Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 26 (2009) 1483–1515

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

The symplectic structure of curves in three dimensional spaces of constant curvature and the equations of mathematical physics

V. Jurdjevic

University of Toronto, Toronto, Ontario, Canada

Received 6 November 2007; received in revised form 22 November 2008; accepted 27 December 2008

Available online 7 February 2009

Abstract

The paper defines a symplectic form on an infinite dimensional Fréchet manifold of framed curves over the three dimensional space forms. The curves over which the symplectic form is defined are called horizontal-Darboux curves. It is then shown that the projection on the Lie algebra of the Hamiltonian vector field associated with the functional $f = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$ satisfies Heisenberg's magnetic equation (HME), $\frac{\partial \Lambda}{\partial t}(s,t) = \frac{1}{i}[\Lambda(s), \frac{\partial^2 \Lambda}{\partial s^2}(s,t)]$ in the space of Hermitian matrices for the hyperbolic and the Euclidean case, and $\frac{\partial \Lambda}{\partial t}(s,t) = [\Lambda(s), \frac{\partial^2 \Lambda}{\partial s^2}(s,t)]$ in the space of skew-Hermitian matrices for the spherical case. It is then shown that the horizontal-Darboux curves are parametrized by curves in SU_2, which along the solutions of (HME) satisfy Schroedinger's non-linear equation (NSL)

$$-i\frac{\partial \psi}{\partial t}(t,s) = \frac{\partial^2 \psi}{\partial s^2}(t,s) + \frac{1}{2}\big(|\psi(t,s)|^2 + c\big)\psi(t,s)$$

It is also shown that the critical points of $\frac{1}{2}\int_0^L \kappa^2(s)\,ds$, known as the elastic curves, correspond to the soliton solutions of (NSL). Finally the paper shows that the modifed Korteweg–de Vries equation or the curve shortening equation are Hamiltonian equations generated by $f_1 = \int_0^L \kappa^2(s)\tau(s)\,ds$ and $f_2 = \int_0^L \tau(s)\,ds$ and that $f_0 = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$, f_1 and f_2 are all in involution with each other.

Keywords: Lie groups; Lie algebras; Symmetric spaces; Orthonormal frame bundles; Fréchet spaces; Symplectic forms; Hamiltonian vector fields

1. Introduction

This paper defines a symplectic form on an infinite dimensional Fréchet manifold of framed curves of fixed length over a three dimensional simply connected Riemannian manifold of constant curvature. The framed curves are anchored at the initial point and are further constrained by the condition that the tangent vector of the projected curve coincides with the first leg of the orthonormal frame. Such class of curves are called anchored Darboux curves and in particular include the Serret-Frenet framed curves.

E-mail address: jurdj@math.toronto.edu.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2008.12.006

The symplectic form ω is defined on the space of "horizontal" curves of fixed length in the universal covers of the orthonormal frame bundles of the underlying manifolds: $SL_2(C)$ for the hyperboloid $\mathbb{H}^3$ and $SU_2 \times SU_2$ for the sphere S^3. The form ω is left invariant and is induced by the Poisson–Lie bracket on the appropriate Lie algebra. More precisely, the form ω in each of the above cases is defined over the curves whose tangents take values in the Cartan space $\mathfrak{p}$ corresponding to the decomposition

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$$

of the Lie algebra $\mathfrak{g}$ subject to the usual Lie algebraic relations

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}, \qquad [\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}, \qquad [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}.$$

In the case of the hyperboloid $\mathfrak{g}$ is equal to $sl_2(C)$ and the Cartan space $\mathfrak{p}$ is equal to the space of the Hermitian matrices, while in the case of the sphere $\mathfrak{g}$ is equal to $su_2 \times su_2$ and the Cartan space is isomorphic to the space of skew-Hermitian matrices $\mathfrak{h}$. The symplectic forms in each of these two cases are isomorphic to each other as a consequence of the isomorphism between $\mathfrak{p}$ and $\mathfrak{h}$ given by $i\mathfrak{h} = \mathfrak{p}$.

The Euclidean space $\mathbb{E}^3$ is identified with $\mathfrak{p}$ equipped with the metric defined by the trace form, and its framed curves are represented in the semidirect product $\mathfrak{p} \rtimes SU_2$. The Euclidean Darboux curves inherit the hyperbolic symplectic form ω which is isomorphic to the symplectic form used by J. Millson and B. Zombro in [16].

Each group G mentioned above is a principal SU_2-bundle over the underlying symmetric space with a natural connection defined by the left invariant vector fields that take values in the Cartan space $\mathfrak{p}$. The vertical distribution is defined by the left invariant vector fields that take their values in $\mathfrak{k}$. In this setting then, anchored Darboux curves are the solutions in G of a differential equation

$$\frac{dg}{ds}(s) = g(s)\big(E_1 + u_1(s)A_1 + u(s)A_2 + u_3(s)A_3\big) \tag{1}$$

with $g(0) = I$, where E_1 is a fixed unit vector in the Cartan space $\mathfrak{p}$. The matrices A_1, A_2, A_3 denote the skew-Hermitian Pauli matrices, and $u_1(s), u_2(s), u_3(s)$ are arbitrary real valued functions on a fixed interval $[0, L]$. Each anchored Darboux curve defines a horizontal-Darboux curve $h(s) \in G$ that is a solution of the differential equation

$$\frac{dh}{ds}(s) = h(s)\Lambda(s), \qquad \Lambda(s) = R(s)E_1R^{-1}(s) \tag{2}$$

with $R(s)$ the solution curve in SU_2 of the equation

$$\frac{dR}{ds} = R(s)\big(u_1(s)A_1 + u_2(s)A_2 + u_3(s)A_3\big) \tag{3}$$

that satisfies $R(0) = I$. The symplectic form for the hyperbolic Darboux curves is given by

$$\omega_\Lambda(V_1, V_2) = \frac{1}{i}\int_0^L \big\langle \Lambda(s), \big[U_1(s), U_2(s)\big]\big\rangle ds \tag{4}$$

with $U_1(s)$ and $U_2(s)$ Hermitian matrices orthogonal to the tangent vector $\Lambda(s)$, that further satisfy $U_j(0) = 0$ and $\frac{dV_j}{ds}(s) = U_j(s)$ for $j = 1, 2$.

In the spherical case the symplectic form has the same form as in the hyperbolic case, except for the factor $\frac{1}{i}$, which is omitted. The matrices U_j in this case take values in $\mathfrak{k}$ and satisfy

$$\frac{dV_j}{ds}(s) = \big[\Lambda(s), V_j(s)\big] + U_j(s)$$

for $j = 1, 2$.

The second part of the paper is devoted to the Hamiltonian flow associated with the function

$$f\big(g(s)\big) = \frac{1}{2}\int_0^L \left\|\frac{d\Lambda}{ds}(s)\right\|^2 ds = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$$

where g denotes a frame-periodic horizontal-Darboux periodic curve, i.e., a Darboux curve for which the solution $R(s)$ of Eq. (3) is periodic. Here $\kappa(s)$ denotes the curvature of the projected curve $x(s)$ in the underlying symmetric space.

It is shown that the Hamiltonian flow induced by the symplectic form ω generates Heisenberg's magnetic equation in the Cartan space $\mathfrak{p}$ given by

$$\frac{\partial \Lambda}{\partial t}(s,t) = \frac{1}{i}\left[\Lambda(s), \frac{\partial^2 \Lambda}{\partial s^2}(s,t)\right] \tag{5}$$

in the hyperbolic and the Euclidean case, and by

$$\frac{\partial \Lambda}{\partial t}(s,t) = \left[\Lambda(s), \frac{\partial^2 \Lambda}{\partial s^2}(s,t)\right]$$

in the spherical case.

It is also shown that the corresponding matrix $R(s,t)$ defines a complex function

$$\psi(s,t) = u(s,t)\exp\left(i\int_0^s u_1(x,t)\,dx\right) \tag{6}$$

with $u(s,t) = u_2(s,t) + iu_3(s,t)$ that is a solution of the non-linear Schroedinger's equation

$$-i\frac{\partial \psi}{\partial t}(t,s) = \frac{\partial^2 \psi}{\partial s^2}(t,s) + \frac{1}{2}\left(\left|\psi(t,s)\right|^2 + c(t)\right)\psi(t,s) \tag{7}$$

where $c(t) = -|u(0,t)|^2$ (Theorem 5).

This finding clarifies a remarkable observation of H. Hasimoto [7] that the function

$$\psi(s,t) = \kappa(s,t)\exp\left(i\int_0^s \tau(x,t)\,dx\right)$$

where $\kappa(t,s)$ and $\tau(t,s)$ are the curvature and the torsion of a curve $\gamma(t,s)$ that evolves according to the filament equation

$$\frac{\partial \gamma}{\partial t}(t,s) = \kappa(t,s)B(t,s) \tag{8}$$

is a solution of the non-linear Schroedinger equation (7). Indeed, when the frame $R(s)$ in Eq. (3) is a Serret–Frenet frame then ψ given by (6) coincides with Hasimoto's function in the hyperbolic and the Euclidean case but not in the spherical case since $u_1(t) = \tau + \frac{1}{2}$.

The curves that correspond to the critical points of $f = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$ are called elastic. The material in Section 5 shows that the elastic curves with periodic curvatures always generate soliton solutions for the non-linear Schroedinger's equation. The elastic curves that generate solitons reside on a fixed energy level and propagate with the speed equal to H_1, where H_1 is a conserved quantity for the elastic problem. The fact that the equations for the heavy top form an invariant subsystem of the equations for the elastic curves makes the connection between elastic curves and solitons even more intriguing: the speed of the soliton corresponds to the angular momentum along the axis of symmetry for the top of Lagrange.

The formalism of this paper suggests that there is a class of functions $f_0, f_1, f_2, \ldots$ over the space of Darboux curves that begins with $f_0 = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$ having the property that any two functions Poisson commute. It is shown in the paper that f_1 and f_2 given by

$$f_1 = i\int_0^L \left\langle\left[\Lambda(s), \frac{d\Lambda}{ds}(s)\right], \frac{d^2\Lambda}{ds^2}(s)\right\rangle ds, \qquad f_2 = \int_0^L \left(\left\|\frac{d^2\Lambda}{dt^2}\right\|^2 - \frac{5}{4}\left\|\frac{d\Lambda}{dt}\right\|^4\right) ds$$

are in this class.

The above functions can be expressed either in terms of the geometric invariants of the underlying Darboux curve as:

$$f_1 = \int_0^L \kappa^2(s)\tau(s)\,ds, \qquad f_2 = \int_0^L \left(\frac{\partial \kappa}{\partial s}^2(s) + \kappa^2(s)\tau^2(s) - \frac{1}{4}\kappa^4(s)\right) ds$$

in which case they agree with the first three functions on the list presented by J. Langer and R. Perline in [14], or they can be expressed in terms of the complex function $u(s)$ defined by Eq. (6) as $f_0 = \frac{1}{2}\int_0^L |u(s)|^2\,ds$ and

$$f_1 = \frac{1}{2i}\int_0^L (\bar{u}\dot{u} - u\dot{\bar{u}})\,ds, \qquad f_2 = \int_0^L \left(\left|\frac{\partial u}{\partial s}(s,t)\right|^2 - \frac{1}{4}\left|u(s,t)\right|^4\right) ds$$

in which case they correspond to the first three conserved quantities, the number of particles, the momentum and the energy, in the paper by C. Shabat and V. Zakharov in [17].

The paper is organized as follows. Section 2 consists of geometric preliminaries leading up to the basic principal bundles in terms of which horizontal-Darboux curves are defined. Section 3 describes the symplectic structure for the space of horizontal-Darboux curves.

Section 4 is devoted to the Hamiltonian flow corresponding to $f_0 = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$. This section also contains a discussion of the Euclidean symplectic form and its connection to the existing results in the literature. Section 5 deals with elastic curves and the soliton solutions for the non-linear Schroedinger's equation. The final section (Section 6) contains a brief discussion of the conservation laws associated with f_0 and their connections to the hierarchies of functions presented in [17] and [14].

2. Darboux curves and their symplectic forms

2.1. Notations and geometric preliminaries

For the purposes of this paper it will be most convenient to realize the three dimensional sphere S^3 and the three dimensional hyperboloid $\mathbb{H}^3$ as subsets of $SL_2(C)$ via the identification of points $z = (z_0, z_1, z_2, z_3)$ in $\mathbb{C}^4$ with the matrices Z in $SL_2(C)$ through

$$Z = \begin{pmatrix} z_0 + iz_1 & z_2 + iz_3 \\ -z_2 + iz_3 & z_0 - iz_1 \end{pmatrix}, \quad z_0^2 + z_1^2 + z_2^2 + z_3^2 = \mathrm{Det}(Z) = 1.$$

Then $S^3 = \{x \in \mathbb{R}^4\colon x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$ is identified with matrices $X = \left(\begin{smallmatrix} u & v \\ -\bar{v} & \bar{u} \end{smallmatrix}\right)$ in SU_2 when z is restricted to $\mathbb{R}^4$ while the hyperboloid $\mathbb{H}^3 = \{x \in \mathbb{R}^4\colon x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1, x_0 > 0\}$ is identified with positive definite Hermitian matrices

$$P = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix}, \quad \mathrm{Det}(P) = 1,$$

by setting $z_0 = x_0, z_1 = -ix_1, z_2 = ix_3, z_3 = -ix_2$.

For notational convenience $SL_2(C)$ will be denoted by G and its Lie algebra by $\mathfrak{g}$. Then $\mathfrak{g}$ is the direct sum of the space of Hermitian matrices $\mathfrak{p}$ of trace zero and the subalgebra of skew-Hermitian matrices $\mathfrak{h}$, the Lie algebra of SU_2. The following relations are basic:

$$[\mathfrak{p},\mathfrak{p}] = \mathfrak{h}, \qquad [\mathfrak{p},\mathfrak{h}] = \mathfrak{p}, \qquad [\mathfrak{h},\mathfrak{h}] = \mathfrak{h}, \qquad i\mathfrak{p} = \mathfrak{h}. \tag{9}$$

Matrices

$$B_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad B_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

known as the Hermitian Pauli matrices, form a basis for $\mathfrak{p}$ while the skew-Hermitian Pauli matrices

$$A_1 = iB_1, \qquad A_2 = iB_2, \qquad A_3 = iB_3$$

form a basis for $\mathfrak{h}$. Together these matrices form a basis for $\mathfrak{g}$ and conform to the following Lie bracket table:

Table 1

$[\,,\,]$	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	$-A_3$	A_2	0	$-B_3$	B_2
A_2	A_3	0	$-A_1$	B_3	0	$-B_1$
A_3	$-A_2$	A_1	0	$-B_2$	B_1	0
B_1	0	$-B_3$	B_2	0	A_3	$-A_2$
B_2	B_3	0	$-B_1$	$-A_3$	0	A_1
B_3	$-B_2$	B_1	0	A_2	$-A_1$	0

Nota bene. In this paper the Lie bracket is defined as $[A, B] = BA - AB$.

Definition 2.1. The quadratic form on $\mathfrak{g}$ defined by $\langle A, B\rangle = 2\,\mathrm{Trace}(AB)$ will be called the trace form.

The trace form is invariant in the sense that

$$\bigl\langle A, [B, C]\bigr\rangle = \bigl\langle [A, B], C\bigr\rangle, \quad \text{and} \quad \bigl\langle gAg^*, gBg^*\bigr\rangle = \langle A, B\rangle \tag{10}$$

for any matrices A, B, C in $\mathfrak{g}$, and any g in SU_2.

It follows that

$$\langle A, B\rangle = a_1b_1 + a_2b_2 + a_3b_3 \tag{11}$$

for any Hermitian matrices $A = \sum_{i=1}^3 a_iB_i$ and $B = \sum_{i=1}^3 b_iB_i$. Since $\langle iA, iB\rangle = -\langle A, B\rangle$ similar formula holds on $\mathfrak{h}$ with the sign reversed.

Definition 2.2. The restriction of the trace form to $\mathfrak{p}$ will be denoted by $\langle,\rangle_h$ and $\langle,\rangle_s$ will denote the negative of the restriction of the trace form to $\mathfrak{h}$. Then $\|\ \|_h$ and $\|\ \|_s$ will denote the induced norms on $\mathfrak{p}$ and $\mathfrak{h}$.

It follows that Hermitian Pauli matrices form an orthonormal basis for $\mathfrak{p}$ relative to $\langle,\rangle_h$ and that skew-Hermitian Pauli matrices form an orthonormal basis on $\mathfrak{h}$ relative to $\langle,\rangle_s$.

