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Sujet : **ETUDE MATHEMATIQUE DE MODELES ASYMPTOTIQUES POUR LES ONDES D'ALFVEN**

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Abstract :

Alfven waves can be seen in several magnetised plasmas, in the presence of a strong magnetic field B_o. These phenomena are described by the general MHD equations. As this equations are very complex from a mathematical point of view, a great interest has been given in the last few years to its asymptotical limits.
In this thesis, we study some of those asymptotical models. We begin by considering the multi-dimensional version of the DNLS equation, derived by Myolhus and Wyller (1988). After obtaining some Stricharz-type estimates for the linear part, we prove the existence of local solutions in an analytical Sobolev space, for the general system. Then, we show the local well-posedness of the Cauchy problem for a DNLS equation with non-local term, in the Sobolev space $H^{1^+}(\mathbb{R})$.
We deal afterwards with the Zakharov-Rubenchick equation. This equation describes the interaction between the high-frequency wave with low-frequency waves of acoustic type, excited by the modulation of the high frequency wave, by means of ponderomotive forces. After showing that this problem is locally well-posed in $H^2(\mathbb{R})$, we get global solutions via energy estimates. We then showed the existence and orbital stability of solitary wave solutions.
Finally, we proved rigourously that in a sens the DNLS equation can be approached by the cubic non-linear Schrödinger equation. A numerical illustration of this fact is given via a finite-difference scheme.

Key words : Alfven waves -Oscillating integrals- the DNLS equation - Analytical Sobolev spaces - the Zakharov-Rubenchick equation- Solitary waves- Orbital stability - Asymptotical limits.

AMS 1991 Mathematics Subject Classification : 35Q55, 35A07, 35A35, 35A05, 35B35.

Ni dans le ciel, ni au milieu de l'océan, ni en pénétrant dans la plus profonde des cavernes, il n'y a pas un endroit sur la terre où l'on puisse fuir les conséquences de nos actions.

Dharmappada

-Vincent ! Comment fais-tu ça, Vincent ?
Comment as-tu réussi à faire tout ça ?
-Tu veux connaitre mon secret ? Voilà mon secret, Anton :
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in "GATTACA"-Andrew Niccol

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Chapitre 1

Introduction : Aspects physiques

On appelle "plasma" un fluide ionisé, de densité suffisamment forte pour que les forces exercées entre elles par les particules le constituant ne soient pas négligeables face aux forces induites par des champs extérieurs. On peut en quelque sorte considérer l'état de plasma comme un quatrième état de la matière, après les états solide, liquide et gazeux.

Historiquement, quatre domaines de la physique eurent un rôle prépondérant dans le développement de la physique des plasmas : les décharges électriques dans les gaz, la théorie cinétique des particules chargées, la propagation d'ondes radio dans la ionosphère et l'astrophysique du soleil.

Michael Faraday, en 1830, fût le premier à s'intéresser à des décharges électriques dans des gaz, suivi de Ampère qui étudia le mouvement d'un liquide conducteur, le mercure, soumis à un champ magnétique extérieur. La théorie des décharges dans des milieux gazeux continua ensuite avec Hittorf en 1869 et Sir William Crookes en 1879, ce dernier ayant été le premier à suggérer l'existence d'un quatrième état de la matière. En 1903, Towsend définit le coefficient d'ionisation d'un gaz, qui mesure son niveau d'ionisation dû aux chocs entre ses particules. Un travail soutenu de plusieurs décennies ont fait de Towsend et Thompson les pionniers de la théorie moderne des décharges électriques en milieu gazeux.

D'un point de vue de la théorie cinétique des particules chargées, Lorentz appliqua avec succès en 1903 la théorie générale de Boltzmann au problème de la conduction électrique dans un gaz d'électrons. Le cas de la distribution des vitesses des électrons dans un gaz soumis à un champ électrique fût traité par Dryvesteyn en 1930, et par Möse, Allis et Lamar en 1935. La théorie cinétique de Boltzman et Maxwell se montra particulièrement perfomante dans la description du comportement d'un fluide n'étant pas en équilibre thermique avec l'extérieur. Dans cette direction, l'article "Motion of ions and electrons" (Allis, 1956) constitue un point essentiel dans la construction d'un modèle de fluide pour les plasmas.

Quant à la propagation d'ondes radio, Marconi démontra en 1901 que celles-ci pouvaient parcourir de très longues distances, notamment traverser l' Atlantique. Ceci fût expliqué par Heaviside et Kennelly, qui attribuèrent le phénomène à certaines propriétés de la ionosphère, créée par des radiations ionisantes issues du soleil. En 1925, Nichols et Schelling étudièrent le phénomène de propagation d'ondes dans un plasma soumis à un champ magnétique extérieur (la ionosphère soumise au champ magnétique terrestre) et lancèrent, avec Breit, la théorie "magnéto-ionique".

Finalement, concernant l'astrophysique, une première observation fût faite par Carrington dans les années 1850, qui constata l'existence d'un lien entre certaines perturbations du champ magnétique terrestre et les taches solaires. En 1908, Hale démontra qu'un intense champ magnétique était associé à ces taches. En utilisant la théorie des tempêtes géo-magnétiques, Chapman et Ferraro ont décrit, dans les années 30, l'intéraction mutuelle entre les forces hydrodynamiques et magnétiques dans les gaz ionisés. C'est dans ce contexte que se situent les travaux d' Alfvén, datant ds années 40, et concernant les ondes magnéto-hydro-dynamiques.

On peut ainsi séparer la théorie des plasmas en deux pôles distincts. Le premier traite des décharges électriques et des fortes intéractions entre les particules neutres et les électrons. Le deuxième correspond aux plasmas astro et géophysiques ou aux plasmas de fusion, et relève plutôt des intéractions entre champs magnétiques et plasmas.

Les effets des plasmas dans l'univers sont presque toujours liés à leur interaction avec des champs magnétiques, dont l'abondante existence dans la galaxie a été à maintes reprises vérifiée, et dont l'intensité peut varier entre 10^{-5} et 10^4 Gauss. Un plasma hautement conducteur et un champ magnétique tendent à se figer dans la situation où ils ont initialement interagi. Ceci signifie par exemple que le plasma éjecté lors d'une explosion solaire va non seulement suivre le champ magnétique mais aussi le "tordre" dans la direction de son propre flot. Pour comprendre ce phénomène, il est utile de considérer le mouvement individuel des particules en présence d'un champ magnétique. Le mouvement dans le plan transverse au champ est contraint à être circulaire autour des lignes de champ alors que le mouvement dans la direction du champ reste inchangé. Ainsi le plasma tend à former dans son mouvement des zones plus lumineuses coïncidant avec les lignes de champ([38],[23]).

Les ondes d'Alfvén sont un phénomène de propagation observé lorsque l'on soumet un plasma magnétisé à un fort champ magnétique extérieur $\mathbf{B_o}$. Ces ondes, de type hydromagnétique, se dissipent plus lentement que les ondes magnéto-acoustiques, et peuvent ainsi parcourir de plus longues distances. Leur présence a été observée dans le vent solaire (un flot de particules chargées émanant du soleil qui vient affecter le champ magnétique terrestre) et plus généralement dans divers plasmas astrophysiques([41]).
Ces phénomènes sont décrits par les équations générales de la magnétohydrodynamique (MHD). Ces équations étant très complexes d'un point de vue mathématique, un intérêt

croissant a été porté ces dernières années à ses limites asymptotiques. En tenant compte de l'effet Hall dans la loi d'Ohm généralisée , le système (MHD) devient dispersif, et il est possible de distinguer, dans le cas de la propagation d'ondes d'Alfvén, deux modèles asymptotiques distincts([9]).

Premièrement, dans le cas où la dispersion est d'un ordre de grandeur comparable aux effets non linéaires, les ondes d'Alvén sont décrites essentiellement par l'équation de Schrödinger Non-Linéaire Dérivée, et non par le modèle "canonique" que constitue l'équation de Korteweg-de-Vries (KdV). Ceci est dû à l'égalité entre la vitesse de phase de l'onde et la vitesse du son, dans la limite de dispersion nulle.

Deuxièment, dans le cas où la dispersion est gardée finie, la dynamique des ondes d'Alfvén monochromatiques et de petite amplitude est régie par l'équation de Schrödinger Non Linéaire, avec des éventuels couplages à des champs de basse fréquence.

Dans la suite de cette introduction, on décrira les méthodes permettant de déduire ces différents modèles à partir des équations générales de la MHD.

1.1 Cas d'une dispersion comparable aux effets non linéaires

Dans ce paragraphe, on se placera systématiquement dans la limite des ondes à grande échelle([36],[49]).
En se restreignant au cas uni-dimensionnel, des ondes d'Alfvén faiblement dispersives et faiblement non-linéaires, se propageant dans la direction du champ extérieur $\mathbf{B_o}$ sont gouvernées par l'équation de Schrödinger Non Linéaire Dérivée (SNLD):

$$\begin{cases} q_t + iq_{xx} + (|q|^2 q)_x = 0 \\ q(x) \to 0, \\ |x| \to +\infty. \end{cases} \tag{1.1}$$

Ici,

$$q = B_y + iB_z$$

représente les composantes du champ magnétique transverses à la direction de propagation (0_x), dans un référentiel se déplaç suivant à la vitesse de l'onde.
Afin de représenter des ondes se déplaçant de manière quasi-parallèle au champ extérieur, on peut également se donner une condition de frontière non nulle à l'infini :

$$q(x) \to q_o \neq 0 \ , \ |x| \to +\infty$$

Pour un plasma chaud sans collisions, on obtient l'équation non-locale

$$q_t + iq_{xx} + (|q|^2 q)_x = \sigma(q[v.p. \int_{-\infty}^{+\infty} \frac{|q|^2(x')}{x - x'} dx'])_x \tag{1.2}$$

où le terme de droite représente l'effet des particules résonantes sur les modulations de l'onde.

Tous ces modèles ne décrivent pas d'éventuelles variations du champ dans le plan transverse.
Afin de tenir compte de ces mêmes effets, Myollhus et Wyller ([36]) proposent une version multidimensionnelle de l'équation SNLD :

$$\begin{cases} q_t + iq_{xx} + [(|q|^2 + 2A)q]_x - \tilde{\nabla}(|q|^2 + 2A) = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y + \frac{1}{2i}(q - \overline{q})_z = 0, \\ \tilde{\nabla} = \frac{\partial}{\partial y} + i\frac{\partial}{\partial z} \end{cases} \tag{1.3}$$

Ce modèle représente la propagation d'ondes d'Alfvén localisées multi-directionnelles, faiblement non-linéaires, faiblement dispersives, faiblement diffractives, se propageant quasi parallèllement au champ magnétique ambiant dans un plasma sans collisions.
Pour des solutions multidimensionnelles de type trains d'onde, un système plus général doit être considéré ([33]).
On commence ce paragraphe par décrire le procédé qui permet de déduire les équations (1.1), (1.2) et (1.3) à partir des équations générales de la MHD (voir [36]).

1.1.1 Equations de la Magnétohydrodynamique avec effet Hall

L'équation de continuité s'écrit

$$\frac{\partial}{\partial t}\rho + \nabla.(\rho \mathbf{u}) = 0 \tag{1.4}$$

où ρ est la densité de masse et $\mathbf{u}$ la vitesse du fluide.
On a par la suite l'équation du mouvement

$$\rho \frac{d}{dt}\mathbf{u} - \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla.(\Pi + \mathbf{P}) \tag{1.5}$$

où $\mathbf{B}$ est le champ magnétique et $\Pi + \mathbf{P}$ le tenseur de pression totale avec :

- $\mathbf{P}$ la pression extérieure donnée par

$$\mathbf{P} = \frac{1}{B^2}(p^{||} - p^{\perp})\mathbf{B}^t\mathbf{B} + p^{\perp}\mathbf{I} \tag{1.6}$$

– Π le terme correctif de Larmor (Finite-Larmor-Radius). On utilisera les composantes

$$\begin{cases} \pi_{xy} = \frac{1}{\Omega_i}[p^{i\perp}(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}) - 2p^{i\|}\frac{\partial u_z}{\partial x}] \\ \pi_{xz} = -\frac{1}{\Omega_i}[p^{i\perp}(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}) - 2p^{i\|}\frac{\partial u_y}{\partial x}] \end{cases} \tag{1.7}$$

Ici, $p^{i\perp,\|}$ représente la pression partielle (perpendiculaire et parallèle) des ions.

Finalement, en combinant la loi d'Ohm généralisée

$$\mathbf{E} = -\frac{1}{c}\mathbf{u} \times \mathbf{B} + \frac{m_i}{\rho q_i}(\frac{1}{c}\mathbf{j} \times \mathbf{B} - \nabla.\mathbf{P}^e)$$

à l'équation d'induction magnétique, on obtient

$$\frac{\partial}{\partial t}\mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{m_i c}{q_i}\nabla \times (\frac{1}{4\pi\rho}(\nabla \times \mathbf{B}) \times \mathbf{B} - \frac{1}{\rho}\nabla.\mathbf{P}^e) \tag{1.8}$$

avec $\mathbf{E}$ le champ magnétique, $\mathbf{j}$ la densité de courant, $\mathbf{P}^e$ le tenseur de pression des électrons, et q_i, m_i la charge et la masse des ions.

1.1.2 Méthode de perturbation réductive

On considère une base orthonormée (e_x, e_y, e_z).
L'état ambiant constant sera caractérisé par :

$$\rho = \rho_o,$$

$$\mathbf{B} = B_o e_x,$$

$$\mathbf{u} = 0, \text{ et}$$

$$\mathbf{P_o} = (p_o^{\|} - p_o^{\perp})e_x^t e_x + p_o^{\perp}\mathbf{I}$$

Au niveau du plasma, on écrira

$$\mathbf{B} = (B_x, B_y, B_z) \text{ et } \mathbf{u} = (u_x, u_y, u_z).$$

Aussi, il sera commode de prendre une notation complexe pour décrire les composantes de champ magnétique et de vitesse dans le plan transverse à la direction de propagation :

$$\tilde{B} = B_y + iB_z \text{ et } \tilde{u} = u_y + iu_z$$

La méthode de perturbation réductive est basée sur le changement de variables :

$$\begin{cases} \xi = \epsilon(x - \lambda t) \\ \eta = \epsilon^{\frac{3}{2}} y \\ \zeta = \epsilon^{\frac{3}{2}} z \\ \tau = \epsilon^2 t \end{cases} \tag{1.9}$$

où λ est la vitesse de l'onde d'Alfvén à l'ordre 0, à déterminer plus tard.

