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THE SPECTRA ASSOCIATED TO PERMUTATIVE CATEGORIES

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IN [1], FIEDOROWICZ axiomatized the passage from rings to spectra. In [5], Thomason and I axiomatized the passage from $\mathscr{G}$-spaces to spectra for a suitable category of operators $\mathscr{G}$. In this sequel to [5], I shall axiomatize the passage from permutative categories to spectra. The idea of such an axiomatization is due to Fiedorowicz, who contemplated a proof parallel to his argument in [1]. All unexplained concepts and notation are taken from [5], and the definition of a permutative category will be recalled below.

Given the techniques of [5], the only really subtle point is the explanation of just why this further consistency statement is of interest. By [5], we know that the infinite loop space machines of Segal [6] and myself [3] turn out equivalent spectra when fed the same data. Here the "same data" means the same $\mathscr{G}$-space for some category of operators $\mathscr{G}$. Many of the most interesting examples come from permutative categories, via functorial associations of $\mathscr{G}$-spaces to such categories. The point is that there are two essentially different functors known. As we shall explain shortly, ideas of Segal [6] lead to a functor which associates an $\mathscr{F}$-space $B\bar{\mathscr{A}}$ to a permutative category $\mathscr{A}$. Here $\mathscr{F}$ is the category of finite based sets, $B\bar{\mathscr{A}}$ is a functor from $\mathscr{F}$ to the category $\mathscr{T}$ of based spaces, and the n^{th} space of $B\bar{\mathscr{A}}$ is equivalent but not equal to the n-fold Cartesian product of its first space, which is the classifying space $B\mathscr{A}$. On the other hand, I showed in [3, §4] that $B\mathscr{A}$ is a $\mathscr{D}$-space for a suitable E_∞ operad $\mathscr{D}$. As explained in [5, §4], $\mathscr{D}$ determines a category of operators $\hat{\mathscr{D}}$ and $B\mathscr{A}$ determines a $\hat{\mathscr{D}}$-space whose nth space is precisely $(B\mathscr{A})^n$. A direct comparison between these functors seems to be surprisingly difficult, and each of them has distinct advantages over the other. For example, Friedlander [2] has shown how the machinery of étale homotopy leads to $\mathscr{F}$-spaces, and his proof of the stable Adams conjecture requires naturality arguments based on the use of Segal's functor. Here the freedom to use products up to equivalence is vital. On the other hand, in multiplicative infinite loop space theory as developed in [4], the use of $\mathscr{D}$-spaces and precise products is certainly convenient and probably essential for at least some of the arguments. Our present result shows that these functors become equivalent after passage to spectra. Precisely, we have the following definitions and theorem.

Let $\mathscr{O}\mathscr{A}$ and $\mathscr{M}\mathscr{A}$ denote the object and morphism spaces of a small topological category $\mathscr{A}$; we require the identity function $\mathscr{O}\mathscr{A} \to \mathscr{M}\mathscr{A}$ to be a cofibration.

Definition 1. A permutative category $\mathscr{A} = (\mathscr{A}, \Box, *, c)$ is a small topological category $\mathscr{A}$, a nondegenerate basepoint $* \in \mathscr{O}\mathscr{A}$, a continuous product $\Box: \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, and a natural commutativity isomorphism $c: A\Box B \to B\Box A$ such that $\Box$ is associative with unit $*$ and the following diagrams commute:

$$\begin{array}{ccc} A\Box * & \xrightarrow{c} & *\Box A, \\ \| & & \| \\ A & \xrightarrow{1} & A \end{array} \qquad \begin{array}{ccc} A\Box B & \xrightarrow{1} & A\Box B, \\ & {\scriptstyle c}\searrow \quad \nearrow{\scriptstyle c} & \\ & B\Box A & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A\Box B\Box C & \xrightarrow{c} & C\Box A\Box B. \\ & {\scriptstyle 1\Box c}\searrow \quad \nearrow{\scriptstyle c\Box 1} & \\ & A\Box C\Box B & \end{array}$$

A continuous functor $f: \mathscr{A} \to \mathscr{A}'$ is a map of permutative categories if $f(*) = *$, $f \cdot \Box = \Box \cdot (f \times f): \mathscr{A} \times \mathscr{A} \to \mathscr{A}'$, and $f(c) = c$ on $f(A \Box B) = f(A) \Box f(B)$; f is said to be an equivalence if Bf is an equivalence (that is, weak equivalence) of spaces. Let $\mathscr{P}\mathscr{C}$ denote the cate the category of permutative categories.