Throughout the paper g^* denotes the Hermitian transpose of a matrix g. Then $g \in SU_2$ whenever $g^* = g^{-1}$. We now pass to the universal covers $SU_2 \times SU_2$ and $SL_2(C)$ of the orthonormal frame bundles of the sphere or the hyperboloid.

These groups will be considered principal SU_2-bundles over S^3 and $\mathbb{H}^3$ respectively via the following constructions.

For $G = SU_2 \times SU_2$ the action of SU_2 is $R(p, q) = (pR^*, qR^*)$ for each (p, q) in $SU_2 \times SU_2$ and each $R \in SU_2$, and the projection map π is given by $X = \pi(p, q) = pq^*$.

For $G = SL_2(C)$ the action is $(R, g) \to gR^*$ for all $g \in G$ and $R \in SU_2$, and the projection map π is given by $\pi(g) = gg^*$. In the material below we will rely on the notion of a connection on a principal bundle (in that context see [18]).

Definition 2.3. Curves $g(t) = (p(t), q(t))$ in $SU_2 \times SU_2$ will be called spherical horizontal if

$$p^*\frac{dp}{dt}(t) = P(t), \qquad q(t)^*\frac{q}{dt}(t) = -P(t)$$

for some curve $P(t)$ in $\mathfrak{h}$. The left invariant distribution $\mathcal{H}_s((p, q)) = \{(pP, q(-P))\colon P \in \mathfrak{h}\}$ in $SU_2 \times SU_2$ will be called the spherical connection.

Definition 2.4. Curves $g(t)$ in $SL_2(C)$ will be called hyperbolic horizontal if

$$g^{-1}(t)\frac{dg}{dt}(t) = B(t)$$

for some curve of matrices $B(t)$ in $\mathfrak{p}$. The left invariant distribution $\mathcal{H}_h(g) = \{gB\colon B \in \mathfrak{p}\}$ will be called the hyperbolic connection.

Definition 2.5.

(a) The length of any spherical horizontal curve $(g_1(t), g_2(t))$ in an interval $[0, T]$ is equal to $\int_0^T \|P(t)\|_s \, dt$.
(b) The length of a hyperbolic horizontal curve $g(t)$ in $[0, T]$ is equal to $\int_0^T \|B(t)\|_h \, dt$.

It can be easily shown that the projection $X(t) = \begin{pmatrix} x_0+ix_1 & x_2+ix_3 \\ -x_2+ix_3 & x_0-ix_1 \end{pmatrix}$ on S^3 of any spherical horizontal curve $(p(t), q(t))$ is a solution of $\frac{dX}{dt}(t) = X(t)(2q(t)P(t)q(t)^*)$ and

$$\int_0^T \sqrt{\frac{dx_0}{ds}^2 + \frac{dx_1}{ds}^2 + \frac{dx_2}{ds}^2 + \frac{dx_3}{ds}^2} \, dt = \int_0^T \|P(t)\|_s \, dt.$$

Similarly the length of a hyperbolic horizontal curve coincides with the Riemannian length

$$\int_0^T \sqrt{-\frac{dx_0}{ds}^2 + \frac{dx_1}{ds}^2 + \frac{dx_2}{ds}^2 + \frac{dx_3}{ds}^2} \, dt$$

of the projected curve $X(t) = g(t)g^*(t)$.

It can also be shown that every curve in the base manifold can be lifted to a horizontal curve and that any two liftings differ by an element in SU_2, consistent with the general theory of principal bundles.

Remark 1. The preceding paragraphs reveal that the metric induced by trace form differs by a factor of 2 from the natural metric on the base manifold inherited from either the Euclidean or the Lorentzian metric in $\mathbb{R}^4$. The present choice of the metric offers some conveniences on the level of Lie algebras that the other choice does not do. For instance, Pauli matrices A_1, A_2, A_3 form an orthonormal basis relative to the trace metric and the coordinates of the matrices in $\mathfrak{h}$ relative to the Pauli matrices satisfy the property that the Lie bracket coincides with the cross product—a fact that is important for this paper. Relative to the natural metric on the sphere, however, vectors $E_i = 2A_i$, $i = 1, 2, 3$, are orthonormal but then $[E_i, E_j] = -2E_k$ and the correspondence with the cross product is changed.

2.2. Darboux curves

It follows from above that the Riemannian metric of the base manifold is induced by the left invariant metric defined on the connection distributions in terms of the trace form. On the sphere each pair (p, q) in $SU_2 \times SU_2$ defines an orthonormal frame (v_1, v_2, v_3) at $X = pq^*$ where

$$v_1 = 2pA_1q^* = 2pq^*(qA_1q^*), \qquad v_2 = 2pA_2q^* = 2q^*(qA_2q^*), \qquad v_3 = 2pA_3q^* = 2pq^*(qA_3q^*).$$

Conversely, every orthonormal frame at a point $X \in SU_2$ can be represented by the tangent vectors $v_1 = 2XU_1$, $v_2 = 2XU_2$, $v_3 = 2XU_3$ for some matrices U_1, U_2, U_3 in $\mathfrak{h}$ that are orthonormal relative to the trace form. There are exactly two matrices $\pm q \in SU_2$ such that

$$U_1 = qA_1q^*, \qquad U_2 = qA_2q^*, \qquad U_3 = qA_3q^*.$$

Having found q, p is uniquely defined by $p = Xq$.

In the case of the hyperboloid $SO(1, 3)$ is the orthonormal frame bundle of $\mathbb{H}^3$ and $SL_2(C)$ is its double cover. We will identify each g in $SL_2(C)$ with the frame

$$v_1 = 2gB_1g^*, \qquad v_2 = 2gB_2g^*, \qquad v_3 = 2gB_3g^* \tag{12}$$

at $X = gg^*$. Conversely every orthonormal frame v_1, v_2, v_3 at a point $X \in \mathbb{H}^3$ can be identified with exactly two matrices $\pm g \in SL_2(C)$ via the above relations.

Definition 2.6. Curves $g(t)$ in the universal covers of the orthonormal frame bundle will be called framed curves. Framed curves defined on a fixed interval $[0, L]$ which define an orthonormal frame v_1, v_2, v_3 at the base curve $X(s)$ in the underlying symmetric space such that $\frac{dX}{ds} = v_1(s)$ will be called Darboux. Darboux curves which satisfy $g(0) = I$ will be called anchored.

Remark 2. Condition $\frac{dX}{ds} = v_1(s)$ implies that $X(s)$ is parametrized by arc length and therefore L is the length of $X(s)$. The fact that the orthonormal bundles are replaced by their universal covers does not matter in the subsequent exposition since all Darboux curves will be anchored.

In the spherical case anchored Darboux curves $g(s) = (p(s), q(s))$ are the solutions of

$$\frac{dp}{ds}(s) = p(s)P(s), \qquad \frac{dq}{ds}(s) = q(s)Q(s), \qquad P(s) - Q(s) = 2A_1$$

satisfying the initial conditions $p(0) = I$, $q(0) = I$. Condition $P(s) - Q(s) = A_1$ can be expressed also as

$$P(s) = U(s) + A_1, \qquad Q(s) = U(s) - A_1, \quad \text{where } U(s) = \frac{1}{2}\big(P(s) + Q(s)\big). \tag{13}$$

Definition 2.7. Anchored spherical Darboux curves are said to be reduced if the curve $U(s)$ in (13) is of the form $U(s) = \left(\begin{smallmatrix} 0 & u(s) \\ -\bar{u}(s) & 0 \end{smallmatrix}\right)$ for some complex curve $u(s)$.

Every anchored Darboux curve $(p(s), q(s))$ can be transformed into a reduced Darboux curve $(\tilde{p}(s), \tilde{q}(s))$, without altering the base curve $X(s)$, by taking $\tilde{p} = ph$, $\tilde{q} = qh$ with $h(s)$ the solution of $\frac{dh}{ds} = -h(s)\left(\begin{smallmatrix} iu_1(s) & 0 \\ 0 & -iu_1(s) \end{smallmatrix}\right)$, $h(0) = I$ where the matrix $\left(\begin{smallmatrix} iu_1(s) & 0 \\ 0 & -iu_1(s) \end{smallmatrix}\right)$ denotes the diagonal part of $U(s)$. Thus $X(s)$ can be lifted also to a reduced Darboux curve. On the other hand, reduced Darboux framed curves exclude the Serret–Frenet frames as we will see later on. The significance of these observations will become clear further on in the paper.

Definition 2.8. Curves $(p(s), q(s))$ in $SU_2 \times SU_2$ which are the solutions of

$$\frac{d}{ds}\big(p(s), q(s)\big) = \big(p(s), q(s)\big)\big(\Lambda(s), -\Lambda(s)\big), \qquad \Lambda(0) = A_1, \qquad \big\|\Lambda(s)\big\| = 1, \qquad p(0) = q(0) = I \tag{14}$$

will be called spherical horizontal-Darboux curves.

Every anchored spherical Darboux curve $g(s) = (p(s), q(s))$ can be transformed into a spherical horizontal-Darboux curve

$$\tilde{p}(s) = p(s)R^*(s), \qquad \tilde{q}(s) = q(s)R^*(s)$$

for some matrix $R(s) \in SU_2$, $R(0) = I$ without altering the projected curve $X(s) = p(s)q^*(s)$. In fact, $R(s)$ is a solution of $\frac{dR}{ds} = \frac{1}{2}R(s)(P(s) + Q(s))$, and $\tilde{p}$ and $\tilde{q}$ are the solutions of

$$\frac{d\tilde{p}}{ds} = \tilde{p}\left(\frac{1}{2}R(P - Q)R^*\right) = \tilde{p}\big(RA_1R^*\big), \qquad \frac{d\tilde{q}}{ds} = \tilde{q}\left(\frac{1}{2}R(Q - P)R^*\right) = \tilde{q}\big(-RA_1R^*\big). \tag{15}$$

Conversely, every curve $\Lambda(s) \in \mathfrak{h}$ with $\|\Lambda(s)\| = 1$, $\Lambda(0) = A_1$ can be written as $\Lambda(s) = R(s)A_1R^*(s)$ for some curve $R(s)$ in SU_2 with $R(0) = I$ because SU_2 acts transitively by conjugations on the sphere $\|\Lambda\| = 1$. The correspondence between Λ and R is not bijective: if $R_0 \to \Lambda$ then $R_0h \to \Lambda$ for any $h = \left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$, $\|z\| = 1$.

Curves $R(s)$ defined by $\Lambda(s) = R(s)A_1R^*(s)$ with $R(0) = I$ define spherical Darboux curves $(p(s), q(s))$ via the relations (13) where $U(s) = R^*(s)\frac{dR}{ds}(s)$. If the diagonal part of $U(s)$ is equal to zero then $(p(s), q(s))$ is a reduced Darboux curve. It follows that such curves set up a bijective correspondence between the horizontal-Darboux curves and the reduced Darboux curves. Thus every curve $X(s)$ parametrized by arc length on the interval $[0, L]$ with boundary conditions $X(0) = I$ and $\frac{dX}{ds}(0) = 2A_1$ can be lifted to a unique spherical horizontal-Darboux curve and also to a unique reduced anchored spherical Darboux curve.

In the subsequent exposition we will be less formal and refer to the spherical horizontal-Darboux curves as the solutions of the initial value problem

$$\frac{dp}{ds}(s) = p(s)\Lambda(s), \qquad \big\|\Lambda(s)\big\| = 1, \qquad p(0) = I, \tag{16}$$

since then the second factor $q(s)$ is defined by of $\frac{dq}{ds} = q(s)(-\Lambda(s))$, $q(0) = I$.

Definition 2.9. Spherical horizontal-Darboux curves $p(s)$ for which $\Lambda(s) = R(s)A_1R^*(s)$ for some curve $R(s) \in SU_2$ such that $R(L) = R(0) = I$ are called frame-periodic.

Remark 3. Frame-periodicity implies not only that $\Lambda(s)$ is periodic but also implies that the corresponding Darboux curve $(p(s), q(s))$ is a solution of an equation with periodic right-hand side since the matrix $U(s)$ is periodic. However, if $U(s)$ is periodic its diagonal part $D = \left(\begin{smallmatrix} iu_1 & 0 \\ 0 & -iu_1 \end{smallmatrix}\right)$ is periodic and therefore $h(s)$, the solution of $\frac{dh}{ds} = -h(s)D(s)$, $h(0) = I$ satisfies $h(L) = I$, from which it follows that the reduced Darboux curve that corresponds to the horizontal-Darboux curve has periodic right-hand side as well.

Definition 2.10. The set of anchored spherical Darboux curves will be denoted by $\mathcal{D}_s(L)$. The set of spherical horizontal-Darboux curves will be denoted by $\mathcal{HD}_s(L)$ and the set of frame-periodic horizontal curves by $\mathcal{PHD}_s(L)$.

In the case of the hyperboloid an anchored Darboux curve $g(s) \in SL_2(C)$ defines frames $v_1(s) = 2g(s)B_1g^*(s)$, $v_2 = 2g(s)B_2g^*(s)$, $v_3(s) = 2g(s)B_3g^*(s)$ over the projected curve $X(s) = g(s)g^*(s)$ such that $\frac{dX}{ds} = 2g(s)B_1g^*(s)$. It then follows that

$$\frac{dg}{ds} = g(s)\big(B_1 + A(s)\big) \tag{17}$$

for some matrix curve $A(s)$ in $\mathfrak{h}$ for the following reasons:

If $\frac{dg}{ds} = g(s)(B(s) + A(s))$ with $B(s) \in \mathfrak{p}$ and $A(s) \in \mathfrak{h}$, and if $\tilde{g}(s) = g(s)R^{-1}(s)$ for some $R(s) \in SU_2$ then both g and $\tilde{g}$ project onto the same curve $X(s)$. In particular if $\frac{dR}{ds} = R(s)A(s)$ then $\frac{d\tilde{g}}{ds} = \tilde{g}(s)(R(s)B(s)R^*(s))$. Hence

$$\frac{dX}{ds} = 2\tilde{g}(s)\big(R(s)B(s)R^*(s)\big)\tilde{g}^*(s) = 2g(s)B(s)g^*(s) = 2g(s)B_1g^*(s),$$

and therefore $B(s) = B_1$.

Similar to the spherical case, hyperbolic Darboux curves for which the diagonal part of the matrix A is equal to zero will be called reduced. It follows that any base curve $X(s)$ of an anchored Darboux curve is initially fixed at $X(0) = I$ and has a fixed initial tangent vector $\frac{dX}{ds}(0) = 2B_1$. Furthermore, it follows from above that $X(s)$ is the projection of a horizontal curve $\tilde{g}(s)$ such that

$$\tilde{g}^*\frac{d\tilde{g}}{ds}(s) = \Lambda(s) = R(s)B_1R^*(s)$$

for some curve $R(s)$ in SU_2.

Definition 2.11. Hyperbolic horizontal curves $g(s)$ will be called hyperbolic horizontal-Darboux if $g(0) = I$ and

$$g^{-1}(s)\frac{dg}{ds}(s) = \Lambda(s), \qquad \Lambda(s) \in \mathfrak{p}, \qquad \big\|\Lambda(s)\big\|_h = 1, \qquad \Lambda(0) = B_1. \tag{18}$$

It follows that every curve $X(s)$ on the hyperboloid parametrized by arc length on $[0, L]$ that satisfies $X(0) = I$ and $\frac{dX}{ds}(0) = 2B_1$ is the projection of a unique hyperbolic horizontal-Darboux curve $g(s)$. Moreover, the relation $\Lambda(s) = R(s)B_1R^*(s)$, $R(0) = I$ defines an anchored hyperbolic curve $\tilde{g} = gR$ over X. As in the spherical case, the correspondence between hyperbolic horizontal-Darboux curves and reduced hyperbolic anchored Darboux curves is bijective.

It follows from above that the horizontal-Darboux curves in both the spherical and the hyperbolic case are parametrized by matrices $R(s)$ in SU_2 which are solutions of $\frac{dR}{ds} = R(s)U(s)$, $R(0) = I$, with $U(s) = \left(\begin{smallmatrix} 0 & u(s) \\ -\bar{u}(s) & 0 \end{smallmatrix}\right)$ for some complex curve $u(s)$ through the relations

$$\frac{dg}{ds} = \Lambda(s) = R(s)CR^*(s), \qquad R(0) = I \tag{19}$$

where $C = A_1$ in the spherical case and $C = B_1$ in the hyperbolic case.

Definition 2.12. Hyperbolic horizontal-Darboux curves $g(s)$ are frame-periodic if $\Lambda(s)$ in (2.11) satisfies $\Lambda(s) = R(s)B_1R^*(s)$ for some curve $R(s) \in \mathfrak{h}$ such that $R(0) = R(L) = I$.

