$
\sum_{\substack{a \le x/q \\ (a,r)=1}} \frac{\mu^{2}(a)\psi^{k}(a)}{k^{2}} = \sum_{\substack{a=1 \\ (a,r)=1}}^{\infty} \frac{\mu^{2}(a)\psi^{k}(a)}{k^{2}} + \mathcal{O}\left(\frac{a}{x}\right)
$$

$$
= \prod_{\substack{p \mid r}} \left(1 + \frac{(p+1)^{k}}{p^{k+2}c_{k}(p)}\right) + \mathcal{O}\left(\frac{a}{x}\right) \tag{3.14}
$$

 \mathcal{L}

on expanding the infinite series as an infinite product of Euler type. From (3.14) and (3.13), we obtain

$$
\sum_{7}^{k+1} = \frac{\psi^{k+1}(r)}{rc_{k}(r)} \prod_{p \nmid r} \left(1 + \frac{(p+1)^{k}}{p^{k+2}c_{k}(p)}\right) + \mathcal{O}\left(\frac{\theta(r)\psi^{k}(r)}{x}\right) .
$$

Substituting this into (3.12), we obtain

$$
\sum_{\substack{m \le x \\ r|m}} \left(\frac{\psi(m)}{m}\right)^{k+1} = \frac{xc_k}{r^{k+2}c_k(r)} \prod_{p \nmid r} \left(1 + \frac{\left(p+1\right)^k}{p^{k+2}c_k(p)}\right) + \prod_{r|m} \left(1 + \frac{\psi(m)}{p^{k+2}c_k(p)}\right)
$$

$$
+ \theta \left(\frac{\theta(r) \psi^{k}(r)}{r^{k+1}} \right) + \theta \left(s(r) (\log x)^{\frac{3k+2}{3}} \right)
$$

=
$$
\frac{ {}^{xc} k + 1 \left(s(r) (\log x)^{\frac{3k+2}{3}} \right)}{r^{k+2} c_{k+1}(r)}
$$

The induction is complete and hence Theorem 3.2 follows.

THEOREM 3.3. For any positive integers r and $k \ge 2$, we have

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{m} \leq \mathbf{x}}} \left(\frac{\psi(\mathbf{m})}{\mathbf{m}}\right)^{k} = \frac{\mathbf{x}\zeta(2)(\psi(\mathbf{r}))^{k-1}}{\mathbf{r}^{k}} D_{k}(\mathbf{r}) + \mathcal{O}(\mathbf{S}(\mathbf{r})(\log \mathbf{x})^{\frac{3k-1}{3}}),
$$
 (3.15)

where

$$
D_{k}(r) = \sum_{\substack{a=1 \ (a,r)=1}}^{\infty} \frac{\mu^{2}(a)(\psi(a))^{k-2}D_{k-1}(ar)}{a^{k}}, \ k = 2,3,... \qquad (3.16)
$$

and

$$
D_{k}(r) = \mathcal{O}(A_{1}(r)), \qquad (3.17)
$$

where $A_1(r)$ is given by (2.9) .

PROOF: For k = 2, Theorem 3.3 follows from Lemma 2.16. We assume Theorem 3.3 for some $k \ge 2$ and all r . By (2.1) and by our induction hypothesis, we have

$$
\sum_{m \leq x} \left(\frac{\psi(m)}{m}\right)^{k+1} = \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{m \leq x} \left(\frac{\psi(m)}{m}\right)^k
$$
\n
$$
r|m \qquad \{r,d\}|m
$$
\n
$$
= \frac{x\zeta(2)}{r^k} \sum_{d \leq x} \frac{\mu^2(d)(\psi(\text{rd}/(r,d))^{k-1}D_k(\text{rd}/(r,d))}{d^{k+1}}
$$
\n
$$
+ \mathcal{O}(s(r)(\log x)^{\frac{3k+2}{3}})
$$
\n
$$
= \frac{x\zeta(2)}{r^k} \sum_{d \leq x} \left(\frac{y}{r} + \mathcal{O}(s(r)(\log x)^{\frac{3k+2}{3}})\right), \text{ say,}
$$

We h ave

$$
\sum_{\alpha=0}^{n} f(x) = (\psi(r))^{k-1} \sum_{\substack{q \text{ odd}}} \frac{\mu^2(q)}{q} \sum_{\substack{a \le x/q \\ (a, r) = 1}} \frac{\mu^2(a) (\psi(a))^{k-1} D_k(ar)}{a^{k+1}} \tag{3.19}
$$

