Classification of Steiner quadruple systems of order 16 and rank 14. *

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Abstract

A Steiner quadruple system $S(v,4,3)$ of order v is a 3-design $T(v,4,3,\lambda)$ with $\lambda = 1$. In the previous paper [1] we classified all such Steiner systems $S(16,4,3)$ of order 16 with rank 13 or less over $\mathbb{F}_2$. In particular, we have proved that there is one $S(16,4,3)$ of rank 11 (the points and planes of affine geometry AG(4, 2)), fifteen systems $S(16,4,3)$ of rank 12 and 4131 systems of rank 13. In this paper we describe all non-isomorphic $S(16,4,3)$ of rank 14 over $\mathbb{F}_2$. All these Steiner systems $S(16,4,3)$ can be obtained by the generalized doubling construction, which we give here. Our main result is that there are exactly 684764 non-isomorphic Steiner quadruple systems $S(16,4,3)$ of order 16 with rank 14. We found all non-isomorphic homogenious systems with rank 14 over $\mathbb{F}_2$.

*The paper has been written under the partial financial support of the Russian fund for the fundamental research (the number of project 03 - 01 - 00098)

§ 1. Introduction

A Stener system $S(n, k, t)$ is a pair (X, B) where X is a v-set and B is a collection of k-subsets of X such that every t-subset of X is contained in exactly one member of B.

A system $S(v, 3, 2)$ is called a Steiner triple system (briefly STS(v)) and a system $S(v, 4, 3)$ is called a Steiner quadruple system (briefly SQS(v)). The necessary condition for existence of an SQS(v) is that $v \equiv 2$ or 4 (mod 6). Hanani [2] proved that the necessary condition $v \equiv 2$ or 4 (mod 6) for the existence of an $S(v, 4, 3)$ is also sufficient.

Two systems SQS(X, B) and SQS(X', B') are *isomorphic*, if there is a bijection $\alpha : X \to X'$ that maps the quadruples of B to those of B'. An *automorphism* of SQS(X, B) is an isomorphism of (X, B) to itself. The determination of number of the non-isomorphic SQS(v), which we will denote by $N(v)$, is the major problem in this area. Barrau [3] proved that $N(v) = 1$ for $v \leq 10$ and Mendelson and Hung [4] derived with the help of a computer that $N(14) = 4$.

In [5] it was shown that $N(16) \geq 8$. Using computer assisted computations, Gibbons, Mathon and Corneil [6] proved that $N(16) \geq 282$. The knowledge of all non-isomorphic 1-factorizations of K_8 (the complete graph on 8 vertices) together with their automorphism groups allowed Lindner and Rosa [7], using the classical doubling construction, obtained the bound $N(16) \geq 31021$ (for the number of systems with rank exactly 14 over $\mathbb{F}_2$) . They slightly improved this bound in [8]: $N(16) \geq 31301$ (adding systems with rank less or equal 13 over $\mathbb{F}_2$). No progress has been made in this regard since this result of Lindner and Rosa (see [9], [10]).

Our result of [1] can be formulated as follows. *Among the non-isomorphic Steiner systems* $S(16, 4, 3)$ *of order* $v = 16$ *there are:*

– *one* $S(16, 4, 3)$ *of rank* 11 *(the points and planes of* 4*-dimensional affine geometry* $AG(4, 2)$ *over* $\mathbb{F}_2$*);*

– 15 *systems* $S(16, 4, 3)$ *of rank* 12*;*

– 4131 *systems* $S(16, 4, 3)$ *of rank* 13.

This paper is a natural continuation of our previous paper [1] where we started the systematic investigation of Steiner systems $S(16, 4, 3)$ of order 16 with given rank over the field $\mathbb{F}_2$. Here we classified all Steiner systems $S(16, 4, 3)$ of order 16 with rank 14 over the field $\mathbb{F}_2$. All such systems can be obtained by the generalized doubling construction, which we introduce here.

Our main result here can be formulated as follows. *Among the non-isomorphic Steiner systems* $S(16, 4, 3)$ *of order* $v = 16$ *there are:*

– 684764 *systems* $S(16, 4, 3)$ *of rank* 14 *over* $\mathbb{F}_2$.

The paper is organized as follows. Preliminary results and terminology are given in § 2. In § 3 we describe the classical doubling construction of SQS($2n$) using given SQS(n). In § 4 we consider the general properties of SQS(n) with rank $n-2$ over $\mathbb{F}_2$. Section § 5 is dedicated to the generalized doubling construction of Steiner systems $S(n, 4, 3)$ of arbitrary order n. The paragraph § 6 contains the main result of the paper: classification of all non-isomorphic Steiner systems $S(16, 4, 3)$ with rank 14 over $\mathbb{F}_2$. In § 7 we give some results concerning the Steiner triple systems $S(15, 3, 2)$ which occur as derivative of all these non-isomorphic

$S(16,4,3)$ with rank 14. In particular, we found only such triple systems with numbers $1,2,...,22$ and 62. We also found all homogeneous Steiner systems $S(16,4,3)$ of rank 14 (and from [1] we know such systems with ranks $11,12$ and 13). We give also the distribution of the number β (the number of non-somorphic derivative $S(15,3,2)$ of given $S(16,4,3)$) over all these systems $S(16,4,3)$ with rank 14.

§ 2. Preliminary results

Let E be a binary alphabet of size $2:\ E=\{0,1\}$. A binary code of length n is an arbitrary subset of E^n. Denote such binary code C with length n, with the minimal distance d and cardinality N as (n,d,N)-code. Denote by $\mathrm{wt}(\boldsymbol{x})$ the Hamming weight of vector $\boldsymbol{x}$ over E. For a (binary) code C denote by $\langle C\rangle$ the linear envelope of words of C over $\mathbb{F}_2$. The dimension of space $\langle C\rangle$ is called the *rank* of C over $\mathbb{F}_2$ and is denoted $\mathrm{rank}(C)$.

Denote by (n,w,d,N) a binary constant weight code W of length n, with weight of all codewords w, with minimal distance d and cardinality N.

For any two subsets Y and Z of E^n denote by $d(Y,Z)$ the minimal distance between Y and Z:

$$d(Y,Z) \;=\; \min\{d(\boldsymbol{y},\boldsymbol{z}):\ \boldsymbol{y}\in Y,\ \boldsymbol{z}\in Z\}.$$

For vector $\boldsymbol{v}=(v_1,...,v_n)\in E^n$ denote by $\mathrm{supp}(\boldsymbol{v})$ its support, i.e. the set of indices with nonzero positions:

$$\mathrm{supp}(\boldsymbol{v})=\{i:\ v_i\neq 0\}.$$

Denote by $\bar{\boldsymbol{v}}$ a vector, which is a complementary to $\boldsymbol{v}$, i.e. $\bar{v_i}=v_i+\mathbf{1}$.