Definition 2.13. The space of all anchored hyperbolic Darboux curves will be denoted by $\mathcal{D}_h(L)$, the space of hyperbolic horizontal-Darboux, respectively frame-periodic hyperbolic horizontal-Darboux curves will be denoted by $\mathcal{HD}_h(L)$ and $\mathcal{PHD}_h(L)$.

In both the spherical and the hyperbolic case frame-periodicity implies that the matrix $U(s) = R(s)^* \frac{dR}{ds}$ is smoothly periodic. The same applies to the matrix $\Lambda(s) = R(s)A_1R^*(s)$ (respectively $\Lambda(s) = R(s)B_1R^*(s)$). This implies that the projections of frame-periodic curves necessarily have periodic curvature and torsion, but need not be closed.

Conversely, all smoothly periodic curves have periodic curvature and torsion. However, it might not be true that smooth periodic curves in the base space lift to frame periodic curves in the orthonormal frame bundle.

3. Darboux curves as Fréchet manifolds

On the basis of the general theory developed in [6] each space of anchored or frame-periodic Darboux curves and their horizontal projections can be considered as an infinite-dimensional Fréchet manifold. Recall that a topological Hausdorff vector space V is called a Fréchet space if its topology is induced by a countable family of semi-norms p_n, and if it is complete relative to the semi-norms in $\{p_n\}$. A Fréchet manifold is defined as follows:

Definition 3.1. A Fréchet manifold is a topological Hausdorff space equipped with an atlas whose charts take values in open subsets of a Fréchet space V such that any change of coordinate charts is smooth.

The paper of R.S. Hamilton [6] singles out an important class of Fréchet manifolds, called tame, in which the implicit function theorem is true. One of the main theorems in [6] is that the set of smooth mappings from a compact interval into a finite-dimensional Riemannian manifold M is a tame Fréchet manifold. It therefore follows from the implicit function theorem that closed subsets of tame Fréchet manifolds $\mathcal{M}$, defined by the zero sets of finitely many smooth functions on $\mathcal{M}$ are tame sub-manifolds of $\mathcal{M}$. Since the anchored Darboux curves are particular cases of the above situation, it follows that each of them is a tame Fréchet manifold and the same applies to their horizontal projections. Tangent vectors and tangent bundles of Fréchet manifolds are defined in the same manner as for finite dimensional manifolds. In particular tangent vectors at a point x in a Fréchet manifold $\mathcal{M}$ are the equivalence classes of curves $\sigma(t)$ in $\mathcal{M}$ all emanating from x (i.e., $\sigma(0) = x$), and all having the same tangent vector $\frac{d\sigma}{dt}(0)$ in each equivalence class. The set of all tangent vectors at x denoted by $T_x\mathcal{M}$ constitutes the tangent space at x.

The tangent bundle of a Fréchet manifold $\mathcal{M}$ is a Fréchet manifold. A vector field X on $\mathcal{M}$ is a smooth mapping from $\mathcal{M}$ into the tangent bundle $T\mathcal{M}$ such that $X(x) \in T_x\mathcal{M}$ for each $x \in \mathcal{M}$. On tame Fréchet manifolds vector fields can be defined as derivations in the space of smooth functions on $\mathcal{M}$.

3.0.1. Tangent spaces for horizontal-Darboux curves

The calculations in this section make use of covariant derivatives which are recalled below for reader's convenience.

Definition 3.2.

(a) The covariant derivative of a curve of tangent vectors $v(s) = X(s)U(s)$ along a curve $X(s)$ in SU_2 is given by

$$\frac{D_X}{ds}(v)(s) = X(s)\left(\frac{dU}{ds} + \frac{1}{2}\big[U(s), \Lambda(s)\big]\right) \tag{20}$$

where $\Lambda(s) = X^*(s)\frac{dX}{ds}(s)$.

(b) The covariant derivative $\frac{D_g}{ds}(v)$ of a curve of tangent vectors $v(s) = g(s)U(s)$, $U(s) \in \mathfrak{p}$, along a horizontal curve $g(s)$ in $SL_2(C)$, is defined by

$$\frac{D_g}{ds}(v)(s) = g(s)\frac{dU}{ds}(s) \tag{21}$$

for all $s \in [0, L]$.

The reader can easily verify that the covariant derivative on SU_2 is equal to the orthogonal projection of the ordinary derivative in $\mathbb{R}^4$ onto the tangent space of the sphere when the sphere is considered a submanifold of $\mathbb{R}^4$. On the hyperboloid, however, the notion of covariant derivative for vectors in the horizontal distribution $\mathcal{H}_h$ coincides with the usual notion of covariant derivative in the base manifold $\mathbb{H}^3$ in the sense that

$$\frac{D_{\pi(g)}}{ds}\big(\pi_*(gV)\big)(s) = \pi_*\left(\left(g(s)\frac{dV}{ds}\right)\right)(s).$$

The subsequent material also makes use of the following

Lemma 1. *Suppose that $X(s,t)$ is a field of curves in SU_2 with its infinitesimal directions*

$$A(s,t) = X^*(s,t)\frac{\partial X}{\partial s}(s,t) \quad \textit{and} \quad B(s,t) = X^*(s,t)\frac{\partial X}{\partial t}(s,t).$$

Then

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial s} + [A,B] = 0. \tag{22}$$

Proof. On any Riemannian manifold $\frac{D_X}{ds}(\frac{\partial X}{\partial t}) = \frac{D_X}{ds}(\frac{\partial X}{\partial s})$. Hence,

$$X\left(\frac{\partial B}{\partial s} + \frac{1}{2}[B,A]\right) = X\left(\frac{\partial A}{\partial t} + \frac{1}{2}[A,B]\right)$$

and therefore,

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial s} + [A,B] = 0. \qquad \square$$

Eq. (22) is also known as the zero-curvature equation [5].

Theorem 1.

(a) *The tangent space $T_p(\mathcal{HD}_s)(L)$ at a spherical horizontal-Darboux curve $p(s)$ with $\Lambda(s) = p^*(s)\frac{dp}{ds}(s)$ consists of curves $v(s) = X(s)V(s)$ with $V(s)$ the solution of*

$$\frac{dV}{ds}(s) = \big[\Lambda(s), V(s)\big] + U(s) \tag{23}$$

such that $V(0) = 0$, where $U(s)$ is a curve in $\mathfrak{h}$ subject to the conditions that $U(0) = 0$ and $\langle \Lambda(s), U(s)\rangle_s = 0$.

(b) *Tangent vectors $v(s) = X(s)V(s)$ at frame-periodic horizontal-Darboux curves $X(s)$ are generated by smoothly periodic curves $U(s)$ whose period is equal to L.*

Proof. Let $Y(s,t)$ denote a family of anchored horizontal-Darboux curves such that $Y(s,0) = p(s)$. Then, $v(s) = \frac{\partial Y}{\partial t}(s,t)_{t=0}$ is a tangent vector at $X(s)$ for which $v(0) = 0$ since the curves $Y(s,t)$ are anchored.

Let $Z(s,t)$ and $W(s,t)$ denote the matrices defined by

$$Z(s,t) = Y(s,t)^*\frac{\partial Y}{\partial s}(s,t), \qquad W(s,t) = Y(s,t)^*\frac{\partial Y}{\partial t}(s,t).$$

It follows that $\Lambda(s) = Z(s,0)$, $V(s) = W(s,0)$. Equation

$$\frac{\partial Z}{\partial t} - \frac{\partial W}{\partial s} + [Z,W] = 0$$

for $t = 0$ reduces to

$$\frac{dV}{ds}(s) = \big[\Lambda(s), V(s)\big] + U(s)$$

where $U(s) = \frac{\partial Z}{\partial t}(s,0)$.

Since the curves $s \to Y(s,t)$ are Darboux for each t, $\langle Z(s,t), Z(s,t)\rangle_s = 1$ and $Z(0,t) = A_1$. Therefore,

$$\left\langle Z(s,t), \frac{\partial Z}{\partial t}(s,t)\right\rangle = 0, \quad \text{and} \quad \frac{\partial Z}{\partial t}(0,t) = 0$$

which implies that $\langle \Lambda(s), U(s)\rangle_s = 0$ and $U(0) = 0$.

It remains to show that any curve $V(s)$ in $\mathfrak{h}$ that satisfies (23) can be realized by the perturbations $Y(s,t)$ used above. So assume that $V(s)$ be any solution of (23) generated by a curve $U(s)$ with $U(0) = 0$ that satisfies $\langle \Lambda(s), U(s)\rangle_s = 0$.

Let $\phi(t)$ denote any smooth function such that $\phi(0) = 0$ and $\frac{d\phi}{dt}(0) = 1$. Define

$$Z(s,t) = \frac{1}{1 + \phi^2(t)\langle U(s), U(s)\rangle_s}\big(\Lambda(s) + \phi(t)U(s)\big).$$

Evidently $Z(0,t) = A_1$ for all t, and an easy calculation shows that $\langle Z(s,t), Z(s,t)\rangle_s = 1$. Therefore $Y(s,t)$, the solution of

$$\frac{\partial Y}{\partial s}(s,t) = Y(s,t)Z(s,t)$$

with $Y(0,t) = I$ corresponds to an anchored horizontal-Darboux curve for each t. Since $U(s) = \frac{\partial Z}{\partial t}(s,0)$ our proof of part (a) is finished.

To prove part (b) assume that the curves $Y(s,t)$ used above belong to $\mathcal{PHD}_s(L)$. Then, $s \to Z(s,t)$ are L-periodic for each t, and therefore, $U(s) = \frac{\partial Z}{\partial t}(s,0)$ is also L-periodic. □

Tangent spaces of $\mathcal{HD}_h(L)$ and $\mathcal{PHD}_h(L)$, obtained along similar lines as in the spherical case, are described by the following

Theorem 2.

(a) *The tangent space $T_g(\mathcal{HD}_h)(L)$ at a hyperbolic horizontal-Darboux curve $g(s)$ with $\Lambda(s) = g^{-1}(s)\frac{dg}{ds}(s)$ consists of curves $v(s) = g(s)V(s)$ such that*

$$\frac{dV}{ds} = U(s), \qquad V(0) = 0, \tag{24}$$

where $U(s)$ is a Hermitian curve subject to the following conditions:

$$U(0) = 0, \qquad \big\langle \Lambda(s), U(s)\big\rangle_h = 0.$$

(b) *For frame-periodic horizontal-Darboux curves the curve $\frac{dV}{ds}(s)$ must be smoothly periodic having the period equal to L.*

Proof. Let $h(s,t)$ denote a family of anchored horizontal-Darboux curves such that $h(s,0) = g(s)$. Then $v(s) = \frac{\partial h}{\partial t}(s,t)_{t=0}$ is a tangent vector at $g(s)$ such that $v(0) = 0$ since the curves $h(s,t)$ are anchored.

Let $Z(s,t)$ and $W(s,t)$ denote the matrices defined by

$$Z(s,t) = h(s,t)^{-1}\frac{\partial h}{\partial s}(s,t), \qquad W(s,t) = h(s,t)^{-1}\frac{\partial h}{\partial t}(s,t).$$

It follows that $\Lambda(s) = Z(s,0)$ and $v(s) = g(s)V(s)$ with $V(s) = W(s,0)$. Then

$$\frac{\partial Z}{\partial t}(s,t) = \frac{\partial W}{\partial s}(s,t)$$

is the hyperbolic analogue of the zero-curvature equation (22). For $t = 0$ the above equation reduces to

$$\frac{dV}{ds}(s) = \frac{\partial W}{\partial s}(s,0) = U(s),$$

where $U(s) = \frac{\partial Z}{\partial t}(s,0)$. Then

$$\big\langle Z(s,t), Z(s,t)\big\rangle_h = 1, \quad \text{and} \quad Z(0,t) = B_1$$

imply that $\langle \Lambda(s), U(s)\rangle_h = 0$ and $U(0) = 0$.

Conversely any curve $V(s)$ in $\mathfrak{h}$ that satisfies (24) can be realized by the perturbations $h(s,t)$ defined in the first part of the proof. The argument is the same as in the spherical case and will be omitted. The same applies to the proof of part (b). $\square$

3.1. The symplectic structure of horizontal-Darboux curves

The basic notions of symplectic geometry of infinite-dimensional Fréchet manifolds are defined through differential forms in the same manner as for the finite-dimensional situations. In particular, differential forms ω of degree n are mappings

$$\omega : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{n} \to C^\infty(\mathcal{M})$$

that are $C^\infty(\mathcal{M})$ multilinear and skew-symmetric. Here $\mathcal{X}(M)$ denotes the space of all smooth vector fields on $\mathcal{M}$.

Definition 3.3. The exterior derivative $d\omega$ of a form of degree n is a differential form of degree $n+1$ defined by

$$\begin{aligned} d\omega(X_1,\ldots,X_{n+1}) = {} & \sum_{i=1}^{n+1}(-1)^{i+1}X_i\big(\omega(X_1,\ldots,\hat{X}_i,\ldots,X_n)\big) \\ & - \sum_{i<j}(-1)^{i+j}\omega\big([X_i,X_j],\ldots,\hat{X}_i,\ldots,\hat{X}_j,X_{n+1}\big) \end{aligned}$$

where the roof sign above an entry indicates its absence from the expression (i.e., $w(\hat{X}_1, X_2) = w(X_2)$ and $w(X_1, \hat{X}_2) = w(X_1)$).

A differential form ω is said to be closed if its exterior derivative $d\omega$ is equal to zero.

Definition 3.4. A differential form ω of degree 2 is said to be symplectic whenever it is closed and non-degenerate, in the sense that the induced form $(i_X\omega)(Y) = \omega(X,Y)$ is non-zero for each non-zero vector field X.

The differential df of a smooth function f is a form of degree 1 defined by $df(v) = \frac{d}{dt} f \circ \sigma(t)|_{t=0}$ for any smooth curve in $\mathcal{M}$ such that $\sigma(0) = x$, and $\frac{d\sigma}{dt}(0) = v$.

In finite dimensional symplectic manifolds with a symplectic form ω there is a unique vector field X_f such that $df = i_{X_f}\omega$. X_f is called the Hamiltonian vector field induced by f, and f is called the Hamiltonian of X_f. However, in infinite dimensional manifolds it may happen that the form df is not equal to $i_X w$ for any $X \in \mathcal{X}(M)$. This is due to the fact that the cotangent bundle of an infinite dimensional Fréchet space is never a Fréchet manifold. Nevertheless,

Definition 3.5. A vector field X is said to be Hamiltonian if there exists a smooth function f such that

$$df(Y) = \omega(X,Y)$$

for all vector fields Y on $\mathcal{M}$. The dependence of X on f shall be noted explicitly by X_f.

The manifold consisting of horizontal-Darboux curves admits a natural 2-form ω which will be defined first for spherical horizontal-Darboux curves. In the process it will become clear how to adapt the results to hyperbolic horizontal-Darboux curves. Let $v_1(s) = X(s)V_1(s)$ and $v_2(s) = X(s)V_2(s)$ denote any tangent vectors at a horizontal-Darboux curve $p(s)$ that is defined by $\frac{dp}{ds}(s) = p(s)\Lambda(s)$. According to (23) there exist unique curves $U_1(s)$ and $U_2(s)$ such that

$$U_i(0) = 0, \qquad \big\langle \Lambda(s), U_i(s)\big\rangle_s = 0$$

and

$$U_i(s) = \frac{dV_i}{ds}(s) - \left[\Lambda(s), V_i(s)\right], \quad i = 1, 2. \tag{25}$$

Then ω is given by

$$\omega_\Lambda(V_1, V_2) = -\int_0^L \left\langle \Lambda(s), \left[U_1(s), U_2(s)\right]\right\rangle_s ds. \tag{26}$$

Remark. As in finite dimensional situations the choice of sign is a matter of convention. The justification for the above choice of sign will be given later in the paper.

Theorem 3. *Both $\mathcal{HD}_s(L)$ and $\mathcal{PHD}_s(L)$ are symplectic Fréchet manifolds relative to ω defined by* (26).

The following lemmas will be useful for the proof of the theorem.

Lemma 2.

$$\left[A, [B, C]\right] = \langle A, C\rangle_s B - \langle A, B\rangle_s C$$

for any elements A, B, C in $\mathfrak{h}$.

We leave the proof to the reader.