On effectue les développements a priori :

$$\begin{cases} \tilde{B} = \epsilon^{\frac{1}{2}}(\tilde{B}^{(1)} + \epsilon\tilde{B}^{(2)} + ...) \\ \tilde{u} = \epsilon^{\frac{1}{2}}(\tilde{u}^{(1)} + \epsilon\tilde{u}^{(2)} + ...) \end{cases} \tag{1.10}$$

et

$$\begin{cases} B_x = B_o + \epsilon B_x^{(1)} + ... \\ u_x = B_o + \epsilon u_x^{(1)} + ... \\ \rho = \rho_o + \epsilon\rho^{(1)} + ... \\ p^{||} = p_o^{||} + \epsilon p^{||(1)} + ... \\ p^{\perp} = p_o^{\perp} + \epsilon p^{\perp(1)} + ... \end{cases} \tag{1.11}$$

Le petit paramètre ϵ et ses puissances caractériseront l'amplitude de l'onde, ainsi que les échelles caractéristiques dans les directions perpendiculaire et parallèle.
Les changements d'échelle (1.9) ont été choisis de sorte que les divers effets contribuent tous au même ordre et s'équilibrent.
Le choix du temps τ impose une condition d'évolution lente, "bloquant" ainsi des éventuels modes se propageant à une vitesse différente de λ.
En injectant ces développements dans les équations (1.4), (1.5) et (1.8), on obtient :

-à l'ordre $\epsilon^{\frac{1}{2}}$:
On a les contributions de (1.5) et (1.8).
On obtient :

$$\lambda = \lambda_o \equiv= \sqrt{v_A^2 + v_\perp^2 - v_{||}^2},$$

où

$$v_A^2 = \frac{B_o^2}{4\pi\rho_o} \text{ (vitesse de Alfvén) et } v_{(\perp,||)}^2 = \frac{p_o^{\perp,||}}{\rho_o}.$$

En supposant la condition de stabilité du plasma :

$$v_A^2 + \frac{p_o^{\perp} - p_o^{||}}{\rho_o} > 0,$$

on peut supposer λ réel.

De plus,

$$\tilde{u}^{(1)} = -\frac{\lambda_o}{B_o}\tilde{B}^{(1)}. \tag{1.12}$$

-à l'ordre ϵ :
En prenant la première composante de (1.8),

$$-\lambda_o \frac{\partial B_x^{(1)}}{\partial \xi} = -B_o \nabla_\perp . u_\perp^{(1)} \tag{1.13}$$

avec

$$\nabla_\perp = (\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta})$$

En combinant ceci avec (1.12),

$$\frac{\partial}{\partial \xi} B_x^{(1)} + \nabla_\perp . \mathbf{B}_\perp^{(1)} = 0. \tag{1.14}$$

Par l'équation de continuité,

$$\frac{1}{\lambda_o} u_x^{(1)} = \frac{1}{\lambda_o^2} (\frac{1}{2} v_A^2 + v_\perp^2 - v_\parallel^2) \frac{|\tilde{B}^{(1)}|^2}{B_o^2} + \frac{1}{\lambda_o^2} (v_\perp^2 - v_\parallel^2) \frac{|B_x^{(1)}|^2}{B_o^2} + \frac{1}{\rho_o \lambda_o^2} p^{\|(1)}. \tag{1.15}$$

-à l'ordre $\epsilon^{\frac{3}{2}}$:
A cet ordre, on obtient après quelques calculs simples :

$$\frac{\partial}{\partial \tau} \tilde{B}^{(1)} + i c_3 \frac{\partial^2}{\partial \xi^2} \tilde{B}^{(1)} + \frac{\partial}{\partial \xi} (N \tilde{B}^{(1)}) - B_o \tilde{\nabla} N = 0, \tag{1.16}$$

où

$$\left\{ \begin{array}{l} N = \frac{v_A^2}{4\lambda_o} (\frac{|B^{(1)}|^2}{B_o^2} + 2 \frac{B_x^{(1)}}{B_o}) + \frac{1}{2\lambda_o \rho_o} p^{\perp(1)} \\ c_3 = \frac{1}{2\Omega_i} (\lambda_o^2 + 3 v_{i\|}^2 - 2 v_{i\perp}^2). \end{array} \right. \tag{1.17}$$

Il reste uniquement à exprimer $p^{\perp(1)}$ en fonction de $\tilde{B}^{(1)}$ et $B_x^{(1)}$, ce qui sera fait en considérant la cinétique du plasma.

1.1.3 Aspect cinétique

On considère des particules de type "k", de masse m_k, de charge q_k, de vitesse v_k et de moments magnétique et conjugué respectivement μ_k et p_k.
Par les travaux de Grad ([18]), l'équation vérifiée par la fonction de distribution des particules $f_k(p_k, \mu_k)$ dans l'approximation "centre-guide"est

$$\frac{\partial}{\partial t} f_k + \frac{1}{m_k^2} \chi^2 p_k \frac{\partial}{\partial \sigma} (\mu_k B + q_k \phi - \frac{1}{2} m_k v_{k\|}^2 + \frac{1}{2 m_k} \chi^2 p_k^2) \frac{\partial}{\partial p_k} f_k = 0 \tag{1.18}$$

σ étant la coordonnée le long d'une ligne de champ et χ un facteur d'étirement de ces lignes de champ.

On reconnait entre parenthèses les diverses forces exercées :

-La force miroir de potentiel $\mu_k B$ et le champ électrique parallèle de potentiel $q_k\phi$. Ces deux termes sont ceux qui contribuent pour le terme d'amortissement de Landau dans le résultat final.

-Les termes $-\frac{1}{2}m_k v_{k\|}^2$ et $\frac{1}{2m_k}\chi^2 p_k^2$ qui se simplifient à la résonance onde-particule caractérisée par $\frac{p_k}{m_k} = \lambda_o$ puisque

$$p_k = \frac{m_k v_{k\|}}{\chi} \text{ et } \mu_k = \frac{1}{2B} m_k v_{k\|}^2.$$

Comme précédemment, on effectue les développements :

$$\left\{ \begin{array}{l} f_k = f_k^{(0)} + \epsilon f_k^{(1)} + ... \\ \phi = \epsilon\phi^{(1)} + ... \end{array} \right. \tag{1.19}$$

De plus, on peut supposer

$$\left\{ \begin{array}{l} \frac{\partial}{\partial\sigma} = \epsilon\frac{\partial}{\partial\xi} + O(\epsilon^2) \\ \chi = 1 + \frac{1}{2}\epsilon\frac{|\tilde{B}^{(1)}|^2}{B_o^2} + O(\epsilon^2). \end{array} \right. \tag{1.20}$$

En termes de la variable $\overline{v_k} = \frac{p_k}{m_k}$ on peut écrire :

$$(\overline{v_k} - \lambda_o)\frac{\partial}{\partial\xi}f_k^{(1)} - \frac{\partial}{\partial\xi}[\frac{V_k}{m_k} + \frac{1}{2}(\overline{v_k}^2 - \lambda_o^2)\frac{|\tilde{B}^{(1)}|^2}{B_o^2}]\frac{\partial}{\partial\overline{v_k}}f_k^{(0)} = 0 \tag{1.21}$$

avec

$$\left\{ \begin{array}{l} V_k = B_o\mu_k A^{(1)} + q_k\phi^{(1)} \\ A^{(1)} = \frac{B_x^{(1)}}{B_o} + \frac{1}{2}\frac{|\tilde{B}^{(1)}|^2}{B_o^2}. \end{array} \right.$$

On obtient

$$f_k^{(1)} = [\frac{V_k}{m_k(\overline{v_k} - \lambda_o)} + \frac{1}{2}(\overline{v_k} + \lambda_o)\frac{|\tilde{B}^{(1)}|^2}{B_o^2}]\frac{\partial}{\partial\overline{v_k}}f_k^{(0)} \tag{1.22}$$

ce qui n'est pas bien défini pour $\overline{v_k} = \lambda_o$. On peut cependant obtenir une limite asymptotique pour l'intégrale de $f_k^{(1)}$:

$$\int_{-\infty}^{+\infty} \frac{V_k(\xi)}{m_k(\overline{v_k} - \lambda_o)}\frac{\partial}{\partial\overline{v_k}}f_k^{(0)}d\overline{v_k} \to \overline{G_k}V_k$$

où l'opérateur $\overline{G_k}$ est donné par

$$G_k = G_k^{(r)} I + G_k^{(i)} H.$$

Ici, I est l'identité, H la transformée de Hilbert, et

$$G_k^{(r)} + iG_k^{(i)} = p.p.\left(\int_{-\infty}^{+\infty} \frac{1}{\overline{v_k} - \lambda_o} \frac{\partial}{\partial \overline{v_k}} f_k^{(0)} d\overline{v_k} + i\pi \frac{\partial}{\partial \overline{v_k}} f_k^{(0)} d\overline{v_k}\right)|_{\overline{v}_k = \lambda_o}$$

Finalement, en écrivant la condition de neutralité

$$\sum_k q_k n_k = 0,$$

on obtient à l'ordre ϵ :

$$\sum_k \int_0^{+\infty} q_k \overline{G}_k V_k d\mu_k = 0.$$

A partir de cette relation, on peut exprimer V_k en fonction de $A^{(1)}$.
En calculant $p^{\perp(1)}$ en tant que somme des pressions partielles des espèces présentes, on obtient

$$p^{\perp(1)} = [2p_o^{\perp} + B_o^3(\overline{N} - \overline{M}^2 \overline{L}^{-1})] A^{(1)} \tag{1.23}$$

où

$$\begin{cases} \overline{L} = \sum_k \int_0^{\infty} q_k^2 \overline{G}_k(\lambda_o, \mu_k) d\mu_k \\ \overline{M} = \sum_k \int_0^{\infty} q_k \mu_k \overline{G}_k(\lambda_o, \mu_k) d\mu_k \\ \overline{N} = \sum_k \int_0^{\infty} \mu_k^2 \overline{G}_k(\lambda_o, \mu_k) d\mu_k \end{cases}$$

1.1.4 L'équation générale

Par (1.14), et en combinant (1.15) et (1.23), on obtient

$$\begin{cases} \frac{\partial}{\partial \tau}\tilde{B} + ic_3 \frac{\partial^2}{\partial \xi^2}\tilde{B} + \frac{\partial}{\partial \xi}[(c_1 + c_2\overline{H})(|\tilde{B}|^2 + 2B_oB_x)\tilde{B}] - B_o\tilde{\nabla}(c_1 + c_2\overline{H})(|\tilde{B}|^2 + 2B_oB_x) = 0 \\ \frac{\partial}{\partial \xi}B_x + \nabla_{\perp}B_{\perp} = 0 \end{cases} \tag{1.24}$$

les c_i étant des constantes réelles.
Ce modèle général permet ainsi de trouver l'équation (1.1) (en négligeant les effets transverses et de résonance), l'équation (1.2) (en négligeant les effets transverses) et finalement le modèle multi-dimensionnel (1.3), lorsque $c_2 = 0$.

1.2 Cas de la dispersion finie

En négligeant les effets dissipatifs, et en tenant compte de l'effet Hall au niveau de la loi d'Ohm généralisée, les équations de la MHD s'écrivent, en unités sans dimension,

$$\left\{\begin{array}{l} \rho_t + \nabla.(\rho\mathbf{u}) = 0 \\ \rho(\mathbf{u}_t + \nabla\mathbf{u}.\mathbf{u}) = -\frac{\beta}{\gamma}\nabla.\rho^\gamma + (\nabla\times\mathbf{B})\times\mathbf{B} \\ \mathbf{B}_t = \nabla\times(\mathbf{u}\times\mathbf{B}) - \frac{1}{R_i}\nabla\times(\frac{1}{\rho}(\nabla\times\mathbf{B})\times\mathbf{B}) = 0 \\ \nabla.\mathbf{B} = 0 \end{array}\right. \tag{1.25}$$

où R_i est la fréquence ion-cyclotron normalisée, γ la constante polytropique des gaz et β est le paramètre fini caractérisant le plasma:

$$\beta = \sqrt{\frac{v_{son}}{v_A}}.$$

En faisant l'hypothèse unidimensionnelle, (1.25) devient [9]:

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0 \\ \frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = -\frac{1}{\rho}\frac{\partial}{\partial x}(\frac{\beta}{\gamma}\rho^\gamma + \frac{1}{2}|b|^2) \\ \rho(\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v) = \frac{\partial}{\partial x}b \\ \frac{\partial}{\partial t}b + \frac{\partial}{\partial x}(ub - v) = i\frac{\sigma}{R_i}\frac{\partial}{\partial x}(\frac{1}{\rho}\frac{\partial b}{\partial x}), \end{array}\right. \tag{1.26}$$

où l'on a noté $u = u_x$ la première composante de u et avec la représentation complexe

$$\left\{\begin{array}{l} b = B_y - i\sigma B_z \\ v = u_y - i\sigma u_z \ , \ (\sigma = 1 \text{ ou } \sigma = -1). \end{array}\right.$$

La composante B_x, supposée constante à cause de la condition $\nabla.B = 0$, a été absorbée dans le champ extérieur.
Le système (1.26) admet pour solutions exactes des ondes d'Alfvén monochromatiques polarisées circulairement (à droite ou à gauche suivant que $\sigma = 1$ ou $\sigma = -1$) de la forme

$$\left\{\begin{array}{l} b = -\frac{\omega}{k}v = B_o e^{i(kx-\omega t)} \\ u = 0 \\ \rho = 1 \end{array}\right.$$

et avec la relation de dispersion reliant nombre d'onde et fréquence:

$$\omega = \frac{\sigma}{2R_i}k^2 + k\sqrt{1 + \frac{k^2}{4R_i^2}}.$$

1.2.1 Equations de modulation

Dans cette partie, on déduit de (1.26) les équations d'enveloppe pour les modulations de l'onde d'Alfvén, par la méthode multi-échelle habituelle ([8]) à partir de la solution exacte

décrite plus haut.
On définit le petit paramètre ϵ et les nouvelles variables lentes

$$\begin{cases} X = \epsilon x \\ \tau = \epsilon t \end{cases}$$

et on fait les développements

$$\begin{cases} v = \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \ldots \\ b = \epsilon b_1 + \epsilon^2 b_2 + \epsilon^3 b_3 + \ldots \\ \rho = 1 + \epsilon^2 \rho_2 + \epsilon^3 \rho_3 + \ldots \\ u = \epsilon^2 u_2 + \epsilon^3 u_3 + \ldots \end{cases}$$