If $E=\mathbb{F}_2$ is a field of order 2, the binary (n,d,N)-code A which is a linear k-dimensional space over $\mathbb{F}_q$ is denoted by $[n,k,d]$-code. For binary vectors $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\boldsymbol{y}=(y_1,\cdots,y_n)$ denote by $(\boldsymbol{x}\cdot\boldsymbol{y})=x_1y_1+\cdots+x_ny_n$ their inner product over $\mathbb{F}_2$. For a linear $[n,k,d]$-code A denote by $A^\perp$ its dual code:

$$A^\perp \;=\; \{\boldsymbol{v}\in\mathbb{F}_2^n:\ (\boldsymbol{v}\cdot\boldsymbol{c})=0,\ \forall\ \boldsymbol{c}\in A\}.$$

It is clear that $A^\perp$ is a linear $[n,n-k,d^\perp]$ code with some minimal distance $d^\perp$.

Denote by E_2^n the set of all binary vectors of length n of weight 2. Let $J_n=\{1,2,...,n\}$ be the coordinate set of E^n and let S_n be the full group of permutations of n elements. For any $i\in J_n$ and $\pi\in S_n$, define the image of i under the action of π by $\pi(i)$. For any set X of E^n and any $\pi\in S_n$ denote $\pi X=\{\pi(\boldsymbol{x}):\ \boldsymbol{x}\in X\}$.

A binary incidence matrix of a Steiner system $S(v,4,3)$ is the binary constant weight code $(v,4,4,v(v-1)(v-2)/24)$, denoted by S which is strongly optimal [15]. In our notation the connection between the system (X,B) and the code S looks as follows:

$$B \;=\; \{\mathrm{supp}(\boldsymbol{v})\subset X:\ \boldsymbol{v}\in S\}.$$

For any Steiner system $S(v,4,3)$ denote by $\mu_s(\boldsymbol{c})$, where $\boldsymbol{c}\in S$ and $s\in\{0,1,2\}$, the number of codewords $\boldsymbol{x}\in S$ with distance $2\,(k-s)$ at $\boldsymbol{c}$, i.e.

$$\mu_s(\boldsymbol{c})=|\{\boldsymbol{x}\in S:\ |\mathrm{supp}(\boldsymbol{c})\cap\mathrm{supp}(\boldsymbol{x})| \;=\; s\}|,\quad s\in\{0,1,2\}.$$

The numbers $\mu_s(\boldsymbol{c})$ do not depend on the choice of $\boldsymbol{c}$ and can be computed explicitly (see Theorem 5 in [15]). In particular, for $S(16,4,3)$ we have:

$$\mu_0 \;=\; 39, \quad \mu_1 \;=\; 64, \quad \mu_2 \;=\; 36. \tag{1}$$

For the case of Steiner systems the definition of equivalence can be formulated as follows.

Definition 1 *Two Steiner systems (X, B) and (X', B') of order 16 are isomorphic, if their incidence matrices S and S' are equivalent as constant weight codes, i.e. if there exists some permutation $\tau \in S_{16}$ such that S and $\tau\, S'$ coincide up to the permutation of rows.*

§ 3. SQS($2n$) obtained by the doubling construction from SQS(n)

In this section, we describe the classical doubling construction of SQS($2n$) from given SQS(n). Both constructions were described in [8], which we give here almost without changes. Denote by $F = F_1, F_2, \ldots, F_{n-1}$ a *full partition* of E_2^n into subcodes with distance 4, i.e. for any i, $i = 1, \ldots, n-1$ the set F_i is a constant weight $(n, 2, 4, n/2)$-code. Let F and H be any such partitions of E_2^n, where $H = H_1, \ldots, H_{n-1}$.

Construction A^*. Let (X, A) and (Y, B) be any two Steiner systems $S(n) = S(n, 4, 3)$ where $X \cap Y = \emptyset$. Let F and H where $F = F_1, \ldots, F_{n-1}$ and $H = H_1, \ldots, H_{n-1}$ be any full partitions of E_2^n and let α be any permutation from S_n. Define a constant weight code S on coordinate set $Q = X \cup Y$ as follows:
(1) Any codeword belonging to A or B belongs to S;
(2) if $i_1, i_2 \in X$ and $j_1, j_2 \in Y$ then $\boldsymbol{c}$ with $\mathrm{supp}(\boldsymbol{c}) = \{i_1, i_2, j_1, j_2\}$ is a codeword of C, if and only if $\in \mathbf{F_i}$ with $\mathrm{supp}(\boldsymbol{f}) = \{i_1, i_2\}$, $\boldsymbol{h} \in H_j$ with $\mathrm{supp}(\boldsymbol{h}) = \{j_1, j_2\}$ and $\alpha(i) = j$.

Proposition 1 [8] *Under construction, described above the set (Q, S) is a Steiner system $S(2n, 4, 3)$.*

In [7] these authors, using this construction and knowledge of all automorphisms groups of these partitions, derived the lower bound for $N(16) \geq 31021$.

Denote by $F_1, F_2, \ldots, F_6$ the all non-isomorphic 1-partitions of E_2^8, obtained in [22,23]. Agree that F_5 and F_6 are two partitions, not containing a sub-partitions of index 2 (see [24]), i.e. subcodes of partitions F_5 (respectively, F_6) do not form a partition of E_2^4 for any choice of 4 positions from E_2^8.