Lemma 3. *Suppose that $v(s) = g(s)V(s)$ is a tangent vector at a horizontal-Darboux curve $g(s)$. Let $U(s)$ be defined by* (23). *Then there exists a curve $C(s)$ in $\mathfrak{k}$ such that*

$$U(s) = \left[\Lambda(s), C(s)\right].$$

Proof. The mapping $C \to [\Lambda(s), C(s)]$ restricted to the orthogonal complement of $\Lambda(s)$ is surjective. Since $U(s)$ is orthogonal to $\Lambda(s)$ the proof follows. □

Proof of the theorem. The proof is the same for each of $\mathcal{HD}_s(L)$ and $\mathcal{PHD}_s(L)$ and will be presented formally without any reference to the underlying space.

Evidently ω is skew-symmetric. To show that it is non-degenerate, assume that $\omega_\Lambda(V_1, V) = 0$ for all tangent vectors gV. Let $U_1(s)$ correspond to $V_1(s)$ defined by Eqs. (25). Then $U(s) = [\Lambda(s), U_1(s)]$ satisfies $U(0) = 0$, and $\langle \Lambda(s), U(s)\rangle_s = 0$. Therefore the corresponding vector gV with V the solution of Eq. (25) belongs to the tangent space at g. It follows from Lemma 2 that

$$\left[U_1, [\Lambda, U_1]\right] = \langle U_1, U_1\rangle_s \Lambda = \|U_1\|_s^2 \Lambda.$$

Therefore,

$$\left\langle \Lambda(s), \left[U_1(s), U(s)\right]\right\rangle_s = \left\|\Lambda(s)\right\|_s^2 \left\|U_1(s)\right\|_s^2 = \left\|U_1(s)\right\|_s^2$$

which implies that $U_1(s) = 0$ since $0 = \omega_\Lambda(V_1, V) = \int_0^L \|U_1(s)\|^2 ds$. But then (25) implies that $V_1(s) = 0$. Hence, ω is non-degenerate.

To show that ω is closed let $v_i(s) = g(s)V_i(s)$, $1 \leqslant i \leqslant 3$ denote any three tangent vectors at a fixed Darboux curve $g(s)$. It is required to show (Definition 3.3) that

$$d\omega(X_1, X_2, X_3) = \sum_{\text{cyclic}} X_i\big(\omega(X_j, X_k)\big) + \sum_{\text{cyclic}} \omega\big([X_i, X_j], X_k\big) = 0 \tag{27}$$

where X_i denote any vector fields such that $X_i(g) = v_i$ for each $i = 1, 2, 3$.

Let $X_i(z) = zZ_i$, $i = 1, 2, 3$ denote vector fields over Darboux base curves z with Z_i the solutions of

$$\frac{dZ_i}{ds}(s) = \left[\Lambda_z(s), Z_i(s)\right] + U_i(s) \quad \text{where } \Lambda_z = z^* \frac{dz}{ds}.$$

Let $C_i(s)$ denote the curves such that $U_i(s) = [\Lambda(s), C_i(s)]$ (Lemma 3). Then, $\frac{dV_i}{ds} = [\Lambda, V_i + C_i]$, and an easy calculation based on Jacobi's identity yields

$$\frac{d}{ds}\big([V_i, V_j]\big) = \big[\Lambda, [V_i, V_j]\big] + \big[[\Lambda, C_i], V_j\big] + \big[V_i, [\Lambda, C_j]\big].$$

Therefore,

$$\begin{aligned}\sum_{\text{cyclic}} \omega\big([X_i, X_j], X_k\big) &= \sum_{\text{cyclic}} \int_0^L \big\langle \Lambda, \big[\big[[\Lambda, C_i], V_j\big] + \big[V_i, [\Lambda, C_j]\big], [\Lambda, C_k]\big]\big\rangle_s ds \\ &= -\sum_{\text{cyclic}} \int_0^L \big\langle \Lambda, \big[\big(\langle V_i, \Lambda\rangle_s C_j - \langle V_j, \Lambda\rangle_s C_i\big) + \big(\langle V_j, C_i\rangle_s \Lambda - \langle V_i, C_j\rangle_s \Lambda\big), [\Lambda, C_k]\big]\big\rangle_s ds \\ &= \sum_{\text{cyclic}} \int_0^L \langle V_j, \Lambda\rangle_s \langle C_i, C_k\rangle_s - \langle V_i, \Lambda\rangle_s \langle C_j, C_k\rangle_s \, ds = 0.\end{aligned}$$

The calculations involving $X_i(\omega(X_j, X_k))$ in (27) require additional notations. Let $t \to z_i(s,t)$ denote the integral curves of the vector field X_i that originate at $g(s)$ for $t = 0$, and let

$$\frac{\partial z_i}{\partial t}(s,t) = z_i(s,t) Z_i\big(z_i(s,t)\big) \quad \text{and} \quad \frac{\partial z_i}{\partial s}(s,t) = z_i(s,t) \Lambda_i\big(z_i(s,t)\big).$$

For simplicity of notation let $Z_i(z_i(s,t))$ and $\Lambda_i(z_i(s,t))$ be denoted by $Z_i(s,t)$ and $\Lambda_i(s,t)$ Then

$$\frac{\partial \Lambda_i}{\partial t} - \frac{\partial Z_i}{\partial s} + [\Lambda_i, Z_i] = 0, \tag{28}$$

which at $t = 0$ reduce to

$$U_i - \frac{dV_i}{ds} + [\Lambda, V_i] = 0.$$

As in the preceding calculation, U_i will be represented by $U_i = [\Lambda, C_i]$.

Then,

$$\begin{aligned}X_i\big(\omega(X_j, X_k)\big) &= \frac{\partial}{\partial t} \int_0^L \big\langle \Lambda_i(s,t), \big[[\Lambda_j(s,t), C_j], [\Lambda_k(s,t), C_k]\big]\big\rangle_s ds\Big|_{t=0} \\ &= \int_0^L \left\langle \frac{\partial \Lambda_i}{\partial t}(s,t), \big[[\Lambda_j(s,t), C_j], [\Lambda_k(s,t), C_k]\big]\right\rangle_s ds\Bigg|_{t=0} \\ &\quad + \int_0^L \left\langle \Lambda_i(s,t), \left[\left[\frac{\partial \Lambda_j}{\partial t}(s,t), C_j\right], [\Lambda_k(s,t), C_k]\right]\right\rangle_s ds\Big|_{t=0} \\ &\quad + \int_0^L \left\langle \Lambda_i(s,t), \left[[\Lambda_j(s,t), C_j], \left[\frac{\partial \Lambda_k}{\partial t}(s,t), C_k\right]\right]\right\rangle_s ds\Bigg|_{t=0} \\ &= \int_0^L \big\langle U_i, [U_j, U_k]\big\rangle_s ds + \int_0^L \big\langle \Lambda, \big(\big[[U_j, C_j], [\Lambda, C_k]\big] + \big[[\Lambda, C_j], [U_k, C_k]\big]\big)\big\rangle_s ds.\end{aligned}$$

The first integral $\int_0^L \langle U_i, [U_j, U_k]\rangle_s \, ds$ is equal to zero because $\langle U_i, [U_j, U_k]\rangle_s$ is the volume of the parallelepiped with sides U_i, U_j, U_k each of which is in the plane orthogonal to Λ.

The second integral $\int_0^L \langle \Lambda, ([[U_j, C_j], [\Lambda, C_k]] + [[\Lambda, C_j], [U_k, C_k]])\rangle_s \, ds$ is also equal to zero because

$$\begin{aligned}
&\big\langle \Lambda, \big(\big[[U_j, C_j], [\Lambda, C_k]\big] + \big[[\Lambda, C_j], [U_k, C_k]\big]\big)\big\rangle_s \\
&\quad = -\big\langle \Lambda, \big(\big\langle [U_j, C_j], \Lambda\big\rangle_s C_k - \big\langle [U_j, C_j], C_k\big\rangle_s \Lambda - \big\langle [U_k, C_k], \Lambda\big\rangle_s C_j - \big\langle [U_k, C_k], C_j\big\rangle_s \Lambda\big)\big\rangle_s \\
&\quad = \big\langle [U_k, C_k], C_j\big\rangle_s - \big\langle [U_j, C_j], C_k\big\rangle_s.
\end{aligned}$$

Since U_k, C_k, C_j are all in the plane orthogonal to Λ the preceding expression is equal to zero.

Thus ω is closed, and hence symplectic. □

Corollary 1. *The 2-form ω defined on the space of anchored hyperbolic horizontal-Darboux curves given by*

$$\omega_\Lambda(V_1, V_2) = \frac{1}{i} \int_0^L \left\langle \Lambda(s), \left[\frac{DV_1}{ds}(s), \frac{dV_2}{ds}(s)\right]\right\rangle_h ds \tag{29}$$

for any tangent vectors $V_2(s), V_2(s)$ at a horizontal curve $g(s)$ is symplectic.

Proof. Recall that tangent vectors at a horizontal curve $g(s)$, defined by Theorem 2, are of the form $v(s) = g(s)V(s)$ with $V(s)$ a Hermitian curve that satisfies the following conditions:

$$V(0) = 0, \qquad \frac{dV}{ds}(0) = 0, \qquad \left\langle \Lambda(s), \frac{dV}{ds}(s)\right\rangle_h = 0.$$

Let

$$\tilde{\Lambda} = i\Lambda, \qquad \widetilde{U_1} = i\frac{dV_1}{ds}, \qquad \widetilde{U_2} = i\frac{dV_2}{ds}.$$

Since $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{h}$ and $i\mathfrak{p} = \mathfrak{h}$, matrices $\tilde{\Lambda}$, $\widetilde{U_1}$, $\widetilde{U_2}$ belong to $\mathfrak{h}$ and satisfy $\langle \tilde{\Lambda}, \widetilde{U_i}\rangle_h = 0$ for $i = 1, 2$. Therefore,

$$\begin{aligned}
\omega_\Lambda(V_1, V_2) &= \frac{1}{i} \int_0^L \left\langle \Lambda(s), \left[\frac{dV_1}{ds}(s), \frac{dV_2}{ds}(s)\right]\right\rangle_h ds \\
&= -\int_0^L \big\langle \tilde{\Lambda}(s), \big[\widetilde{U_1}(s), \widetilde{U_2}(s)\big]\big\rangle_s ds
\end{aligned}$$

coincides with the form given in Definition 26. □

The isomorphism $i\mathfrak{p} = \mathfrak{h}$, apart from justifying the choice of sign in (26), also makes transparent the proof of the following theorem.

Theorem 4. *Both $\mathcal{HD}_h(L)$ and $\mathcal{PHD}_h(L)$ are symplectic manifolds relative to ω defined by Corollary* 1.

Remark 4. It may be appropriate to point out that both the spherical and the hyperbolic symplectic form defined above are isomorphic to the symplectic structure of anchored loops on the sphere S^2 given explicitly by

$$\omega_\lambda(u_1, u_2) = \int \lambda(s) \cdot \big(u_1(s) \times u_2(s)\big)\, ds$$

where $u_1(s)$ and $u_2(s)$ are tangent vectors at $\lambda(s)$ on S^2. For when λ, $u_1$and u_2 are identified with coordinates vectors of Λ, U_1 and U_2 relative to the Pauli matrices, then the coordinate vector of $[U_1, U_2]$ is given by the cross product $u_1 \times u_2$. Therefore,

$$\int \big\langle \Lambda, [U_1, U_2]\big\rangle_s ds = \int \lambda(s) \cdot \big(u_1(s) \times u_2(s)\big)\, ds.$$

In each of the two cases the right action of SU_2 extends to the space of anchored horizontal-Darboux curves with $(g,a) \to ag(s)a^*$ for each horizontal curve $g(s)$ and each $a \in SU_2$. This action is symplectic relative to the forms used in this paper and $J(g) = \int_0^L \Lambda(s)\,ds$ is the moment map associated with this action (under the implicit assumption that the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is identified with $\mathfrak{g}$ via the trace form).

The moment map induces a function $J_A(g) = \int_0^L \langle \Lambda(s), A\rangle\,ds$ on the space of horizontal anchored curves for each element $A \in \mathfrak{g}$. The Hamiltonian vector field induced by this function coincides with the infinitesimal generator of the action-induced one-parameter group of transformations $\{e^{tA}\gamma e^{-tA}\}$. Then it is well known [1] that J is an integral of motion for each Hamiltonian function which is invariant under the action.

The moment map will be taken up again in the problems of mathematical physics further down in the text. There is another symplectic form on the space of anchored curves given by the following expression:

$$\Omega_\Lambda(V_1,V_2) = \int_0^L \big\langle \Lambda, [V_1,V_2]\big\rangle\,ds.$$

Such a form is mentioned elsewhere in the literature (see for instance [2,3,14]). This form is compatible with the form used in this paper and can be used to get the integrability results for systems which are bi-Hamiltonian [15]. However, such investigations seem too particular for the scope of this paper and will not be pursued here.

4. The Hamiltonian flow of $\frac{1}{2}\int_0^L k^2(s)\,ds$

Recall now that the geodesic curvature $\kappa(s)$ of a curve $x(s)$ on a Riemannian manifold M is defined by $\|\frac{D_x}{ds}(\frac{dx}{ds})\|$ provided that $\|\frac{dx}{ds}\| = 1$. Spherical curves which are the projections of Darboux curves are parametrized by arc length and

$$\frac{D_X}{ds}\left(\frac{dX}{ds}\right)(s) = 2X(s)\left(\frac{d\Lambda}{ds} + \frac{1}{2}\big[\Lambda(s),\Lambda(s)\big]\right) = 2X(s)\frac{d\Lambda}{ds}.$$

Therefore, $\kappa(s) = \|\frac{d\Lambda}{ds}\|_s$. In the hyperbolic case $\kappa(s) = \|\frac{d\Lambda}{ds}\|_h$ where $\Lambda(s)$ is the matrix that defines the horizontal lift of a base curve $X(s)$. Since every curve in the base manifold with proper initial conditions can be lifted to a horizontal-Darboux curve the above formulas are valid for any base curve.

Consider now any curve $R(s)$ in SU_2 that satisfies either $\Lambda(s) = R(s)A_1R^*(s)$ or $\Lambda(s) = R(s)B_1R^*(s)$. Each such $R(s)$ defines an anchored Darboux curve over $X(s)$ provided that $R(0) = I$. Suppose that

$$A(s) = R^*(s)\frac{dR}{ds} = u_1(s)A_1 + u_2(s)A_2 + u_3(s)A_3.$$

The corresponding Darboux curve is reduced precisely when $u_1 = 0$. In the spherical case

$$\frac{d\Lambda}{ds} = \frac{d}{ds}\big(R(s)A_1R^*(s)\big) = u_3R(s)A_2R^*(s) - u_2R(s)A_3R^*(s),$$

as can be easily read from Table 1, and therefore

$$\kappa^2(s) = u_2^2(s) + u_3^2(s). \tag{30}$$

This formula remains unchanged in the hyperbolic case as can be verified by an analogous argument.

The expression for the torsion τ associated with $X(s)$ depends on the choice of the frame above X. Recall now that the Serret–Frenet frame $v_1(s), v_2(s), v_3(s)$ along a curve $X(s)$ is defined through the following relations:

$$\frac{dX}{ds} = v_1, \qquad \frac{D_X}{ds}(v_1) = \kappa v_2, \qquad \frac{D_X}{ds}(v_2) = -\kappa v_1 + \tau v_3, \qquad \frac{D_X}{ds}(v_3) = -\tau v_2.$$

In the case that $R(s)$ defines the Serret–Frenet frame, $u_2 = 0$, $u_1 + \frac{1}{2} = \tau$ on the sphere and $u_1 = \tau$ on the hyperboloid, and $u_3 = \kappa$. These details will be left to the reader.

Consider now the functional

$$\frac{1}{2}\int_0^L k^2(s)\,ds = \frac{1}{2}\int_0^L \left\|\frac{d\Lambda}{ds}\right\|^2 ds$$

on the space of frame-periodic horizontal-Darboux curves. Here it will be understood that $\Lambda(s)$ is either a periodic Hermitian matrix and the metric is defined by trace form, or $\Lambda(s)$ is periodic skew-Hermitian and the metric is defined by the negative of the trace form.

It will be shown below that the Hamiltonian vector field $\mathcal{X}_f$ obtained through the symplectic structure on the space of frame-periodic horizontal-Darboux curves leads to Heisenberg's magnetic equation and the non-linear Schroedinger's equation. A derivation of this fact, together with the connections to the known results in the literature constitute the subject matter for the remaining part of the paper.

4.1. Heisenberg's magnetic equation

Although conceptually alike, the calculations in the spherical setting are different in several aspects from those in the hyperbolic setting and will be done separately in each of the above mentioned cases. For notational simplicity the inner products in both cases will be denoted by the single symbol $\langle,\rangle$ and the same will apply to the induced norms.