En annullant les divers coefficients de ϵ^j et en effectuant des moyennes sur les variables rapides, on obtient, avec $B = e^{i(kx-\omega t)}(b_1 + \epsilon b_2)$:

$$\begin{cases} i(B_T + v_g B_X) + \epsilon \frac{\omega''}{2} B_{XX} - \epsilon k(u - \frac{v_g}{2}\rho)B = 0 \\ \rho_T + u_X = 0 \\ u_T + (\beta\rho + \frac{1}{2}|B|^2)_X = 0 \end{cases} \tag{1.27}$$

où $v_g = \frac{2\omega''^3}{k(k^2+\omega''^2)}$ représente la vitesse de groupe et $\omega'' = \omega''(k)$.
En choisissant un référentiel se déplaçant à la vitesse de l'onde

$$\begin{cases} \xi = X - v_g T \\ \tau = \epsilon T, \end{cases}$$

$$\begin{cases} iB_\tau + \frac{\omega''}{2} B_{\xi\xi} - k(u - \frac{vg}{2}\rho)B = 0 \\ \epsilon\rho_\tau + (u - v_g\rho)_\xi = 0 \\ \epsilon u_\tau + (\beta\rho - v_g u + \frac{1}{2}|B|^2)_\xi = 0 \end{cases} \tag{1.28}$$

1.2.2 L'approximation adiabatique

Cette approximation consiste à négliger les termes en ϵ dans (1.28).
On obtient alors l'équation de Schrödinger Non Linéaire :

$$iB_\tau + \frac{\omega''}{2} B_{\xi\xi} + \frac{kv_g}{\beta - v_g^2}|B|^2 B = 0$$

Cette formulation devient evidemment singulière à la résonance $\beta = v_g^2$ (i.e. $v_{son} = v_A$).
Près de cette valeur critique, et via un nouveau changement d'échelle

$$\begin{cases} \beta^{\frac{1}{2}} - v_g = \epsilon^{\frac{2}{3}}\lambda \\ \xi = \epsilon^{\frac{2}{3}}(x - v_g t) \\ \tau = \epsilon^{\frac{4}{3}} t \end{cases} \tag{1.29}$$

et

$$u - \frac{v_g}{2}(\rho - 1) = \epsilon^{\frac{4}{3}}\phi,$$

on obtient l'équation de Benney :

$$\begin{cases} iB_\tau + \omega'' B_{\xi\xi} - k\phi B = 0 \\ \phi_\tau + \lambda\phi_\xi = -\frac{1}{8}|B|^2_\xi \end{cases} \tag{1.30}$$

1.2.3 L'équation de Zakharov-Rubenchick

Loin de la résonance, une formulation hamiltonienne peut remplacer l'equation (1.28). Elle est obtenue en posant

$$\tilde{u} = u - (k + \frac{v_g}{2}\frac{kv_g - 1}{\beta - v_g^2})B \text{ et } \tilde{\rho} = \rho - \frac{kv_g - 1}{\beta - v_g^2}|B|^2,$$

$$\begin{cases} iB_\tau + \omega'' B_{\xi\xi} - k(\tilde{u} - \frac{v_g}{2}\tilde{\rho} + q|B|^2)B = 0 \\ \epsilon\tilde{\rho}_\tau + (\tilde{u} - v_g\tilde{\rho}) = -k|B|^2_\xi \\ \epsilon\tilde{u}_\tau + (\beta\tilde{\rho} - v_g\tilde{u}) = \frac{kv_g}{2}|B|^2_\xi \end{cases} \tag{1.31}$$

où l'on a posé

$$q = k + \frac{v_g}{4}\frac{kv_g - 1}{\beta - v_g^2}.$$

Cette équation décrit de façon générique l'intéraction d'une onde haute-fréquence avec des ondes de basse-fréquence du type acoustique, excitée par la modulation de l'onde haute fréquence au travers de forces pondéromotrices.([62])

Chapitre 2

L'équation SNLD multidimensionnelle

Dans ce chapitre, on étudie l'équation (1.3) en dimension 2 :

$$\begin{cases} q_t + iq_{xx} + [(|q|^2 + 2A)q]_x - (|q|^2 + 2A)_y = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0, \\ q(x, y, 0) = q_o(x, y). \end{cases} \tag{2.1}$$

2.1 L'équation linéarisée

Le problème linéarisé en 0 s'écrit

$$\begin{cases} q_t + iq_{xx} - 2A_y = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0, \\ q(x, y, 0) = q_o(x, y). \end{cases} \tag{2.2}$$

Plus généralement, sous sa forme dérivée, (2.2) devient

$$(q_t + iq_{xx})_x + (q + \overline{q})_{yy} = 0. \tag{2.3}$$

Ce chapitre est organisé comme suit :
Dans une première partie, on étudiera les problèmes d'évolution associés à (2.2) et (2.3). Puis, en établissant des estimations uniformes d'intégrales oscillantes, et par interpolation complexe, on déduit les estimations du type "$L^p - L^q$".
Finalement, dans la troisième partie, on donnera les estimations du type Strichartz qui en découlent.

2.1.1 Le groupe unitaire

La conjugaison complexe n'étant pas linéaire vis-à-vis de la transformée de Fourier, on va poser

$$\begin{cases} a(x,y,t) = Re(q(x,y,t)) \\ b(x,y,t) = Im(q(x,y,t)), \end{cases}$$

et on obtient le système

$$\begin{cases} (a_t - b_{xx})_x + 2a_{yy} = 0 \\ b_{tx} + a_{xxx} = 0 \end{cases} \tag{2.4}$$

On commence cette étude par un premier calcul (formel) :
En prenant la transformée de Fourier en espace,

$$\frac{\partial}{\partial t}\begin{pmatrix} \hat{a}(\xi,\eta,t) \\ \hat{b}(\xi,\eta,t) \end{pmatrix} = M(\xi,\eta)\begin{pmatrix} \hat{a}(\xi,\eta,t) \\ \hat{b}(\xi,\eta,t) \end{pmatrix},$$

où (ξ,η) sont les variables duales de (x,y) et

$$M(\xi,\eta) = \begin{pmatrix} \frac{2\eta^2}{i\xi} & -\xi^2 \\ \xi^2 & 0 \end{pmatrix}.$$

La matrice M est diagonalisable :

$$P^{-1}(\xi,\eta)M(\xi,\eta)P(\xi,\eta) = \begin{pmatrix} iL_1(\xi,\eta) & 0 \\ 0 & iL_2(\xi,\eta) \end{pmatrix},$$

avec

$$L_1(\xi,\eta) = \sqrt{\frac{\eta^4}{\xi^2} + \xi^4} - \frac{\eta^2}{\xi},\ L_2(\xi,\eta) = -\sqrt{\frac{\eta^4}{\xi^2} + \xi^4} - \frac{\eta^2}{\xi}, \tag{2.5}$$

et

$$P(\xi,\eta) = \begin{pmatrix} iL_1(\xi,\eta) & iL_2(\xi,\eta) \\ \xi^2 & \xi^2 \end{pmatrix}.$$

Ainsi, en posant

$$\begin{pmatrix} \phi(\xi,\eta,t) \\ \psi(\xi,\eta,t) \end{pmatrix} = P^{-1}(\xi,\eta)\begin{pmatrix} \hat{a}(\xi,\eta,t) \\ \hat{b}(\xi,\eta,t) \end{pmatrix},$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \phi(\xi,\eta,t) \\ \psi(\xi,\eta,t) \end{pmatrix} = \begin{pmatrix} iL_1(\xi,\eta) & 0 \\ 0 & iL_2(\xi,\eta) \end{pmatrix}\begin{pmatrix} \phi(\xi,\eta,t) \\ \psi(\xi,\eta,t) \end{pmatrix},$$

i.e.

$$\begin{pmatrix} \phi(\xi,\eta,t) \\ \psi(\xi,\eta,t) \end{pmatrix} = \begin{pmatrix} e^{itL_1} & 0 \\ 0 & e^{itL_2} \end{pmatrix}\begin{pmatrix} \phi_o(\xi,\eta) \\ \psi_o(\xi,\eta) \end{pmatrix}.$$

En revenant aux variables initiales :

$$\begin{pmatrix} \hat{a}(\xi,\eta,t) \\ \hat{b}(\xi,\eta,t) \end{pmatrix} = S(\xi,\eta,t) \begin{pmatrix} \hat{a_o}(\xi,\eta) \\ \hat{b_o}(\xi,\eta) \end{pmatrix},$$

avec

$$\begin{aligned} S(\xi,\eta,t) &= P \begin{pmatrix} e^{itL_1} & 0 \\ 0 & e^{itL_2} \end{pmatrix} P^{-1} \\ &= \frac{1}{2i\sqrt{\eta^4+\xi^6}} \begin{pmatrix} |\xi|(iL_1e^{itL_1} - iL_2e^{itL_2}) & \xi^2|\xi|(e^{itL_2}-e^{itL_1}) \\ \xi^2|\xi|(e^{itL_1}-e^{itL_2}) & |\xi|(iL_1e^{itL_1}-iL_2e^{itL_2}) \end{pmatrix}. \end{aligned}$$

En remarquant que

$$|\frac{\xi}{\sqrt{\eta^4+\xi^6}}L_{1,2}| = |-\frac{\eta^2}{\sqrt{\eta^4+\xi^6}} \pm 1| \leq 2$$

et

$$|\frac{\xi^3}{\sqrt{\eta^4+\xi^6}}| \leq 1,$$

on a pour tout $t \in \mathbb{R}$,

$$S(\xi,\eta,t) \in L^\infty(\mathbb{R}^2).$$

Par conséquent :

Theorème 2.1.1 *Soit $s \in \mathbb{R}$.*
Le problème d'évolution (2.3) définit un groupe à un paramètre $\{S(t)\}_{t\in\mathbb{R}}$ agissant sur $H^s(\mathbb{R}^2)$ par :

$$\begin{aligned} S(t) : H^s(\mathbb{R}^2) &\to H^s(\mathbb{R}^2) \\ q_o &\to q(x,y,t) = \mathcal{F}^{-1}(S(\xi,\eta,t)\hat{q_o}(\xi,\eta)). \end{aligned}$$

Remarque 2.1.2 *$S(t)$ est unitaire. En effet un calcul simple permet de montrer que*

$$\overline{S(t)}^t S(t) = Id.$$

Il s'agit à présent de montrer que $q(t) = S(t)q_o$ est effectivement une solution pour les problèmes d'évolution (2.2) et (2.3). Il est clair que pour celà il suffit de justifier la dérivation par rapport au temps de $q(t)$.
Supposons que, pour $s \in \mathbb{R}$, $q_o = a_o + ib_o \in H^s(\mathbb{R}^2)$. On a :

$$a(x,y,t) = \frac{1}{8i\pi^2}\int \frac{e^{ix\xi+iy\eta}}{\sqrt{\eta^4+\xi^6}}[i|\xi|(L_1e^{itL_1} - L_2e^{itL_2})\hat{a_o} + |\xi|^3(e^{itL_2}-e^{itL_1})\hat{b_o}]d\xi d\eta, \quad (2.6)$$

et

$$b(x,y,t) = \frac{1}{8i\pi^2}\int \frac{e^{ix\xi+iy\eta}}{\sqrt{\eta^4+\xi^6}}[|\xi|^3(e^{itL_1}-e^{itL_2})\hat{a_o} + i|\xi|(L_1e^{itL_2}-L_2e^{itL_1})\hat{b_o}]d\xi d\eta. \quad (2.7)$$

Afin de justifier la dérivation sous le signe de somme, il suffit de montrer que les intégrales dérivées restent absolument convergentes, ce qui n'est pas le cas ici puisque la différentiation par rapport au temps fait apparaitre une singularité en $\frac{1}{\xi}$. Cependant, remarquons que

$$a_x(x,y,t) = \frac{1}{8\pi^2}\int \frac{e^{ix\xi+iy\eta}}{\sqrt{\eta^4+\xi^6}}[i|\xi|(L_1e^{itL_1}-L_2e^{itL_2})\xi\hat{a_o} + |\xi|^3(e^{itL_2}-e^{itL_1})\xi\hat{b_o}]d\xi d\eta,$$

et

$$b_x(x,y,t) = \frac{1}{8\pi^2}\int \frac{e^{ix\xi+iy\eta}}{\sqrt{\eta^4+\xi^6}}[|\xi|^3(e^{itL_1}-e^{itL_2})\xi\hat{a_o} + i|\xi|(L_1e^{itL_2}-L_2e^{itL_1})\xi\hat{b_o}]d\xi d\eta.$$

Puisque

$$|\xi L_{1,2}| = |-\eta^2 \pm \sqrt{(\eta^4+\xi^6)}| \leq (1+|\xi|^2+|\eta|^2)^{\frac{3}{2}},$$

la différentiation par rapport au temps devient possible, et

$$a_{xt}, b_{xt} \in H^{s-3}(\mathbb{R}^2).$$

Ainsi,

Theorème 2.1.3 *Soit $s \in \mathbb{R}$ et $q_o \in H^s(\mathbb{R}^2)$.*
Alors il existe

$$q \in C(\mathbb{R}; H^s(\mathbb{R}))$$

solution au problème d'évolution

$$\left(\begin{array}{l} (q_t)_x + iq_{xxx} + (q+\overline{q})_{yy} = 0 \\ q(0,x,y) = q_o(x,y). \end{array}\right) \quad (2.8)$$

Pour finir, on s'intéresse au système "non-dérivé" (2.2).
On remarque que si l'on suppose l'existence de $\phi \in H^s$ tel que $a_o = \partial_x\phi$,

$$\hat{a}_o(\xi,\eta) = i\xi\hat{\phi}(\xi,\eta),$$

et la dérivation sous le signe de somme des intégrales (2.6) et (2.7) est justifiée. De plus, notons que pour tout $t \in \mathbb{R}$,

$$i\eta\hat{a}(\xi,\eta,t) = i\xi[\frac{|\xi|\eta}{2\sqrt{\eta^4+\xi^6}}(iL_1e^{itL_1} - iL_2e^{itL_2})\hat{\phi} + \frac{|\xi|^2 sgn(\xi)\eta}{2i\sqrt{\eta^4+\xi^6}}(e^{itL_2}-e^{itL_2})\hat{b_o}].$$

Ainsi, en posant

$$\hat{A}(\xi,\eta,t) = \frac{|\xi|\eta}{2\sqrt{\eta^4+\xi^6}}(iL_1e^{itL_1} - iL_2e^{itL_2})\hat{\phi} + \frac{|\xi|^2 sgn(\xi)\eta}{2i\sqrt{\eta^4+\xi^6}}(e^{itL_2} - e^{itL_2})\hat{b_o} \in \mathcal{F}^{-1}(H^{s-1}(\mathbb{R}^2)), \quad (2.9)$$

on obtient la proposition suivante:

Theorème 2.1.4 *Soit $s \in \mathbb{R}$ et $q_o \in H^s(\mathbb{R}^2)$, tel que $q_o + \overline{q_0} = 2\phi_x$, $\phi \in H^s(\mathbb{R}^2)$. Alors il existe (q, A) solution du problème d'évolution*

$$\begin{cases} q_t + iq_{xx} - 2A_y = 0 \\ A_x + \frac{1}{2}(q+\overline{q})_y = 0, \\ q(x,y,0) = q_o(x,y). \end{cases} \quad (2.10)$$

avec

$$q \in C(\mathbb{R}; H^s(\mathbb{R}))\ ,\ A \in C(\mathbb{R}; H^{s-1}(\mathbb{R})).$$

