HOW MANY ORDERS ARE THERE?

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Abstract. The number of preferential arrangements of n labelled elements (also known as the ordered Bell numbers, and listed as sequence A000670 in Sloane) is investigated. A generating function and a recurrence relation are obtained and the asymptotic behaviour of the sequence is described.

Let us count two assignments of values to n items $\{x_i\}_{i=1}^{n}$ as being distinct if different order-relations result. For instance, we might be looking at assigning scores to people who have sat an exam, or (the case that originally drew me to this problem) assigning priorities to operators in a parser for a computer language. We're not interested in the particular value that is assigned to each item, only to its position relative to other items: first, second, or whatever.

If we don't allow two items to have equal values then there are exactly $n!$ distinct assignments, because we have n choices for the first item, $n-1$ choices for the second item, and so on until there is only one item left.

What if we do allow some items to be the same, so that items can be "first equal" as well as "first" or "second"? Let's denote the number of distinct assignments in this case by C_n. So the question is: how big is C_n?

Obviously $C_n > n!$, since we get $n!$ choices even if we don't allow items to be equal, so anything beyond that is a bonus. Let's look at C_3. First of all let's assume that all the x_i are distinct: that gives us 3! choices. Then let's assume that two of the x_i are the same and one is different: we have 3 choices for the different one, and 2 choices as to its position (it can either be greater than the other two or less than the other two): so that gives us 3×2 choices. Finally, let's assume that all the x_i are identical: that gives us one further choice. The total is $3! + 3 \times 2 + 1 = 13$, so $C_3 = 13$.

What can we say about C_n in general?

0.1. Some results.

$$\frac{1}{2-e^z} = \sum_0^\infty C_n \frac{z^n}{n!} \tag{1}$$

$$C_n = \sum_0^{n-1} \binom{n}{k} C_k \tag{2}$$

or, symbolically,

$$2C_n = (C+1)^n \tag{3}$$

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$C_n$is sequence no. A000670 in N.J.A. Sloane's *On-Line Encyclopedia of Integer Sequences*: http://www.research.att.com/~njas/sequences/A000670.

which is similar to the $B_n = (B+1)^n$ that defines the Bernoulli numbers.

Denoting ln 2 by α and defining $f(z) \equiv \frac{z}{e^z - 1}$,

$$\frac{\alpha^n}{n!} C_n = \frac{1}{2\alpha} \sum_{k=0}^{n} \frac{\alpha^k}{k!} f^{(k)}(-\alpha) \tag{4}$$

whence

$$f^{(n)}(-\ln 2) = 2(C_n \ln 2 - nC_{n-1}) \tag{5}$$

As $n \to \infty$,

$$\sum_{k=0}^{n} \frac{(\ln 2)^k}{k!} f^{(k)}(-\ln 2) \to 1 \tag{6}$$

so that

$$C_n \sim \frac{n!}{2(\ln 2)^{n+1}} \tag{7}$$

1. The generating function

We can classify order assignments by the number of equal items that they contain and how those equal items are distributed.

For instance, if we have $a = b > c > d > e = f = g > h = i > j$, we can encode this as <2,1,1,1,3,1,2,1>. Every order assignment belongs to exactly one such "equality class". Note, too, that the order of numbers in the encoding of the equality class is significant: <1,2,1,1,3,2,1,1> contains different order assignments. It should be clear by now that if you're talking about order assignments for n items, the relevant equality classes are exactly those lists of positive integers that add up to n: all the way from $<n>$ on its own to <1,1,1...>.

How many order assignments belong to each individual equality class? Taking the example given earlier, we can write the letters a to j in any order we like: this gives us 10! possibilities. But the number 2 occurs in the encoding of the equality class, which means that the first two items are equal and can be swapped: $b = a$ means the same as $a = b$. So we divide the number of possibilities by 2. Later on, the number 3 occurs, meaning that there are three equal items, which can be rearranged in any way you like: there are 3! ways, so the number of possibilities gets divided by 3!.