The reader is referred to [4, VI] for discussion and examples. Suffice it to say that such categories play a vital role in algebraic K-Theory. We have the following analog of [5, 2.1].

Definition 2. An infinite loop space machine defined on permutative categories is a functor E from $\mathscr{P}\mathscr{C}$ to connective spectra, written $E\mathscr{A} = \{E_i\mathscr{A}, \sigma_i\}$, together with a natural group completion $\iota: B\mathscr{A} \to E_0\mathscr{A}$.

Our conventions on spectra are explained in [5, §2]. Let S denote Segal's infinite loop space machine defined on $\mathscr{F}$-spaces, as constructed in [5, §3]. The following uniqueness theorem is our main result.

THEOREM 3. *For any infinite loop space machine E defined on permutative categories, there is a natural equivalence of spectra between $E\mathscr{A}$ and $SB\bar{\mathscr{A}}$.*

For example, this applies to $E\mathscr{A} = MB\mathscr{A}$, the composite of the May machine defined on $\mathscr{D}$-spaces and the functor $B: \mathscr{P}\mathscr{C} \to \mathscr{D}[\mathscr{T}]$.

We assume given a fixed machine E defined on $\mathscr{P}\mathscr{C}$. The proof of Theorem 3 begins with the following analogs of [5, 2.2–2.4], which admit the same proofs. Recall that the classifying space functor B from based categories to based spaces preserves equivalences and products.

LEMMA 4. *If $\pi_0 B\mathscr{A}$ is a group, then $\iota: B\mathscr{A} \to E_0\mathscr{A}$ is an equivalence.*

LEMMA 5. *If $f: \mathscr{A} \to \mathscr{A}'$ is an equivalence of permutative categories, then $Ef: E\mathscr{A} \to E\mathscr{A}'$ is an equivalence.*

LEMMA 6. *For permutative categories $\mathscr{A}$ and $\mathscr{A}'$, the projections specify an equivalence $E(\mathscr{A} \times \mathscr{A}') \to E\mathscr{A} \times E\mathscr{A}'$.*

The proof of the uniqueness theorem in [5] depended on use of the notion of an $\mathscr{F}\mathscr{G}$-space [5, 3.1], or $\mathscr{F}$-object in the category of $\mathscr{G}$-spaces. We require the following analog.

Definition 7. An $\mathscr{F}$-permutative category is a functor $\mathscr{B}: \mathscr{F} \to \mathscr{P}\mathscr{C}$, written $\mathbf{n} \to \mathscr{B}_n$ on objects, such that the following properties hold.

(1) $\mathscr{B}_0$ is equivalent to the trivial category.

(2) For $n > 1$, the functor $\mathscr{B}_n \to \mathscr{B}_1{}^n$ with coordinates δ_i is an equivalence of permutative categories.

Let $\mathscr{F}\mathscr{P}\mathscr{C}$ denote the category of $\mathscr{F}$-permutative categories, its morphisms being the natural transformations under $\mathscr{F}$.

The following lemmas are analogs of [5, 3.2 and 3.3].

LEMMA 8. *Let D be any functor from permutative categories to based spaces which satisfies the following properties.*

(i) *If $\mathscr{A}$ is equivalent to the trivial category, then $D\mathscr{A}$ is aspherical.*

(ii) *If $f: \mathscr{A} \to \mathscr{A}'$ is an equivalence of permutative categories, then $Df: D\mathscr{A} \to D\mathscr{A}'$ is an equivalence.*

(iii) *The map $D(\mathscr{A} \times \mathscr{A}') \to D\mathscr{A} \times D\mathscr{A}'$ given by the projections is an equivalence.*

Then for any $\mathscr{F}$-permutative category $\mathscr{B}$, the spaces $D\mathscr{B}_n$ and maps $D\phi$ for $\phi: \mathbf{m} \to \mathbf{n}$ in $\mathscr{F}$ specify an improper $\mathscr{F}$-space $D\mathscr{B}$.

Obviously the classifying space functor B satisfies the specified properties, and Lemmas 4–6 show that each functor E_i also does so.