Remarque 2.1.5 *Par (2.9), on remarque que*

$$\phi_y(\xi,\eta) = -A(\xi,\eta,0)$$

2.1.2 Etude des propriétés de dispersion

Estimations d'intégrales oscillantes

On commence par montrer le résultat suivant:

Theorème 2.1.6 *Soit $n \geq 2$, et $\phi \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}/\{0\})$ avec:*
$|\phi^{(2)}(\xi)| \geq C_1 > 0$ *pour* $|\xi| \geq \xi_o > 0$

$|\phi^{(n)}(\xi)| \geq C_2 > O$, *et* $lim_{\xi\to 0}|\phi^{(n)}| = +\infty$

Alors, en posant

$$I(t,x) = \int_{\mathbb{R}} e^{it(\phi(\xi)+x\xi)}d\xi,$$

on a, pour $T > 0$ fixé:

$|I(t,x)| \leq C(T)t^{\frac{-1}{n}}$ *(resp.* $|I(t,x)| \leq C(T)t^{\frac{-1}{2}}$*) pour* $t \geq T > 0$ *(resp.* $O < t \leq T$*).*

Remarque 2.1.7 *La proposition (2.1.6) est une généralisation du cas connu où $\phi(\xi)$ est un polynôme de degré n ([2]), et semble particulièrement intéréssante lorsque, par exemple, toutes les dérivées jusqu'à l'ordre (n-1) de ϕ s'annulent en un point où la dérivée d'ordre n explose, puisque dans cette situation il n'y a pas manière d'utiliser l'estimation "universelle" de Van der Corput.*

Remarque 2.1.8 *Par exactement la même méthode, on peut montrer des estimations pour*

$$I(t,x) = \int_{\mathbb{R}} \xi^{\alpha} e^{it(\phi(\xi)+x\xi)} d\xi$$

où $\alpha > 0$ *si* $\phi''(\xi) \sim \xi^{\alpha}$ *pour* ξ *grand.*

Preuve de la proposition:

On commence par énoncer la version du lemme de Van der Corput qui nous sera utile dans la suite :

Pour

$$I = \int_a^b e^{i\phi(\xi)} d\xi,$$

$\phi \in C^k[a;b]$,$\phi^{(k)} \neq 0$ sur $[a;b]$, on a :

$$|I| \leq \frac{C(k)}{Min(|\phi^{(k)}|)^{\frac{1}{k}}}.$$

On prouve maintenant la proposition 2.1.6 pour

$$I(t,x) = \int_0^{\infty} e^{it(\phi(\xi)+x\xi)} d\xi,$$

l'étude sur $[-\infty, 0]$ étant analogue. Nous allons raisonner comme dans ([28],[2]) :

i)Estimation sur $[\xi_o; \infty[$
On choisit ξ_o assez grand, de sorte que

$\phi''(\xi) \geq C_1 > O$ et $\phi'(\xi) > 0$ pour $|\xi| \geq |\xi_o|$.
L'équation $\phi'(\xi) + x = 0$ a au plus une solution ξ_x puisque ϕ' est strictement croissante.
On pose $B = [\alpha\xi_x; \beta\xi_x]$, avec $\alpha < 1$ et $\beta > 1$ de sorte que $B \subset [\xi_o; \infty[$.

Directement par Van der Corput :

$$|\int_B e^{it(\phi(\xi)+x\xi)} d\xi| \leq \frac{C}{(C_2 t)^{\frac{1}{2}}} \leq \frac{C}{t^{\frac{1}{2}}} \tag{2.11}$$

On pose maintenant $B' = [\xi_o; \infty[/B$: Pour $\xi \in B'$:

$$|\phi'(\xi) + x| = |\phi'(\xi) - \phi(\xi_x)| = |\int_{\xi_x}^{\xi} \phi''(\eta) d\eta| \geq |\xi - \xi_x| C_1 \geq C\xi \geq C\xi_o.$$

Ainsi,

$$\begin{aligned}\int_{B'} e^{it(\phi(\xi)+x\xi)}d\xi &= \frac{1}{it}\int_{B'}\frac{1}{\phi'(\xi)+x}\partial_\xi e^{it(\phi(\xi)+x\xi)}d\xi\\ &= \frac{1}{it}\int_{B'}\frac{\phi''(\xi)}{(\phi'(\xi)+x)^2}e^{it(\phi(\xi)+x\xi)}d\xi + \frac{1}{it}[\frac{1}{\phi'(\xi)+x}e^{it(\phi(\xi)+x\xi)}]_{\partial B'}.\end{aligned}$$

d'où

$$|\int_{B'} e^{it(\phi(\xi)+x\xi)}d\xi| \leq \frac{C}{t}Sup_{B'}\frac{1}{|\phi'(\xi)+x|} \leq \frac{C(\xi_o)}{t}. \tag{2.12}$$

ii)Estimation sur $[\epsilon; \xi_o]$

Soit $\epsilon > 0$ fixé:

Directement par Van Corput:

$$|\int_\epsilon^{\xi_o} e^{it(\phi(\xi)+x\xi)}d\xi| \leq \frac{C(\xi_o)}{t^{\frac{1}{n}}}. \tag{2.13}$$

iii)Estimation sur $[0, \epsilon]$
Soit $\{v_j\}_{j\geq 0}$ une suite strictement croissante, avec

$$lim(v_j) = +\infty \text{ , } v_o = |\phi^{(n)}(\epsilon)|$$

et

$$\sum_{j\geq 0}\frac{1}{v_j^{\frac{1}{n}}} < \infty.$$

Soit de plus une suite $\{\epsilon_j\}_{j\geq 0}$ strictement décroissante et de limite nulle à l'infini, de sorte que

$$\forall\xi \in [\epsilon_{j+1}; \epsilon_j], |\phi^{(n)}(\xi)| \geq v_j.$$

Posons

$$I_j = \int_{\epsilon_{j+1}}^{\epsilon_j} e^{it(\phi(\xi)+x\xi)}d\xi.$$

Alors

$$|I_j| \leq \frac{C(n)}{(tv_j)^{\frac{1}{n}}}$$

et

$$|\int_0^\epsilon e^{it(\phi(\xi)+x\xi)}d\xi| = |\sum_{j\geq 0} I_j| \leq \frac{C}{t^{\frac{1}{n}}}. \tag{2.14}$$

En combinant (2.11),(2.12),(2.13) et (2.14) on obtient l'estimation pour t grand.

Pour l'estimation pour t petit, on va poser

$$I_\lambda(x,t) = \int_0^\infty e^{it(\phi_\lambda(\xi)+x\xi)}d\xi$$

où $\phi_\lambda(\xi) = \frac{1}{(\lambda)^2}\phi(\lambda\xi)$ et $\lambda \geq 1$.

On cherche des estimations de $I_\lambda(x,t)$ pour t grand, uniformes en x e λ.

i)Sur $[\xi_o, \infty[$

Le choix de ξ_o et la minoration de ϕ_λ sont indépendantes de λ puisque

$$\phi''_\lambda(\xi) = \phi''(\lambda\xi) et \lambda\xi \geq \xi.$$

Soit $\epsilon > 0$ et $\lambda \geq 1$ fixées:
ii)Sur $[\frac{\epsilon}{\lambda}; \xi_o]$, on a

$$\phi_\lambda^{(n)}(\xi) = \lambda^{n-2}\phi^{(n)}(\lambda\xi) \geq |\phi^{(n)}|(\lambda\xi) \geq C_2 > 0,$$

d'où encore une estimation uniforme sur cet intervalle.

On pose

$$I_j(x,t) = \int_{\frac{\epsilon_{j+1}}{\lambda}}^{\frac{\epsilon_j}{\lambda}} e^{it(\phi_\lambda(\xi)+x\xi)}d\xi.$$

(on reprend $\{\epsilon_j\}$ et $\{v_j\}$ définis précedemment).

Pour $\xi \in [\frac{\epsilon_{j+1}}{\lambda}; \frac{\epsilon_j}{\lambda}]$, on a

$$|\phi_\lambda^{(n)}(\xi)| \geq |\phi^{(n)}(\lambda\xi) \geq v_j,$$

d'où

$$|I_j(x,t)| \leq \frac{C}{(tv_j)^{\frac{1}{n}}}$$

et

$$I_\lambda(x,t) = \int_0^{\frac{\epsilon}{\lambda}} e^{it(\phi_\lambda(\xi)+x\xi)}d\xi \leq \frac{C}{t^{\frac{1}{n}}},$$

C indépendant de λ.

On a ainsi montrer que $|I_\lambda(x,t)| \leq \frac{C(T)}{t^{\frac{1}{n}}}$, pour $t \geq T > 0$.

En effectuant le changement de variables $\eta = \lambda\xi$,

$$|I(x,t)| = |\lambda I_\lambda(\lambda^2 t, \lambda x)| \leq \lambda \frac{C(T)}{(\lambda^2 t)^{\frac{1}{n}}}$$

pour $\lambda^2 t \geq T$.
On a maintenant fini en choisissant $\lambda = \sqrt{\frac{T}{t}}$.

Application à l'équation 2.2

D'après la première partie, il suffit de borner notre étude à

$$\begin{cases} q_t + iL_j(D) = 0 \\ q(x,y,0) = q_o(x,y), \end{cases} \tag{2.15}$$

où $L_j(D)$, $j = 1, 2$ sont les opérateurs donnés par :

$$[L_j(\hat{D})\phi](\xi,\eta) = L_j(\xi,\eta)\phi(\xi,\eta). \tag{2.16}$$

Les solutions de ce système s'écrivent

$$q(x,y,t) = q_o * K(x,y,t) \tag{2.17}$$

où K est le noyau de convolution

$$K(x,y,t) = \frac{1}{4\pi^2}\int_{R^2} e^{i(tL_j(\xi,\eta)+x\xi+y\eta)}d\xi d\eta, \tag{2.18}$$

qui peut être vu comme le Fourier inverse d'une distribution tempérée ou bien en tant qu'intégrale oscillante.

Remarque 2.1.9

$$L_1(\xi,\eta) = \sqrt{\frac{\eta^4}{\xi^2} + \xi^4} - \frac{\eta^2}{\xi}$$

A ξ fixé, ce symbole n'oscille pas à l'infini en η. Il comporte par conséquent un défaut d'oscillation en y.
De fait, un calcul permet de montrer que

$$\mathbf{D}(\xi,\eta) = Det\begin{pmatrix} \partial_\xi^2 L_1 & \partial_\xi\partial_\eta L_1 \\ \partial_\xi\partial_\eta L_1 & \partial_\eta^2 L_1 \end{pmatrix} \tag{2.19}$$

est croissant en η, et on s'attend donc á une "perte' de dérivée en y. C'est là une grande différence avec le groupe associé à l'équation KP, où ce déterminant vaut 1.

En remarquant que

$$tL_j(\xi,\eta) = L_j(\xi t^{\frac{1}{2}}, \eta t^{\frac{3}{4}}) \tag{2.20}$$

on obtient après le changement de variables $(\xi', \eta') = (t^{\frac{1}{2}}\xi, t^{\frac{3}{4}}\eta)$

$$q = \frac{1}{t^{\frac{5}{4}}} K(\frac{x}{t^{\frac{1}{2}}}, \frac{y}{t^{\frac{3}{4}}}, 1) * q_0.$$

Cette "pseudo-homogénéité" nous permet déjà de mettre en facteur une puissance négative de t.
Plus,

$$q = q_1 + q_2 = \frac{C}{t^{\frac{5}{4}}} q_0 * K_1(\frac{x}{t^{\frac{1}{2}}}, \frac{y}{t^{\frac{3}{4}}}) + \frac{C}{t^{\frac{5}{4}}} q_0 * K_2(\frac{x}{t^{\frac{1}{2}}}, \frac{y}{t^{\frac{3}{4}}}) \tag{2.21}$$

avec

$$K_1(X,Y) = \int_{\xi=0}^{+\infty} \int_{\mathbb{R}} e^{iX\xi + iY\eta + iL_j(\xi,\eta)} d\xi d\eta \tag{2.22}$$

$$K_2(X,Y) = \int_{\xi_1=-\infty}^{0} \int_{\mathbb{R}} e^{iX\xi + iY\eta + iL_j(\xi,\eta} d\xi d\eta. \tag{2.23}$$

Dans tout ce qui suit, on prendra $L = L_1$,le cas $L = L_2$ étant analogue.
i)Estimation de q_2

$$K_2(x,y) = \int d\eta e^{i\frac{y\eta}{t^{\frac{3}{4}}}} \int_0^{\infty} e^{i\frac{x\xi}{t^{\frac{1}{2}}} + \frac{\eta^2}{\xi} - \sqrt{\frac{\eta^4}{\xi^2} + \xi^4}} d\xi_1 \tag{2.24}$$

On note

$$\phi = \phi_{X,\eta}(\xi) = \xi X + \frac{\eta^2}{\xi} - \sqrt{\frac{\eta^4}{\xi^2} + \xi^4},$$

et on cherche à estimer

$$I(\eta, X) = \int_0^{\infty} e^{i\phi_{X,\eta}(\xi)} d\xi.$$

Par un calcul simple, ϕ à les propriétés suivantes :

$$\phi''(\xi) = 2Z - \frac{2+13Z^2+2Z^4}{(1+Z^2)^{\frac{3}{2}}},$$

où $Z = \frac{\eta^2}{\xi^3}$. De plus $\phi''(0) = 2$ et $\phi''(\xi) \neq 0$ pour toute valeur de ξ. Donc, pour $Z \leq Z_o$ fixé, $\phi'' \geq C(Z_o) > 0$, et on peut écrire

$$|\int_{Z<Z_o} e^{i\phi_{X,\eta}(\xi)} d\xi| = |\int_{\xi > \frac{\eta^{\frac{2}{3}}}{Z_o^{\frac{1}{3}}}} e^{i\phi_{X,\eta}(\xi)} d\xi| \leq C(Z_o).$$

Sur $Z > Z_o$, on remarque que

$$\phi^{(6)}(\xi) = \frac{1}{\xi^4}\theta(Z),$$

où $\theta \sim \frac{C}{Z}$ à l'infini. Ainsi $\phi^{(6)}(\xi) = \frac{1}{\eta^{\frac{8}{3}}}\theta_o(Z)$,où θ_o a une limite infinie a l'infini.