Denote by $\{F_i\}$ the orbit by action of S_8 on F_i:

$$\{F_i\} \;=\; \mathrm{Orb}_{S_8}(F_i), \quad i = 1, \ldots, 6.$$

Simple arguments show that [7] for any two fixed $S(8,4,3)$ systems (X, A) and (Y, B) there are at least $|\{F_i\}| \cdot |\{F_j\}| \cdot 7!$ distinct $S(16,4,3)$ obtained by Construction A^* by taking any $F_i \in \{F_i\}$ for F and any $F_j \in \{F_j\}$ for G with $j \in \{5,6\}$ and $i \neq j$. In addition, every such $S(16,4,3)$ has exactly two subsystems $S(8,4,3)$, namely (X, A) and (Y, B). It follows [7] that there are at least

$$N_{A^*} = \frac{7!}{1344^2} \left(\sum_{i=1}^{5} |\{F_i\}| \cdot |\{F_6\}| + \sum_{i=1}^{4} |\{F_i\}| \cdot |\{F_5\}| \right).$$

By this construction it can be seen [7] that all resulting systems $S(16,4,3)$ have rank exacxtly 14 over $\mathbb{F}_2$. The exact computation shows [7] that $N_{A^*} = 31021$. Using 280 systems found in [25], for which the number of sub-systems $S(8,4,3)$ different from two (which means that these 280 systems $S(16,4,3)$ have the rank less or equal to 13 over $\mathbb{F}_2$), one can get [7] that $N(16) \geq 31301$.

§ 4. General properties of SQS(16) with rank 14 over $\mathbb{F}_2$

Let S be an arbitrary Steiner system $S(16,4,3)$ of rank 14 over $\mathbb{F}_2$. We consider the general properties of such system.

Applying the appropriate permutation of coordinates, S can be presented in the form, when the $[16,8,2]$-code $S^\perp$, dual to S, looks as follows:

$$S^\perp \;=\; \{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_1 + \boldsymbol{u}_2\}, \tag{2}$$

where $\boldsymbol{u}_0$ is the zero vector, $\boldsymbol{u}_1 = (1111111100000000)$, and $\boldsymbol{u}_2 = (0000000011111111)$. Thus we split coordinates of S into two blocks of eight coordinates such that any $\boldsymbol{c} \in S$ consists of two vectors $\boldsymbol{c} = (\boldsymbol{c}_1 \,|\, \boldsymbol{c}_2)$ where each vector $\boldsymbol{c}_i$ satisfies to the overall parity checking:

$$\mathrm{wt}(\boldsymbol{c}_i) \;\equiv 0 \pmod 2, \quad i = 1,2$$

(we call it a *parity rule*).

Definition 2 *Let S be a Steiner system $(16,4,3)$ of rank 14 over $\mathbb{F}_2$ with dual code (2). Define the subset S_{uv} of S where $u,v \in \{0,2,4\}$ as follows:*

$$S_{uv} \;=\; \{\boldsymbol{c} = (\boldsymbol{a} \,|\, \boldsymbol{b}) \in S: \; \mathrm{wt}(\boldsymbol{a}) = u, \; \mathrm{wt}(\boldsymbol{b}) = v\}.$$

These words are called (u,v)-words.

Lemma 1 *Let S be a Steiner system $(16,4,3)$ of rank 14 over $\mathbb{F}_2$ with dual code (2). Then S is a union of three subsets*

$$S \;=\; S_{40} \bigcup S_{04} \bigcup S_{22}$$

where S_{40} (respectively S_{04}) is a Steiner system $S(8,4,3)$ and S_{22} has cardinality 112.

Proof. Follows from definition of Steiner system $S(16,4,3)$. ▲

The group (subgroup of S_{16}) of two elements which permutes the blocks is identified with S_2. An element $\tau_1 \times \tau_2 \in S_8 \times S_8 \subset S_{16}$ acts on $(\boldsymbol{x} \,|\, \boldsymbol{y})$ in the natural way:

$$(\tau_1 \times \tau_2)(\boldsymbol{x} \,|\, \boldsymbol{y}) \;=\; (\tau_1(\boldsymbol{x}) \,|\, \tau_2(\boldsymbol{y})).$$

We have the following statement.

Lemma 2 *Let S be an arbitrary Steiner system $S(16,4,3)$ of rank 14 over $\mathbb{F}_2$ with dual code (2). Suppose there exists a permutation $\sigma \in S_{16}$ so that σS satisfies the parity rule. Then $\sigma \in S_2 \rtimes (S_8 \times S_8)$.*

Proof. Since S satisfies parity rule, we have that

$$(\boldsymbol{x} \cdot \boldsymbol{u}_1) \;=\; 0, \tag{3}$$

for any $\boldsymbol{x} \in S$. Similarly, since σS satisfies the parity rule, we have that

$$(\sigma(\boldsymbol{x}) \cdot \boldsymbol{u}_1) = 0, \quad \text{for any } \boldsymbol{x} \in S.$$

Multiplying both vectors $\sigma(\boldsymbol{x})$ and $\boldsymbol{u}_1$ by σ^{-1}, we obtain

$$(\boldsymbol{x} \cdot \sigma^{-1}(\boldsymbol{u}_1)) = 0, \quad \text{for any } \boldsymbol{x} \in S. \tag{4}$$

Let $\boldsymbol{u}' = \boldsymbol{u}_1 + \sigma^{-1}(\boldsymbol{u}_1)$. From (3) and (4) we have that

$$(\boldsymbol{x} \cdot \boldsymbol{u}') = 0, \quad \text{for any } \boldsymbol{x} \in S.$$

Thus $\boldsymbol{u}' \in S^{\perp}$ and consequently (recall that $S^{\perp}$ is a vector space) $\sigma^{-1}(\boldsymbol{u}_1) \in S^{\perp}$. Taking into account that $\sigma^{-1}(\boldsymbol{u}_1)$ is of weight 8, we obtain that $\sigma^{-1}(\boldsymbol{u}_1)$ is equal to either $\boldsymbol{u}_1$ or $\boldsymbol{u}_2$. So $\sigma(\boldsymbol{u}_1) = \boldsymbol{u}_1$ or $\sigma(\boldsymbol{u}_2) = \boldsymbol{u}_1$, in other words, σ either stabilizes the blocks or permutes them. ▲

Recall that E^8_2 is the subset of E^8, formed by the all vectors of weight 2. Denote any codeword of S by $\boldsymbol{c} = (\boldsymbol{a} \,|\, \boldsymbol{b})$.

