AN INTRODUCTION TO AFFINE SCHEMES

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Abstract. This paper gives a basic introduction to modern algebraic geometry. The goal of this paper is to present the basic concepts of algebraic geometry, in particular affine schemes and sheaf theory, in such a way that they are more accessible to a student with a background in commutative algebra and basic algebraic curves or classical algebraic geometry. This paper is based on introductions to the subject by Robin Hartshorne, Qing Liu, and David Eisenbud and Joe Harris, but provides more rudimentary explanations as well as original proofs and numerous original examples.

Contents

1. Sheaves in General

Before we discuss schemes, we must introduce the notion of a sheaf, without which we could not even define a scheme.

Definitions 1.1. Let X be a topological space. A *presheaf* $\mathscr{F}$ of commutative rings on X has the following properties:

(1) For each open set $U \subseteq X$, $\mathscr{F}(U)$ is a commutative ring whose elements are called the *sections* of $\mathscr{F}$ over U,
(2) $\mathscr{F}(\emptyset)$ is the zero ring, and
(3) for every inclusion $U \subseteq V \subseteq X$ such that U and V are open in X, there is a *restriction map*

$$\mathrm{res}_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$$

such that
  (a) $\mathrm{res}_{V,U}$ is a homomorphism of rings,
  (b) $\mathrm{res}_{U,U}$ is the identity map, and
  (c) for all open $U \subseteq V \subseteq W \subseteq X$, $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$.

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In order to simplify notation, we will refer to $\mathrm{res}_{V,U}(f)$ for $f \in \mathscr{F}(V)$ as the restriction of f to U, or simply as $f|_U$, and we refer to the elements of $\mathscr{F}(V)$ as the sections over V.

We can similarly define presheaves of modules, abelian groups, or even just sets. However, for our purposes, we will only be dealing with presheaves of commutative rings, and from now on, we will assume that all rings mentioned are commutative and have a multiplicative identity 1.

In order for a presheaf to be a sheaf, it needs to satisfy an additional condition, called the *sheaf axiom*. We state this as a definition:

Definition 1.2. Let $\mathscr{F}$ be a presheaf on a topological space X, and let $U \subseteq X$ be open. Then $\mathscr{F}$ is a *sheaf* if it satisfies the following condition, known as the *sheaf axiom*:

If $U = \bigcup_{i \in J} U_i$ is an open covering of U and $\{f_i\}_{i \in J}$ is a set of elements with $f_i \in \mathscr{F}(U_i)$ for all $i \in J$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for each pair $i, j \in J$, then there exists a unique element $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in J$.

Now we look at a simple example of sheaves over discrete spaces.

Example 1.3. Consider the set $X = \{0, 1\}$ given the discrete topology, and let $\mathscr{F}$ be a sheaf over X. The two single-element open sets only contain themselves in their respective open coverings, so they give us no information about $\mathscr{F}$. Instead, let's consider the covering $\{\{0\}, \{1\}\}$ of X. Let $f_0 \in \mathscr{F}(0)$ and $f_1 \in \mathscr{F}(1)$. Then, since there is only one section over the empty set, we have

$$f_0|_{\{0\} \cap \{1\}} = f_0|_{\emptyset} = f_1|_{\emptyset} = f_1|_{\{0\} \cap \{1\}}.$$

Thus by the sheaf axiom, there is a unique section g over X such that $g|_{\{0\}} = f_0$ and $g|_{\{1\}} = f_1$. That is, $\mathscr{F}(X)$ is set theoretically equal to $\mathscr{F}(\{0\}) \times \mathscr{F}(\{1\})$, and the restriction maps are simply the projection maps.

More generally, if Y is any space given the discrete topology, it is clear that $\mathscr{F}(Y) = \prod_{y \in Y} \mathscr{F}(\{y\})$.

Another important feature of a sheaf is its stalks. The stalks describe the space and its sheaf locally, near a given point. We give a more precise definition.

Definition 1.4. Let X be a topological space and $\mathscr{F}$ a presheaf on X. Let $x \in X$. Then the *stalk* of $\mathscr{F}$ at x, denoted $\mathscr{F}_x$, is defined to be

$$\mathscr{F}_x = \varinjlim_{\substack{x \in U \subseteq X \\ U \text{ open}}} \mathscr{F}(U) = \left(\bigsqcup_{\substack{x \in U \subseteq X \\ U \text{ open}}} \mathscr{F}(U) \right) \Big/ \sim,$$

where $\sim$ is an equivalence relation such that $a \sim b$ if $a \in \mathscr{F}(U), b \in \mathscr{F}(V)$ and there is an open neighborhood $W \subseteq U \cap V$ such that $a|_W = b|_W$.