For $(a,r) = 1$, by (3.17) ,

$$
D_k(ar) = \hat{U}(A_1(a)A_1(r)).
$$

Therefore by Lemma 2.1 and partial summation,

$$
\sum_{\substack{\lambda \to \lambda/q \\ (a, r) = 1}} \frac{\mu^2(a) (\psi(a))^{k-1} D_k(ar)}{a^{k+1}} = \mathcal{O}(A_1(r) \frac{q}{x}),
$$

so that

$$
\sum_{\substack{a \le x/q \\ (a, r) = 1}} \frac{\mu^{2}(a)(\psi(a))^{k-1}D_{k}(ar)}{a^{k+1}} = D_{k+1}(r) + \theta (A_{1}(r) \frac{q}{x})
$$

Hence from (3.19) we get that

$$
\sum_{8} r = \frac{(\psi(r))^{k}}{r} D_{k+1}(r) + \mathcal{O}\left(\frac{\theta(r)\psi(r)\right)^{k-1} A_{1}(r)}{x}\right) .
$$

Substituting this into (3.18), we get

$$
\sum_{\substack{m\leq x\\ m\leq x}} \left(\frac{\psi(m)}{m}\right)^{k+1} = \frac{x\zeta(2)(\psi(r))^{k}}{r^{k+1}} + \mathcal{O}\left(\frac{\theta(r)(\psi(r))^{k-1}A_{1}(r)}{r^{k}}\right) + \mathcal{O}\left(s(r)\cdot(\log x)^{\frac{3k+2}{3}}\right)
$$
\n
$$
r|m
$$
\nClearly

$$
D_{k+1}(r) = U(A_1(r)).
$$

Thus the induction is complete and hence Theorem 3.3 follows.

COROLLARY 3.1. We have for $k \geq 1$,

$$
\sum_{\substack{m\leq x\\r|m}} (\varphi(m))^k = \frac{x^{k+1}(\varphi(r))^k B_k}{(k+1)r^{k+1} B_k(r)} + \mathcal{O}\left(\sigma_{-1+e}^x(r) \left(\frac{\psi(r)}{r}\right)^{k-1} x^k \lambda(x) (\log x)^{k-1}\right) \tag{3.20}
$$

where B_k and $B_k(r)$ are as given in Theorem 3.1.

PROOF: Follows from Theorem 3.1 and partial summation. COROLLARY 3.2. We have

(i) For square-free r and $k \ge 1$,

$$
\sum_{\substack{m\leq x\\ m\leq x}} (\psi(m))^k = \frac{x^{k+1}(\psi(r))^k c_k}{(k+1)r^{k+1} c_k(r)} + \mathcal{O}(s(r)x^k(\log x)^{\frac{3k-1}{3}}), \qquad (3.20a)
$$

where c_k and $c_k(r)$ are as given in Theorem 3.2.

(ii) For any positive integer r and $k > 2$,

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{n}\leq \mathbf{x} \\ \mathbf{r}\mid \mathbf{m}}} (\psi(\mathbf{m}))^{k} = \frac{\mathbf{x}^{k+1}\zeta(2)(\psi(\mathbf{r}))^{k-1}D_{k}(\mathbf{r})}{(k+1)\mathbf{r}^{k}} + \mathcal{O}\big(\mathbf{S}(\mathbf{r})\mathbf{x}^{k}(\log \mathbf{x})^{\frac{3k-1}{3}}\big), \qquad (3.20b)
$$

where $D_k(r)$ is as given in Theorem 3.3.

PROOF: Follows from Theorems 3.2, 3.3 and partial summation.

REMARK 3.1: Theorem 3.1 in case $r = 1$ was originally established by S.D. Chowla [3] with a weaker 0 -estimate of the error term: $0((\log x)^k)$. Taking $r = 1$ and $k = 2$ in (3.20) and (3.20a) we obtain results due to D. Suryanarayana ($[17]$), Theorems 3.6 and 3.7) who established them using the identitites

$$
\varphi^2(n) = \sum_{d\delta = m} \mu^*(d) \varphi(d) \varphi(\delta) \delta \quad \text{and} \quad \psi^2(n) = \sum_{d\delta = m} \lambda^*(d) \mu^*(d) \psi(d) \psi(\delta) \delta
$$

where $\lambda'(d) = (-1)^{\Omega(d)}$, $\Omega(m)$ being the total number of prime factors of m.