Definition 3 *Let S be a Steiner system $(16,4,3)$ of rank 14 over $\mathbb{F}_2$ with dual code (2). Let $\boldsymbol{c} = (\boldsymbol{a} \,|\, \boldsymbol{b})$ be any codeword of S such that $\mathrm{wt}(\boldsymbol{a}) = \mathrm{wt}(\boldsymbol{b})$. Denote by $A_l(\boldsymbol{b})$ (respectively, by $A_r(\boldsymbol{a})$) the sets obtained by fixing vector $\boldsymbol{b}$ (respectively $\boldsymbol{a}$) :*

$$A_r(\boldsymbol{a}) \;=\; \{\boldsymbol{b} : \; (\boldsymbol{a} \,|\, \boldsymbol{b}) \in S\}, \quad A_l(\boldsymbol{b}) \;=\; \{\boldsymbol{a} : \; (\boldsymbol{a} \,|\, \boldsymbol{b}) \in S\}.$$

Lemma 3 *Suppose the conditions of lemma 2 are satisfied. Let $\boldsymbol{c} = (\boldsymbol{a} \,|\, \boldsymbol{b})$ be any codeword of S such that $\mathrm{wt}(\boldsymbol{a}) = \mathrm{wt}(\boldsymbol{b})$. Then the set $A_l(\boldsymbol{b})$ (respectively $A_r(\boldsymbol{a})$) is a Steiner system $S(8,2,1)$ (or, equivalently, a constant weight $(8,2,4,4)$ code).*

Proof. The fact that $A_l(\boldsymbol{b})$ (respectively, $A_r(\boldsymbol{a})$) is a constant weight code $(8,2,4,N_l(\boldsymbol{b}))$ with minimal distance 4 follows from definition of such set. Indeed, any two words of S have distance not less than 4, implying that any two distinct words $\boldsymbol{x}$ and $\boldsymbol{x}'$ of $A_l(\boldsymbol{b})$ have distance not less than 4. From the other side, since S is a 3-design, nonzero positions of vectors $\boldsymbol{x}$ from $A_l(\boldsymbol{b})$ should cover all 8 positions of the first coordinate block of S. This means that for any $\boldsymbol{b} \in E^8_2$ the set $A_l(\boldsymbol{b})$ is a 1-design or $S(8,2,1)$. This follows also from counting arguments. In average, over all $\boldsymbol{b} \in E^8_2$, we have that

$$|\bar{A}_l| \;=\; \frac{1}{|E^8_2|} \times \sum_{\boldsymbol{b} \in E^8_2} |A_l(\boldsymbol{b})| \;=\; \frac{|C_{(2)}|}{|E^8_2|} \;=\; 4.$$

From the other side, $|A_l(\boldsymbol{b})|$ can not be more than 4 for any $\boldsymbol{b} \in E^8_2$. Thus $|A_l(\boldsymbol{b})| = 4$. Similarly, the same equality is valid for $|A_r(\boldsymbol{a})|$. ▲

Definition 4 *Define the sphere $W_i \subset E_2^8$, $i = 1, 2, ..., 8$ of radius two as a set of seven vectors $\boldsymbol{e}_1(i), ..., \boldsymbol{e}_7(i)$ from E_2^8, which satisfy to the following properties:*
1). $\{i\} \in \mathrm{supp}(\boldsymbol{e}_j(i),\ j = 1, ...7$.
2). $d(\boldsymbol{e}_j(i), \boldsymbol{e}_s(i)) = 2$, for any $j \neq s$.
For example, the sphere W_8, which we use very often, consists of the following vectors, which we denote for short $\boldsymbol{e}_s(8) = \boldsymbol{e}_s$:

$$\begin{array}{rclrcl} \boldsymbol{e}_1 & = & (00000011), & \boldsymbol{e}_2 & = & (00000101), \\ \boldsymbol{e}_3 & = & (00001001), & \boldsymbol{e}_4 & = & (00010001), \\ \boldsymbol{e}_5 & = & (00100001), & \boldsymbol{e}_6 & = & (01000001), \\ \boldsymbol{e}_7 & = & (10000001). & & & \end{array}$$

Note that the stabilizer of W_8 in S_8 fixes the last nonzero coordinate of $\boldsymbol{e}_i(8)$ and is isomorphic to S_7.

Lemma 4 *Suppose we are in conditions of lemma* 2 *and let $(\boldsymbol{a}_1 \,|\, \boldsymbol{b}_1)$ and $(\boldsymbol{a}_2 \,|\, \boldsymbol{b}_2)$ be any two codewords of $C_{(2)}$. Let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (respectively, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$) be such that $d(\boldsymbol{a}_1, \boldsymbol{a}_2) = 2$ (respectively, $d(\boldsymbol{b}_1, \boldsymbol{b}_2) = 2$). Then the corresponding codes $A_r(\boldsymbol{a}_1)$ and $A_r(\boldsymbol{a}_2)$ (respectively, $A_l(\boldsymbol{b}_1)$ and $A_l(\boldsymbol{b}_2)$) do not intersect each other, i.e. $A_r(\boldsymbol{a}_1) \cap A_r(\boldsymbol{a}_2) = \emptyset$ (respectively, $A_l(\boldsymbol{b}_1) \cap A_l(\boldsymbol{b}_2) = \emptyset$).*

Proof. In contrary, assume that there is $\boldsymbol{x}$ such that $\boldsymbol{x} \in A_r(\boldsymbol{a}_1) \cap A_r(\boldsymbol{a}_2)$. Then we have

$$d((\boldsymbol{a}_1 \,|\, \boldsymbol{x}), (\boldsymbol{a}_2 \,|\, \boldsymbol{x})) \;=\; d(\boldsymbol{a}_1, \boldsymbol{a}_2) \;=\; 2,$$

i.e. a contradiction, since $(\boldsymbol{a}_1 \,|\, \boldsymbol{x})$ and $(\boldsymbol{a}_2 \,|\, \boldsymbol{x})$ are distinct codewords of C. The proof of the second statement is similar. ▲

Lemma 5 *Suppose we are in conditions of lemma 2 and let $W_i = \{\boldsymbol{e}_1(i), ..., \boldsymbol{e}_7(i)\}$ be any sphere, $i = 1, 2, \ldots, 8$. Then the set of codes $A_l(\boldsymbol{e}_1(i))$, $A_l(\boldsymbol{e}_2(i))$, ..., $A_l(\boldsymbol{e}_7(i))$ forms a partition of E_2^8.*

Proof. Since

$$|W_i| \times |A_l(\boldsymbol{e}_s(i))| \;=\; |E_2^8| \;=\; 28,$$

we have to check only that any two distinct codes $A_l(\boldsymbol{e}_j(i))$ and $A_l(\boldsymbol{e}_s(i))$ where $j \neq s$ and $j, s \in \{1, ..., 7\}$ have empty intersection. But this follows from lemma 4, since for any $\boldsymbol{e}_j(i), \boldsymbol{e}_s(i)$ from W_i we have that $d(\boldsymbol{e}_j(i), \boldsymbol{e}_s(i)) = 2$. ▲

Remark 1 *It is easy to see that the results above, which we derived for Steiner system $S(16, 4, 3)$ of rank* 14 *over $\mathbb{F}_2$, are valid for any $S(n, 4, 3)$ of arbitrary order $n \geq 16$ with rank $n - 2$ over $\mathbb{F}_2$ such that $n/2 \equiv 2$ or* 4 (mod 6).