We illustrate this concept with a few more simple examples.

Example 1.5. We again consider the space of two elements given the discrete topology. We call it $X = \{0, 1\}$. By our definition, we have

$$\mathscr{F}_0 = \mathscr{F}(\{0\}) \sqcup \mathscr{F}(\{0, 1\}) / \sim .$$

Clearly $\{0\} \cap \{0, 1\} = \{0\}$, and if $a \in \mathscr{F}(\{0, 1\})$ then $a|_{\{0\}} \in \mathscr{F}(\{0\})$ so that $a \sim b$ for some $b \in \mathscr{F}(\{0\})$. Thus $\mathscr{F}_0 \subseteq \mathscr{F}(\{0\})$. However, if $f, f' \in \mathscr{F}(\{0\})$

such that $f \sim f'$, then $f = f'$ since $\mathrm{res}_{\{0\},\{0\}}$ is the identity on $\mathscr{F}(\{0\})$. That is, $\mathscr{F}_0 = \mathscr{F}(\{0\})$, and similarly $\mathscr{F}_1 = \mathscr{F}(\{1\})$.

It is again clear that we can generalize to any space Y given the discrete topology: for $y \in Y$, $\mathscr{F}_y = \mathscr{F}(\{y\})$. Moreover, if any single point in a space is open, the stalk at the point is simply the sheaf on the set containing only that point.

Example 1.6. Now we consider a non-discrete, but still simple, example. Let $X = \{0, 1\}$, but this time let the open sets be only $\emptyset$, $\{0\}$, and $\{0, 1\}$. From the previous example we see that $\mathscr{F}_0 = \mathscr{F}(\{0\})$. Now, it is clear that the stalk at 1 is simply $\mathscr{F}_1 = \mathscr{F}(\{0, 1\})$, since $\{0, 1\}$ is the only open neighborhood of 1.

After defining sheaves and stalks, it is natural to define maps between them.

Definition 1.7. Let X be a topological space and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Then a *morphism* $\phi : \mathscr{F} \to \mathscr{G}$ of sheaves is a collection of maps $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ where $U \subseteq X$ is open and for every V open in X such that $U \subseteq V$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \\ \downarrow{\scriptstyle \mathrm{res}_{V,U}} & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \end{array}$$

In the case where $\mathscr{F}$ and $\mathscr{G}$ are sheaves of rings, the maps $\phi(U)$ are ring homomorphisms.

A morphism of sheaves also induces a morphism of stalks, in this case a ring homomorphism, of the respective sheaves. We denote this morphism $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$ for $x \in X$.

2. The Structure Sheaf and Affine Schemes

Now that we have described some of the basics of sheaves, we present a specific sheaf, which we will use throughout the rest of the paper: the structure sheaf. First, we must quickly review a few definitions from commutative algebra:

The *spectrum* of a ring R, denoted $\operatorname{Spec} R$, is a topological space whose underlying set is the set of prime ideals of R. We give $\operatorname{Spec} R$ the *Zariski Topology.* The closed sets of this topology are of the form

$$V(S) := \{\mathfrak{p} \mid S \subseteq \mathfrak{p}\}$$

for S an arbitrary subset of R.

Given $f \in R$, we define the *distinguished* open set of $X = \operatorname{Spec} R$ associated with f to be

$$X_f := \operatorname{Spec} R - V(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}.$$

The set of distinguished open sets of X, $\{X_f \mid f \in R\}$, form a basis for the topology on X.

Now that we have a topology on X, we can define the structure sheaf $\mathscr{O}_X$ on X. When the space over which we are dealing is clear, we will denote the structure sheaf simply as $\mathscr{O}$. It turns out that it suffices to define the sections of $\mathscr{O}$ over distinguished open sets and the restriction maps between basic open sets. In fact, every sheaf on basis elements of a space Y satisfying the sheaf axiom with respect to inclusions and coverings can be extended uniquely to a sheaf on Y. For the purposes of this paper, we will omit the proof of this statement.