In general, given an encoding of an equality class as $\{a_i\}_1^m$, that equality class represents a set of order assignments of $\sum_1^m a_i$ items. This gives $(\sum_1^m a_i)!$ possibilities if equality is ignored. In order to take account of equality we have to divide the total by $a_i!$ for each a_i. Thus the total contribution of that equality class is

$$\frac{(\sum_1^m a_i)!}{\prod_1^m (a_i!)} \tag{8}$$

and C_n is the sum of the contributions of all equality classes for which $\sum_1^m a_i = n$.

Now consider the expansion

$$(e^z - 1)^1 = \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \tag{9}$$

$n!$ times the coefficient of z^n is 1, which is the contribution to C_n of the equality class that has just one number in its encoding: $<n>$.

Next, consider the expansion of $(e^z - 1)^2$. The coefficient of z^n in this is the sum of $\frac{1}{i!}\Delta\frac{1}{j!}$, taken over all pairs of positive integers which satisfy $i + j = n$. This means that $n!$ times the coefficient of $z^n$is the contribution to C_n of the equality classes that have just two numbers in their encoding: $<i, j>$.

So in general, $n!$ times the coefficient of z^nin the expansion of $(e^z - 1)^k$ is the contribution to C_n of the equality classes that have exactly k numbers in their encoding. Taking the sum over all values of k, we find that C_n is $n!$ times the coefficient of z^nin

$$\sum_0^\infty (e^z - 1)^k \tag{10}$$

(the coefficient of z^n is finite for any finite n because it is zero for all terms with $k > n$).

Summing the series, we get the result

$$\frac{1}{2 - e^z} = \sum_0^\infty C_n \frac{z^n}{n!} \tag{11}$$

2. The recurrence relation

Let us look at a set of $n + 1$ items, which can be ordered in C_{n+1}ways, and let's see how to relate that to smaller sets.

Suppose that the first item is *unique*. If we remove it from the set then we get a set of n items, which can be ordered in C_n ways. However, we should be aware that this first item could have been any of the $n + 1$ items in the list. So we count $(n + 1)C_n$.

Suppose that the first item is *one of a pair*. If we remove them both from the set then we get a set of $n - 1$ items, which can be ordered in C_{n-1} ways. However, we should be aware that the first two items could have been any two items from the list: there are $n(n + 1)$ such ordered pairs, but the order of the items in the pair doesn't matter, so we count $\frac{1}{2}n(n + 1)C_{n-1}$.

Continuing this reasoning, we get

$$C_{n+1} = \binom{n+1}{1} C_n + \binom{n+1}{2} C_{n-1} + \binom{n+1}{3} C_{n-2} + ... \tag{12}$$

or

$$C_{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} C_k \tag{13}$$

which can be written symbolically as

$$C_n = (C + 1)^n - C^n \tag{14}$$

or even

$$2C_n = (C + 1)^n \tag{15}$$

which is interestingly similar to the $B_n = (B + 1)^n$of the Bernoulli numbers.

3. The Bernoulli numbers

In this section we shall denote ln 2 by $\alpha$and $z/(e^z-1)$ by $f(z)$. One of the definitions of Bernoulli numbers is that

$$f(z) \equiv \frac{z}{e^z-1} = \sum_0^\infty \frac{B_n}{n!} z^n \tag{16}$$

We have already established that

$$\sum_0^\infty C_n \frac{z^n}{n!} = \frac{1}{2-e^z} \tag{17}$$

Now

$$\begin{aligned}
\frac{1}{2-e^z} &= \frac{z-\alpha}{e^{z-\alpha}-1} \cdot \frac{1/2\alpha}{1-z/\alpha} \\
&= \frac{1}{2\alpha} \cdot \frac{1}{1-z/\alpha} \sum_0^\infty \frac{B_n}{n!}(z-\alpha)^n && \text{(definition of Bernoulli numbers)} \\
&= \frac{1}{2\alpha} \cdot \frac{1}{1-z/\alpha} \sum_{n=0}^\infty \frac{B_n}{n!} \sum_{m=0}^n (-\alpha)^m z^{n-m} \binom{n}{m} && \text{(Binomial Theorem)} \\
&= \frac{1}{2\alpha} \cdot \frac{1}{1-z/\alpha} \sum_{n=0}^\infty B_n \sum_{m=0}^n (-\alpha)^m z^{n-m} \frac{1}{m!(n-m)!}
\end{aligned}$$