§ 5. Generalized doubling construction of $S(16, 4, 3)$ with rank 14 over $\mathbb{F}_2$

Now we describe the general doubling construction of Steiner systems $S(16,4,3)$ with rank 14 over $\mathbb{F}_2$. This construction is induced by the general doubling construction of the extended binary perfect nonlinear $(16,4,2^{11})$-codes of rank 14 over $\mathbb{F}_2$, which we described in [15]. Indeed, the set of codewords of weight four of any such $(16,4,2^{11})$-code with zero codeword forms a Steiner system $S(16,4,3)$.

It is convenient for us to present such a system $S(16,4,3)$ by the corresponding constant weight $(16,4,4,140)$ code, which uniquely defines this system [16], and which we denote here by S. Denote by $\mathcal{S}$ the set of all such distinct $(16,4,4,140)$ codes S. Our purpose now is to parameterize all these Steiner systems, using the canonical partitions of E_2^8. We can do it using the special subsets of S, called headings, formed by the two partitions, connected with the two spheres $W_8 = \{\boldsymbol{e}_s : \; s = 1, ..., 7\}$ which occur on the left and right hand sides (the first and the second blocks) of the codewords. We start with the definition of *heading* of a code. Clearly when $\boldsymbol{c} = (\boldsymbol{a} \,|\, \boldsymbol{b})$ runs over S, each of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ run over the set E_2^8. In particular, when $\boldsymbol{a}$ runs over the sphere W_8 the corresponding codes $A_r(\boldsymbol{a})$ form a partition of E_2^8,

$$E_2^8 \;=\; \bigcup_{\boldsymbol{a} \in W_8} A_r(\boldsymbol{a}) \;=\; \bigcup_{s=1}^{7} A_r(\boldsymbol{e}_s).$$

Similarly, when $\boldsymbol{b}$ runs over the set W_8, the codes $A_l(\boldsymbol{b})$ also form a partition of E_2^8.

Denote by Ω the set of all distinct partitions $L_i = (L_{i,1}, L_{i,2}, L_{i,3}, L_{i,4})$ of E_2^8 into (binary constant weight) $(8,2,4,4)$ codes $L_{i,s}, \; s = 1,2,3,4$. Moreover the following result holds.

Proposition 2 (*Computational result*). *There exist exactly* 6240 *different partitions of* E_2^8 *which can be arranged under action of* S_8 *into six orbits* $\mathrm{Orb}_{S_8}(L_i)$, *ordered according to the indices* i *of* $\mathrm{Orb}_{S_8}(L_i)$.

We assume that the unique Steiner system $S(8,4,3)$ is formed by the following vectors (in addition to words of all zeroes and ones):

$$\begin{array}{ll}
(1111|0000), & (0000|1111), \\
(1100|1100), & (0011|0011), \\
(1100|0011), & (0011|1100), \\
(1010|0110), & (0101|1001), \\
(1010|1001), & (0101|0110), \\
(1001|1010), & (0110|0101), \\
(1001|0101), & (0110|1010).
\end{array}$$

Denote by P its stabilizer in S_8 and by P' its stabilizer in the group S_7.

Definition 5 *Define the group:*

$$G = S_2 \rtimes (P \times P) \subset S_{16}$$

It is known (see, for example, [8]) that $|P| = 1344$. Recall Lemma 1 that any system S of rank 14 (with dual code $S^{\perp}$ given by (2)) is partitioned into three subsets S_{40}, S_{04} and S_{22}. Without loss of generality, we can assume from now that all our systems S from $\mathcal{S}$ are such that the subsets $S_{40} = A_l(\mathbf{0})$ and $S_{04} = A_r(\mathbf{0})$ of 14 elements are obtained from the Steiner system given above. This condition increases the number of non-equivalent partitions of E_2^8 since we consider the P-equivalence and P'-equivalence.

Proposition 3 (*Computational result*). *Let Ω be the set of all 6240 different partitions of E_2^8 into $(8, 2, 4, 4)$ codes. Then Ω splits into 43 P-orbits $\mathrm{Orb}_P(L_i)$ $(i = 1, \ldots, 43)$ and 62 P'-orbits $\mathrm{Orb}_{P'}(L'_i)$. We assume that the 62 non-equivalent partitions L'_i are chosen so that $L_i = L'_i$, where $i = 1, \ldots, 43$.*

We denote L'_i via L_i, $i = 1, \ldots, 62$ and call them *canonical partitions* of E_2^8.

For any such canonical partition L_i, denote by $\mathrm{Stab}_P(L_i)$ the stabilizer of L_i in the group P and by $Q_i \subset S_7$ a group of permutations of its seven components $L_{i,1}, L_{i,2}, \ldots, L_{i,7}$ induced by the automorphisms of P:

$$Q_i \;=\; \{\pi \in S_7 :\; \exists\, \boldsymbol{g} \in \mathrm{Stab}_P(L_i) :\; \boldsymbol{g} L_{i,s} = L_{i,\pi^{-1}(s)},\; i = 1, \ldots, 7\}.$$

For an element $\boldsymbol{a} \in E^8$ and set $X \subseteq E^8$ denote:

$$\boldsymbol{a} \times X \;=\; \{(\boldsymbol{a} \,|\, \boldsymbol{x}) :\; \boldsymbol{x} \in X\}, \quad X \times \boldsymbol{a} \;=\; \{(\boldsymbol{x} \,|\, \boldsymbol{a}) :\; \boldsymbol{x} \in X\}.$$

Definition 6 *Let S be a $(16, 4, 4, 140)$ code with rank 14 over $\mathbb{F}_2$. Define the following subset $F = F(S)$ of S (of 56 words), consisting of two partitions (with 7 common words counted twice)*

$$F(S) \;=\; \bigcup_{s=1}^{7} \{(\boldsymbol{e}_s \,|\, \boldsymbol{y}) :\; \boldsymbol{y} \in A_r(\boldsymbol{e}_s)\} \;\bigcup\; \bigcup_{s=1}^{7} \{(\boldsymbol{x} \,|\, \boldsymbol{e}_s) :\; \boldsymbol{x} \in A_l(\boldsymbol{e}_s)\}. \tag{5}$$