Going back to our space $X = \operatorname{Spec} R$, with basis $\{X_f \mid f \in R\}$, we set $\mathscr{O}(X_f) = R_f$, the localization of R at f. If $X_f \subseteq X_g$ (that is $f \in \sqrt{(g)}$, or $f^n \in (g)$ for some power n), then we define the restriction map $\operatorname{res}_{X_g, X_f} : R_g \to R_f$, as the localization map $R_g \to R_{gf} = R_f$. This leads us to the following simple observations:

Observation 2.1. *If $X = \operatorname{Spec} R$, then $\mathscr{O}(X) = R$.*

Proof. Consider $X_1 = \operatorname{Spec} R - V(1) = X - \emptyset = X$. Then we have $\mathscr{O}(X) = \mathscr{O}(X_1) = R_1 = R$, as desired. □

Observation 2.2. *A point $\mathfrak{p} \in X = \operatorname{Spec} R$ is closed if and only if $\mathfrak{p}$ is a maximal ideal of R.*

Proof. The point $\mathfrak{p}$ is closed in X if and only if $\{\mathfrak{p}\} = V(S)$ for some $S \subseteq R$ if and only if there is no prime ideal properly containing $\mathfrak{p}$ if and only if $\mathfrak{p}$ is a maximal ideal of R. □

Now we are able to construct the stalks of $\mathscr{O}$:

Lemma 2.3. *Let $\mathscr{O}$ be the structure sheaf on $\operatorname{Spec} R = X$, and let $\mathfrak{p} \in X$. Then the stalk at $\mathfrak{p}$ is $\mathscr{O}_\mathfrak{p} = R_\mathfrak{p}$, the localization at the prime ideal $\mathfrak{p}$.*

Proof. We begin by simply using our definition of a stalk. We have

$$\mathscr{O}_\mathfrak{p} = \varinjlim_{\substack{\mathfrak{p} \in U \subseteq X \\ U \text{ open}}} \mathscr{O}(U) = \left(\bigsqcup_{\substack{\mathfrak{p} \in U \subseteq X \\ U \text{ open}}} \mathscr{O}(U) \right) \Big/ \sim$$

where $a \sim b$ if there exists an open neighborhood X_h containing $\mathfrak{p}$ such that $a|_{X_h} = b|_{X_h}$. However, we simplify this by showing that it suffices to take the direct limit of $\mathscr{O}(U)$ where U is a basis element:

Suppose we have U open in X, and $f \in \mathscr{O}(U)$. Let V be a distinguished open set such that $x \in V \subseteq U$. Then $f|_V \sim f$, so it represents the same element of the stalk. Thus, we can write

$$\mathscr{O}_\mathfrak{p} = \varinjlim_{\mathfrak{p} \in X_f} \mathscr{O}(X_f) = \varinjlim_{f \notin \mathfrak{p}} R_f.$$

There is a canonical ring homomorphism

$$\phi : \varinjlim_{f \notin \mathfrak{p}} R_f \to R_\mathfrak{p}$$

that sends each element to its equivalence class. Thus it suffices to show that ϕ is an isomorphism. Let $\alpha \in R_\mathfrak{p}$. Then $\alpha = \frac{a}{f}$ for some $f \notin \mathfrak{p}$. Thus α is mapped to by some element of R_f, so ϕ is surjective. Now, suppose $\phi\left(\frac{a}{f^n}\right) = 0$, for some $f \notin \mathfrak{p}$. This implies that there is a $g \notin \mathfrak{p}$ such that $ga = 0$, so we have $\frac{a}{f^n} = 0$ in R_{gf}. Since $gf \notin \mathfrak{p}$, this implies that $ker\phi = 0$, so ϕ is injective and thus an isomorphism. Thus $\mathscr{O}_\mathfrak{p} \cong R_\mathfrak{p}$, as desired. □

Since all the stalks of the structure sheaf are local rings, we call X along with the sheaf $\mathscr{O}$ a *locally ringed space*, which is simply a topological space together with a sheaf of commutative rings on the space such that all the stalks of the sheaf are local rings.

We now have the machinery necessary to define an affine scheme.

Definition 2.4. Let the topological space X along with the sheaf $\mathscr{O}_X$ be a locally ringed space, where $\mathscr{O}_X(X) = R$. Then X along with its sheaf $\mathscr{O}_X$, which we will denote $(X, \mathscr{O}_X)$, is an *affine scheme* if it is isomorphic to $(\operatorname{Spec}(R), \mathscr{O}_{\operatorname{Spec} R})$, which is true when the following conditions hold:

(1) $\mathscr{O}_X(X_f) = R_f$, and
(2) X and $\operatorname{Spec}(R)$ are homeomorphic as topological spaces.