Remark 3.2. Formula (3.20) (k=1) was established by 0. Holder [6] and S.S. Pillai [9] with error term \hat{U} (x log x) which does not appear to be uniform in r. In 1961, E. Cohen $([2]$, lemma 3.2, s=1) obtained the formula (3.20) $(k=1)$ with error term θ ($\theta(r)r^{-1}x$ log x). In 1977, Suryanarayana and Subrahmanyam (cf. [20], lemma 3.1) established (3.20) (k=1) with error term \hat{U} (x λ (x)) which they stated to be uniform in r. We may mention here that in view of Remark 2.1, in [20] they would get (3.20) (k=1) with error term $\hat{U}(x\lambda(x)a^{\omega(r)})$ where $a = (\lambda(3))$ ⁻¹ (For example, see the proof of lemma 2.1 in [14]) and the error term in (3.20) (k=1) is clearly better than this since $a > 2$.

THEOREM 3.4. For each fixed integer $k \geq 1$,

$$
\sum_{\substack{m\leq x\\(3,21)}} \left(\frac{\varphi(m)}{\psi(m)}\right)^{k} = xA_{k}B_{k} + \mathcal{O}(\lambda(x)(\log x))^{N+k-1},
$$

where

$$
N = N_k = \begin{cases} {k \choose k/2} (k-1) + 1 & , \text{if } k \text{ is even} \\ {k \choose (k+1/2)} (k-1) + 1 & , \text{if } k \text{ is odd} \end{cases}
$$
(3.22)

and

$$
A_{k} = \prod_{p} \left\{ 1 + \frac{(p-1)^{k} ((p+1)^{k} - p^{k})}{p^{k+1} (p+1)^{k} B_{k}(p)} \right\},
$$
\n(3.23)

where B_k and $B_k(p)$ are as given in Theorem 3.1. PROOF: Let $g(m) = m^{k}/(\psi(m))^{k}$ for any m. We write

 \mathbf{I}

$$
g(m) = \sum_{d \mid m} f(d) , \qquad (3.24)
$$

for any m , so that by the Mobius inversion formula, we get $f(m) = \{ (d)g(m|d) \}.$ d m **|** Therefore, if p is a prime and α is a positive integer, we have

$$
f(p^{\alpha}) = g(p^{\alpha}) - g(p^{\alpha-1})
$$

=
$$
\begin{cases} g(p)-1, & \text{if } \alpha = 1 \\ 0, & \text{if } \alpha \ge 2, \end{cases}
$$
 (3.25)

since $g(p^{\alpha}) = g(p)$ for any prime p and $\alpha \geq 1$. We have

$$
|f(p)| = |g(p)-1| = 1 - \frac{p^{k}}{(p+1)^{k}} = \frac{(p+1)^{k} - p^{k}}{(p+1)^{k}}
$$

$$
= \frac{1 + {k \choose 1}p + {k \choose 2}p^{2} + \dots + {k \choose k-1}p^{k-1}}{(p+1)^{k}}
$$

$$
\leq \frac{1 + (k-1)M_{k}p^{k-1}}{(p+1)^{k}} \leq \frac{(1 + (k-1)M_{k})p^{k-1}}{(p+1)^{k}},
$$

where M_k is the maximum of the binomial coefficients $\begin{pmatrix} k \\ 1 \end{pmatrix}$, $\begin{pmatrix} k \\ 2 \end{pmatrix}$,..., $\begin{pmatrix} k \\ k-1 \end{pmatrix}$ for $k \ge 2$, with $M_1 = 1$, so that

$$
M_k = \begin{cases} \begin{array}{c} k \\ k/2 \end{array} \end{cases}
$$
, if k is even

$$
\begin{pmatrix} k \\ (k+1)/2 \end{pmatrix}
$$
, if k is odd.