Donc, en choisissant Z_o assez grand,on applique la proposition 2.1.6 pour montrer que :

$$|\int_{Z>Z_o} e^{i\phi_{X,\eta}(\xi)} d\xi| = |\int_{\xi < \frac{\eta^{\frac{2}{3}}}{Z_o^{\frac{1}{3}}}} e^{i\phi_{X,\eta}(\xi)} d\xi| \leq \eta^{\frac{4}{9}} C(Z_o).$$

D'où

$$I(\eta, X) = \int_0^\infty e^{i\phi_{X,\eta}(\xi)} d\xi \leq C(1+\eta^{\frac{4}{9}}),$$

et, pour $\epsilon > 0$,

$$q_2 = \frac{C}{t^{\frac{5}{4}}} q_o * \int_{|\eta|<A} I(\eta, X) e^{i\frac{y\eta}{t^{\frac{3}{4}}}} d\eta + \frac{C}{t^{\frac{1}{6}-\frac{3\epsilon}{4}}} \partial_y^{\frac{13}{9}+\epsilon} q_o * \int_{|\eta|>A} \eta^{-\frac{13}{9}-\epsilon} I(\eta, X) e^{i\frac{y\eta}{t^{\frac{3}{4}}}} d\eta. \quad (2.25)$$

Finalement,

$$||q_2(.,t)||_\infty \leq \frac{C}{t^{\frac{5}{4}}} ||q_o||_1 + \frac{C}{t^{\frac{1}{6}-\frac{3\epsilon}{4}}} ||\partial_y^{\frac{13}{9}+\epsilon} q_o||_1, \forall \epsilon > 0. \quad (2.26)$$

ii)Estimation de q_1

Cette estimation est nettement plus facile :

$$K_1(x,y) = \int d\eta\, e^{i\frac{y\eta}{t^{\frac{3}{4}}}} \int_0^\infty e^{i\frac{x\xi}{t^{\frac{1}{2}}} + \frac{\eta^2}{\xi} + \sqrt{\frac{\eta^4}{\xi^2}+\xi^4}} d\xi.$$

Ici,

$$\phi = \phi_{X,\eta}(\xi) = \xi X + \frac{\eta^2}{\xi} + \sqrt{\frac{\eta^4}{\xi^2} + \xi^4},$$

et

$$\phi''(\xi) = 2Z + \frac{2 + 13Z^2 + 2Z^4}{(1+Z^2)^{\frac{3}{2}}},$$

où une fois de plus, $Z = \frac{\eta^2}{\xi^3}$. Par conséquent,$\phi''(\xi) \geq C > 0$ pour tout $\xi > 0$ et on peut écrire :

$$||q_1(.,t)||_\infty \leq \frac{C}{t^{\frac{5}{4}}}||q_o||_1 + \frac{C}{t^{\frac{1}{2}-\frac{3\epsilon}{4}}}||\partial_y^{1+\epsilon} q_o||_1, \forall \epsilon > 0 \tag{2.27}$$

Conclusion-Estimations $L^p - L^q$

En combinant (2.26) et (2.27) on peut énoncer le résultat suivant :

Soit $q_o \in W(\mathbb{R}^2)$, où

$$W(\mathbb{R}^2) = \{f \in L^1(\mathbb{R}^2)/\partial_y^{1+\epsilon} f \in L^1(\mathbb{R}^2), et \partial_y^{\frac{13}{9}+\epsilon} f \in L^1(\mathbb{R}^2)\}.$$

Alors, si $q(x,y,t) = q_o * K(x,y,t) = S(t)q_o$,

$$\forall t > 0, ||q(.,t)||_\infty \leq \frac{C}{t^{\frac{5}{4}}}||q_o||_1 + \frac{C'}{t^{\frac{1}{2}-\frac{3\epsilon}{4}}}||\partial_y^{1+\epsilon} q_o||_1 + \frac{C''}{t^{\frac{1}{6}-\frac{3\epsilon}{4}}}||\partial_y^{\frac{13}{9}+\epsilon} q_o||_1 \tag{2.28}$$

ce qui est une estimation pour les temps longs, ou, pour tout t, $0 < t \leq T$ (où $T > 0$ fixé) ,

$$||q(.,t)||_\infty \leq \frac{C(T)}{t^{\frac{5}{4}}}(||q_o||_1 + ||\partial_y^{1+\epsilon} q_o||_1 + ||\partial_y^{\frac{13}{9}+\epsilon} q_o||_1). \tag{2.29}$$

Remarque 2.1.10 *Bien qu'il soit difficile de voir si cette estimation est optimale, on a nécessairement perte d'au moins une dérivée en y(cf. Remarque 2.1.9).*
En posant

$$W_\theta(\mathbb{R}^2) = \{f \in L^{\frac{2}{1+\theta}}(\mathbb{R}^2)/||f||_{W_\theta} = ||f||_{L^{\frac{2}{1+\theta}}} + ||\partial_y^{\theta(1+\epsilon)} f||_{L^{\frac{2}{1+\theta}}} + ||\partial_y^{\theta(\frac{13}{9}+\epsilon)} f||_{L^{\frac{2}{1+\theta}}} < \infty\}, \tag{2.30}$$

pour tout $\theta \in [0,1]$.

$$S(t) : W_o = L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \text{ avec norme } 1.$$

$$S(t) : W_1(\mathbb{R}^2) = W(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2) \text{ avec norme } \frac{C(T)}{t^{\frac{5}{4}}}.$$

Par interpolation complexe,

$$S(t) : W_\theta(\mathbb{R}^2) \to L^{\frac{2}{1-\theta}}(\mathbb{R}^2) \text{ avec norme } \frac{C(T)}{t^{\frac{5\theta}{4}}},$$

autrement dit, on vient d'établir les estimations "$L^p - L^q$" :

Theorème 2.1.11 *Soit $T > 0$ fixé, $0 < t < T$.*
Alors, pour tout $\theta \in [0,1]$ et pour tout $\epsilon > 0$,

$$||q(.,t)||_{L^{\frac{2}{1-\theta}}}(\mathbb{R}^2) \leq \frac{C(T)}{t^{\frac{5\theta}{4}}}(||q_o||_{L^{\frac{2}{1+\theta}}} + ||\partial_y^{\theta(1+\epsilon)} q_o||_{L^{\frac{2}{1+\theta}}} + ||\partial_y^{\theta(\frac{13}{9}+\epsilon)} q_o||_{L^{\frac{2}{1+\theta}}}). \quad (2.31)$$

De même, pour des temps "grands"

Theorème 2.1.12 *Soit $T > 0$ fixé, $T \leq t$.*
Alors, pour tout $\theta \in [0,1]$ et pour tout $\epsilon > 0$,

$$||q(.,t)||_{L^{\frac{2}{1-\theta}}}(\mathbb{R}^2) \leq \frac{C(T)}{t^{\frac{\theta}{6}-\frac{3\theta\epsilon}{4}}}(||q_o||_{L^{\frac{2}{1+\theta}}} + ||\partial_y^{\theta(1+\epsilon)} q_o||_{L^{\frac{2}{1+\theta}}} + ||\partial_y^{\theta(\frac{13}{9}+\epsilon)} q_o||_{L^{\frac{2}{1+\theta}}}). \quad (2.32)$$

2.1.3 Estimations du type "Strichartz"

Grâce aux estimations $L^p - L^q$ déduites dans la partie précédente, on peut construire les estimations de Strichartz associées :

Theorème 2.1.13 *Soit $\theta \in [0,1]$ et*

$$A = L_t^q L_x^r W_y^{s,r}(\mathbb{R}^2) \text{ où } s = \frac{13}{9} + \epsilon \text{ et } r = \frac{2}{1+\theta}$$

$$A^* = L_t^{q'} L_x^{r'} W_y^{-s,r'}(\mathbb{R}^2) \text{ son espace dual.}$$

Alors

$$||S(t)\phi||_{A^*} \leq ||\phi||_{L^2}$$

pour $q' = \frac{4}{5\theta} \geq 1$.

Preuve :
Par dualité, il s'agit de montrer que pour tout ψ,

$$|\int_{t,x} S(t)\phi\psi| \leq C||\phi||_{L^2}||\psi||_A.$$

Or, par Plancherel,

$$\begin{aligned} |\int_{t,x} S(t)\phi\psi| &= |\int_{\tau,\xi} \delta(\tau - L(\xi))\hat{\phi}(\xi)\hat{\psi}(\tau,\xi)| \\ &\leq ||\phi||_{L^2}||\hat{\psi}(L(\xi),\xi)||_{L^2}. \end{aligned}$$

Reste donc à montrer le lemme de restriction :

Lemme 2.1.14 *Pour* $q' = \frac{4}{5\theta} \geq 1$,

$$||\hat{\psi}(L(\xi),\xi)||_{L^2} \leq C||\psi||_A.$$

En effet, par Plancherel,

$$\begin{aligned} ||\hat{\psi}(L(\xi),\xi)||^2_{L^2} &= \int_{\tau,\xi} \hat{K}(\tau,\xi)|\hat{\psi}(\tau,\xi)^2 \\ &= \int_{t,x} (K * \psi)(t,x)\psi(-t,-x) \\ &\leq ||K * \psi||_{A^*}||\psi||_A. \end{aligned}$$

De plus,

$$||K * \psi||_{A^*} \leq C \int_0^T ||(||F^{-1}e^{-i(t-s)L(\xi)}\hat{\psi}(s,\xi)||_{L_x^{r'}W_y^{r',-s}})||_{L_t^{q'}}ds$$

Compte tenu du fait que

$$W_y^{r',-s} \to L_y^{r'},$$

d'après (2.32) et par Young-généralisé :

$$||K * \psi||_{A^*} \leq C \int_0^T ||(. - s)^{\frac{-5\theta}{4}}||_{L_t^{q'}}ds$$

ce qui est une quantité finie pour $q' = \frac{4}{5\theta} \geq 1$.

2.2 Les données initiales analytiques

Local analytic solutions to the DNLS equation in higher dimension

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Résumé

We prove the local well-posedness of the evolution system

$$\begin{cases} q_t + (q(|q|^2 + 2A))_x - (|q|^2 + 2A)_y + iq_{xx} = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0 \\ q(0, X) = q_o(X) \end{cases}$$

in the analytic Sobolev spaces $X^m(r(T))$, $m \geq 7$, with the initial compatibility conditions $(q_o + \overline{q_o})_y = -2A_{ox}$ and $|q_o|^2 + 2A_o = \psi_{ox}$, where $A_o, \psi_o \in X^m(r(T))$. This model, due to E. Mjolhus and J. Wyller ([44]), describes in the long-wave limit the evolution of localized nonlinear Alfvén waves propagating in a parallel (or quasi-parallel) direction to the ambient magnetic field and is a generalisation of the DNLS equation.

2.2.1 Introduction

When studying weakly nonlinear waves in fluids, the reductive perturbation method ([45],[15]) appears to be a useful unifying technique. In fact, it leads to well known partial differential equations such as the Burgers equation, the Korteweg de Vries (KdV) equation and the Nonlinear Schrödinger Equation. In particular, for weakly nonlinear, weakly dispersive MHD waves, propagating perpendicular or obliquely to the ambient magnetic field, this reductive perturbation method allows to derive the KdV equation for magnetosonic waves ([27],[31]):

$$u_t + u_{xxx} + uu_x = 0. \tag{2.33}$$

Then, by taking into account weak dependence on the transverse direction, this KdV equation was generalized in [25] by Kadomtsev and Petviashvili:

$$\begin{cases} u_t + u_{xxx} + v_y + uu_x = 0 \\ v_x = u_y. \end{cases} \tag{2.34}$$

Similarly, when dealing with the singular case of parallel propagation, the above mentioned reduction technique results in the Derivative Non Linear Schrödinger equation (DNLS) (see for example [16]):

$$q_t + iq_{xx} + (|q|^2 q)_x = 0. \tag{2.35}$$

The DNLS equation has been studied by many authors (see for instance [40], [46],[22], [35]).

Then, by allowing weak dependence on the transverse directions, Mjolhus and Wyller derived a 2-dimensional model for localized parallel propagating MHD waves, with the assumptions of weak nonlinearity, weak dispersion, weak diffraction and slow evolution (see [44], [39] and [49]), namely

$$\left\{ \begin{array}{l} \frac{\partial \overline{B}}{\partial \tau} + \alpha(\frac{\partial}{\partial \xi}((|\overline{B}|^2 + 2B_oB_x)\overline{B}) - B_o\frac{\partial}{\partial \eta}(|\overline{B}|^2 + 2B_oB_x)) + i\beta\frac{\partial^2 \overline{B}}{\partial \xi^2} = 0 \\ \frac{\partial B_x}{\partial \xi} + \frac{\partial B_y}{\partial \eta} = 0, \end{array} \right. \tag{2.36}$$

where B_o is the ambient magnetic field, and the magnetic field B is such that, to leading order

$$B = (B_o + \epsilon B_x, \epsilon^{\frac{1}{2}} B_y, \epsilon^{\frac{1}{2}} B_z),$$

(weak nonlinearity)

$$\overline{B} = B_y + iB_z \ ,$$

$$\alpha = \frac{v_A}{4B_o^2} \text{ and } \beta = \frac{\Omega_i}{2v_A},$$

with $v_A = \frac{B_o^2}{4\pi\rho_o}$ the Alfven velocity, Ω_i the ions gyrofrequency and L a lenght of reference, and finally

$$\left\{ \begin{array}{l} \xi = \epsilon(x - v_A t) \text{ (weak dispersion)} \\ \eta = \epsilon^{\frac{3}{2}} y \text{ (weak diffraction)} \\ \tau = \epsilon^2 t \text{ (slow evolution).} \end{array} \right. \tag{2.37}$$

Note that the small parameter ϵ does not appear in (2.36) since this equation is invariant by the transformation

$$(\overline{B'}, \overline{B'_x}, \xi', \eta', \tau') \to (\epsilon^{\frac{1}{2}}\overline{B}, \epsilon\overline{B_x}, \epsilon^{-1}\xi, \epsilon^{\frac{-3}{2}}\eta, \epsilon^{-2}\tau).$$

Finally by normalisation of the physical constants, (2.36) can easily be brought to the canonical form:

$$\left\{ \begin{array}{l} q_t + (q(|q|^2 + 2A))_x - (|q|^2 + 2A)_y + iq_{xx} = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0, \end{array} \right. \tag{2.38}$$

which is clearly a 2-dimensional extension of the DNLS equation, in a similar way that the KP equation is a 2-dimensional extension of the KdV equation.