Hyperbolic Darboux curves. To calculate the directional derivative $df_\Lambda(V)$, let $\hat{g}(s,t)$ be a family of anchored horizontal-Darboux curves that are the solutions of

$$\frac{\partial \hat{g}}{\partial s} = \hat{g}(s,t)\hat{\Lambda}(s,t)$$

such that $\hat{g}(s,0) = g(s)$, $\hat{\Lambda}(s,0) = \Lambda(s)$, $\frac{\partial \hat{\Lambda}}{\partial t}(s,0) = \frac{dV}{ds}(s)$. The directional derivative $df_\Lambda(V)$ is given by

$$df_\Lambda(V) = \frac{1}{2}\frac{\partial}{\partial t}\int_0^L b\left\langle \frac{\partial \hat{\Lambda}}{\partial s}(s,t), \frac{\partial \hat{\Lambda}}{\partial s}(s,t)\right\rangle ds\bigg|_{t=0.}$$

Then

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_0^L \left\langle \frac{\partial \hat{\Lambda}}{\partial s}(s,t), \frac{\partial \hat{\Lambda}}{\partial s}(s,t)\right\rangle ds\bigg|_{t=0} &= \int_0^L \left\langle \frac{\partial \hat{\Lambda}}{\partial s}(s,t), \frac{\partial}{\partial s}\frac{\partial \hat{\Lambda}}{\partial t}(s,t)\right\rangle ds\bigg|_{t=0} \\
&= \int_0^L \left\langle \frac{d\Lambda}{ds}, \frac{d}{ds}\left(\frac{dV}{ds}\right)\right\rangle ds \\
&= -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, \frac{dV}{ds}\right\rangle ds + \left\langle \frac{d\Lambda}{ds}, \frac{dV}{ds}\right\rangle\bigg|_{s=0}^{s=L}.
\end{aligned}$$

In the space of frame-periodic horizontal-Darboux curves the boundary terms $\langle \frac{d\Lambda}{ds}, \frac{dV}{ds}\rangle|_{s=0}^{s=L}$ are equal to 0 because of periodicity. Consequently,

$$df_\Lambda(V) = -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, \frac{dV}{ds}\right\rangle ds.$$

The Hamiltonian vector field is of the form $\mathcal{X}_f(g) = gF$ for some Hermitian matrix $F(s)$ that satisfies

$$df_\Lambda(V) = \frac{1}{i}\int_0^L \left\langle \Lambda(s), \left[\frac{dF}{ds}, \frac{dV}{ds}\right]\right\rangle ds$$

for an arbitrary tangential direction $V(s)$. The above is equivalent to

$$\int_0^L \left\langle \left(\frac{d^2\Lambda}{ds^2} + \frac{1}{i}\left[\Lambda(s), \frac{dF}{ds}\right]\right), \frac{dV}{ds}\right\rangle ds = 0. \tag{31}$$

Since the mapping $U \to i[\Lambda, U]$ is bijective on the space of Hermitian matrices orthogonal to $\Lambda(s)$ there exists a Hermitian matrix $U(s)$ such that $U(0) = 0$ and $\frac{dV}{ds} = i[\Lambda(s), U(s)]$. Then Eq. (31) becomes

$$\int_0^L \left\langle \left(i\left[\frac{d^2\Lambda}{ds^2}, \Lambda(s)\right] + \left[\left[\Lambda(s), \frac{dF}{ds}\right], \Lambda(s)\right]\right), U(s)\right\rangle ds = 0.$$

It follows that

$$i\left[\frac{d^2\Lambda}{ds^2}, \Lambda(s)\right] + \left[\left[\Lambda(s), \frac{dF}{ds}\right], \Lambda(s)\right] = 0 \tag{32}$$

because $U(s)$ is sufficiently arbitrary. Since

$$\big[[A, B], C\big] = \langle B, C\rangle A - \langle A, C\rangle B$$

for any Hermitian matrices A, B, C, as can be readily verified through Table 1, $[\Lambda(s), [\Lambda(s), \frac{dF}{ds}]] = -\frac{dF}{ds}$ and Eq. (32) becomes $i[\frac{d^2\Lambda}{ds^2}, \Lambda(s)] - \frac{dF}{ds} = 0$.

It follows that

$$\mathcal{X}_f(g) = g(s)F(s) \quad \text{with } F(s) = i\int_0^s \left[\frac{d^2\Lambda}{dx^2}(x), \Lambda(x)\right] dx$$

is the Hamiltonian vector field that corresponds to f.

The integral curves $t \to g(s,t)$ of $\mathcal{X}_f$ are the solutions of the following partial differential equations

$$\frac{\partial g}{\partial t}(s,t) = g(s,t)i\int_0^s \left[\frac{d^2\Lambda}{dx^2}(x,t), \Lambda(x,t)\right] dx, \tag{33}$$

$$\frac{\partial g}{\partial s}(s,t) = g(s,t)\Lambda(s,t). \tag{34}$$

The equality of mixed partial derivatives $\frac{D_g}{ds}(\frac{\partial g}{\partial t}) = \frac{D_g}{dt}(\frac{\partial g}{\partial s})$ implies that the matrices $\Lambda(s,t)$ evolve according to

$$\frac{\partial \Lambda}{\partial t}(s,t) = i\left[\frac{\partial^2\Lambda}{\partial s^2}, \Lambda(s,t)\right]. \tag{35}$$

Eq. (35) when expressed in terms of the coordinates $\lambda(s,t)$ of $\Lambda(s,t)$ relative to the basis of Hermitian Pauli matrices becomes:

$$\frac{\partial \lambda}{\partial t}(s,t) = \lambda(s,t) \times \frac{\partial^2 \lambda}{\partial s^2}(s,t). \tag{36}$$

Eq. (36) is well known in the literature in applied mathematics. L.D. Faddeev and L.A. Takhtajan refer to it as the continuous isotropic Heisenberg ferromagnetic model [5, Part II, Chapter 1] which they treat in an ad hoc manner as an equation in the space of Hermitian matrices. V.I. Arnold and B. Khesin [2] connect (36) to the filament equation which they further consider as a special type of a Landau–Lifschitz equation on $so_3(R)$.

Spherical Darboux curves. The derivation of the corresponding Hamiltonian equations on the sphere is similar to the preceding case except for the details related to the covariant derivative. Recall that the tangent space $T_X(Horiz(\mathcal{D}_s)(L))$ at an anchored spherical horizontal-Darboux curve $X(s)$ consists of tangent curves $v(s) = X(s)V(s)$ with $V(s)$ the solution of

$$\frac{dV}{ds}(s) = \big[\Lambda(s), V(s)\big] + U(s) \tag{37}$$

with $V(0) = 0$. The matrix $\Lambda(s)$ is the tangent vector of X, i.e.,

$$\frac{dX}{ds}(s) = X(s)\Lambda(s)$$

and $U(s)$ is a curve in $\mathfrak{h}$ subject to $U(0) = 0$ and $\langle \Lambda(s), U(s)\rangle = 0$.

Let $v(s) = X(s)V(s)$ be a fixed tangent vector at an anchored horizontal-Darboux curve $X(s)$. To find the appropriate expression for the directional derivative $df_X(V)$ at X in the direction V, let $Y(s,t)$ denote a family of anchored horizontal-Darboux curves such that $Y(s,0) = X(s)$ and such that $v(s) = \frac{\partial Y}{\partial t}(s,t)_{t=0}$.

Let $Z(s,t)$ denote the matrices defined by

$$\frac{\partial Y}{\partial s}(s,t) = Y(s,t)Z(s,t).$$

It follows that $\Lambda(s) = Z(s,0)$, and that $V(s)$ is the solution of (37) with $U(s) = \frac{\partial Z}{\partial t}(s,0)$.

Then,

$$\begin{aligned} df_\Lambda(V) &= \frac{1}{2}\frac{\partial}{\partial t}\int_0^L \left\langle \frac{\partial Z}{\partial s}(s,t), \frac{\partial Z}{\partial s}(s,t)\right\rangle ds\bigg|_{t=0} \\ &= \int_0^L \left\langle \frac{d\Lambda}{ds}(s), \frac{dU}{ds}(s)\right\rangle ds \\ &= -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}(s), U(s)\right\rangle ds + \left\langle \frac{d\Lambda}{ds}(s), U(s)\right\rangle_{s=0}^{s=L}. \end{aligned}$$

Analogous to the hyperbolic case the boundary terms vanish in the frame-periodic case, and therefore

$$df_X(V) = -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, U(s)\right\rangle ds. \tag{38}$$

The Hamiltonian vector field $\mathcal{X}_f$ that corresponds to f is of the form

$$\mathcal{X}_f(X)(s) = X(s)F(s)$$

for some curve $F(s) \in \mathfrak{h}$. Since $\mathcal{X}_f(X) \in T_X(Horiz(\mathcal{PD})_s(L))$, $F(s)$ is the solution of

$$\frac{dF}{ds}(s) = \left[\Lambda(s), F(s)\right] + U_f(s), \quad F(0) = 0$$

for some curve $U_f(s) \in \mathfrak{h}$ that satisfies

$$U_f(0) = 0 \quad \text{and} \quad \left\langle \Lambda(s), U_f(s)\right\rangle = 0.$$

The curve $U_f(s)$ is determined by the symplectic form ω with

$$df_\Lambda(U) = -\int_0^L \left\langle \Lambda(s), \left[U_f(s), U(s)\right]\right\rangle ds \tag{39}$$

where $U(s)$ is an arbitrary curve in $\mathfrak{h}$ that satisfies $U(0) = 0$ and $\langle \Lambda(s), U(s)\rangle = 0$. Eqs. (38) and (39) yield

$$\int_0^L \left\langle \frac{d^2\Lambda}{ds^2} - \left[\Lambda(s), U_f(s)\right], U(s)\right\rangle ds = 0. \tag{40}$$

Then $U(s)$ can be written as $U(s) = [\Lambda(s), C(s)]$ where $C(s)$ is any curve that satisfies $C(0) = 0$, in which case (40) becomes

$$\int_0^L \left\langle \left[\frac{d^2\Lambda}{ds^2}(s), \Lambda(s)\right] - \left[\left[\Lambda(s), U_f(s)\right], \Lambda\right], C(s)\right\rangle ds = 0. \tag{41}$$

Since $C(s)$ is arbitrary,

$$\left[\frac{d^2\Lambda}{ds^2}, \Lambda\right] - \big[[\Lambda, U_f], \Lambda\big] = 0. \tag{42}$$

Lemma 2 implies that

$$\big[[\Lambda, U_f], \Lambda\big] = U_f$$

and therefore,

$$U_f = -\left[\Lambda, \frac{d^2\Lambda}{ds^2}\right].$$

The integral curves $t \to X(s,t)$ of $\mathcal{X}_f$ are the solutions of

$$\frac{\partial X}{\partial t}(s,t) = X(s,t)F(s,t), \quad \text{and} \quad \frac{\partial X}{\partial s} = X(s,t)\Lambda(s,t) \tag{43}$$

where $F(s,t)$ is the solution of $\frac{\partial F}{\partial s}(s,t) = [\Lambda(s,t), F(s,t)] - [\Lambda(s,t), \frac{\partial^2\Lambda}{\partial s^2}(s,t)]$. But then zero-curvature equation (22) implies that

$$\frac{\partial \Lambda}{\partial t} - \frac{\partial F}{\partial s} + [\Lambda, F] = 0,$$

and therefore

$$\frac{\partial \Lambda}{\partial t}(s,t) = \left[\frac{\partial^2\Lambda}{\partial s^2}(s,t), \Lambda(s,t)\right]. \tag{44}$$

Thus in both the hyperbolic and the spherical case $\Lambda(s,t)$ evolves according to the same equation; in the hyperbolic case Λ is Hermitian, while in the spherical case, Λ is skew-Hermitian. To pass from the hyperbolic case to the spherical case multiply $\Lambda(s,t)$ in Eq. (35) by i.

Definition 4.1. Eqs. (35) and (44) will be referred to as Heisenberg's magnetic equations.

4.2. The non-linear Schroedinger equation

Each solution $\Lambda(s,t)$ of Heisenberg's magnetic equation generate a family of matrices $R(s,t)$ periodic in s for each t through the relations $\Lambda(s,t) = R(s,t)B_1R^*(s,t)$ in the hyperbolic case, and through $\Lambda(s,t) = R(s,t)A_1R^*(s,t)$ in the spherical case. Curves $R(s,t)$ then evolve according to

$$\frac{\partial R}{\partial s}(s,t) = R(s,t)U(s,t), \quad \text{and} \quad \frac{\partial R}{\partial t} = R(s,t)V(s,t)$$

for some matrices $U(s,t)$ and $V(s,t)$ in $\mathfrak{h}$, which further conform to the zero-curvature equation

$$\frac{\partial U}{\partial t}(s,t) - \frac{\partial V}{\partial s}(s,t) + \big[U(s,t), V(s,t)\big] = 0. \tag{45}$$

Moreover, $V(0,t) = 0$ for all t because horizontal-Darboux curves are anchored at $s = 0$.

The results that follow make a use of the following formulas:

$$\big[A, [A,B]\big] = \langle A,B\rangle A - \langle A,A\rangle B, \quad \text{for } A \text{ and } B \text{ in } \mathfrak{p}, \tag{46}$$

and

$$\big[[A,B], B\big] = \langle B,B\rangle A - \langle A,B\rangle B, \quad \text{for } A \text{ in } \mathfrak{h} \text{ and } B \text{ in } \mathfrak{p}. \tag{47}$$

Theorem 5. *Let $U(s,t) = \sum u_j(s,t)A_j$ generate a solution of Heisenberg's magnetic equation and let $u(s,t) = u_2(s,t) + iu_3(s,t)$. Then,*

$$\psi(s,t) = u(s,t)\exp\left(i\int_0^s u_1(x,t)\,dx\right)$$

is a solution of the non-linear Schroedinger's equation

$$\frac{\partial}{\partial t}\psi(s,t) = i\frac{\partial^2\psi}{\partial s^2}(s,t) + i\left(\frac{1}{2}\left|\psi(s,t)\right|^2 + c\right)\psi(s,t), \tag{48}$$

where $c(t) = -\frac{1}{2}|u(0,t)|^2$.

Proof. The proof of the theorem will be done for the hyperbolic case although the arguments are the same in both cases, as will become clear below.