From the definition of N given in (3.22), we get

$$
|f(p)| = |g(p)-1| \leq \frac{Np^{k-1}}{(p+1)^k} \tag{3.26}
$$

From (3.25) and (3.26) , we obtain

$$
|f(m)| \leq \frac{\mu^{2}(m) m^{k-1} N^{\omega(m)}}{(\psi(m))^{k}} \leq \frac{N^{\omega(m)} m^{k-1}}{(\psi(m))^{k}}, \qquad (3.27)
$$

for any m. Now, by (3.24) and Theorem 3.1, we get

$$
\sum_{m\leq x} \left(\frac{\varphi(m)}{\psi(m)}\right)^k = \sum_{m\leq x} \left(\frac{\varphi(m)}{m}\right)^k d\Big|_m^{\frac{1}{2}} d\Big|_m^{\frac
$$

Since $\psi(m) \geq m$, from (3.27) we obtain,

and

$$
\sum_{m \leq x} |f(m)| \leq \sum_{m \leq x} \frac{N^{\omega(m)}}{m} = \hat{\mathcal{O}}((\log x)^{N}),
$$
(3.29)

$$
\sum_{d \leq x} \frac{|f(d)| \sigma_{-1+\varepsilon}^{\star}(d) (\psi(d))^{k-1}}{d^{k-1}} \leq \sum_{d \leq x} \frac{N^{\omega(d)} \sigma_{-1+\varepsilon}^{\star}(d)}{d}
$$

$$
= \hat{\mathcal{O}}((\log x)^{N}),
$$
(3.30)

which follows from lemma 2.2 and induction on N. From (3.29) and partial summation, we get that

$$
\sum_{m \ge x} \frac{|f(m)|}{m} = \mathcal{O}\left(\frac{(\log x)^N}{x}\right) \tag{3.31}
$$

Also, the series $\sum_{i=1}^{n} \frac{f(u_i - u_i)}{h_i}$ converges absolutely. Expanding this as an d=1 d^{h +}B_k(d)

225

infinite product of Euler-type, we obtain from (3.23) that

$$
A_k = \sum_{d=1}^{\infty} \frac{f(d)(\varphi(d))}{d^{k+1}B_k(d)}.
$$

From this, (3.31) , (3.30) and (3.28) , we obtain Theorem 3.4 . Taking $k = 1$ in Theorem 3.4, we obtain

Corollary 3.3

$$
\sum_{m \leq x} \frac{\varphi(m)}{\psi(m)} = x\beta + \mathcal{O}(\lambda(x) \log x),
$$
\n(3.32)\n
$$
\beta = \prod_{p} \left(1 - \frac{2}{p(p+1)}\right).
$$

where

Remark 3.2. Formula (3.32) has been established by D. Suryanarayana ([19], Theorem 3.5) with a weaker $\mathcal{O}-$ estimate of the error term: (log²x). This formula was originally established by S. Wigert ([22], [23]) with much weaker \hat{U} -estimate of the error term, namely $\mathcal{O}(x^{1/2} \log^{3/2} x)$.

On lines similar to that of Theorem 3.4, we can prove the following:

Theorem 3.5. Let g be a multiplicative function satisfying (i) $g(p^m) = g(p)$, for all prime powers p^m , $m \ge 1$ (ii) $|g(p)^{-1}| \leq Np$ /(p+l), for some positive integers k and N, for all primes p.

Then

$$
\sum_{m\leq x} g(m)(\varphi(m)m^{-1})^k = xA_k^B_{k} + \mathcal{O}(\lambda(x)(\log x)^{N+k-1}),
$$

 $\ddot{}$

where

$$
A'_{k} = \prod_{p} \left(1 + \frac{(p-1)^{k} (g(p)-1)}{p^{k+1} B_{k}(p)} \right) ,
$$

where $B_k(p)$ is as given in Theorem 3.1.

Corollary 3.4. Let t be an non-integral real number > I. Let T be the integral part and s be the fractional part of t. Then we have

$$
\sum_{m\leq x} \left(\frac{\varphi(m)}{m}\right)^{t} = xA_{T}^{B} + \mathcal{O}\left(\lambda(x)\left(\log x\right)^{N+T-1}\right) ,
$$

where N is any positive integer with $N \geq s(\frac{3}{2})^{T+1}$, and

$$
A_{T}^{*} = \prod_{p} (1 - \frac{(p-1)^{T} (p^{s} - (p-1)^{s})}{p^{t+1} B_{T}(p)} .
$$

<u>Proof</u>. We take $k = T$ and $g(m) = (\frac{\varphi(m)}{m})^S$ in Theorem 3.5. Then

$$
g(p) = (1 - \frac{1}{p})^s = 1 - \frac{s}{p} + \sum_{a=2}^{\infty} {s \choose a} (-1)^a (\frac{1}{p})^a
$$