We say that S has a heading F and for the sake of simplicity write as:

$$F \;=\; \bigcup_{s=1}^{7} \boldsymbol{e}_s \times A_r(\boldsymbol{e}_s) \;\bigcup\; \bigcup_{s=1}^{7} A_l(\boldsymbol{e}_s) \times \boldsymbol{e}_s.$$

Assume that the partition $A_l(\boldsymbol{e}_1)$, ..., $A_l(\boldsymbol{e}_7)$ is equivalent to L_i for some i, $i = 1, ..., 43$ and the partition $A_r(\boldsymbol{e}_1)$, ..., $A_r(\boldsymbol{e}_7)$ is equivalent to L_j for some j, $j = 1, ..., 62$. Recall that L_i (respectively, L_j) are among of the 43 (respectively 62) canonical (non-equivalent) partitions, given by proposition 3. All these partitions L_i, $i = 1, ..., 62$ are ordered, according to the vectors $\boldsymbol{e}_s$ of the ball W_8:

$$L_i = (L_{i,1}, \ldots, L_{i,7}) \;\text{ where }\; \boldsymbol{e}_s \in L_{i,s} \;\text{ for }\; s = 1, \ldots, 7.$$

Without loss of generality we can assume that $i \le j$ (if not we can consider the Steiner system S' obtained from S by switching the sides). Furthermore, by the corresponding permutation of coordinates we can obtain the following ordering of L_i:

$$L_i \;=\; (L_{i,1}, ..., L_{i,7}), \quad L_{i,s} = A_l(\boldsymbol{e}_s), \tag{6}$$

where the vectors $\boldsymbol{e}_s$ $(s = 1, ..., 7)$ are given by definition 4. In such way we arrive to the following natural *canonical heading*.

Definition 7 (*Canonical* (i, j, k) *heading*). *Let* $1 \le i \le 43$ *and* $i \le j \le 62$ *and* L_i, L_j *are two canonical partitions. Define the set of* 56 (*where* 7 *words are counted twice*) *elements as follows*:

$$\begin{aligned} F_{i,j}^{(k)} &= \bigcup_{s=1}^{7} L_{i,\pi(s)} \times \boldsymbol{e}_s \bigcup \bigcup_{s=1}^{7} \boldsymbol{e}_{\pi(s)} \times L_{j,s} \\ &= \bigcup_{s=1}^{7} \{(\boldsymbol{x} \,|\, \boldsymbol{e}_s) : \; \boldsymbol{x} \in L_{i,\pi(s)}\} \bigcup \bigcup_{s=1}^{7} \{(\boldsymbol{e}_{\pi(s)} \,|\, \boldsymbol{y}) : \; \boldsymbol{y} \in L_{j,s}\}. \end{aligned}$$

where $\pi = \pi_k^{-1}$, $k = 1, 2, ..., m(i, j)$, *and*

$$\{\pi_1, , \pi_2, ..., \pi_{m(i,j)}\}$$

is a fixed set of the $(Q_j$-$Q_i)$*-double-coset representatives of* S_8.

We know all canonical headings (i, j, k).

Proposition 4 (*Computational result*). *There exist* 339716 *different canonical headings* (i, j, k).

Using canonical headings, now we can define *canonical Steiner systems* C.

Definition 8 (*Canonical Steiner system*). *Let* S *be any Steiner system from* $\mathcal{S}$. *We say that* S *is a canonical* (i, j, k) *code, denoted by* $S_{i,j}^{(k)}$ *if* S *has a canonical heading*

$$F(S_{i,j}^{(k)}) = F_{i,j}^{(k)}.$$

Now the important question is *does any system* S *from* $\mathcal{S}$ *equivalent to a canonical one* $S_{i,j}^{(k)}$?

Proposition 5 *Let* $S \in \mathcal{S}$ *and let* $F = F(S)$ *be the heading of* S. *Then* S *is* G*-equivalent to a canonical Steiner system* $S_{i,j}^{(k)} \in \mathcal{S}$ *with heading* $F_{i,j}^{(k)}$, *where* $1 \le i \le 43$ *and* $i \le j \le 62$ *and where the permutation* π_k *is defined by definition* 7.

Proof. Let S be any Steiner system from $\mathcal{S}$. Define the following subset of S

$$Y_2 = \bigcup_{s=1}^{7} A_l(\boldsymbol{e}_s) \times \boldsymbol{e}_s \;=\; \{(\boldsymbol{y}\,|\,\boldsymbol{e}_s):\; \boldsymbol{e}_s \in W_8,\; \boldsymbol{y} \in A_l(\boldsymbol{e}_s)\}, \tag{7}$$

where $A_l(\boldsymbol{e}_s)$, $s = 1, \ldots, 7$ is a partition A_l of E_2^8. Assume that A_l is P-equivalent to L_i for some i. Thus there exists a permutation $\tau_2 \in P$ such that $\tau_2 A_l = L_i$ and in particular

$$\tau_2 A_l(\boldsymbol{e}_s) \;=\; L_{i,\tau_2^{-1}(s)}. \tag{8}$$

Let 1_8 be the identity element of S_8. Applying the element $\tau_2 \times 1_8$ to S, its subset defined (7), and taking into account (8), we have

$$\begin{aligned}
(\tau_2 \times 1_8)Y_2 &= (\tau_2 \times 1_8)\Big\{\bigcup_{s=1}^{7} A_l(\boldsymbol{e}_s) \times \boldsymbol{e}_s\Big\} \\
&= \bigcup_{s=1}^{7} (\tau_2 A_l(\boldsymbol{e}_s)) \times \boldsymbol{e}_s \\
&= \bigcup_{s=1}^{7} L_{i,\tau_2^{-1}(s)} \times \boldsymbol{e}_s \\
&= \bigcup_{s=1}^{7} L_{i,s} \times \boldsymbol{e}_{\tau_2(s)}.
\end{aligned}$$