LEMMA 9. *The group completions $\iota: B\mathscr{B}_n \to E_0\mathscr{B}_n$ and equivalences $\sigma_i: E_i\mathscr{B}_n \to \Omega E_{i+1}\mathscr{B}_n$ specify maps $\iota: B\mathscr{B} \to E_0\mathscr{B}$ and $\sigma_i: E_i\mathscr{B} \to \Omega E_{i+1}\mathscr{B}$ of improper $\mathscr{F}$-spaces.*

As in [5], all improper $\mathscr{F}$-spaces of interest are in fact $\mathscr{F}$-spaces, but we shall exploit [5, 1.6] rather than assume conditions on E which ensure this.

The only $\mathscr{F}$-permutative categories we shall need arise via the following analog of [5, 3.4], which is also abstracted from ideas of Segal [6]. I am indebted to Richard Steiner for a very helpful discussion of these ideas.

Construction 10. Construct a functor $\mathscr{P}\mathscr{C} \to \mathscr{F}\mathscr{P}\mathscr{C}$, written $\mathscr{A} \to \bar{\mathscr{A}}$ on objects, in two steps as follows.

Step 1. Construction of the nth permutative category $\bar{\mathscr{A}}_n$: The objects of $\bar{\mathscr{A}}_n$ are the systems $\langle A_s; i_{(s,t)}\rangle$, where s runs through those subsets of $\mathbf{n} = \{0, 1, \ldots, n\}$ which contain 0, (s, t) runs through those pairs of subsets with $s \cap t = \{0\}$, the A_s are objects of $\mathscr{A}$ and $A_0 = *$, and the $i_{(s,t)}$ are isomorphisms $A_{s\cup t} \to A_s \square A_t$ such that the following diagrams commute:

$$\begin{array}{ccc} & & A_s \\ & \swarrow^{i_{(s,0)}} & \downarrow 1 \\ A_s\square * & = & A_s \end{array} \qquad \begin{array}{ccc} A_{r\cup s\cup t} & \xrightarrow{i_{(r,s\cup t)}} & A_r\square A_{s\cup t} \\ \downarrow i_{(r\cup s,t)} & & \downarrow 1\square i_{(s,t)} \\ A_{r\cup s}\square A_t & \xrightarrow{i_{(r,s)}\square 1} & A_r\square A_s\square A_t \end{array} \quad \text{and} \quad \begin{array}{ccc} A_{r\cup s} & \xrightarrow{i_{(r,s)}} & A_r\square A_s \\ \| & & \downarrow c \\ A_{s\cup r} & \xrightarrow{i_{(s,r)}} & A_s\square A_r \end{array}$$

the morphisms $\langle A_s; i_{(s,t)}\rangle \to \langle A'_s; i'_{(s,t)}\rangle$ are the systems $\langle \alpha_s\rangle$ of morphisms $\alpha_s: A_s \to A'_s$ such that $\alpha_0 = 1$ on $*$ and the following diagrams commute:

$$\begin{array}{ccc} A_{s\cup t} & \xrightarrow{i_{(s,t)}} & A_s\square A_t \\ \downarrow \alpha_{s\cup t} & & \downarrow \alpha_s\square\alpha_t \\ A'_{s\cup t} & \xrightarrow{i'_{(s,t)}} & A'_s\square A'_t. \end{array}$$

Composition and identities are inherited from $\mathscr{A}$, and the sets $\mathscr{O}\bar{\mathscr{A}}_n$ and $\mathscr{M}\bar{\mathscr{A}}_n$ are topologized as subspaces of

$$\prod_s \mathscr{O}\mathscr{A} \times \prod_{(s,t)} \mathscr{M}\mathscr{A} \qquad \text{and} \qquad \prod_s \mathscr{M}\mathscr{A}.$$

The unit $*$ of $\bar{\mathscr{A}}_n$ is the object with each $A_s = *$ and each $i_{(s,t)} = 1$. The product $\square: \bar{\mathscr{A}}_n \times \bar{\mathscr{A}}_n \to \bar{\mathscr{A}}_n$ is specified on objects by

$$\langle A_s; i_{(s,t)}\rangle \square \langle A'_s; i'_{(s,t)}\rangle = \langle A_s\square A'_s; (1\square c\square 1)(i_{(s,t)}\square i'_{(s,t)})\rangle$$

and on morphisms by $\langle \alpha_s\rangle \square \langle \alpha'_s\rangle = \langle \alpha_s \square \alpha'_s\rangle$. The commutativity isomorphism c is inherited from that of $\mathscr{A}$.