For $a \geq 2$,

$$
|{s \choose a}| = \frac{s(1-s)(2-s) \cdots ((a-1)-s)}{a!} < \frac{s \cdot (1 \cdot 2 \cdots (a-1))}{a!} = \frac{s}{a},
$$

so that

$$
|g(p)-1| < \frac{s}{p} + s \sum_{a=2}^{\infty} \frac{1}{ap^a} \le \frac{s}{p} + \frac{s}{2} + \frac{1}{p^2} + \frac{p}{p-1} \le \frac{s}{p} + \frac{s}{p^2} = \frac{s(p+1)}{p^2}.
$$

Further

$$
\frac{s(p+1)}{p^2} \leq N \frac{p^{T-1}}{(p+1)^T} \iff N \geq s(1+\frac{1}{p})^{T+1}.
$$

Since

$$
\left(1+\frac{1}{p}\right)^{T+1} \leq \left(1+\frac{1}{2}\right)^{T+1} = \left(\frac{3}{2}\right)^{T+1},
$$

 $N \geq s(\frac{3}{2})^{T+1}$ implies that

$$
|g(p)-1| \leq N \frac{p^{T-1}}{(p+1)^T}
$$
.

Now, Corollary 3.4 follows from Theorem 3.5.

As special cases, taking $t = \frac{3}{2}$ and $t = \frac{5}{4}$ successively in Corollary 3.4, we obtain the following:

$$
\sum_{\substack{\mathbf{m}\leq \mathbf{x} \\ \mathbf{m}\leq \mathbf{x}}} \left(\frac{\varphi(\mathbf{m})}{\mathbf{m}}\right)^{3/2} = \frac{6\mathbf{x}\mathbf{A}_1^{\mathbf{u}}}{\pi^2} + \mathcal{O}\left(\lambda(\mathbf{x})(\log \mathbf{x})^2\right),
$$

and

$$
\sum_{m\leq x} \left(\frac{\varphi(m)}{m}\right)^{5/4} = \frac{6xB_1^{\prime}}{\pi^2} + \mathcal{O}\left(\lambda(x)(\log x)\right),
$$

where

$$
A_1^* = \frac{1}{p} \left(1 - \frac{(\sqrt{p} - \sqrt{p-1})}{p^{1/2} (p+1)} \right)
$$

$$
\quad\text{and}\quad
$$

$$
B_1' = \frac{1 - \frac{p^{1/4} - (p-1)^{1/4}}{p^{1/4}(p+1)}}{p^{1/4}(p+1)}.
$$

Remark 3.3. Let $0 \le s \le 1$. Taking $k = 1$ and $g(m) = \left(\frac{y(m-1)}{m}\right)^{\infty}$ in Theorem 3.5 we get

$$
\sum_{\substack{m\leq x\\ m\leq x}} \left(\frac{\varphi(m)}{m}\right)^s = \frac{6x}{\pi^2} \left[1 + \frac{\left(p-1\right)^s (p^{1-s} - (p-1)^{1-s}}{(p^2-1)}\right] + \mathcal{O}(\lambda(x) (\log x)^N)
$$

for any integer $N > 3(1-s)$. However I.I. Iljasov [6] proved a better result that for $0 < s < 1$

$$
\sum_{n \leq x} \left(\frac{\varphi(n)}{n}\right)^s = cx + \mathcal{O}(\lambda(x)).
$$

where c is a positive constant.

Lemma 3.1. Let N and k be fixed positive integers. Then we have

(a)
$$
\sum_{m \leq x} \frac{N^{\omega(m)} S(m)}{m} = \mathcal{O}((\log x)^{N})
$$

(b)
$$
\sum_{m \leq x} \frac{N^{\omega(m)} S(m) m^{k-1}}{(\varphi(m))^{k}} = \mathcal{O}((\log x)^{N})
$$

(c)
$$
\sum_{m\leq x} \left(\frac{\psi(m)}{\varphi(m)}\right)^k = \theta(x).
$$

(d)
$$
\sum_{m \leq x} N^{\omega(m)} \left(\frac{\psi(m)}{\varphi(m)} \right)^k = \mathcal{O}(x(\log x)^{N-1}),
$$

where $S(m)$ is as given in (2.10) .