Now we give a few basic examples of affine schemes.

Example 2.5. Let K be a field. Then $\operatorname{Spec} K$ consists of exactly one point, corresponding to the zero ideal, and the structure sheaf on $\operatorname{Spec} K$ is simply K. That is, these are the sections over the unique point.

Example 2.6. Let K be a field, and let $R = K[x]_{(x)}$, the localization at the maximal ideal (x). Then R is a local ring with only two prime ideals, (0) and (x). The ideal (x) is the only closed point of $X = \operatorname{Spec} R$ by observation 2.2, so the only nonempty open sets of X are $\{(0)\}$ and X, which is the situation given in example 1.6. Thus we have

$$\mathscr{O}(X) = \mathscr{O}_{(0)} = R$$

and

$$\mathscr{O}((0)) = \mathscr{O}_{(0)} = R_{(0)} = K(x),$$

the field of rational functions.

Example 2.7. Let $R = \mathbb{Z}/6\mathbb{Z}$. R has exactly two prime ideals, (2) and (3), both of which are maximal. Thus $X = \operatorname{Spec} R$ is the discrete space on two points, as in Examples 1.3 and 1.5. Using these examples, we see that

$$\mathscr{O}((2)) = \mathscr{O}_{(2)} = R_{(2)} = \mathbb{Z}/6\mathbb{Z}_{(2)}.$$

Localizing at (2), we notice that $\frac{1}{1} = \frac{3}{1} = \frac{5}{1}$, since $3 \cdot 1 = 3 \cdot 3 = 3 \cdot 5 = 3$ and similarly, $\frac{0}{1} = \frac{2}{1} = \frac{4}{1}$. Thus, $R_{(2)} = \mathbb{Z}/2\mathbb{Z}$.

From example 1.3, we see that

$$\mathscr{O}((2)) \times \mathscr{O}((3)) = \mathscr{O}(X) = \mathbb{Z}/6\mathbb{Z}.$$

Thus

$$\mathbb{Z}/2\mathbb{Z} \times \mathscr{O}((3)) = \mathbb{Z}/6\mathbb{Z},$$

which means that

$$\mathscr{O}((3)) = \mathscr{O}_{(3)} = \mathbb{Z}/3\mathbb{Z}.$$

3. Affine n-Space Over Algebraically Closed Fields

Though the simple examples of finite schemes at the end of the previous section give relatively interesting algebraic results, we now turn to schemes that have much more interesting geometric structures, which generalize the notion of an affine variety in classical algebraic geometry.

Definition 3.1. Let K be an algebraically closed field. Then the scheme

$$\mathbb{A}^n_K := \operatorname{Spec} K[x_1, x_2, \ldots, x_n]$$

is called *affine n-space* over K.

In order to understand this space geometrically, we state a useful result from commutative algebra, a corollary to the weak Nullstellensatz:

Proposition 3.2. *Let K be an algebraically closed field. Then the maximal ideals of $K[x_1, x_2, \ldots, x_n]$ are of the form*

$$\mathfrak{m} = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n),$$

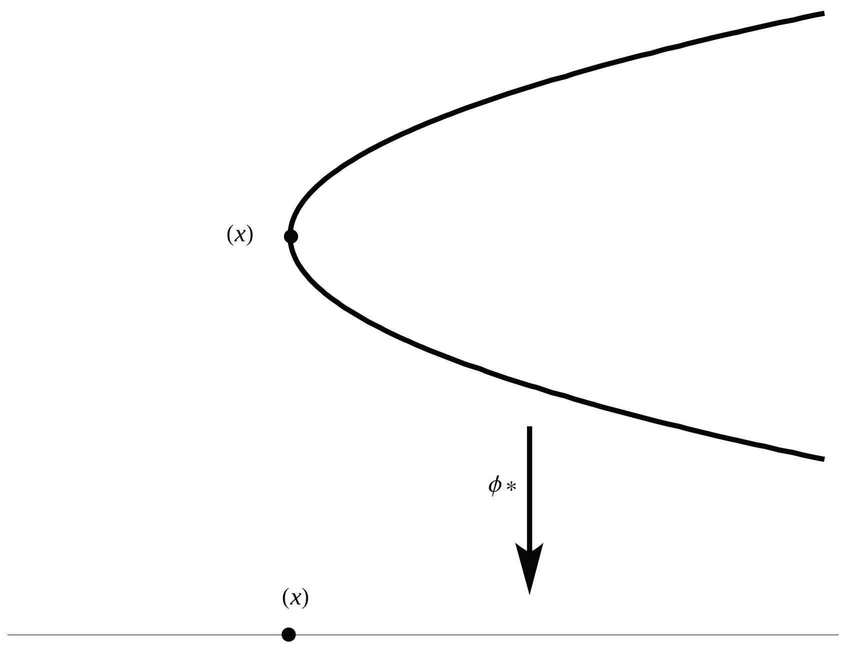
where $a_i \in K$. That is, the maximal ideals are in one-to-one correspondence with the points of K^n.