Since $\Lambda(s,t) = R(s,t)B_1R^*(s,t)$

$$\frac{\partial\Lambda}{\partial t} = \frac{\partial}{\partial t}\bigl(R(s,t)B_1R^*(s,t)\bigr) = R[B_1,V]R^*, \qquad \frac{\partial\Lambda}{\partial s} = R[B_1,U]R^* \quad \text{and}$$
$$\frac{\partial^2\Lambda}{\partial s^2} = R\left(\bigl[[B_1,U],U\bigr] + \left[B_1,\frac{\partial U}{\partial s}\right]\right)R^*.$$

The fact that $\Lambda(s,t)$ evolves according to Heisenberg's magnetic equation implies that

$$[B_1,V] = i\left(\bigl[\bigl[[B_1,U],U\bigr],B_1\bigr] + \left[\left[B_1,\frac{\partial U}{\partial s}\right],B_1\right]\right). \tag{49}$$

Relations (46) imply that

$$\bigl[[B_1,U],U\bigr] = \langle U,B_1\rangle U - \langle U,U\rangle B_1 = -\bigl(u_2^2+u_3^2\bigr)B_1 + u_1u_2B_2 + B_3u_1u_3,$$

hence

$$\bigl[\bigl[[B_1,U],U\bigr],B_1\bigr] = u_1u_3A_2 - u_1u_2A_3.$$

Similarly,

$$\left[B_1,\frac{\partial U}{\partial s}\right] = \frac{\partial u_3}{\partial s}B_2 - \frac{\partial u_2}{\partial s}B_3, \quad \text{and} \quad \left[\left[B_1,\frac{\partial U}{\partial s}\right],B_1\right] = -\frac{\partial u_3}{\partial s}A_3 - \frac{\partial u_2}{\partial s}A_2.$$

Eq. (49) then reduces to

$$[B_1,V] = i\left(u_1(u_3A_2-u_2A_3) - \frac{\partial u_3}{\partial s}A_3 - \frac{\partial u_2}{\partial s}A_2\right) = -u_1(u_3B_2-u_2B_3) + \frac{\partial u_3}{\partial s}B_3 + \frac{\partial u_2}{\partial s}B_2.$$

If $V = v_1A_1+v_2A_2+v_3A_3$ then $[B_1,V] = v_3B_2 - v_2B_3$, which, when combined with the above, yields

$$v_2 = -u_1u_2 - \frac{\partial u_3}{\partial s}, \quad \text{and} \quad v_3 = -u_1u_3 + \frac{\partial u_2}{\partial s},$$

or

$$v(s,t) = -u_1(s,t)u(s,t) + i\frac{\partial u}{\partial s}(s,t), \tag{50}$$

where $u = u_2 + iu_3$ and $v = v_2 + iv_3$. The zero curvature equation implies that

$$\frac{\partial u_1}{\partial t} = \frac{\partial v_1}{\partial s} + \frac{1}{2}\frac{\partial}{\partial s}\bigl(u_2^2+u_3^2\bigr), \tag{51}$$

and that

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - \frac{\partial u_1}{\partial s}u - i\bigl(v_1+u_1^2\bigr)u. \tag{52}$$

Eq. (51) implies that

$$\frac{\partial}{\partial t}\int_0^s u_1(x,t)\,dx = v_1(s,t) + \frac{1}{2}\bigl(u_2^2(s,t)+u_3^2(s,t)\bigr) + c(t)$$

where $c(t) = -v_1(0,t) - \frac{1}{2}(u_2^2(0,t) + u_3^2(0,t)) = -\frac{1}{2}(u_2^2(0,t) + u_3^2(0,t))$, since $V(0,t) = 0$. The substitution of $v_1(s,t) = \frac{\partial}{\partial t}\int_0^s u_1(x,t)\,dx - \frac{1}{2}|u(s,t)|^2 - c$ into (52) leads to

$$\frac{\partial u}{\partial t} + iu\frac{\partial}{\partial t}\int^s u_1(t,x)\,dx = i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - i\left(-\frac{1}{2}|u|^2 - c + u_1^2\right)u. \tag{53}$$

After the multiplication by $\exp(i\int_0^s u_1(x,t)\,dx)$ Eq. (53) can be expressed as

$$\frac{\partial}{\partial t}\psi(s,t) = \left(i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - i\left(u_1^2 - \frac{1}{2}|u|^2 - c\right)u\right)\exp\left(i\int_0^s u_1(x,t)\,dx\right)e^{-ict}$$

where

$$\psi(s,t) = u(s,t)\exp\left(i\int_0^s u_1(x,t)\,dx\right). \tag{54}$$

Since

$$\frac{\partial\psi}{\partial s} = \left(\frac{\partial u}{\partial s} + iuu_1\right)\exp\left(i\int_0^s u_1(x,t)\,dx\right)$$

and

$$\frac{\partial^2\psi}{\partial s^2} = \left(\frac{\partial^2 u}{\partial s^2} + 2iu_1\frac{\partial u}{\partial s} + iu\frac{\partial u_1}{\partial s} - u_1^2 u\right)\exp\left(i\int_0^s u_1(x,t)\,dx\right).$$

It follows that

$$i\frac{\partial^2\psi}{\partial s^2} = \left(i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - iu_1^2 u\right)\exp\left(i\int_0^s u_1(x,t)\,dx\right),$$

and therefore

$$\frac{\partial}{\partial t}\psi(t,s) = i\frac{\partial^2\psi}{\partial s^2} + i\left(\frac{1}{2}|\psi|^2 + c(t)\right)\psi. \qquad \square$$

In the spherical case the evolution along Heisenberg's magnetic equation leads to

$$[A_1, V] = \left(\big[\big[[A_1, U], U\big], A_1\big] + \left[\left[A_1, \frac{\partial U}{\partial s}\right], A_1\right]\right). \tag{55}$$

The preceding equation is the same as Eq. (49) because $A_1 = iB_1$. Therefore the calculations that led to the non-linear Schroedinger equation in the hyperbolic case are equally valid in the spherical case with the same end result.

The steps taken in the passage from Heisenberg's equation to the Schroedinger's equation are reversible. Any solution $\psi(s,t)$ of (48) generates matrices

$$U = \frac{1}{2}\begin{pmatrix} 0 & \psi \\ -\bar{\psi} & 0 \end{pmatrix} \quad \text{and} \quad V = \frac{1}{2}\begin{pmatrix} -\frac{1}{2}i(|\psi|^2 + c(t)) & i\frac{\partial\psi}{\partial s} \\ i\frac{\partial\bar{\psi}}{\partial s} & \frac{1}{2}i(|\psi|^2 + c(t)) \end{pmatrix}$$

that satisfy the zero-curvature equation. Therefore, there exist unique curves $R(s,t)$ in SU_2 with boundary conditions $R(0,t) = I$ that evolve according to the differential equations:

$$\frac{\partial R}{\partial s}(s,t) = R(s,t)U(s,t), \qquad \frac{\partial R}{\partial t}(s,t) = R(s,t)V(s,t).$$

Such curves define $\Lambda(s,t)$ through familiar formulas $\Lambda(s,t) = R(s,t)B_1R^*(s,t)$ or $\Lambda(s,t) = R(s,t)A_1R^*(s,t)$ depending on the case. It then follows that Λ is a solution of the Heisenberg's magnetic equation because $\psi = u$ and $v = i\frac{\partial u}{\partial s}$, and each Eqs. (49) and (55) is satisfied.

Remark 5. Theorem 5 reveals that $SO_2 = \{\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, |z| = 1\}$ is a symmetry group for the non-linear Schroedinger equation, reflected in the fact that $\psi(s,t) = u(s,t)\exp(i\int_0^s u_1(x,t)\,dx)$ is a solution of the non-linear Schroedinger equation independently of u_1. That fact, together with the results presented in [7], motivated calculations with a general $A(s,t)$ rather than the one with diagonal part equal to zero.

To correlate the findings of this paper with the related existing literature, which almost exclusively deals with curves in $\mathbb{R}^3$, it seems appropriate to include a discussion of the only remaining simply connected three dimensional symmetric space, namely the Euclidean space.

4.3. Euclidean Darboux curves

The most convenient way to pass to Euclidean Darboux curves is to realize the Euclidean group of motions as the semi direct product $S_H(\mathfrak{p}) = \mathfrak{p} \rtimes H$ where $H = SU_2$.

Recall that the semi direct product of a vector space V and a group H which acts linearly on V consists of pairs (x, R) with $x \in V$ and $R \in H$ with the group operation given by $(x, R)(y, T) = (x + Ry, RT)$. In this general setting the Lie algebra $s_H(V)$ of $S_H(V)$ consists of pairs (a, A) with $a \in V$ and $A \in \mathfrak{h}$ with the Lie bracket

$$\big[(a, A), (b, B)\big] = \big(A(b) - B(a), [A, B]\big).$$

In our specific situation $H = SU_2$ acts on the space of Hermitian matrices $\mathfrak{p}$ by conjugations. In this context, $A(a) = [a, A]$ for $a \in \mathfrak{p}$ and $A \in \mathfrak{h}$, and the Lie bracket is given by

$$\big[(a, A), (b, B)\big] = \big([b, A] - [a, B], [A, B]\big).$$

As a vector space $\mathfrak{p} \rtimes \mathfrak{h}$ can be identified with $sl_2(C)$ via the embedding $(a, A) \to a + A$ for any (a, A) in $\mathfrak{p} \ltimes su_2$. With this identification, $s_H(V) = \mathfrak{p} \oplus \mathfrak{h}$ and

$$[\mathfrak{p}, \mathfrak{p}] = 0, \qquad [\mathfrak{p}, \mathfrak{h}] = \mathfrak{p}, \qquad [\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}.$$

The group $S_H(\mathfrak{p})$ acts on $\mathfrak{p}$ by $(x, R)(y) = R(y) + x$ for each $(x, R) \in S_H(\mathfrak{p})$ and each $y \in \mathfrak{p}$. The action is transitive, and H is equal to the isotropy group of the orbit through the origin $y = 0$. Then $\mathfrak{p}$, when identified with the orbit through the origin, becomes the coset space $S_H(\mathfrak{p})/H$.

The space of Hermitian matrices endowed with the metric induced by the trace form becomes a three dimensional Euclidean space $\mathbb{E}^3$. The preceding action extends to an action on the tangent bundle of $\mathbb{E}^3$ in which a tangent vector v at y is taken to the tangent vector $R(v)$ at x under the action by an element $(x, R) \in S_H(\mathbb{E}^3)$. The action on the tangent bundle extends further to an action on the orthonormal frame bundle of $\mathbb{E}^3$ such that a frame (v_1, v_2, v_3) at a point $y \in \mathbb{E}^3$ is taken to the frame $(R(v_1), R(v_2), R(v_3))$ at x under the action by an element $(x, R) \in S_H(\mathbb{E}^3)$. The kernel of this action consists of $\pm I$, and hence $S_H(\mathbb{E}^3)/\{\pm I\}$ can be identified with the positively oriented orthonormal frame bundle of $\mathbb{E}^3$ as the orbit through the standard frame (B_1, B_2, B_3) at the origin.

In the left-invariant representation of the tangent bundle of $S_H(\mathbb{E}^3)$ tangent vectors at a point (x, R) are given by pairs $(R(a), RA)$ with $a \in \mathbb{E}^3$ and $A \in \mathfrak{h}$. Hence curves $(x(s), R(s))$ in $S_H(\mathbb{E}_3)$ are represented by differential equations

$$\frac{dx}{ds}(s) = R(s)\big(a(s)\big), \qquad \frac{dR}{ds}(s) = R(s)A(s). \tag{56}$$

The terminology concerning Darboux curves in non-Euclidean cases extends naturally to the Euclidean setting. In particular, curves $(x, R) \in S_H(\mathbb{E}^3)$ are Euclidean anchored Darboux curves if $\frac{dx}{ds}(s) = R(s)(B_1)$, i.e., whenever $a(s) = B_1$, subject to further boundary conditions $x(0) = 0$, $R(0) = I$. Euclidean horizontal-Darboux curves are the projections $x(s)$ of Euclidean anchored Darboux curves, i.e., they are the solutions of

$$\frac{dx}{ds}(s) = R(s)(B_1), \qquad x(0) = 0$$

with $R(s)$ an arbitrary curve in H such that $R(0) = I$. Frame-periodic Darboux curves (x, R) conform to the periodicity of $R(s)$ with its period equal to the length of $x(s)$.

For any horizontal-Darboux curve $x(s)$

$$\frac{d^2x}{ds^2}(s) = R(s)\big(\big[B_1, A(s)\big]\big),$$

and therefore

$$\kappa^2(s) = \left\| \frac{d^2x}{ds^2}(s) \right\|^2 = u_2^2(s) + u_3^2(s)$$

where $A(s) = \sum u_i(s)A_i$. The frame $R(s)$ is a Serret–Frenet frame if $A(s) = \tau(s)A_1 + \kappa(s)A_3$, in which case the frame vectors $T(s)$, $N(s)$, $B(s)$ are given by

$$T(s) = R(s)(B_1), \qquad N(s) = R(s)(B_2), \qquad B(s) = R(s)(B_3).$$

The reader may easily verify that the tangent space at each anchored horizontal-Darboux curve $x(s)$ consists of curves $v(s)$ such that

(a) $v(0) = \frac{dv}{ds}(0) = 0$, and
(b) $\langle \frac{dx}{ds}(s), \frac{dv}{ds}(s) \rangle = 0$.

The space of frame-periodic Euclidean horizontal-Darboux curves inherits the symplectic structure given by Corollary 1. This symplectic structure is isomorphic to the structure used by J. Millson and B.A. Zombro in [16]. More precisely, in the Millson–Zombro paper Euclidean space $\mathbb{E}^3$ is identified with $so_3(R)$ and their symplectic form is isomorphic to the one given by Eq. (26) in view of the isomorphism between su_2 and $so_3(R)$.

It can be shown by arguments identical to the ones already presented in this paper that the Hamiltonian flow induced by the function $f(x(s)) = \frac{1}{2}\int_0^l \kappa^2(s)\,ds$ leads to Heisenberg's magnetic equation, and that the passage to the non-linear Schroedinger's equation is the same as the one presented for the non-Euclidean cases.

The present formalism clarifies Hasimoto's first observation that $\psi = \kappa \exp(i \int \tau\,dx)$ of a curve $\gamma(s,t)$ that satisfies the filament equation

$$\frac{\partial \gamma}{\partial t}(s,t) = \kappa(s,t)B(s,t) \tag{57}$$

is a solution of the non-linear Schroedinger equation [7]. In this notation, it is understood that $t \to \gamma(s,t)$ denotes a family of curves in R^3 parametrized by t and that $B(s,t)$ denotes the binormal vector along the curve $s \to \gamma(s,t)$.

When the solution curves of the filament equation are restricted to curves parametrized by arc-length, i.e., curves $\gamma(s,t)$ such that $\|\frac{\partial \gamma}{\partial s}(s,t)\| = 1$, then

$$T(t,s) = \frac{\partial \gamma}{\partial s}(t,s) \quad \text{and} \quad \frac{\partial T}{\partial s}(s,t) = \kappa(s,t)N(s,t) = \frac{\partial^2 \gamma}{\partial s^2}(s,t).$$

Moreover, $B(s,t) = T(s,t) \times N(s,t)$. It then follows that in the space of arc-length parametrized curves the filament equation can be written as

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \times \frac{\partial^2 \gamma}{\partial s^2}. \tag{58}$$

For each solution curve $\gamma(s,t)$ of (58) the tangent vector $T(s,t)$ satisfies

$$\frac{\partial T}{\partial t} = T \times \frac{\partial^2 T}{\partial s^2} \tag{59}$$

as can be easily verified by differentiating with respect to s.

Any solution $T(s,t)$ of the preceding equation may be interpreted as the coordinate vector of $\Lambda(s,t)$ relative to an orthonormal basis in either $\mathfrak{h}$ or $\mathfrak{p}$. In the first case $\Lambda(s,t)$ evolves according to

$$\frac{\partial \Lambda}{\partial t}(s,t) = i\left[\frac{\partial^2 \Lambda}{\partial s^2}(s,t), \Lambda(s,t)\right]$$

and in the second case Λ evolves according to

$$\frac{\partial \Lambda}{\partial t}(s,t) = \left[\frac{\partial^2 \Lambda}{\partial s^2}(s,t), \Lambda(s,t)\right].$$

The function $\psi(s,t) = u(s,t)\exp(i\int_0^s u_1(x,t)\,dx)$ generated by $R(s,t)$ that is associated with $\Lambda(s,t)$ is a solution of the non-linear Schroedinger's equation independently of the choice of the symmetric space (Theorem 5). When the frame $R(s,t)$ is a Serret–Frenet frame then:

$u_1 = \tau$, $u_2 = 0$, $u_3 = \kappa$ in the Euclidean and the hyperbolic case, while
$u_1 = \tau + \frac{1}{2}$, $u_2 = 0$, $u_3 = \kappa$ in the spherical case.

Hasimoto's function $\kappa(s,t)\exp(i\int_0^s \tau(x,t)\,dx)$ coincides with $u(s,t)\exp(i\int_0^s u_1(x,t)\,dx)$ in the hyperbolic and the Euclidean case, but not in the spherical case. Of course, the most natural frame is the reduced frame $u_1 = 0$ which bypasses these inessential connections with the torsion.

The geometry of the underlying space becomes visible only when the integration of the Hamiltonian equations is carried out on the full tangent bundle of the Lie group and not just on the part of the equations that resides in the Lie algebra $\mathfrak{g}$.

5. Elastic curves and solitons

For mechanical systems the Hamiltonian function represents the total energy of the system and its critical points correspond to the equilibrium configurations. In an infinite-dimensional setting the behavior of a Hamiltonian system at a critical point of a Hamiltonian seems not to lend itself to such simple characterizations.

For the Hamiltonian function $f = \frac{1}{2}\int_0^L k^2\,ds$ the critical points are the elastic curves. The solutions of the associated Hamiltonian system that originate at an elastic curve, instead of being stationary, form traveling waves known as solitons (first noticed in [8]). Soliton solutions of either Heisenberg's magnetic equation or the non-linear Schroedinger's equation are waves that travel at constant speeds with an elastic curves at their wave fronts. To explain these statements in some detail it will be necessary to make a small detour into the geometry of elastic curves [10–12].

5.1. Elastic curves and their Hamiltonian systems

To maintain continuity with the material already presented and yet to keep the detour at a minimum, the discussion will be confined to the Euclidean and the hyperbolic case. The spherical case requires adjustments in notation but is otherwise similar to the other two cases (as demonstrated in [10], [12], and [13]).