Proof. The result in part (a) can be proved by induction on $S(m)m - U (log x)$ and $(N+1)^{w(m)} = \int u^2(d)N^{w(m)}$ $m \leq x$ d | m N, using the results

Result (b) follows using induction on k and the result in (a).

Result (c) can be obtained from the identity $\frac{x \cdot 2}{\varphi(m)} = \int_{d/m}^{\infty} u \cdot (d) \theta(d) / \varphi(d)$ and induction on k.

Result (d) follows by induction on N and the result in (c).

Hence Lemaa 3.1 follows.

Theorem 3.6. Suppose h is a multiplicative function satisfying

(1) $h(p^{m}) = h(p)$, for all primes p and $m \ge 1$.

(ii) $|h(p) - 1| \leq Np^{k-1}/(p-1)^k$, for some fixed positive integers N and for all primes p.

Then we have

$$
\sum_{m \leq x} h(m) (\psi(m) m^{-1})^{k} = x A_{k}^{1} C_{k} + \mathcal{O}((\log x)^{\frac{3k-1}{3}})^{k}
$$

where

$$
A_{k}^{\dagger \dagger} = \prod_{p} \left(1 + \frac{(p+1)^{k} (h(p)-1)}{p^{k+1} c_{k}(p)} \right) ,
$$

 c_k and $c_k(p)$ being as given in Theorem 3.2.

Proof. The proof is similar to that of Theorem 3.5 if we make use of Lemma 3.1 and Theorem 3.2.

Corollary 3.5. Let t be a real number > 1 and t be not an integer. Let T be the integer part and s be the fractional part of t . Then for any positive integer $N \geq \frac{3s}{2}$, we have

$$
\sum_{m \leq x} \left(\frac{\psi(m)}{m} \right)^t = x A_{T}^{\prime \prime} C_{T} + \mathcal{O} \left((\log x)^{\frac{3T-1}{3}} \right) ,
$$

where

$$
A_{T}^{\dagger \dagger} = \prod_{p} \left(1 + \frac{(p+1)^{T}((p+1)^{S}-p^{S})}{p^{t+1}c_{T}(p)} \right) .
$$

Proof. This follows from Theorem 3.6, by taking $k = T$ and $h(m) = \frac{N \pm \infty}{m}$. The condition $N > \frac{3s}{2}$ ensures that h satisfies condition (ii) of Theorem 3.6.

Remark 3.5. Taking h(m) = (m/9(m)) k in Theorem 3.6, we obtain an asymptotic 3k-i formula for $\left(\frac{\psi(x)-y}{\psi(x)}\right)^{k}$ with error term $\mathcal{O}((\log x)^{3}$) where N_k is as give in Theorem 3.4. For $k = 1$, this formula becomes $\sum_{m \le x} \frac{1.2m}{\varphi(m)} = ax +U((\log x)^{-1}),$ a being an absolute constant.

Remark 3.6. Taking $h(m) = \left(\frac{m(m+1)}{2}\right)^m$, where $0 \leq s \leq 1$ and $k = 1$ in Theorem 3.6, we obtain the following asymptotic formula: For $0 < s < 1$

$$
\sum_{\substack{m\leq x\\ m\leq x}} \left(\frac{\psi(m)}{m}\right)^s = \frac{15x}{\pi^2} \prod_{p} \left(1 - \frac{(p+1)^8((p+1)^{1-s}-p^{1-s})}{p^2+1}\right) + \mathcal{O}((\log x)^{5/3}).
$$

§4. Some Remarks. Using Theorems 3.1, 3.2 and [p(d) = i or 0 according as d | n $n = 1$ or $n > 1$, it is not difficult to prove the following: For any positive integers k and n, we have

$$
\sum_{\substack{m \le x \\ (m,n)=1}} \left(\frac{\varphi(m)}{m}\right)^k = xB_k \cdot B_k^*(n) + \hat{U}(\log x)^{k-1} S_{k,\epsilon}(n)
$$
 (4.1)

and

$$
\sum_{\substack{m \le x \\ (m,n)=1}} \left(\frac{\psi(m)}{m}\right)^k = xC_k \cdot C_k^*(n) + \mathcal{O}((\log x)^{\frac{3k-1}{3}} S'(n))
$$
\n(4.2)

where

$$
B_{k}^{\star}(n) = \sum_{d|n} \frac{\mu(d)(\varphi(d))^{k}}{d^{k+1}B_{k}(d)},
$$

$$
C_{k}^{\star}(n) = \sum_{d|n} \frac{\mu(d)\psi^{k}(d)}{d^{k+1}C_{k}(d)},
$$

$$
S_{k,\epsilon}(n) = \sum_{d|n} \frac{\mu^{2}(d)(\psi(d))^{k-1}\sigma_{-1+\epsilon}^{\star}(d)}{d^{k-1}}
$$

$$
S'(n) = \sum_{d|r} \mu^{2}(d)S(d) = \frac{\theta(n)n}{\phi(n)}.
$$

and

The -estimates in (4.1) and (4.2) are uniform in x and n.