This proposition, along with observation 2.2, implies that the closed points of $\mathbb{A}^n_K$ are precisely the points corresponding to prime ideals of the form $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$. That is, the closed points correspond to vectors in K^n.

We will first look at the simplest example of an affine n-space, the case in which $n = 1$. We refer to $\mathbb{A}^1_K = \operatorname{Spec} K[x]$, in this case, as the affine line. According to the proposition, the maximal ideals of $\mathbb{A}^1_K$ are ideals of the form $(x - a)$, $a \in K$. In fact, since K is algebraically closed, these are the only nontrivial prime ideals of $\mathbb{A}^1_K$. Thus the scheme $\mathbb{A}^1_K$ is almost identical to its classical algebraic geometry counterpart, with one notable difference: since $K[x]$ is a domain, (0) is a prime ideal, so $\mathbb{A}^1_K$ actually has one additional, non-closed point. This point has an interesting property: its closure, $\overline{\{(0)\}}$, is the whole affine line, since 0 is contained in every ideal. We call this point the *generic point* of $\mathbb{A}^1_K$. We now look at a simple example of a map from the affine line to itself.

Example 3.3. Let $\phi^* : \mathbb{A}^1_K \to \mathbb{A}^1_K$ be the map induced by the ring homomorphism $\phi : K[x] \to K[x]$ defined by mapping x to x^2. We now determine what the prime ideals are mapped to under ϕ^*. First, let's look at $(x - a)$. Elements of $(x - a)$ are of the form $(x - a)f(x)$. Thus $x^2 - a^2 \in (x - a)$. So $x^2 - a^2 \in \phi((x - a^2)) \subseteq (x - a)$. Therefore, since $(x - a^2)$ is maximal, and the preimage of prime ideals under a homomorphism is prime, we have that the preimage of $(x - a)$ under ϕ must be precisely $(x - a^2)$. That is, $\phi^*((x - a)) = (x - a^2)$. Similarly, $\phi^*((x + a)) = (x - a^2)$ as well, so that the fiber of $(x - a^2)$ under ϕ^* is $\{(x + a), (x - a)\}$. Thus (x) only has one point, (x), in its fiber under ϕ^*. We know that ϕ is an injective homomorphism, so the preimage of 0 under ϕ is simply 0. Thus $\phi^*((0)) = (0)$.