For notational simplicity G_ϵ will denote $S_H(\mathbb{E}^3)$ for $\epsilon = 0$, and $SL_2(C)$ for $\epsilon = -1$. The Lie algebra of G_ϵ will be denoted by $\mathfrak{g}_\epsilon$. As vector spaces $\mathfrak{g}_0$ and $\mathfrak{g}_{-1}$ are equal to each other, but as Lie algebras they are different. Their Lie brackets conform to the following table

Table 2

[,]	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	$-A_3$	A_2	0	$-B_3$	B_2
A_2	A_3	0	$-A_1$	B_3	0	$-B_1$
A_3	$-A_2$	A_1	0	$-B_2$	B_1	0
B_1	0	$-B_3$	B_2	0	$-\epsilon A_3$	ϵA_2
B_2	B_3	0	$-B_1$	ϵA_3	0	$-\epsilon A_1$
B_3	$-B_2$	B_1	0	$-\epsilon A_2$	ϵA_1	0

Definition 5.1. The problem of finding the minimum of the integral

$$\frac{1}{2}\int_0^L \left(u_2^2(s) + u_3^2(s)\right) ds$$

over all curves $g(s)$ in G_ϵ that are the solutions of

$$\frac{dg}{ds}(s) = g(s)\big(B_1 + u_2(s)A_2 + u_3(s)A_3\big) \tag{60}$$

and satisfy fixed boundary conditions at $s = 0$ and $s = L$ shall be called the elastic problem on G_ϵ.

Since the elastic problem is left-invariant, the initial point can always be taken at the identity. It is evident from the first part of the paper that

$$\frac{1}{2}\int_0^L \big(u_2^2(s) + u_3^2(s)\big)\,ds = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$$

where $\kappa(s)$ is the curvature of the projected curve $\pi_\epsilon(g(s))$. The set of curves (60) may be considered as a "reduced" Darboux space for the function $f(g) = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$ for the following reasons:

For any Darboux curve $g(s)$ that is a solution of

$$\frac{dg}{ds}(s) = g(s)\big(B_1 + u_1(s)A_1 + u_2(s)A_2 + u_3(s)A_3\big)$$

$g_0(s) = g(s)\exp(-A_1\int_0^s u_1(x)\,dx)$ projects onto the same base point as $g(s)$ and satisfies (60). Consequently,

$$f\big(g(s)\big) = f\big(g_0(s)\big).$$

Definition 5.2. The projections $x(s) = \pi_\epsilon(g(s))$ of the "extremal curves" $g(s)$ on the underlying space G_ϵ/K are called elastic curves.

It is known that the elastic problem has a solution for any pair of boundary points provided that L is sufficiently large. Moreover, the solutions are the projections of integral curves of a single Hamiltonian system on the cotangent bundle T^*G_ϵ of G_ϵ [10]. Since the setting in [10] is sufficiently different from the one adopted in this paper ($S^3 = SO_4(R)/SO_3(R)$ and $\mathbb{H}^3 = SO(1,3)/SO_3(R)$ rather than $S^3 = SU_2$ and $\mathbb{H}^3 = SL_2(C)/SU_2$) it seems more expedient to re-do the basic results rather than quote them from [10].

To take advantage of the left-invariant symmetries, the cotangent bundle T^*G_ϵ will be realized as $G_\epsilon \times \mathfrak{g}_\epsilon^*$ via the left-translations, where $\mathfrak{g}_\epsilon^*$ denotes the dual of $\mathfrak{g}_\epsilon$. Then linear functions in $\mathfrak{g}_\epsilon^*$ will be represented by coordinate functions $h_1, h_2, h_3, H_1, H_2, H_3$ relative to the dual basis $B_1^*, B_2^*, B_3^*, A_1^*, A_2^*, A_3^*$ defined by the Pauli matrices .

An easy application of the Maximum Principle shows that the regular extremal curves of the elastic problem are the projections of the integral curves of the Hamiltonian vector field $\vec{H}$ defined by the Hamiltonian function

$$H = \frac{1}{2}\big(H_2^2 + H_3^2\big) + h_1 \tag{61}$$

induced by extremal controls

$$u_2 = H_2, \qquad u_3 = H_3. \tag{62}$$

Remark 6. The abnormal extremal curves shall be ignored since all optimal solutions are the projections of regular extremal curves [10].

The most direct way to get the equations of $\vec{H}$ is via the Poisson brackets involving the variables $h_1, h_2, h_3, H_1, H_2, H_3$. The Poisson brackets of these variables are isomorphic to the Lie brackets in Table 2, and are reproduced in Table 3 below.

Table 3

{ , }	H_1	H_2	H_3	h_1	h_2	h_3
H_1	0	$-H_3$	H_2	0	$-h_3$	h_2
H_2	H_3	0	$-H_1$	h_3	0	$-h_1$
H_3	$-H_2$	H_1	0	$-h_2$	h_1	0
h_1	0	$-h_3$	h_2	0	$-\epsilon H_3$	ϵH_2
h_2	h_3	0	$-h_1$	ϵH_3	0	$-\epsilon H_1$
h_3	$-h_2$	h_1	0	$-\epsilon H_2$	ϵH_1	0

Remark 7. The reader may readily verify by appealing to the isomorphism $so_4(R) \cong su_2 \times su_2$ that there exists a basis in $(su_2 \times su_2)^*$ that conforms to the above Poisson bracket table for $\epsilon = 1$.

Then Hamiltonian equations associated with (61) are given by:

$$\begin{aligned}
\frac{dH_1}{ds} &= \{H_1, H\} = H_2\{H_1, H_2\} + H_3\{H_1, H_3\} + \{H_1, h_1\} = 0,\\
\frac{dH_2}{ds} &= \{H_2, H\} = H_3\{H_2, H_3\} + \{H_2, h_1\} = -H_3H_1 + h_3,\\
\frac{dH_3}{ds} &= \{H_3, H\} = H_2\{H_3, H_2\} + H_3\{H_3, H_3\} + \{H_3, h_1\} = H_2H_1 - h_2,\\
\frac{dh_1}{ds} &= \{h_1, H\} = H_2\{h_1, H_2\} + H_3\{h_1, H_3\} + \{h_1, h_1\} = H_3h_2 - H_2h_3,\\
\frac{dh_2}{ds} &= \{h_2, H\} = H_2\{h_2, H_2\} + H_3\{h_2, H_3\} + \{h_2, h_1\} = -H_3h_1 + \epsilon H_3,\\
\frac{dh_3}{ds} &= H_2\{h_3, H_2\} + H_3\{h_3, H_3\} + \{h_3, h_1\} = \{h_3, H\} = H_2h_1 - \epsilon H_2.
\end{aligned} \tag{63}$$

It is well known that

$$I_1 = h_1^2 + h_2^2 + h_3^2 + \epsilon\big(H_1^2 + H_2^2 + H_3^2\big)$$

and

$$I_2 = h_1H_1 + h_2H_2 + h_3H_3$$

are Casimir functions on g_ϵ and hence are integrals of motion for (63), which together with H_1 and H account for four independent constants of motion. Therefore, (63) is completely integrable.

Any solution $h_1(s)$, $h_2(s)$, $h_3(s)$, $H_1(s)$, $H_2(s)$, $H_3(s)$ of (63) defines complex functions $u(s) = H_2(s) + iH_3(s)$ and $w(s) = h_2(s) + ih_3(s)$. Then,

Theorem 6. *There exists a number ξ such that $\psi(s,t) = u(s + \xi t)$ is a solution of the non-linear Schroedinger's equation $\frac{\partial\psi}{\partial t} = -i(\frac{\partial^2\psi}{\partial s^2} + \frac{1}{2}|\psi|^2\psi)$ precisely when $H = \epsilon$, and $\xi = -H_1$.*

Proof. It follows from Eqs. (63) that

$$\frac{du}{ds}(s) = iH_1u(s) - iw(s), \quad \text{and} \quad \frac{dw}{ds} = i(h_1 - \epsilon)u(s). \tag{64}$$

Therefore,

$$\frac{\partial\psi}{\partial t} = i\xi(H_1\psi - w), \quad \text{and} \quad \frac{\partial^2\psi}{\partial s^2} = -H_1^2\psi + H_1w + (h_1 - \epsilon)\psi.$$

Since $H = \frac{1}{2}|\psi|^2 + h_1$,

$$\begin{aligned}
-i\frac{\partial\psi}{\partial t} - \left(\frac{\partial^2\psi}{\partial s^2} + \frac{1}{2}|\psi|^2\psi\right) &= \xi(H_1\psi - w) - \big(-H_1^2\psi + H_1w + (h_1 - \epsilon)\psi + \psi(H - h_1)\big)\\
&= -(\xi + H_1)w + \big(\xi H_1 + H_1^2 + \epsilon - H\big)\psi.
\end{aligned}$$

The preceding equation implies that

$$\xi = -H_1 \quad \text{and} \quad H = \epsilon. \qquad \square \tag{65}$$

Thus the extremals which reside on energy level $H = \epsilon$ generate soliton solutions of the non-linear Schroedinger's equation traveling with speed equal to the level surface $H_1 = -\xi$. These soliton solutions degenerate to the stationary solution when $H_1 = 0$. To show that periodic solutions $u(s)$ exist on energy level $H = \epsilon$ requires explicit formula for $u(s)$ in terms of the remaining constants of motion I_1 and I_2.

To begin with, note that

$$(H_2h_3 - H_3h_2)^2 + (H_2h_2 + H_3h_3)^2 = \big(H_2^2 + H_3^2\big)\big(h_2^2 + h_3^2\big).$$

Then,

$$\begin{aligned}\left(\frac{d}{ds}h_1\right)^2 &= \big(H_2h_3 - H_3h_2\big)^2 = \big(H_2^2 + H_3^2\big)\big(h_2^2 + h_3^2\big) - (H_2h_2 + H_3h_3)^2 \\ &= \big(H_2^2 + H_3^2\big)\big(I_1 - \epsilon\big(H_1^2 + H_2^2 + H_3^2\big) - h_1^2\big) - (I_2 - h_1H_1)^2 \\ &= 2(H - h_1)\big(I_1 - \epsilon H_1^2 - 2\epsilon(H - h_1) - h_1^2\big) - (I_2 - h_1H_1)^2 \\ &= 2h_1^3 + c_1h_1^2 + c_2h_1 + c_3 \end{aligned} \tag{66}$$

where c_1, c_2, c_3 are the constants of motion given by the following expressions

$$c_1 = -\big(H_1^2 - 2H - 4\epsilon\big), \qquad c_2 = \big(2I_2H_1 - 2\epsilon H_1^2 + 4\epsilon H - 2I_1\big), \qquad c_3 = 2H\big(I_1 - \epsilon H_1^2 - 2\epsilon H\big) - I_2^2.$$

Therefore, $h_1(s)$ is expressed in terms of elliptic functions, and since $k^2 = H_2^2 + H_3^2 = 2(H - h_1)$ the same can be said for the curvature of the projected elastic curve. The remaining variables $u = H_2 + iH_3$ and $w = h_2 + ih_3$ can be integrated in terms of two angles θ and ϕ defined as follows:

$$\begin{aligned} I_1 &= h_1^2 + |w|^2 + \epsilon\big(H_1^2 + |u|^2\big) = h_1^2 + |w|^2 + \epsilon\big(H_1^2 + 2(H - h_1)\big) \\ &= (h_1 - \epsilon)^2 + |w|^2 + \epsilon H_1^2 + 2\epsilon H - \epsilon^2 \end{aligned}$$

and therefore,

$$(h_1 - \epsilon)^2 + |w|^2 = J^2 \tag{67}$$

where J^2 denotes $I_1 - \epsilon H_1^2 - 2\epsilon H + \epsilon^2$. Since J is constant along each extremal trajectory, Eq. (67) defines a sphere along each extremal curve. The angles θ and ϕ are defined on that sphere by

$$\big(h_1(s) - \epsilon\big) = J\cos\theta(s) \quad \text{and} \quad w(s) = J\sin\theta(s)e^{i\phi(s)}. \tag{68}$$

It follows that

$$\frac{dh_1}{ds} = -J\sin\theta\frac{d\theta}{ds}, \quad \text{and} \quad \frac{dw}{ds} = w\left(\frac{\cos\theta}{\sin\theta}\frac{d\theta}{ds} + i\frac{d\phi}{ds}\right). \tag{69}$$

Furthermore,

$$\begin{aligned}\frac{u}{w} = \frac{u\bar{w}}{|w|^2} &= \frac{H_2h_2 + H_3h_3 + i(H_3h_2 - H_2h_3)}{J^2 - (h_1 - \epsilon)^2} \\ &= \frac{I_2 - h_1H_1 + i\frac{dh_1}{ds}}{J^2\sin^2\theta} \\ &= \frac{I_2 - h_1H_1}{J^2\sin^2\theta} - \frac{i}{J\sin\theta}\frac{d\theta}{ds}.\end{aligned}$$

According to Eqs. (64) and (68) then

$$\begin{aligned} w\left(\frac{\cos\theta}{\sin\theta}\frac{d\theta}{ds} + i\frac{d\phi}{ds}\right) &= \frac{dw}{ds} \\ &= i(h_1 - \epsilon)u = \left(iJ\cos\theta\frac{(I_2 - h_1H_1)}{J^2\sin^2\theta} + \frac{\cos\theta}{\sin\theta}\frac{d\theta}{ds}\right)w \end{aligned}$$

hence,

$$\frac{d\phi}{ds} = J\cos\theta\frac{(I_2 - h_1H_1)}{J^2\sin^2\theta} = \frac{J\cos\theta(I_2 - \epsilon H_1 - H_1J\cos\theta)}{J^2\sin^2\theta}. \tag{70}$$

The first line of Eq. (66) can be written also as

$$\left(\frac{dh_1}{ds}\right)^2 = 2(H - h_1)|w|^2 - (I_2 - H_1h_1)^2.$$

The substitutions from (68) and (69) in the preceding equation define θ as the solution of the following differential equation

$$\left(\frac{d\theta}{ds}\right)^2 = 2(H - \epsilon - J\cos\theta) - \frac{(I_2 - H_1(\epsilon + J\cos\theta))^2}{J^2\sin^2\theta}. \tag{71}$$

The solutions $\theta(s)$ of (71) parametrize the extremal curves: for then ϕ is given by Eq. (70) and u and w by Eqs. (68) and (69).

We now return to the question of periodicity. Evidently, both u and w are periodic whenever $\phi(0) = \phi(L)$ and $\theta(0) = \theta(L)$. Soliton solutions propagate with speed $-\xi = H_1$ on energy level $H = \epsilon$. On this energy level $\phi(0) = \phi(L)$ and $\theta(0) = \theta(L)$ if and only if

$$\int_0^L \frac{J\cos\theta(I_2 + H_1 - H_1 J\cos\theta)}{J^2\sin^2\theta}\,ds = 0 \tag{72}$$

where θ denotes a closed solution of the equation

$$\left(\frac{d\theta}{ds}\right)^2 = -2J\cos\theta - \frac{(I_2 - H_1(\epsilon + J\cos\theta))^2}{\sin^2\theta}. \tag{73}$$

It is known that there are infinitely many closed solutions for suitable constants I_1, I_2, H_1 (for instance, in [9]), however, we will not go into such details here.

6. Complete integrability

There are further connections between elastic curves and solutions of the non-linear Schroedinger equation that were first noticed by J. Langer and R. Perline, namely that some of the integrals of motion for the elastic curves correspond to the integrals of motion for the non-linear Schroedinger's equation [14].

We will illustrate this phenomenon by considering the functional

$$f(\gamma) = \int_0^L \kappa^2\tau\,ds,$$

where κ and τ are the curvature and torsion of a periodic curve γ. To begin with, is well known that $\kappa^2(s)\tau(s)$ is an integral of motion for the elastic problem ([12] or [10]). Then $f(\gamma)$ is an integral of motion for Heisenberg's magnetic equation as described by the following

Theorem 7.

(a) *The Hamiltonian flow of* $f = \int_0^L \kappa^2(s)\tau(s)\,ds$ *is given by*

$$\frac{\partial\Lambda}{\partial t} = 2\left(\frac{\partial^3\Lambda}{\partial t^3} - \left\langle\frac{\partial^3\Lambda}{\partial t^3}, \Lambda\right\rangle\Lambda\right) - 3\left\langle\Lambda, \frac{\partial^2\Lambda}{\partial t^2}\right\rangle\frac{\partial\Lambda}{\partial t}. \tag{74}$$

(b) *Function* f *Poisson commutes with* $f_0 = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$.