We will be concerned here with the Cauchy problem associated with (2.38).
Results in the classical Sobolev spaces $H^s(\mathbb{R}^2)$ do not seem easy to get, essentially for two reasons :

First, when considering the linearized problem

$$\begin{cases} q_t + iq_{xx} - 2A_y = 0 \\ A_x + \frac{1}{2}(q+\overline{q})_y = 0, \end{cases} \tag{2.39}$$

the linear operator which generates the solutions of the Cauchy problem associated to (2.39) has the form, when $D_x = \frac{1}{i}\partial_x$ and $D_y = \frac{1}{i}\partial_y$,

$$L(D_x, D_y) = e^{itS(D_x,D_y)},$$

where $D_x = \frac{1}{i}\partial_x$, $D_y = \frac{1}{i}\partial_y$, and

$$S(\xi_1,\xi_2) = -\frac{\xi_2^2}{\xi_1} + \sqrt{\xi_1^4 + \frac{\xi_2^4}{\xi_1^2}}.$$

The expression $e^{itS(\xi_1,\xi_2)}$ does not oscillate for fixed ξ_1 and letting ξ_2 tend to infinity. Actually, for fixed ξ_1,

$$\lim_{\xi_2\to\infty} \exp itS(\xi_1,\xi_2) = 1.$$

For that reason, one cannot obtain any dispersive estimates with smoothing for the free evolution of (2.38). In particular, local smoothing estimates or Strichartz type inequalities with smoothing do not seem to be available here.
Secondly, we only know three conservation laws for (2.38)

$$\frac{d}{dt}J_i(t) = 0,$$

with

$$J_o(t) = \int q dxdy \ , \ J_1(t) = \int |q|^2 dxdy$$

and

$$J_2(t) = \int [(|q|^2 + 2A)^2 + 2iq\overline{q_x}]dxdy \ ,$$

which are natural extensions of the three first time-invariants for the DNLS equation :

$$J_o = \int q dx \ , \ J_1 = \int |q|^2 dx$$

and

$$J_2 = \int (|q|^4 + 2iq\overline{q_x})dx.$$

One may try to use these invariants in order to derive a local existence result for (2.38) in the classical Sobolev spaces $H^s(\mathbb{R}^2)$, similarly to the case of the KdV equation, the KP equation, and many other dispersive models. This energy approach seems to fail in the case of (2.38) because the nonlinear term contains the formal anti-derivative $\partial_x^{-1}(q+\overline{q})_y$. For this same reason, we were not able to find any a priori estimates in $H^s(\mathbb{R}^2)$ for possible solutions of (2.38).

However, an easy formal computation shows that, if we take the auxiliar evolution variable $u = |q|^2 + 2A$, we obtain

$$\left\{ \begin{array}{l} q_t + iq_{xx} - u_y + (uq)_x = 0 \\ u_t - \partial_x^{-1} u_{yy} + \frac{i}{2}(q-\overline{q})_{xy} - (u(q+\overline{q}))_y + i(\overline{q}q_{xx} - q\overline{q_{xx}}) + 2Re(\overline{q}(uq)_x) = 0 \end{array} \right. \tag{2.40}$$

and we only get an anti-derivative in the linear part. The price to pay for this operation is the appearance of a big "derivative-loss" in the non linear term.
Many authors deal with this kind of problem by introducing the analytic Sobolev spaces, which are essentially the functions $f \in L^2(\mathbb{R}^2)$ that can be extended to holomorphic functions in a band containing the real axis. For instance, those spaces were used by Hayashi (see [20],[21]) to overcome derivative losses in nonlinear Schrödinger equations.

In [6], de Bouard extends this result to nonlinear "non elliptic" Schrödinger equations, by proving the well-posedness in the analytic Sobolev spaces of:

$$\left\{ \begin{array}{l} i\mathbf{U}_t + L(\mathbf{U}) + G(\mathbf{U}, \overline{\mathbf{U}}, \nabla\mathbf{U}, \nabla\overline{\mathbf{U}}) = 0 \\ \mathbf{U}(X,0) = \mathbf{U}_o(X) \end{array} \right. \tag{2.41}$$

where G is a polynomial such that $G(0,0,0,0) = 0$, the self-adjoint operator L is given by

$$L\hat{\mathbf{U}}(\xi) = \mathbf{P}(\xi)\hat{\mathbf{U}}(\xi) \ ,$$

and $\mathbf{P}(\xi)$ is a symmetric matrix symbol with real entries $P_j(\xi) \in L^\infty_{loc}$.

Here, we will generalize this result to the case where the linear part presents a singularity of the kind:

$$L = L_1 + L_2, \tag{2.42}$$

with L_1 self-adjoint as in [6] and

$$L_2 = Diag(i\epsilon_1 \partial_{x_1}^{-1} \partial_{x_{k_1}}^2, ..., i\epsilon_n \partial_{x_n}^{-1} \partial_{x_{k_n}}^2) \ , \ \epsilon_j \in \{0;1\} \text{ and } k_j \in [1,n],$$

since, as we will see later, (2.40) can be reduced to this form.
The rest of this paper is organized as follows: In the second section, we introduce the analytic spaces and equip them with a local structure, which is crucial for our proof. At

the end of this chapter, we will state our main theorem.

In the third section, we prove the local well-posedness of (2.41), with L given by (2.42) : we first build a family of approximate solutions $\{u_\epsilon\}_{\epsilon>0}$ by regularizing the anti-derivatives, and then take its limit as $\epsilon \to 0$.

In the last section, we apply these previous results to equation (2.38). For technical reasons, we will be dealing with a system possessing seven unknowns :

$$(q+\overline{q}, q-\overline{q}, (q+\overline{q})_x, (q-\overline{q})_x, (q+\overline{q})_y, (q-\overline{q})_y, |q|^2+2A).$$

Finally, we will return to the initial system (2.38).

2.2.2 Sobolev analytic spaces

Definitions and first properties

We begin by introducing the analytic spaces used in [20],[21].

Definition 2.2.1 *Let $m \in \mathbb{N}$, $r > 0$ and $n \geq 1$.*
We set(analytic Sobolev space of order m)

$$X^m(r) = \{f \in L^2(\mathbb{R}^n), ||f||^2_{X^m(r)} =< q(r,\xi)\sigma^{2m}(\xi)\hat{f}, \hat{f} >_{L^2(\mathbb{R}^n)} < +\infty\}$$

where $\sigma^m(\xi) = (1+|\xi|^2)^{\frac{m}{2}}$ and

$$q(r,\xi) = q_1(r,\xi) + q_2(r,\xi) := \prod_{j=1}^{n} \cosh(2r\xi_j) + \sum_{k=1}^{n} \xi_k \sinh(2r\xi_k) \prod_{j\neq k} \cosh(2r\xi_j).$$

We also introduce the Hardy analytic spaces :

Definition 2.2.2 *Let $r > 0$ and $n \geq 1$:*
We set $S(r) = \{z \in C^n, \forall 1 \leq j \leq n \ , \ |Imz_j| < r\}$,

$$L_r = \{f \in L^2(\mathbb{R}^n),\ f \textit{ is analytic on the band } S(r),$$

$$\textit{and } ||f||^2_{L_r} = Sup_{y\in]-r,r[^n} ||f(.+iy)||^2_{L^2(\mathbb{R}^n)} < +\infty\},$$

$$Y_m(\mathbb{R}^n) = \{f \in L_r, ||f||^2_{Y_m(\mathbb{R}^n)} = \sum_{|\alpha|\leq m} ||\partial_z^\alpha f||^2_{L_r} < +\infty\},$$

and, for $m \geq 1$,

$$Y^*_m(\mathbb{R}^n) = \{f/\nabla f \in L_r, ||f||^2_{Y^*_m(\mathbb{R}^n)} = \sum_{1\leq|\alpha|\leq m} ||\partial_z^\alpha f||^2_{L_r} < +\infty\}.$$

We now state two results proved by Hayashi in [20]:

Lemma 2.2.3 *Let $f \in L_r$.*
Then

$$\int q_1(r,\xi)|\hat{f^*}(\xi)|^2 d\xi \leq 2^n ||f||^2_{L_r},$$

where f^* is the trace of f on the real axis. Conversely, if $\int q_1(r,\xi)|\hat{f}(\xi)|^2 d\xi$ is finite, then f can be extended to an analytic function over $S(r)$, and

$$||f||^2_{L_r} \leq 2^n \int q_1(r,\xi)|\hat{f}(\xi)|^2 d\xi.$$

Lemma 2.2.4 *We set*

$$L_{r,2} = \{f \in L^2(\mathbb{R}^n),\ f \text{ is analytic in the band } S(r),$$

$$\text{and } ||f||^2_{L_{r,2}} = \sum_{k=1}^{n} Sup_{y/y_k \in]-r,r[^{n-1}} \int_{-r}^{r} ||\partial_{z_k} f(.+iy)||^2_{L^2(\mathbb{R}^n)} dy_k < +\infty\}.$$

Then

$$\int q_2(r,\xi)|\hat{f}(\xi)|^2 d\xi \leq 2^{n-1} ||f||^2_{L_{r,2}}.$$

Conversely, if $\int q_2(r,\xi)|\hat{f}(\xi)^2 d\xi$ is finite, then f admits an analytic extension to the band $S(r)$ and

$$||f||^2_{L_{r,2}} \leq 2^{n-1} \int q_2(r,\xi)|\hat{f}(\xi)|^2 d\xi.$$

This leads to the following corollary:

Corollary 2.2.5 *Let $m \in \mathbb{N}$ and $r > 0$. Then*

$$Y_{m+1}(r) \subset X^m(r) \subset Y_m(r)$$

Finally, we give a last result:

Lemma 2.2.6 *Assume that $m \geq [n/2]+1$. Then, for $w_1, ..., w_k \in Y_m(r)$,*

$$||\prod_{j=1}^{k} w_j||_{Y_m(r)} \leq C \sum_{l=1}^{k} (\prod_{j \neq l} ||w_j||_{Y_{m-1}(r)}) ||w_l||_{Y_m(r)}.$$

Proof: see [20].

We can state now our main result:

Theorem 2.2.7 *Let $(X,t) \in \mathbb{R}^2 \times \mathbb{R}$ and $m \geq 7$.*
Let $q_o \in X^m(r_o)$, $r_o > 0$, such that

$$(q_o + \overline{q_o})_y = -2A_{ox} \text{ where } A_o \in X^m(r_o) \tag{2.43}$$

and

$$|q_o|^2 + 2A_o = \psi_x \text{ with } \psi \in X^m(r_o). \tag{2.44}$$

Then there exists $M > 0$, $T > 0$, depending only on q_o and $\exists r : [-T,T] \to [r_o, r(T)]$ strictly decreasing on $[0,T]$, even, with $r(0) = r_o$ such that the system

$$\left\{ \begin{array}{l} q_t + (q(|q|^2 + 2A))_x - (|q|^2 + 2A)_y + iq_{xx} = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0 \\ q(0,x,y) = q_o(x,y) \end{array} \right. \tag{2.45}$$

has an analytic solution

$$q \in L^\infty(-T,T,X^m(r(T))) \cap L^2(-T,T,Y^*_{m+1}(r(T))),$$

with

$$q' \in L^\infty(-T,T,X^{m-2}(r(T))) \cap L^2(-T,T,Y^*_{m-1}(r(T))).$$

Moreover,

$$q \in C_w([-T,T]; X^m(r(T)))$$

and is unique in the class

$$B^w_m(T) = \{w \in C_w([-T,T]; X^m(r(T))) / \sup_{|t| \le T} ||w||^2_{X^m(r(|t|))} + M \int_{-T}^{T} ||w||^2_{Y^*_{m+1}}(r(t))dt < \infty\}.$$

Here C_w denotes the weak continuity.

Remark 2.2.8

Condition (2.44) is natural, since by differentiating the equation with respect to y and taking the real part, we get:

$$(|q|^2 + 2A)_{yy} = \partial_x[-2A_t + iq_{xy} + (uq)_y].$$

Local Spaces

In this section, we define the local spaces $X^m_B(r)$ where B is a open ball of $\mathbb{R}^2$.

Lemma 2.2.9 *Let $r > 0$ and $m \in \mathbb{N}$.*
There exists a family $\{C_\alpha(r)\}_{\alpha \in \mathbb{N}^n}$ of $\mathbb{R}^*_+$, with $\sum_\alpha C_\alpha < +\infty$ such that

$$f \in X^m(r) \text{ if and only if } \sum_{|\alpha| \ge 0} C_\alpha(r) ||\partial^\alpha_X f||^2_{H^m(\mathbb{R}^n)} < +\infty.$$

Moreover,

$$||f||^2_{X^m(r)} = \sum_{|\alpha| \ge 0} C_\alpha(r) ||\partial^\alpha_X f||^2_{H^m(\mathbb{R}^n)}.$$

Proof:
One can write

$$q(r,\xi)=\prod_{j=1}^{n}(\sum_{k\geq 0}\frac{(2r)^{2k}}{(2k)!}\xi_j^{2k})+\sum_{j_0=1}^{n}\prod_{j\neq j_o}(\sum_{k\geq 0}\frac{(2r)^{2k}}{(2k)!}\xi_j^{2k})(\sum_{k\geq 0}\frac{(2r)^{2k+1}}{(2k+1)!}\xi_{j_0}^{2k+2})$$

and, by Fubini-Tonnelli,

$$q(r,\xi)=\sum_{|\alpha|\geq 0}C_\alpha(r)\xi^{2\alpha}.$$

Thus, for $f\in X^m(r)$,

$$\begin{aligned}||f||^2_{X_m(r)} &= <q(r,\xi)\sigma^{2m}(\xi)\hat{f},\hat{f}>_{L^2(\mathbb{R}^n)}\\ &= \int q(r,\xi)|\hat{\Lambda^m f}(\xi)|^2 d\xi\\ &= \int(\sum_{|\alpha|\geq 0}C_\alpha(r)\xi^{2\alpha})|\hat{\Lambda^m f}(\xi)|^2 d\xi\end{aligned}$$

and again by Fubini-Tonnelli,

$$||f||^2_{X_m(r)}=\sum_{|\alpha|\geq 0}C_\alpha(r)||\partial_X^\alpha f||^2_{H^m(\mathbb{R}^n)}.$$

Note that we have obtained another formulation for the scalar product of $X^m(r)$:

$$<f,g>_{X_m(r)}=<q(r,\xi)\sigma^{2m}(\xi)\hat{f},\hat{g}>_{L^2(\mathbb{R}^n)}=\sum_{|\alpha|\geq 0}C_\alpha(r)<\partial_X^\alpha f,\partial_X^\alpha g>_{H^m(\mathbb{R}^n)}.$$

Definition 2.2.10 *For all ball $B\subset\mathbb{R}^n$, we set*

$$X^m_B(r)=\{f\in L^2(B)/||f||^2_{X^m_B(r)}=\sum_{|\alpha|\geq 0}C_\alpha(r)||\partial_X^\alpha f||^2_{H^m(B)}<+\infty\}.$$

We now shortly prove the following elementary property:
Lemma 2.2.11 *Let $m\in\mathbb{N}$ and $\delta>0$. Then, for all ball $B\subset\mathbb{R}^n$, the embedding*

$$X^{m+\delta}_B(r)\hookrightarrow X^m_B(r)$$

is compact.
Proof:
Let B_1 be the unit ball of $X^{m+\delta}_B(r)$ and $\{f_n\}_{n\geq 0}$ a sequence in B_1.