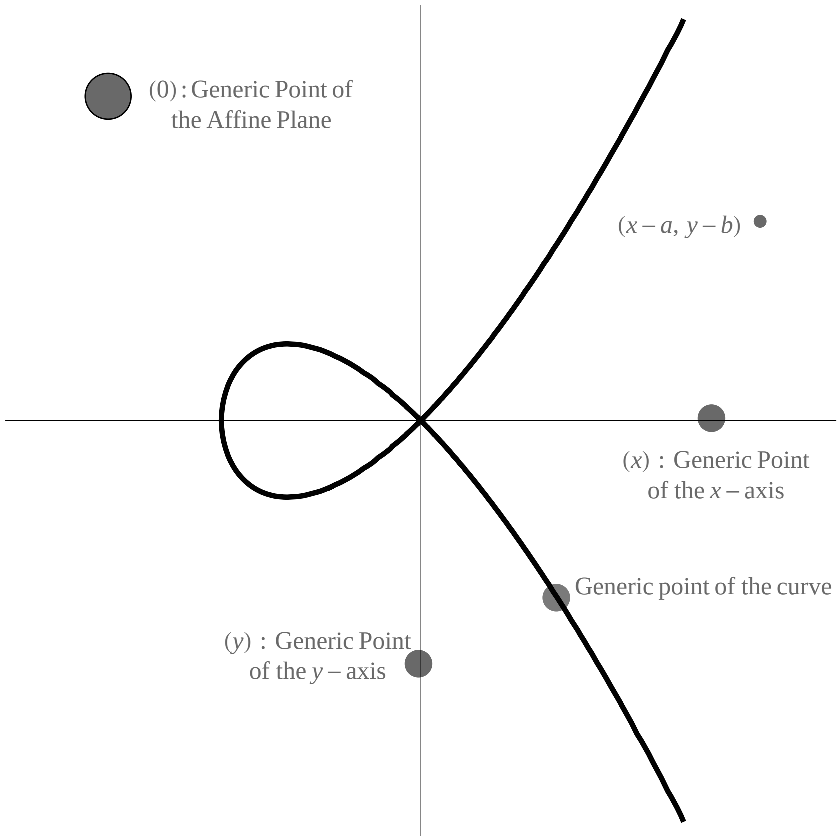
This example is illustrated in the picture below, which is based on a picture from exercise II-2 in Eisenbud and Harris.



It is necessary to keep in mind that K might not actually be a "line" in the usual topological sense (for example, we don't usually think of the visualization of $\mathbb{C}$ as a line), so our scheme diagrams cannot completely illustrate how these spaces behave geometrically. However, they can be useful as a visualization tool.

We now look at the example of the affine plane, $\mathbb{A}^2_K = \operatorname{Spec} K[x,y]$. Again by our proposition, the closed points of $\mathbb{A}^2_K$ are precisely points of the form $(x-a, y-b)$, where $a, b \in K$. However, the affine plane behaves much more interestingly than the line since it has many more prime ideals. Let us classify and examine these ideals. First, we have the obvious maximal ideals that we mentioned, and, just as on the affine line, we have (0) as the generic point. The last class of prime ideals are those of the form $(f(x,y))$, where f is an irreducible polynomial in $K[x,y]$. When we take the closure of (f), for instance, we get the set of maximal ideals $\{(x-a, y-b) \mid f(a,b) = 0\}$ along with the point itself. Again, we call this point the generic point of this set. Although these points are not closed and thus do not have representatives on the coordinate plane, we can think of them as defining the curve that is their closure: the generic point is infinitely "close," in the topological sense, to the curve that it defines. It is clear that these points correspond to the irreducible subvarieties of the classical affine plane.

The picture below, based on one from section II.1.1 of Eisenbud and Harris, illustrates this idea, showing examples of the points of $\mathbb{A}^2_K$.



The example of the affine plane can be easily extended to affine n-space, $\mathbb{A}^n_K$. Just like in the plane, we have three types of points in affine n-space:

(1) Closed points, which are of the form $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$,
(2) Non-closed points whose closures correspond to irreducible subvarieties of classical affine n-space, and
(3) (0), the generic point of $\mathbb{A}^n_K$.

4. Affine n-space Over *Non*-Algebraically Closed Fields

In the previous section, we noticed that affine space over algebraically closed fields behaved very nicely, due to the fact that the maximal ideals of $K[x_1, x_2, \ldots, x_n]$ are in one-to-one correspondence with points in K^n. However, this is not the case when working over non-algebraically closed fields.

We first give the example of the affine line over $\mathbb{R}$, or $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$.

Example 4.1. We can figure out what the closed points of $\mathbb{A}^1_{\mathbb{R}}$ are by looking at the map from $\mathbb{A}^1_{\mathbb{C}}$ to $\mathbb{A}^1_{\mathbb{R}}$ induced by the inclusion map from $\mathbb{R}[x]$ to $\mathbb{C}[x]$. The points of the form $(x-a)$ in $\mathbb{A}^1_{\mathbb{C}}$, where a is real, are simply sent to the same points in $\mathbb{A}^1_{\mathbb{R}}$. However, with points of the form $(x-b)$ such that $b \in \mathbb{C} - \mathbb{R}$, we have

$$(x-b) \longmapsto (x-b)(x-\bar{b}) = x^2 - (b+\bar{b})x + (b\bar{b}) = x^2 - 2\text{Re}(b)x + |b|^2.$$

Since the only irreducible polynomials in $\mathbb{R}[x]$ are of these two forms, along with the fact that $\mathbb{R}[x]$ is a principal ideal domain, we see that these are the only maximal ideals, and that there is only one other prime ideal, the zero ideal.