Set $S' = (\tau_2 \times 1_8)S$, and define the following subset of S'

$$Y_1 = \bigcup_{s=1}^{7} \boldsymbol{e}_s \times A_r(\boldsymbol{e}_s) \;=\; \{(\boldsymbol{e}_s\,|\,\boldsymbol{y}):\; \boldsymbol{e}_s \in W_8,\; \boldsymbol{y} \in A_r(\boldsymbol{e}_s)\},$$

where $A_r(\boldsymbol{e}_s)$, $s = 1, \ldots, 7$ is a partition A_r of E_2^8. Assume that A_r is P'-equivalent to L_j for some j. Thus there exists an element $\tau_1 \in P'$ such that $\tau_1 A_r = L_j$ and $\tau_1 W_8 = W_8$. In particular

$$\tau_1 A_r(\boldsymbol{e}_s) \;=\; L_{j,\tau_1^{-1}(s)}. \tag{9}$$

Applying the element $1_8 \times \tau_1$ to S', its subset Y_1, and taking into account (9), we have

$$\begin{aligned}
(1_8 \times \tau_1)Y_1 &= (1_8 \times \tau_1)\Big\{\bigcup_{s=1}^{7} \boldsymbol{e}_s \times A_r(\boldsymbol{e}_s)\Big\} \\
&= \bigcup_{s=1}^{7} \boldsymbol{e}_s \times (\tau_1 A_r(\boldsymbol{e}_s)) \\
&= \bigcup_{s=1}^{7} \boldsymbol{e}_s \times L_{j,\tau_1^{-1}(s)}.
\end{aligned}$$

Moreover, we have

$$(1_8 \times \tau_1)\Big\{\bigcup_{s=1}^{7} L_{i,s} \times \boldsymbol{e}_{\tau_2(s)}\Big\} \;=\; \bigcup_{s=1}^{7} L_{i,s} \times \tau_1(\boldsymbol{e}_{\tau_2(s)}) \;=\; \bigcup_{s=1}^{7} L_{i,s} \times \boldsymbol{e}_{\tau_3(s)},$$

for some permutation $\tau_3 \in S_7$. Since $\boldsymbol{e}_{\tau_3(s)} \in L_{j,\tau_1^{-1}(s)}$ we conclude that $\tau_3 = \tau_1^{-1}$. Set $S'' = (1_8 \times \tau_1)S'$. Then S'' is equivalent to S and its heading by definition is equal to

$$\bigcup_{s=1}^{7} L_{i,s} \times \boldsymbol{e}_{\tau_1^{-1}(s)} \bigcup \bigcup_{s=1}^{7} \boldsymbol{e}_s \times L_{j,\tau_1^{-1}(s)}.$$

Without loss of generality we can always assume that $i \leq j$ (if not apply the permutation of S_2 from the definition of G, i.e. switch the blocks of coordinates). ▲

It is clear that a Steiner system S can have different headings as well as different Steiner systems may have the same heading.

Now we want to describe the general doubling construction of Steiner systems $S(16, 4, 3)$ of rank 14 over $\mathbb{F}_2$.

Definition 9 *Let $\mathcal{M}_s$, $s = 1, 2, \ldots, 7$ be the set of constant weight $(8, 2, 4, 4)$ codes containing e_s. Let*

$$\mathcal{M} = \bigcup_{s=1}^{7} \mathcal{M}_s$$

be the set of all constant weight $(8, 2, 4, 4)$ codes.

It is easy to check that there are 15 codes in every set $\mathcal{M}_s$ so that the total number of $(8, 2, 4, 4)$ codes is 105. We consider functions from E_2^8 to $\mathcal{M}$.

Definition 10 (*Admissible function*) *We say that a function $\Lambda\colon E_2^8 \to \mathcal{M}$ is admissible if there exist $1 \leq i \leq 43$, $i \leq j \leq 62$, and a permutation π_k such that:*
1). $\Lambda(\boldsymbol{e}_{\pi^{-1}(s)}) = L_{j,s}$, for $s = 1, \ldots, 7$.
2). $\Lambda(\boldsymbol{x}) = M \in \mathcal{M}_s$, where $\boldsymbol{x} \in L_{i,\pi^{-1}(s)}$ and $s = 1, \ldots, 7$.
Such function will be called an (i, j, k)-admissible function.

Admissible functions are used to parameterize canonical Steiner systems. Indeed for any canonical Steiner system $S = S_{i,j}^{(k)}$, and any $\boldsymbol{x} \in E_2^8$, set $\Lambda(\boldsymbol{x}) = A_r(\boldsymbol{x})$ (see Definition 3). Then

$$S_{22} \;=\; \bigcup_{\boldsymbol{x} \in E_2^8} \boldsymbol{x} \times \Lambda(\boldsymbol{x}),$$

where Λ is (i, j, k)-compatible by definition.

§ 6. Derived triple systems

For an SQS(v), given by the pair of sets (X, B), a derived triple system (briefly DTS($v-1$)) of (X, B) is a pair (X_a, B_a), where $X_a = X \setminus \{a\}$ and $B_a = \{b \setminus \{a\} : \ a \in b \in B\}$. It is obvious, that every derived triple system is a Steiner triple system $S(v-1, 3, 2)$. For $v = 16$ we obtain a system $S(15, 3, 2)$. It is known [19] from 1917 that there are exactly 80 non-isomorphic systems $S(15, 3, 2)$. There is a standard numbering of these systems by the indices from 1 to 80, related to the number of Pasch configurations (see [1]).

Given a Steiner system $S = S(v, 4, 3)$, let $\beta = \beta(S)$ denote the number of its pairwise non-isomorphic DTS($v-1$). Clearly $1 \leq \beta \leq v$ for any SQS(v). A system SQS(v) is said to be *homogeneous* (respectively, *heterogeneous*), if $\beta = 1$ (respectively, $\beta = v$). Among all Steiner systems SQS(16) of rank at most thirteen, the only derived systems DTS(15) that we found are those with numbers $1, 2, 3, 4, 5, 6, 7$. All Steiner triple systems with these numbers occur as the DTS(15) in the homogeneous SQS(16).

Denote by $N_{hom}(i)$ the number of non-isomorphic homogeneous systems SQS(16) with rank at most thirteen, whose derived systems are DTS(15) with number i, where $i \in \{1, 2, ..., 7\}$. Denote by $N(\beta)$ the number of such non-isomorphic systems SQS(16) with rank at most thirteen with given β. Denote by $N(\mu(i_1), \mu(i_2), ..., \mu(i_\beta))$ the number of non-isomorphic systems SQS(16) with rank at most thirteen which have $\mu(i_s) > 0$ derived systems with number i_s, where $i_s \in \{1, 2, ..., 7\}$ for $s = 1, ..., \beta$, i.e. in our notation $N_{hom}(i) = N(\mu(i) = 16)$.