Define $n' \equiv n-m$ and reverse the order of summation:

$$\frac{1}{2-e^z} = \frac{1}{2\alpha} \cdot \frac{1}{1-z/\alpha} \sum_{m=0}^\infty \frac{(-\alpha)^m}{m!} \sum_{n'=0}^\infty B_{m+n'} \frac{z^n}{n'!}$$

Expand $\frac{1}{1-z/\alpha}$ into an infinite series:

$$\frac{1}{2-e^z} = \frac{1}{2\alpha} \sum_{m=0}^\infty \frac{(-\alpha)^m}{m!} \sum_{n'=0}^\infty \sum_{j=0}^\infty \frac{z^{n'+j}}{n'!} B_{m+n'} \alpha^{-j}$$

Redefine $n \equiv n'+j$:

$$\frac{1}{2-e^z} = \frac{1}{2\alpha} \sum_{m=0}^\infty \frac{(-\alpha)^m}{m!} \sum_{n=0}^\infty \sum_{j=0}^n \frac{z^n}{(n-j)!} B_{m+n-j} \alpha^{-j}$$

Define $k \equiv n-j$:

$$\begin{aligned}
\frac{1}{2-e^z} &= \frac{1}{2\alpha} \sum_{m=0}^\infty \frac{(-\alpha)^m}{m!} \sum_{n=0}^\infty \frac{z^n}{\alpha^n} \sum_{k=0}^n \frac{B_{m+k}}{k!} \alpha^k \\
&= \frac{1}{2\alpha} \sum_{n=0}^\infty (\frac{z}{\alpha})^n \sum_{k=0}^n \frac{\alpha^k}{k!} \sum_{m=0}^\infty \frac{(-\alpha)^m}{m!} B_{m+k} && \text{(change order of summation)} \\
&= \frac{1}{2\alpha} \sum_{n=0}^\infty (\frac{z}{\alpha})^n \sum_{k=0}^n \frac{\alpha^k}{k!} f^{(k)}(-\alpha) && \text{(Taylor's theorem)} \qquad (18)
\end{aligned}$$

(for experts in series and Bernoulli numbers there is probably a two-line way of deriving all this!).

Equating coefficients of z^n, this gives us

$$\frac{\alpha^n}{n!}C_n = \frac{1}{2\alpha}\sum_{k=0}^{n}\frac{\alpha^k}{k!}f^{(k)}(-\alpha) \tag{19}$$

It follows that

$$\frac{1}{2\alpha}\frac{\alpha^n}{n!}f^{(n)}(-\alpha) = \frac{\alpha^n}{n!}C_n - \frac{\alpha^{n-1}}{(n-1)!}C_{n-1} \tag{20}$$

whence

$$\frac{1}{2n}f^{(n)}(-\alpha) = \frac{\alpha}{n}C_n - C_{n-1} \tag{21}$$

and

$$f^{(n)}(-\alpha) = 2(\alpha C_n - nC_{n-1}) \tag{22}$$

4. Asymptotic behaviour

$\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}f^{(k)}(-\alpha)$ is a Taylor series for $f(\alpha-\alpha) = f(0) = 1$. Therefore as $n \to \infty$, $\sum_{k=0}^{n}\frac{\alpha^k}{k!}f^{(k)}(-\alpha) \to 1$. So as $n \to \infty$,

$$C_n \sim \frac{n!}{2(\ln 2)^{n+1}} \tag{23}$$

And here we end, because the asymptotic behaviour of C_n is what got me interested in this question in the first place.

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