Proof. For simplicity of exposition the proof will be confined to the hyperbolic case. Other cases, which can be analyzed similarly, will be left to the reader. The first part in the proof consists in showing that

$$\kappa^2\tau = -i\left\langle\left[\Lambda, \frac{d\Lambda}{ds}\right], \frac{d^2\Lambda}{ds^2}\right\rangle.$$

Suppose now that $T(s) = \Lambda(s)$ denotes the Hermitian matrix that corresponds to the tangent vector of a horizontal Darboux curve that projects onto a curve $\gamma \in \mathbb{H}^3$. Then $N(s)$ and $B(s)$, the matrices that correspond to the normal and the binormal vectors, are given by

$$N = \frac{1}{\kappa}\frac{d\Lambda}{ds} \quad \text{and} \quad B(s) = \frac{1}{i}\big[T(s), N(s)\big] = -\frac{i}{\kappa}\left[\Lambda, \frac{d\Lambda}{ds}\right].$$

According to the Serret–Frenet equations $\frac{dN}{ds} = -k\Lambda + \tau B$. Therefore,

$$\tau = \left\langle \frac{dN}{ds}, B \right\rangle = -i \left\langle -\frac{1}{\kappa^2}\frac{d\kappa}{ds}\frac{d\Lambda}{ds} + \frac{1}{\kappa}\frac{d^2\Lambda}{ds^2}, \frac{1}{\kappa}\left[\Lambda, \frac{d\Lambda}{ds}\right] \right\rangle = -i\frac{1}{\kappa^2}\left\langle \left[\Lambda, \frac{d\Lambda}{ds}\right], \frac{d^2\Lambda}{ds^2} \right\rangle,$$

and hence

$$\kappa^2\tau = -i\left\langle \left[\Lambda, \frac{d\Lambda}{ds}\right], \frac{d^2\Lambda}{ds^2} \right\rangle.$$

Let $V(s)$ be an arbitrary tangent vector at a frame-periodic horizontal-Darboux curve $g(s)$. Then the directional derivative of f at g in the direction V is given by the following expression:

$$df(V) = -i\frac{\partial}{\partial t}\int_0^L \left\langle \left[Z(s,t), \frac{\partial Z}{\partial s}(s,t)\right], \frac{\partial^2 Z}{\partial s^2}(s,t) \right\rangle ds \bigg|_{t=0}$$

where $Z(s,t)$ denotes a field of Hermitian matrices such that

$$Z(s,0) = \Lambda(s) \quad \text{and} \quad \frac{\partial Z}{\partial t}(s,0) = \frac{dV}{ds}(s).$$

It follows that

$$\begin{aligned}
df(V) &= -i\int_0^L \langle \dddot{V}, [\Lambda, \dot{\Lambda}]\rangle + \langle \ddot{\Lambda}, [\dot{V}, \dot{\Lambda}]\rangle + \langle \ddot{\Lambda}, [\Lambda, \ddot{V}]\rangle\, ds \\
&= -i\int_0^L 2\langle [\ddot{\Lambda}, \Lambda], \ddot{V}\rangle - \langle [\ddot{\Lambda}, \dot{\Lambda}], \dot{V}\rangle\, ds \\
&= i\int_0^L \left\langle 2\left(\frac{d}{ds}([\ddot{\Lambda}, \Lambda]) + [\ddot{\Lambda}, \dot{\Lambda}]\right), \dot{V} \right\rangle ds \\
&= i\int_0^L \langle 2[\dddot{\Lambda}, \Lambda] + 3[\ddot{\Lambda}, \dot{\Lambda}], \dot{V}\rangle\, ds,
\end{aligned}$$

where the dots indicate derivatives with respect to s. In the preceding calculations periodicity of Λ is implicitly assumed to eliminate the boundary terms in the integration by parts.

Let now $V_1(s)$ denote the Hermitian matrix such that $df(V) = \omega_\Lambda(V_1, V)$ for all tangent vectors V.

Then,

$$i\int_0^L \langle 2[\dddot{\Lambda}, \Lambda] + 3[\ddot{\Lambda}, \dot{\Lambda}], \dot{V}\rangle\, ds = \frac{1}{i}\int_0^L \langle [\Lambda, \dot{V}_1], \dot{V}\rangle\, ds$$

which implies

$$\int_0^L \langle [\Lambda, \dot{V}_1] + 2[\dddot{\Lambda}, \Lambda] + 3[\ddot{\Lambda}, \dot{\Lambda}], \dot{V}\rangle\, ds = 0.$$

When $\dot{V} = [\Lambda, C]$ the above becomes

$$\int_0^L \langle [[\Lambda, \dot{V}_1], \Lambda] + 2[[\dddot{\Lambda}, \Lambda], \Lambda] + 3[[\ddot{\Lambda}, \dot{\Lambda}], \Lambda], C(s)\rangle\, ds = 0.$$

Since $C(s)$ can be an arbitrary curve with $C(0)=0$ the preceding integral equality reduces to

$$\big[[\Lambda, \dot{V}_1], \Lambda\big] + 2\big[[\dddot{\Lambda}, \Lambda], \Lambda\big] + 3\big[[\ddot{\Lambda}, \dot{\Lambda}], \Lambda\big] = 0.$$

The Lie bracket relations in Lemma 2 imply that

$$\dot{V}_1 + 2\big(\langle\dddot{\Lambda}, \Lambda\rangle\Lambda - \dddot{\Lambda}\big) + 3\langle\ddot{\Lambda}, \Lambda\rangle\dot{\Lambda} = 0.$$

Now it follows by the arguments used earlier in the paper that the Hamiltonian flow X_f satisfies

$$\frac{\partial\Lambda}{\partial t} = 2\big(\dddot{\Lambda} - \langle\dddot{\Lambda}, \Lambda\rangle\Lambda\big) - 3\langle\Lambda, \ddot{\Lambda}\rangle\dot{\Lambda}.$$

Thus part (a) is proved.

To prove part (b) it is required to show that the Poisson bracket of f_0 and f, given by the formula

$$\{f_0, f\}(\Lambda) = \omega_\Lambda\big(V_0(\Lambda), V_1(\Lambda)\big) = \frac{1}{i}\int_0^L \big\langle\Lambda(s), \big[\dot{V}_0(s), \dot{V}_1(s)\big]\big\rangle\, ds$$

with $\dot{V}_0(\Lambda) = i[\ddot{\Lambda}, \Lambda]$ and $\dot{V}_1 = -(2(\langle\dddot{\Lambda}, \Lambda\rangle\Lambda - \dddot{\Lambda}) + 3\langle\ddot{\Lambda}, \Lambda\rangle\dot{\Lambda})$, is equal to 0.

An easy calculation shows that

$$[\dot{V}_0, \dot{V}_1] = i\big(2\big(\langle\dddot{\Lambda}, \ddot{\Lambda}\rangle\big) - \langle\Lambda, \ddot{\Lambda}\rangle\langle\dddot{\Lambda}, \Lambda\rangle\big) - 3\langle\ddot{\Lambda}, \Lambda\rangle\langle\ddot{\Lambda}, \dot{\Lambda}\rangle\Lambda.$$

Hence,

$$\{f_0, f\} = \int_0^L \big(2\langle\dddot{\Lambda}, \ddot{\Lambda}\rangle - 2\langle\Lambda, \dddot{\Lambda}\rangle\langle\Lambda, \ddot{\Lambda}\rangle - 3\langle\Lambda, \ddot{\Lambda}\rangle\langle\dot{\Lambda}, \ddot{\Lambda}\rangle\big)\, ds.$$

The integral of the first term is zero because $2\langle\dddot{\Lambda}, \ddot{\Lambda}\rangle = \frac{d}{ds}\langle\ddot{\Lambda}, \ddot{\Lambda}\rangle$.

Since

$$2\langle\Lambda, \ddot{\Lambda}\rangle\langle\Lambda, \dddot{\Lambda}\rangle = \frac{d}{ds}\langle\Lambda, \ddot{\Lambda}\rangle^2 - 2\langle\Lambda, \ddot{\Lambda}\rangle\langle\dot{\Lambda}, \ddot{\Lambda}\rangle,$$

the remaining integrand reduces to one term $-\langle\Lambda, \ddot{\Lambda}\rangle\langle\dot{\Lambda}, \ddot{\Lambda}\rangle$. But then $\frac{1}{4}\frac{d}{ds}\langle\dot{\Lambda}, \dot{\Lambda}\rangle^2 = \langle\Lambda, \ddot{\Lambda}\rangle\langle\dot{\Lambda}, \ddot{\Lambda}\rangle$ because $\langle\dot{\Lambda}, \dot{\Lambda}\rangle = -\langle\Lambda, \ddot{\Lambda}\rangle$, and part (b) is proved. □

Theorem 8. *Suppose that $\Lambda(s,t) = R(s,t)B_1R^*(s,t)$ evolves according to Eq.* (74) *where $R(s,t)$ is the solution of*

$$\frac{\partial R}{\partial s}(s,t) = R(s,t)\begin{pmatrix} 0 & u(s,t) \\ -\bar{u}(s,t) & 0 \end{pmatrix}, \quad R(0,t) = I.$$

Then, $u(s,t)$ is a solution of

$$\frac{\partial u}{\partial t} - 3|u|^2\frac{\partial u}{\partial s} - 2\frac{\partial^3 u}{\partial s^3} = 0. \tag{75}$$

This theorem is proved by a calculation similar to the one used in the proof of Theorem 5, the details of which will not be re produced here.

Eq. (75) is similar to the modified Korteweg–de Vries equation (R. Abraham and J. Marsden [1])

$$v_t - 6v^2v_x + v_{xxx} = 0 \tag{76}$$

with some notable differences. Eq. (75) is a complex equation while the modified Korteweg–de Vries equation is a real equation. Because of the difference in sign in front of the third derivative it is not apparent that Eq. (75) is the complexification (modulo some homothetical transformation) of the Eq. (76). It remains an intriguing question if there are any connections between the Korteweg–de Vries equation and the elastic curves.

Functions f_0 and $f_1 = f$ also appear in a paper on integrability of the non-linear Schroedinger's equation by C. Shabat and V. Zacharov [17], as first noticed by J. Langer and R. Perline [14], but in a completely different context.

The first two integrals of motion in the paper of Shabat and Zacharov are up to the constant factors given by the following integrals:

$$C_1=\int_{-\infty}^{\infty}\left|u(s,t)\right|^2 ds,\qquad C_2=\int_{-\infty}^{\infty}\big(u(s,t)\dot{\bar{u}}(s,t)-\bar{u}(s,t)\dot{u}(s,t)\big)\,ds$$

where they are interpreted as the number of particles and the momentum. To see that C_1 and C_2 are in exact correspondence with functions f_0 and f_1 assume that the Darboux curves are expressed by reduced frames $R(s)$ as in Definition 5.1 i.e., as the solutions of

$$\frac{dR}{ds}(s)=R(s)U(s)\quad \text{with } U(s)=u_2(s)A_2+u_3(s)A_3.$$

Then,

$$f_0=\frac{1}{2}\int_0^L\left\|\dot{\Lambda}(s)\right\|^2 ds=\frac{1}{2}\int_0^L\left\|\big[B_1,U(s)\big]\right\|^2 ds=\frac{1}{2}\int_0^L\left|u(s)\right|^2 ds.$$

Hence, C_1 corresponds to $\int_0^L\kappa^2(s)\,ds$. Furthermore,

$$\int_0^L\kappa^2\tau\,ds=\frac{1}{2i}\int_0^L\big(u(s)\dot{\bar{u}}(s)-\bar{u}(s)\dot{u}(s)\big)\,ds$$

because

$$i\big\langle\Lambda,[\dot{\Lambda},\ddot{\Lambda}]\big\rangle=\big\langle\big[[B_1,U],[B_1,\dot{U}]\big],B_1\big\rangle=\operatorname{Im}(\bar{u}\dot{u}),$$

where $\operatorname{Im}(z)$ denotes the imaginary part of a complex number z. Therefore f_1 corresponds to C_2.

In the language of mathematical physics vector $\int_0^L\Lambda(s)\,ds$ is called the total spin [5]. In this paper it appears as the moment map discussed in the previous section. It is a conserved quantity since the Hamiltonian is invariant under the action of SU_2. This fact can be verified directly as follows:

$$\frac{\partial}{\partial t}\int_0^L\Lambda(t,s)\,ds=\int_0^L\frac{\partial\Lambda}{\partial t}(t,s)\,ds=i\int_0^L\left[\frac{\partial^2\Lambda}{\partial s^2},\Lambda\right]ds=i\int_0^L\frac{\partial}{\partial s}[\Lambda,\dot{\Lambda}]\,ds=0.$$

The third integral of motion C_3 in [17], called the energy, is given by

$$C_3=\int_{-\infty}^{\infty}\left(\left|\frac{\partial u}{\partial s}(s,t)\right|^2-\frac{1}{4}\left|u(s,t)\right|^4\right)ds.$$

It corresponds to the function

$$f_2=\int_0^L\left(\left\|\ddot{\Lambda}(s)\right\|^2-\frac{5}{4}\left\|\dot{\Lambda}(s)\right\|^4\right)ds=\int_0^L\left(\frac{\partial\kappa}{\partial s}(s)^2+\kappa^2(s)\tau^2(s)-\frac{1}{4}\kappa^4(s)\right)ds.$$

It can be shown that functions f_0, f_1, f_2 are in involution, i.e., that they Poisson commute with each other. There seems to be a hierarchy of functions that contains f_0, f_1, f_2 such that any two functions in the hierarchy Poisson commute. For instance, D. Krepski (in a personal communication) has shown that $f_3=\int_0^L\tau(s)\,ds$ is in this hierarchy, and he has also shown that the flow of the corresponding Hamiltonian vector field generates the curve shortening equation [4]

$$\frac{\partial\Lambda}{\partial t}(s,t)=\frac{\partial\Lambda}{\partial s}(s,t)=\kappa(s,t)N(s,t).$$

A detailed investigation of this hierarchy of Poisson commuting functions and its relation to the hierarchies obtained by Langer–Perline and Shabat–Zacharov will be deferred to a separate study.

Acknowledgement

This paper grew out of numerous discussions with B. Wagneur during his postdoctoral tenure at the University of Toronto. I am deeply indebted to him for his contributions and critical comments of the early drafts. I am also indebted to the referee for his constructive criticisms of the earlier versions of the present manuscript.

References

[1] R. Abraham, J. Marsden, Foundations of Mechanics, Benjamin–Cummings, Reading, MA, 1978.
[2] V.I. Arnold, B.A. Khesin, Topological Methods in Hydrodynamics, Appl. Math. Sci., vol. 125, Springer-Verlag, New York, 1998.
[3] J.P. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization, Progr. Math., vol. 108, Birkhäuser, Boston, 1993.
[4] C.L. Epstein, M.I. Weinstein, A stable manifold theorem for the curve shortening equation, Comm. Pure Appl. Math. XL (1987) 119–139.
[5] L. Faddeev, L. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag, Berlin, 1980.
[6] R.S. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. 7 (1972) 65–221.
[7] H. Hasimoto, Motion of a vortex filament and its relation to elastica, J. Phys. Soc. Japan 31 (1971) 293–294.
[8] H. Hasimoto, A soliton on a vortex filament, J. Fluid Mech. 51 (1972) 477–485.
[9] T. Ivey, D.A. Singer, Knot types, homotopies and stability of closed elastic curves, Proc. London Math. Soc. 79 (3) (1999) 429–450.
[10] V. Jurdjevic, Hamiltonian systems on complex Lie groups and their homogeneous spaces, Mem. Amer. Math. Soc. 178 (838) (2005).
[11] V. Jurdjevic, Geometric Control Theory, Cambridge Studies in Advanced Mathematics, vol. 51, Cambridge Univ. Press, New York, 1997.
[12] V. Jurdjevic, F. Monroy-Perez, Hamiltonian systems on Lie groups: Elastic curves, tops and constrained geodesic problems, in: Non-Linear Geometric Control Theory and its Applications, World Scientific Publishing Co., Singapore, 2002, pp. 3–52.
[13] V. Jurdjevic, Hamiltonian systems on Lie groups: Kowalewski type, Ann. Math. 150 (1999) 1–40.
[14] J. Langer, R. Perline, Poisson geometry of the filament equation, J. Nonlinear Sci. 1 (1978) 71–93.
[15] F. Magri, A simple model for the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156–1162.
[16] J. Millson, B.A. Zombro, A Kähler structure on the moduli spaces of isometric maps of a circle into Euclidean spaces, Invent. Math. 123 (1) (1996) 35–59.
[17] C. Shabat, V. Zakharov, Exact theory of two dimensional self-focusing and one dimensional self-modulation of waves in non-linear media, Sov. Phys. JETP 34 (1972) 62–69.
[18] S. Sternberg, Lectures on Differential Geometry, Prentice-Hall Inc., Englewood-Cliffs, NJ, 1964.