Step 2. Construction of the $\mathscr{F}$-permutative structure on $\bar{\mathscr{A}}$: For a map $\phi: \mathbf{m} \to \mathbf{n}$ in $\mathscr{F}$, construct a functor $\phi: \bar{\mathscr{A}}_m \to \bar{\mathscr{A}}_n$ as follows. On objects, ϕ sends $\langle A_s; i_{(s,s')}\rangle$ to $\langle B_t; j_{(t,t')}\rangle$, where

$$B_t = A_{s(t)} \qquad \text{and} \qquad j_{(t,t')} = i_{(s(t),s(t'))}$$

with $s(t) = \{0\} \cup \{i \mid \phi(i) \in t - \{0\}\}$. On morphisms, ϕ sends $\langle \alpha_s\rangle$ to $\langle \beta_t\rangle$ where $\beta_t = \alpha_{s(t)}$. With this definition, it is straightforward to verify that ϕ is a map of permutative categories and that $\bar{\mathscr{A}}$ is a functor from $\mathscr{F}$ to $\mathscr{P}\mathscr{C}$. Clearly $\bar{\mathscr{A}}_0$ is the trivial category, and it remains to verify that, for $n > 1$,

$$\delta = \mathop{\times}_{i=1}^{n} \delta_i : \bar{\mathscr{A}}_n \to \bar{\mathscr{A}}_1{}^n = \mathscr{A}^n$$

is an equivalence of topological categories. Clearly δ sends the object $\langle A_s; i_{(s,t)}\rangle$ to $(A_1, \ldots, A_n)$ and the morphism $\langle \alpha_s\rangle$ to $(\alpha_1, \ldots, \alpha_n)$, where the subscripts i, $1 \le i \le n$, refer to the subset $\{0, i\}$ of $\mathbf{n}$. Define a continuous functor $\nu: \mathscr{A}^n \to \bar{\mathscr{A}}_n$ as follows. On objects, ν sends $(A_1, \ldots, A_n)$ to $\langle A_s; c_{(s,t)}\rangle$, where $A_s = A_{s_1}\square \ldots \square A_{s_q}$ if $s = \{0, s_1, \ldots, s_q\}$ with $0 < s_1 < \cdots < s_q$ and $c_{(s,t)}$ is that shuffle (defined in terms of the commutativity isomorphism c) which rearranges $A_{s\cup t}$ in the form $A_s\square A_t$. On morphisms, ν sends $(\alpha_1, \ldots, \alpha_n)$ to $\langle \alpha_s \rangle$, where $\alpha_s = \alpha_{s_1}\square \ldots \square \alpha_{s_q}$ for s as above. Clearly $\delta\nu$: $\mathscr{A}^n \to \mathscr{A}^n$ is the identity functor. Define a natural equivalence ξ: $1 \to \nu\delta$ of functors $\bar{\mathscr{A}}_n \to \bar{\mathscr{A}}_n$ by letting ξ assign to the object $\langle A_s; i_{(s,t)}\rangle$ the isomorphism $\langle \xi_s \rangle$, where $\xi_s: A_s \to A_{s_1}\square \ldots \square A_{s_q}$ for s as above is the isomorphism determined inductively (and uniquely by associativity) by the $i_{(s,t)}$.

By Lemma 8, we now have a well-defined improper $\mathscr{F}$-space $B\bar{\mathscr{A}}$, and it is easy to verify from Step 2 that $B\bar{\mathscr{A}}$ satisfies the cofibration condition required of $\mathscr{F}$-spaces in [5, 1.2]. Thus the Segal spectrum $SB\bar{\mathscr{A}}$ is defined. We now have all the preliminaries necessary for the proof of Theorem 3. Indeed, we have the proof itself: the argument consists of a word for word repetition of the proof of the uniqueness theorem in [5, §3 and App. B], but with the $\mathscr{F}$-space Y, $\mathscr{F}\mathscr{F}$-space $\bar{Y}$, and group completion ι: $Y \to E_0\bar{Y}$ used there replaced by the permutative category $\mathscr{A}$, $\mathscr{F}$-permutative category $\bar{\mathscr{A}}$, and group completion ι: $B\bar{\mathscr{A}} \to E_0\bar{\mathscr{A}}$ present here.

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