Thus every closed point of $\mathbb{A}^1_{\mathbb{R}}$ corresponds either to a point on the real line or to a complex number and its conjugate, so we can think of this affine line as the complex plane with complex conjugates identified, or the closed upper half plane, along with the generic point, (0).

In general, if K is a field and $\overline{K}$ its algebraic closure, the points of the affine n-space over K correspond to the orbits of the action of the corresponding Galois group. In this next example, we look at the natural map from $\mathbb{A}^2_{\overline{\mathbb{Q}}}$ to $\mathbb{A}^2_{\mathbb{Q}}$.

Example 4.2. Let $\phi^* : \mathbb{A}^2_{\overline{\mathbb{Q}}} \to \mathbb{A}^2_{\mathbb{Q}}$ be the map induced by the inclusion map $\phi : \mathbb{Q}[x,y] \to \overline{\mathbb{Q}}[x,y]$. In particular, we will determine the image of the point $\mathfrak{m} = (x-\sqrt{2}, y-\sqrt{2}) \in \mathbb{A}^2_{\overline{\mathbb{Q}}}$ under ϕ^*. Since ϕ is simply the inclusion map, it is clear that $\phi^*(\mathfrak{m}) = \mathfrak{m} \cap \mathbb{Q}[x,y]$. We recall from Galois Theory that the only field automorphism of $\mathbb{Q}(\sqrt{2})$ acting nontrivially on $\sqrt{2}$ sends $\sqrt{2}$ to its conjugate $-\sqrt{2}$. However, this automorphism fixes elements of $\mathbb{Q}$, so we have $\phi^*(\mathfrak{m}) \subseteq \overline{\mathfrak{m}} = (x+\sqrt{2}, y+\sqrt{2})$, and thus $\phi^*(\mathfrak{m}) \subseteq \overline{\mathfrak{m}} \cap \mathfrak{m}$. Now $(x+\sqrt{2}) - (x-\sqrt{2}) = 2\sqrt{2} \in \overline{\mathfrak{m}} + \mathfrak{m}$. $2\sqrt{2}$ is a unit in $\overline{\mathbb{Q}}[x,y]$, so $\overline{\mathfrak{m}} + \mathfrak{m} = \overline{\mathbb{Q}}[x,y]$; that is, $\overline{\mathfrak{m}}$ and $\mathfrak{m}$ are coprime, which means that

$$\phi^*(\mathfrak{m}) \subseteq \overline{\mathfrak{m}} \cap \mathfrak{m} = \overline{\mathfrak{m}}\mathfrak{m} = (x^2-2, y^2-2, xy-\sqrt{2}x+\sqrt{2}y-2, xy+\sqrt{2}x-\sqrt{2}y-2).$$

We obtain from this

$$(xy-\sqrt{2}x+\sqrt{2}y-2) + (xy+\sqrt{2}x-\sqrt{2}y-2) = 2xy-4 \in \overline{\mathfrak{m}}\mathfrak{m},$$

which means that

$$\frac{1}{2}(2xy-4) = xy-2 \in \overline{\mathfrak{m}}\mathfrak{m} \cap \mathbb{Q}[x,y].$$

Thus we have

$$(x^2-2, y^2-2, xy-2) \subseteq \overline{\mathfrak{m}}\mathfrak{m} \cap \mathbb{Q}[x,y] = \phi^*(\mathfrak{m}) \subsetneq \mathbb{Q}[x,y].$$

However, we notice that

$$-\frac{1}{2}y^2(x^2-2) + \frac{1}{2}(xy+6)(xy-2) = y^2-2,$$

so that

$$(x^2-2, y^2-2, xy-2) = (x^2-2, xy-2).$$

Now, modding out by this ideal, we get

$$\mathbb{Q}[x,y]/(x^2-2, xy-2) \cong \mathbb{Q}(\sqrt{2}),$$

which is a field. Thus $(x^2-2, xy-2)$ is maximal in $\mathbb{Q}[x,y]$, which means that $\phi^*(\mathfrak{m}) = (x^2-2, xy-2)$.

5. The Gluing Construction

Just like their classical counterpart, schemes can be glued together to make more complicated schemes. At this point, it is necessary to define a scheme in general (that is, a scheme that is not necessarily affine).