§ 7. Non-isomorphic Steiner systems SQS(16) of rank 14 over $\mathbb{F}_2$

Theorem 1 *There exists* 684764 *non-equivalent Steiner systems* $S(16, 4, 3)$ *of length* 16 *and rank* 14*.*

Proof. Computational result. First, we construct all different Steiner systems using (i, j, k)-admissible functions Λ. Then to any Steiner system SQS(16) we associate a set of 16 indices of the derived triple systems. We note that if the two sets that correspond to an arbitrary two SQS(16) systems are different these systems are non-equivalent. Thus all different Steiner systems are arranged into lists which correspond to the same set of 16 indices. The lists are pair-wise non-equivalent, i.e. two systems belong to the different lists are non-equivalent. ▲

§ 8. Resolvability

The general *resolvability* problem for SQS(v) can be stated as follows. A Steiner system $S(v, 4, 3)$ is called (t, λ)*-resolvable* if its block set B can be partitioned into r subsets $B_1, B_2, ..., B_r$ such that (S, B_i) is a t-design $T(v, 4, t, \lambda)$ for all i. It is clear that

$$\frac{|B|}{r} = \frac{\binom{v}{t}}{\binom{4}{t}} \cdot \lambda.$$

For the case of systems $S(v,4,3)$ there are two possibilities: $t=1$ or $t=2$. Denote (t,λ)-resolvable SQS(v) by RSQS(t,λ,v). If $(t,\lambda)=(1,1)$ such SQS(v) is also called *resolvable*, and if SQS(v) is ((t,1)-resolvable simultaneously for $t=1$ and 2 it is also called *double resolvable*. The first infinite family of double resolvable SQS(v) for all $v=4^m$ was given in [21] (see also [11] and references there). Next, we would like to show that all systems SQS(16) of rank 14 over $\mathbb{F}_2$ are resolvable.

REFERENCES

1. *Zinoviev V.A., Zinoviev D.V.* Classification of Steiner quadruple systems of order 16 and rank at most thirteen// Problems of Information Transmission. 2004. V. 40. N° 4. P. 48–67.

2. *Hanani H.* On quadruple systems// Canad. J. Math. 1960. V. 12. P. 145-157

3. *Barrau J.A.* On the combinatory problem of Steiner // K. Akad. Wet. Amst. Proc. Sect. Sci. 1908. V. 11. P. 352-360.

4. *Mendelsohn N.S., Hung S.H.Y.* On the Steiner Systems $S(3,4,14)$ and $S(4,5,15)$ // Utilitas Math. 1972. V. 1. P. 5-95.

5. *Doyen J., Vandensavel M.* Non-isomorphic Steiner quadruple systems// Bull. Soc. Math. Belg. 1971. V. 23. P. 393-410.

6. *Gibbons P.B., Mathon R., Corneil D.G.* Steiner quadruple systems on 16 symbols// In: Proc. 6-th Southeastern Conf. Combinatorics, Graph Theory and Computing, Boca Raton. 1975. Congressus Numerantium XIV (Utilitas Math. Publ. Winnipeg. 1975). P. 345-365.

7. *Lindner C.C., Rosa A.* There are at least 31021 nonisomorphic Steiner quadruple systems of order 16// Utilitas Math. 1976. V. 10. P. 61-64.

8. *Lindner C.C., Rosa A.* Steiner quadruple systems – A survay// Discrete Mathematics. 1978. V. 21. P. 147-181.

9. *Hartman A., Phelps K.T.* Steiner Quadruple Systems// In: Contemporary Design Theory: A Collection of Surveys. Dinitz J.H., Stinson D.R., Eds. John Wiley & Sons. 1992. Ch. 6. P. 205-240.

10. *C.J. Colbourn C.J., J.H. Dinitz J.H.* The CRC Handbook of Combinatorial Designs. Boca Raton, FL: CRC Press, 1996.

13. *Zinoviev V.A., Zinoviev D.V.* Classification of Steiner quadruple systems of order 16 and rank at most thirteen// Proc. Ninth Int. Workshop on Algebraic and Combinatorial Coding Theory. Kranevo, Bulgaria (June 19-25), 2004, P. 399-403.

14. *Zinoviev V.A., Zinoviev D.V.* Binary extended perfect codes of length 16 by generalized concatenated construction// Problems of Information Transmission. 2002. V. 38. N° 4. P. 56–84.

15. *Zinoviev V.A., Zinoviev D.V.* Binary extended perfect codes of length 16 and rank 14// Problems of Information Transmission. 2004. submitted.

16. *Semakov N.V., Zinoviev V.A.* Constant weight codes and tactical configurations// Problems of Information Transmission. 1969. V. 5. N° 3. P. 29–38.

17. *Zinoviev V.A., Lobstein A.* On generalized concatenated constructions of perfect binary nonlinear codes// Problems of Information Transmission. 2000. V. 36. N° 4. P. 59–73.

18. *Doyen J., Hubaut X., Vandensavel M.* Ranks of Incidence Matrices of Steiner Triple Systems// Mathem. Zeitschr. 1978. V. 163. P. 251-259.

19. *MacWilliams F.J., N.J.A. Sloane N.J.A.*, *The theory of Error Correcting Codes*, North-Holland, 1986.

20. *Cole F.N., Cummings L.D., White H.S.* The complete enumeration of trial systems in 15 elements// Proc. Nat. Acad. Sci. USA. 1917. V. 3. P. 197-199.

21. *de Vries H.L.* Some Steiner quadruple systems $S(3,4,16)$ such that all 16 derived triple systems $S(2,3,15)$ are isomorphic// Ars Combin. 1987. V. 24A. P. 107–129.

22. *Zaitsev G.V., Zinoviev V.A., Semakov N.V.* Interrelation of Preparata and Hamming codes and extension of Hamming codes to new double-error-correcting codes, in: "2nd International Symposium on Information Theory, Tsahkadzer, Armenia, USSR, 1971," Akademiai Kiado-Budapest. 1973. P. 257-263.

23. *Dickson L.E., Safford F.H.* Solution to Problem 8 (Group Theory)// Am. Math. Monthly. 1906. V. 13. P. 150–151.

24. *Wallis W.D.* On one-factorizations of complete graphs// J. Austral. Math. Soc. 1973. V. 16. P. 167–171.

25. *Lindner C.C., Mendelsohn E., Rosa A.* On the number of 1-factorizations of the complete graph// J. Combinatorial Theory. B. 1976. V. 20. P. 265–282.

26. *Gibbons P.B.* Computing techniques for the construction and analysis of block designs// Ph.D Thesis. University of Toronto. 1976.

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