In fact, by somewhat more complicated arguments, we can also establish asymptotic formulae for sums such as

$$
\sum_{\substack{m\leq x\\ \ldots \\ m \equiv a \pmod{b}}} \left(\frac{\varphi(m)}{m}\right)^k \quad \text{and} \quad \sum_{\substack{m\leq x\\ \ldots \\ m \equiv a \pmod{b}}} \left(\frac{\psi(m)}{m}\right)^k,
$$
\n
$$
\sum_{\substack{m\leq x\\ \ldots \\ m \equiv a \pmod{b}}} \left(\frac{\psi(m)}{m}\right)^k,
$$

with $(a,b) = 1$ for any positive integers r and n satisfying $(r,n) = 1$, with uniform 0 -estimates. This would be done later in a separate paper.

§5. Asymptotic Formula Involving the Unitary Analogues of φ and σ Functions. We recall that these unitary analogues ϕ and σ have the evaluation [1]:

$$
\varphi^{*}(n) = \prod_{p^{a} \parallel n} (\varphi^{a} - 1), \sigma^{*}(n) = \prod_{p^{a} \parallel n} (\varphi^{a} + 1),
$$

where p^{a} and denotes $p^{a}|n$ but p^{a+1}/n . Using Lemma 2.11, formula (4.1) (k=1) of this paper and induction on k, we can establish the following results: (using lemmas 2.3 and 2.4 of $[19]$):

THEOREM 5.1. For any positive integer k we have

$$
\sum_{\substack{\underline{m\leq x}\\ m\leq x}} \left(\frac{\sigma^{*}(m)}{m}\right)^{k} = \frac{x\zeta(2)\beta_{k}(n)}{\zeta(3)} + \mathcal{O}\left(\frac{n\theta(n)}{\varphi(n)}\left(\log x\right)^{\frac{6k-1}{3}}\right)
$$

where the -constant depends only on k and

$$
\beta_1(n) = \frac{\varphi(n)J_2(n)}{J_3(n)},
$$

and for $k \geq 2$,

$$
\beta_{k}(n) = \sum_{\substack{\delta=1 \ (\delta,n)=1}}^{\infty} \frac{\sigma^{*}(\delta))^{k-1} \beta_{k-1}(n\delta)}{\delta^{k+1}} = \mathcal{O}(1).
$$

THEOREM 5.2 For integers k > i we have

$$
\sum_{\substack{m\leq x\\(m,n)=1}} \left(\frac{\mathfrak{A}(m)}{m}\right)^k = x\alpha \cdot \alpha_k^*(n) + \mathcal{O}(\lambda(x) (\log x)^{2k-1} S^*(n)).
$$

where

$$
\alpha = \prod_{p} \left(1 - \frac{1}{p(p+1)} \right),
$$
\n
$$
\star \quad \alpha_{k}(n) = \sum_{\substack{m=1 \ \text{ (m,n)=1}}}^{\infty} \frac{\left(\varphi^{*}(m) \right)^{k-1} \mu^{*}(m) \alpha_{k-1}^{*}(mn)}{m^{k+1}}, \quad \text{for} \quad k \geq 2,
$$

with α_1^* (n) = \prod $\frac{p^2-1}{2}$ pln p^r-p-1 Remark. Similar results can be obtained for functions associated with biunitary divisors [16]; however we shall not go into details.

 $\$6$. Concluding Remarks. An estimate for $~\,\rangle~$ ϕ (m) also appeared earlier in 1964 m<x in a paper of S.L. Segal [12], who gave the weaker error term $\mathcal{O}(\mathrm{x}$ log x) which is the same as Chowlas' result for $k = 2$. A probabilistic proof of the formula for ~2(m) without error term has been given by M. Kac [8] who ascribed it to m<x Schur.

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