Definition 5.1. A *scheme* X is a topological space, denoted $|X|$, together with a sheaf $\mathscr{O}_X$ of rings on X such that $(|X|, \mathscr{O}_X)$ is *locally affine.* Locally affine means that $|X|$ has an open cover $\{U_i\}$ such that there exist rings R_i and homeomorphisms $|\operatorname{Spec} R_i| \cong U_i$ such that $\mathscr{O}_X|_{U_i} \cong \mathscr{O}_{\operatorname{Spec} R_i}$.

It now seems natural, seeing this definition, that affine schemes can be glued (a term that we will define below) to create general schemes. Naively it seems reasonable to glue schemes along open subsets of their underlying topological spaces. In fact, we make this more precise by defining an open subscheme:

Definition 5.2. Let U be an open subset of a scheme X. Then the pair $(U, \mathscr{O}_X|_U)$ is again a scheme, called an *open subscheme* of X.

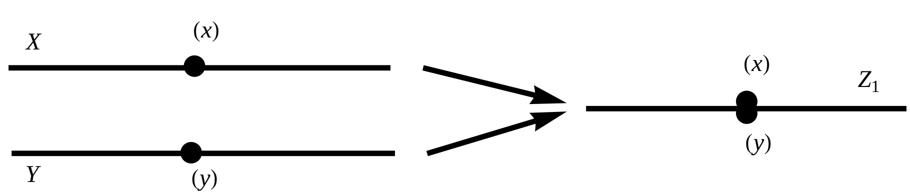
We can finally describe the gluing construction. Suppose we have two schemes X and Y and open sets $U \subseteq X$, $V \subseteq Y$ such that there is an isomorphism of schemes $\psi : U \to V$. We can then glue U and V along ψ in the natural way: we glue the underlying topological spaces along the corresponding homeomorphic open sets, and the sheaves on U and V, that is the restrictions of the original sheaves to U and V, are already identical by our assumption.

We now give two examples of gluing affine lines, the second of which naturally leads to the definition of projective schemes.

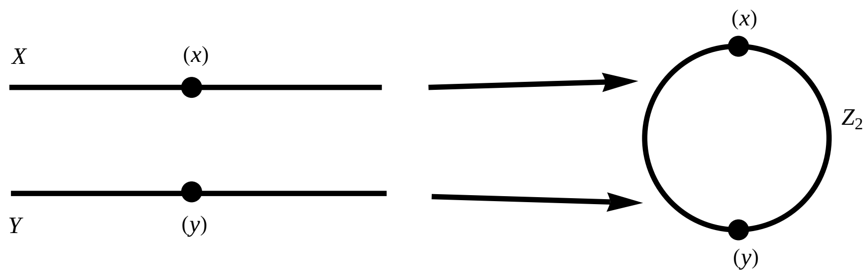
Example 5.3. Let K be an algebraically closed field, and let $X = \operatorname{Spec} K[x]$ and $Y = \operatorname{Spec} K[y]$. Let $U = X_x$ and $V = Y_y$. These are both simply the affine line without the origin. Let $\psi : V \to U$ be the isomorphism corresponding to the map

$$\mathscr{O}_X(U) = K[x, x^{-1}] \to K[y, y^{-1}] = \mathscr{O}_Y(V)$$

sending x to y. The resulting scheme, Z_1, is identical to the affine line everywhere but the origin. In place of the origin, there are two points corresponding to x and y, which are not glued. Thus Z_1 is the affine line with a doubled origin, as illustrated below.



Example 5.4. Let X and Y and U and V be defined as in the previous example. However, now let $\gamma : V \to U$ be the morphism corresponding to the map $K[x, x^{-1}] \to K[y, y^{-1}]$ sending x to y^{-1}. We call the resulting space Z_2, illustrated below.



The scheme Z_2 actually turns out to be isomorphic to the projective line $\mathbb{P}^1_K$. In fact, we construct projective n-space in general in an analogous way. We conclude with a "teaser": let R be a ring. *Projective n-space over R*, denoted $\mathbb{P}^n_K$, is made by gluing $n+1$ copies of affine space $\mathbb{A}^n_R$ over R.

6. Conclusion

Although we have only skimmed the surface of modern algebraic geometry, we can get a feeling for the power and generality of schemes, and the beautiful union of algebra and geometry that is not as obvious when merely dealing with algebraic varieties.

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