For all $n\in\mathbb{N}$,

$$\sum_{|\alpha|\geq 0}C_\alpha(r)||\partial_X^\alpha f_n||^2_{H^{m+\delta}(B)}\leq 1.$$

Thus, for every fixed α, the sequence $||\partial_X^\alpha f_n||_{H^{m+\delta}(B)}$ is bounded. The injection

$$H_B^{m+\delta}(r) \hookrightarrow H_B^m(r)$$

being compact, there exists a subsequence $\{\partial_X^\alpha f_{\phi_\alpha(n)}\}_{n\geq 0}$ which converges to F_α in $H^m(B)$. Plainly,

$$F_\alpha = \partial_X^\alpha F_o,$$

and using a classical diagonal process, it is clear that there exists a subsequence $f_{\phi(n)}$, still denoted f_n, such that for all $\alpha \in \mathbb{N}^n$,

$$\lim_{n\to\infty} ||\partial_X^\alpha f_n - \partial_X^\alpha F_o||_{H_B^m(r)} = 0,$$

and, for example,

$$||\partial_X^\alpha f_n - \partial_X^\alpha F_o||_{H_B^m(r)} \leq 1, \text{ for all n.}$$

Hence, since $C_\alpha||\partial_X^\alpha f_n - \partial_X^\alpha F_o||_{H_B^m(r)} \leq C_\alpha$ (with $\sum_\alpha C_\alpha < +\infty$), we have by dominated convergence that:

$$\lim_{n\to\infty} ||f_n - F_o||_{X_B^m(r)} = 0,$$

and yet the proposition is proved.

We end this section by the following remark:

Lemma 2.2.12 *Let $m > 2$ and $n = 2$.*
Then

$$\forall (f,g) \in X^m(r), fg \in X^m(r) \text{ and } ||fg||_{X^m(r)} \leq C||f||_{X^m(r)}||g||_{X^m(r)}.$$

Proof:

From Lemma 2.2.5, one gets, for $f, g \in X^m(\mathbb{R}^2)$,

$$||fg||_{Y_m(r)} \in C||f||_{Y_m(r)}||g||_{Y_m(r)}.$$

Moreover, note that for $f \in Y_m(\mathbb{R}^n)$,

$$||\partial_z^\alpha f(. + iy)||_{L^2(\mathbb{R}^2)} = ||\partial_X^\alpha f(. + iy)||_{L^2(\mathbb{R}^2)}.$$

Thus, for $\partial_X^\alpha f, \partial_X^\alpha g \in L_{r,2}$ and for all $|\alpha| \leq m$,

$$
\begin{aligned}
\sum_{|\alpha\leq m} ||\partial_X^\alpha(fg)||^2_{L_{r,2}} &= \sum_{|\alpha\leq m}\sum_{k=1}^{n} Sup_{y/y_k\in]-r,r[^{n-1}} \int_{-r}^{r} ||\partial_{z_k}\partial_X^\alpha(fg)(.+iy)||^2_{L^2(\mathbb{R}^n)}dy_k \\
&\leq C\sum_{k=1}^{n} Sup_{y/y_k\in]-r,r[^{n-1}} \int_{-r}^{r} ||\partial_{z_k}(fg)(.+iy)||^2_{H^m(\mathbb{R}^n)}dy_k \\
&\leq C\sum_{k=1}^{n} Sup_{y/y_k\in]-r,r[^{n-1}} \int_{-r}^{r} ||f(.+iy)\partial_{z_k}(g)(.+iy)||^2_{H^m(\mathbb{R}^n)}dy_k \\
&\quad + \int_{-r}^{r} ||g(.+iy)\partial_{z_k}(f)(.+iy)||^2_{H^m(\mathbb{R}^n)}dy_k \\
&\leq C\sum_{k=1}^{n} Sup_{y/y_k\in]-r,r[^{n-1}} \int_{-r}^{r} (||f(.+iy)||^2_{H^m(\mathbb{R}^n)}||\partial_{z_k}(g)(.+iy)||^2_{H^m(\mathbb{R}^n)} \\
&\quad +||g(.+iy)||_{H^m(\mathbb{R}^n)}||\partial_{z_k}(f)(.+iy)||^2_{H^m(\mathbb{R}^n)})dy_k \\
&\leq C(||f||^2_{Y^m(r)}\sum_{|\alpha|\leq m}||\partial_X^\alpha g||_{L^2_{r,2}} + ||g||^2_{Y^m(r)}\sum_{|\alpha|\leq m}||\partial_X^\alpha f||^2_{L_{r,2}})
\end{aligned}
$$

and the lemma is proved by Lemmas 2.2.3 and 2.2.4.

2.2.3 Approximate solutions for the regularized system

In this section we treat the scalar case, the generalisation for systems being straightforward.

For all $\epsilon > 0$, we consider the approximate system S_ϵ:

$$
\left\{ \begin{array}{l} iu_t + L_\epsilon u + G(u,\overline{u},\nabla u,\nabla\overline{u}) = 0 \\ u(X,0) = u_o(X) \end{array} \right. \tag{2.46}
$$

where $(X,t)\in\mathbb{R}^n\times\mathbb{R}$, G is a polynomial such that $G(0,0,0,0)=0$ and

$$
L_\epsilon\hat{(u)}(\xi) = \xi_{j_o}^2 p_\epsilon(\xi) \text{ with } p_\epsilon(\xi) = \frac{\xi_{j_1}}{(\epsilon+\xi_{j_1}^2)},\ (j_o,j_1)\in\{1;n\}.
$$

The system S_ϵ fulfils the conditions of theorem 1 of ([6]), hence:

Lemma 2.2.13 *Let $m = 2[n/2]+5$ and $u_o\in X^m(r_o)$ for some $r_o>0$.*
Then, there exists $M_\epsilon>0$ and $T_\epsilon>0$ such that the system S_ϵ has an analytic solution

$$
u_\epsilon(t)\in C([-T_\epsilon,T_\epsilon];X^m(r(T)))\cap L^2(-T_\epsilon,T_\epsilon;Y^*_{m+1}(r(T)))
$$

with, for all $t\in[-T_\epsilon,T_\epsilon]$, $u(t)\in X^m(r(t))$, $r(t)=r_o e^{\frac{-|t|M_\epsilon T_\epsilon}{r_o}}$.
Moreover, this solution is unique in the class:

$$
B_m(T) = \{w\in C([-T_\epsilon,T_\epsilon];X^m(r(T_\epsilon))),
$$

$$||w||^2_{B_m(T)} = \sup_{|t|\leq T} ||w||^2_{X^m(r(t))} + M_\epsilon \int_{-T_\epsilon}^{T_\epsilon} ||w||^2_{Y^*_{m+1}}(r(t))dt < \infty\}.$$

Uniform estimates for u_ϵ

Lemma 2.2.14 *In the Lemma 2.2.13, one can choose $T = T_\epsilon$ and $M = M_\epsilon$ independently of ϵ.*
Moreover, there exists $C = C(u_o) > 0$ such that

$$||u_\epsilon(t)||_{L^\infty(-T,T;X^m(r(T)))} \leq C.$$

Proof: The proof of this lemma relies essentially on the proof of Theorem 1 of [6]. For clarity, we will shortly present this proof. For details , one can refer to that paper.

We denote B_ρ the ball of radius $\rho = 2||u_o||_{X^m(r_o)}$ in $B_m(T_\epsilon)$, and for every $v \in B_\rho$, we consider the equation

$$\begin{cases} iu_{\epsilon t} + L_\epsilon u_\epsilon = -G(v, \overline{v}, \nabla v, \nabla \overline{v}) \\ u(X,0) = u_o(X). \end{cases} \tag{2.47}$$

By setting $w = w_\epsilon = \partial^\beta_X u_\epsilon$, $|\beta| \leq m$, and taking the Fourier transform of (2.47), we get

$$i\hat{w}_t + L_\epsilon \hat{w} = -\partial^\beta_X \hat{F}(v). \tag{2.48}$$

Multiplying this last expression by $q(r(t),\xi)\overline{w(\xi)}$ and integrating the imaginary part,

$$\frac{\partial}{\partial t}||w(t)||^2_{X_m(r(t))} - 2r'(t) < |\xi^2|q_1(r(t),\xi)\hat{w}(\xi), \hat{w}(\xi) >_{L^2(\mathbb{R}^n)}$$
$$\leq -2Im < \partial^\beta_X \hat{F}(v), q(r(t),\xi)\hat{w}(\xi) >_{L^2(\mathbb{R}^n)},$$

since

$$\partial_t q(r(t),\xi) = \sum_{k=1}^n 2r'(t)\xi_k^2 q_1(r(t),\xi) + h(\xi,t),$$

where $h \leq 0$.

Thus, by Cauchy-Schwarz inequality, and integrating between $-T_\epsilon$ and T_ϵ:

$$||u_\epsilon||^2_{B_m(T)} \leq ||u_o||^2_{X^m(r_o)} + 2\int_{-T_\epsilon}^{T_\epsilon} ||\partial^\beta_X F(v(s))||_{L_{r(t)}} ||w(s)||_{Y_1(r(s))} ds$$

Finally, using the fact that

$$||F(v)||_{Y_m(r)} \leq Q(||v||_{Y_m(r)})||v||_{Y_{m+1}(r)} \text{ if } m \geq [n/2]+1 \tag{2.49}$$

where Q is a polynomial with positive coefficients, we get(see [6] for details)

$$||u_\epsilon||_{B_m(T)} \leq \frac{\rho}{\sqrt{2}} + 2\rho Q(\rho)(2T_\epsilon + \frac{1}{M_\epsilon}). \tag{2.50}$$

In order to get a contraction in B_ρ and therefore obtain a fixed point u_ϵ solution of S_ϵ, it is enough to choose M_ϵ and T_ϵ such that

$$2Q(\rho)(2T_\epsilon + \frac{1}{M_\epsilon}) + \frac{1}{\sqrt{2}} < 1, \tag{2.51}$$

which can be done obviously independently of ϵ. More, for all $t \in [-T, T]$,

$$||u||^2_{B_m(T)} \leq C(u_o)$$

and we have proved Lemmas 2.2.13 and 2.2.14.

Remark 2.2.15 *Uniqueness*
In [6], the uniqueness of the solutions of (2.46) is given in the class

$$B_m(T) = \{w \in C([-T,T]; X^m(r(T)))/ \sup_{|t|\leq T} ||w||^2_{X^m(r(|t|))} + M\int_{-T}^{T} ||w||^2_{Y^*_{m+1}}(r(t))dt < \infty\}.$$

More precisely, it is shown in [6] that if $u_1, u_2 \in B_\rho$ are two fixed points solutions of (2.46) corresponding to the same initial data $\phi \in X^m(r_o)$,

$$||u_1 - u_2||_{B_m(T)} \leq (2Q_1(\rho,\rho) + 2\sqrt{\rho}Q_2(\rho,\rho))(2T + \frac{1}{M}||u_1 - u_2||_{B_m(T)})$$

where Q_1 and Q_2 are two polynomials and once again $\rho = 2||\phi||_{X^m(r_o)}$.
Therefore, by eventually choosing new values for T and M, we get $u_1 = u_2$.

Uniform estimates for u'_ϵ

Lemma 2.2.16 *Assume that* $u_o \in \partial^{-1}_{x_{j_1}} X^m(r_o)$, *i.e.* $u_o = \partial_{x_{j_1}}\phi$, *with* $\phi \in X^m(r_o)$. *Then there exists* $C = C(u_o) > 0$ *such that*

$$||u'_\epsilon||_{L^\infty(-T,T;X^{m-2}r(T))} \leq C.$$

Proof:
We begin by proving the existence of u''_ϵ.

We fix $t \in]-T, T[$.
For h small enough (in order to have $t+h \in]-T, T[$), we set

$$f_\epsilon^{(h)}(t) = \frac{1}{h}(u_\epsilon(t+h) - u_\epsilon(t)).$$

It can be seen from the system S_ϵ that

$$u'_\epsilon \in C([-T,T]; X^{m-2}(r(T))). \tag{2.52}$$

Therefore, a standard argument shows that

$$f_\epsilon^{(h)}(t) \to u'_\epsilon \text{ as } h \to 0 \text{ in } X^{m-2}(r(T)) \text{ weak.}$$

An elementary computation shows that

$$\partial_t f_\epsilon^{(h)}(t) + L_\epsilon f_\epsilon^{(h)}(t) + R(f_\epsilon^{(h)}(t), u_\epsilon(t)) = 0,$$

where

$$R(f_\epsilon^{(h)}(t), u_\epsilon(t)) = f_\epsilon^{(h)}(t) R_1(u_\epsilon) + \overline{f^{(h)}}_\epsilon(t) R_2(u_\epsilon) + \nabla f_\epsilon^{(h)}(t) R_3(u_\epsilon) + \nabla \overline{f^{(h)}}_\epsilon(t) R_4(u_\epsilon)$$

and

$$R_j(u_\epsilon) = R_j(u_\epsilon(t), \overline{u}_\epsilon(t+h), \nabla u_\epsilon(t), \nabla \overline{u}_\epsilon(t+h))$$

are polynomials.

More, for all t,

$$\left\{ \begin{array}{l} u_\epsilon(t+h) \to u_\epsilon(t) \\ \nabla u_\epsilon(t+h) \to \nabla u_\epsilon(t) \text{ in } X^{m-1}(r(T)) \text{ strong.} \end{array} \right.$$

Hence, $u_{\epsilon tt}(t)$ exists in $X^{m-4}(r(T))$, and, setting $v = v_\epsilon = u_{\epsilon t}$,

$$\left\{ \begin{array}{l} v_t + L_\epsilon(v) + R(v(t), u_\epsilon(t)) = 0 \\ v(0) = -L_\epsilon(u_{\epsilon o}) + G(u_{\epsilon o}). \end{array} \right. \tag{2.53}$$

We now consider equation (2.53):
We have $v_o \in X_{m-2}(T)$, $m-2 = 2[n/2]+3$ and

$$||R(v(t), u_\epsilon(t))||_{Y_{m-2}(r)} \leq C(||v||_{Y_{m-2}(r)}|| + ||\nabla v||_{Y_{m-2}(r)}).$$

Therefore, the estimate (2.49) holds for the nonlinear term $R(v, u_\epsilon)$ and we can apply the fixed-point technique described above to equation (21): this equation possesses a unique solution $v_s \in B_{m-2}(T)$. Moreover,

$$||v_s(t)||_{X^{m-2}(r(T))} \leq C||v_o||_{X^{m-2}(r_o)}.$$

Also, $u'_\epsilon \in B_m(T)$ satisfies (2.53). Therefore $u'_\epsilon = v_s$ and

$$||u'_\epsilon(t)||_{X^{m-2}(r(T))} \leq C||u'_o(t)||_{X^{m-2}(r(T))}$$

where C is a positive constant independent of ϵ.

Finally,

$$||u'_o||_{X^{m-2}(r_o)} \leq ||L_\epsilon u_o||_{X^{m-2}(r_o)} + ||G(u_o)||_{X^{m-2}(r_o)}$$

and

$$\begin{aligned} ||L_\epsilon u_o||^2_{X^{m-2}(r_o)} &= \int q(r(0),\xi)\sigma^{2(m-2)}(\xi)\frac{\xi_{j1}^2}{(\epsilon+\xi_{j1}^2)^2}|\hat{u_o}|^2 d\xi \\ &= \int q(r(0),\xi)\sigma^{2(m-2)}(\xi)\frac{\xi_{jo}^4\xi_{j1}^4}{(\epsilon+\xi_{j1}^2)^2}|\hat{\phi}_o|^2 d\xi \\ &\leq ||\phi_o||^2_{X^m(r(T))}. \end{aligned}$$

2.2.4 Limit of the approximate solutions

Lemma 2.2.17 *Let $m \geq 2[n/2]+5$ and $u_o \in X^m(r_o)$ such that $\partial_{x_{j_1}}\phi = u_o$ where $\phi \in X^m(r_o)$.*
Then there exists a sequence $\epsilon_n \to 0$ such that

$$u_{\epsilon_n} \to u \text{ in } L^\infty(-T,T;X^m(r(T))) \text{ weak-*}$$

and

$$u'_{\epsilon_n} \to u' \text{ in } L^\infty(-T,T;X^{m-2}(r(T))) \text{ weak-*}.$$

Moreover, for $\alpha \in]0,2[$,

$$u_{\epsilon_n} \to u \text{ in } C([-T,T],X^{m-\alpha}_{loc}(r(T))) \text{ strong}.$$

Proof:
The first two assertions are consequences of Lemmas 2.2.13 and 2.2.14.
Since the embedding

$$X_B^{m-2}(r) \hookrightarrow X_B^{m-\alpha}(r)$$

is compact, the standard Aubin's compactness lemma yields, up to a subsequence,

$$u_{\epsilon_n} \to u \text{ in } L^2(-T,T;X^{m-\alpha}_{loc}(r(T))) \text{ strong.}$$

Moreover, one can write, for all $t_o, t_1 \in [-T;T]$:

$$\begin{aligned} ||u_{\epsilon_n}(t_1)-u_{\epsilon_n}(t_o)||^2_{X^{m-\alpha}} &\leq \int q(r(T),\xi)\sigma^{2(m-\alpha)}||\hat{u_{\epsilon_n}}(t_1)-\hat{u_{\epsilon_n}}(t_1)|^2 d\xi \\ &\leq ||u_{\epsilon_n}(t_1)-u_{\epsilon_n}(t_o)||^{\frac{\alpha}{2}}_{X^{m-2}(r(T))}||u_{\epsilon_n}(t_1)-u_{\epsilon_n}(t_o)||^{1-\frac{\alpha}{2}}_{X^m(r(T))} \\ &\leq C\int_{t_o}^{t_1}||u'_{\epsilon_n}(\tau)||^{\frac{\alpha}{2}}_{X^{m-2}(r(T))} \\ &\leq C|t_1-t_o| \end{aligned}$$

and u_{ϵ_n} is equicontinuous in $C([-T;T];X^{m-\alpha}(r(T)))$, $0<\alpha<2$.
By the same calculations,

$$||u(t_1)-u(t_o)||^2_{X_{m-\alpha}} \leq C|t_1-t_o|.$$

Now, by setting $f_n(t)=||u(t)-u_{\epsilon_n}(t)||^2_{X^{m-\alpha}_{loc}}$,

$$\int_{-T}^{T} f_n(t)dt \to 0 \text{ as } n\to+\infty$$

and f_n is equicontinuous, hence

$$\sup_{[-T,T]} f_n(t)\to 0 \text{ as } n\to+\infty.$$

Lemma 2.2.18

$$i\partial_{x_{j_1}}L_{\epsilon_n}u_{\epsilon_n}\to -\partial^2_{x_{j_o}}u \textit{ weakly-* in } L^\infty(-T,T;X^{m-2}(r(T))).$$

Proof:
It is clear that $\partial_{x_{j_1}}L_{\epsilon_n}u_{\epsilon_n}\to v$ in $L^\infty(-T,T;X^{m-2}(r(T)))$ weak-*.
Let $f\in L^1(-T,T;X^{m-2}(r(T)))$.
On one hand,

$$\int_{-T}^{T} <\partial_{x_{j_1}}L_{\epsilon_n}u_{\epsilon_n}(\tau),f(\tau)>_{X^{m-2}(r(T))}d\tau \to \int_{-T}^{T} <v(\tau),f(\tau)>_{X^{m-2}(r(T))}d\tau.$$

On the other hand,

$$\int_{-T}^{T} <\partial_{x_{j_1}}L_{\epsilon_n}u_{\epsilon_n}(\tau),f(\tau)>_{X^{m-2}(r(T))}d\tau = -\int_{-T}^{T} <\partial^2_{x_{j_o}}u(\tau),f(\tau)>_{X^{m-2}(r(T))}d\tau$$
$$+\int_{-T}^{T} <\sigma^{2(m-2)}q(r(T),\xi)\hat{f}(\tau),\frac{\epsilon_n}{\epsilon_n+\xi^2_{j_1}}\widehat{\partial_{x_{j_o}}u}(\tau)>_{L^2(\mathbb{R}^n)}.$$

By dominated convergence, it is clear that the above integral tends to 0, and the lemma is proved.
We can already state that

$$(iu_t+G(u,\overline{u},\nabla u,\nabla\overline{u}))_{x_{j_1}} = -\partial_{x_{j_o}}u \text{ in } L^\infty(-T,T;X^{m-1}(r(T))).$$

The nonlinear terms also converge, since

$$u_{\epsilon_n}\to u \text{ in } C([-T,T],X^{m-\alpha}_{loc}(r(T))) \text{ strong}.$$

Finally,

Remark 2.2.19

One has

$$u \in C_w([-T,T], X^m(r(T))),$$

i.e., for all $\psi \in X^m(r(T))$, $t \to < \psi, u(t) >_{X^m(r(T))}$ is continuous.

In fact, it suffices to notice that

$$q \in L^\infty(-T,T; X^m(r(T)))$$

and

$$q' \in L^\infty(-T,T; X^{m-2}(r(T))).$$

(See for instance [37], Vol.1, Chap.1).
We finish this section by the following theorem :

Theorem 2.2.20 *Let $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $m = 2[n/2]+5$, and G a polynomial such that $G(0,0,0,0) = 0$.*
We assume that for some $r_o > 0$, $u_o \in X^m(r_o)$ such that

$$u_o = \phi_{x_{j_1}} \text{ with } \phi := \partial_{x_{j_1}}^{-1} u_o \in X^m(r_o).$$

Then, there exists $M > 0$ and $T > 0$, depending only on u_o and $r : [-T,T] \to [r_o, r(T)]$ strictly decreasing over $[0,T]$, even, $r(0) = r_o$, such that the problem

$$\begin{cases} iu_t + \partial_{x_{j_o}}^2 v + G(u, \overline{u}, \nabla u, \nabla \overline{u}) = 0 \\ v_{x_{j_1}} = u \\ u(0,X) = u_o(X) \end{cases} \tag{2.54}$$

has an analytic solution such that

$$u \in L^\infty(-T,T,X^m(r(T))) \cap L^2(-T,T,Y^*_{m+1}(r(T)))$$

and

$$u' \in L^\infty(-T,T,X^{m-2}(r(T))) \cap L^2(-T,T,Y^*_{m-1}(r(T))).$$

Moreover, $u \in C_w([-T,T], X^m(r(T)))$ and is unique in the class

$$B^w_m(T) = \{w \in C_w([-T,T]; X^m(r(T)))/ \sup_{|t|\leq T} ||w||^2_{X^m(r(|t|))} + M \int_{-T}^{T} ||w||^2_{Y^*_{m+1}}(r(t))dt < \infty\}.$$

Proof :
We first check that $u \in B^w_m(T)$:
It is easy to see that

$$Sup_{|t|\leq T} ||u(t)||_{X^m(r(t))} < \infty$$

by replacing in the proof above t by T.
More, we have, for all $\epsilon > 0$,

$$\int_{-T}^{T} ||u_\epsilon||_{Y^*_{m+1}(r(t))} dt \leq C(u_o).$$

Also,

$$\int_{-T}^{T} ||u_\epsilon||_{Y^*_{m+1}(r(t))} dt = \int_{-T}^{T} ||\sigma_m(\xi)(q_1(r(t),\xi))^{\frac{1}{2}} \xi \hat{u}_\epsilon(\xi,t)||^2_{L^2_\xi} dt.$$

Therefore,

$$\sigma_m(\xi) q_1(r(t),\xi))^{\frac{1}{2}} \xi \hat{u}_\epsilon(\xi,t) \to \psi \text{ in } L^2(-T,T;L^2(\mathbb{R})) \text{ weak.}$$

Since $\sigma_m(\xi)\xi(q_1(r(t),\xi))^{\frac{1}{2}} \in C^\infty([-T,T]\times\mathbb{R})$, a simple argument of uniqueness of the limit in the distributional sense yields

$$\psi = \sigma_m(\xi)(q_1(r(t),\xi))^{\frac{1}{2}} \xi \hat{u}(\xi,t).$$

Uniqueness of this solution can now be obtained as in Remark 3.3, since the same a priori estimate holds here. Therefore, we only need to show that $u(0,.) = u_o(.)$. In fact, for $\phi \in X^m(r(T))$ fixed and for all n,

$$< u_o, \phi >_{X^m(r(T))} = < u_{\epsilon_n}(0), \phi >_{X^m(r(T))} \to < u(0), \phi >_{X^m(r(T))} .$$

2.2.5 Application to Alfvén waves

The case of systems

As also noticed in [6], it is straightforward to generalize these results to the system (2.41):
Theorem 2.2.21 *Let* $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ *et* $m \geq 2[n/2]+5$.
Let $\mathbf{U} = (u_1, u_2, ..., u_d)$ *and* $\{G_j\}_{1\leq j\leq d}$ *d polynomials such that* $G_j(\mathbf{0}) = 0$.
We set $\mathbf{B^w_m}(T) = \{B^w_m\}^d$ *and* $L = L_1 + L_2$,
where L_1 *is a self-adjoint operator with a symmetric matrix symbol with real entries* $P_j \in L^\infty_{loc}$ *and*

$$L_2 = Diag(i\epsilon_1 \partial^{-1}_{x_1} \partial^2_{x_{k_1}}, ..., i\epsilon_n \partial^{-1}_{x_n} \partial^2_{x_{k_n}})$$

where $\epsilon_j \in \{0;1\}$, $k_j \in [1,n]$.
Then, if for each $\epsilon_j = 1$, $u_{oj} \in \partial^{-1}_{x_1} X^m(r_o)$, $r_o > 0$, *there exists* $T > 0$ *and* $M > 0$ *depending only on* $\mathbf{U_o}$ *and* $r : [-T,T] \to [r_o, r(T)]$ *strictly decreasing over* $[0,T]$, *even,* $r(0) = r_o$, *such that the system*

$$\begin{cases} i\mathbf{U}_t + L\mathbf{U} + G\mathbf{U} = 0 \\ \mathbf{U}(x,0) = \mathbf{U_o}(x) \end{cases} \tag{2.55}$$

where

$$G(\mathbf{U}) = \begin{pmatrix} G_1(\mathbf{U}) \\ \dots \\ G_d(\mathbf{U}) \end{pmatrix},$$

has a solution $\mathbf{U}$ *such that*

$$\mathbf{U} \in L^\infty(-T,T;\mathbf{X}^m(r(T))) \cap L^2(-T,T;\mathbf{Y}^*_{m+1}(r(T)))$$

and

$$\mathbf{U}' \in L^\infty(-T,T;\mathbf{X}^{m-2}(r(T))) \cap L^2(-T,T;\mathbf{Y}^*_{m-1}(r(T))).$$

Moreover,

$$\mathbf{U} \in C_w([-T,T];\mathbf{X}^m(r(T)))$$

and is unique in the class $\mathbf{B^w_m}(T)$.

Proof of Theorem (2.2.7)

We consider the following system:

$$\begin{cases} iQ_t + LQ + G(Q) = 0 \\ Q(X,O) = Q_o(X) \end{cases} \tag{2.56}$$

with $Q = (a, b, \alpha, \beta, \gamma, \delta, u)$,

$$\mathbf{L} = \begin{pmatrix} i\partial_x^{-1}\partial_{yy} & -\partial_{xx} & 0 & 0 & 0 & 0 & 0 \\ -\partial_{xx} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_{xx} & 0 & 0 & 0 \\ 0 & 0 & -\partial_{xx} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\partial_x^{-1}\partial_{yy} & -\partial_{xx} & 0 \\ 0 & 0 & 0 & 0 & -\partial_{xx} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\partial_x^{-1}\partial_{yy} \end{pmatrix}$$

and

$$2G = \begin{pmatrix} ua_x + af(a,b) - \frac{1}{2}(a^2-b^2)_y \\ -iub_x - ibf(a,b)) \\ u_xa_x + u\alpha_x + a_xf(a,b) + 2ag(a,\alpha,b,\beta) - (a_xa_y - b_xb_y + a\alpha_y - b\beta_y) + \gamma_y \\ -iu_xb_x - iu\beta_x - ib_xf(a,b) - ibg(a,\alpha,b,\beta) \\ u_ya_x + u\gamma_x + a_yf(a,b) + 2ah(a,\alpha,b,\beta,\gamma) - (a_y^2 - b_y^2 + a\gamma_y - b\delta_y) \\ -iu_yb_x + u\gamma_x - b_yf(a,b) - ibh(a,\alpha,b,\beta,\gamma) \\ 2\alpha_y + (a\alpha_x - b\beta_x) + 2(u_ya) + 2(ua)_y - 2u_x(a^2-b^2) \end{pmatrix}$$

where

$$f(a,b) = \frac{1}{4}(a^2-b^2)_x - a_y \ ,$$

$$g(a,\alpha,b,\beta) = \frac{1}{4}(a_x^2 + a\alpha_x - b_x^2 - b\beta_x + 4\alpha_y) \ ,$$

and

$$h(a,\alpha,b,\beta,\gamma) = \frac{1}{4}(2\gamma_y + a_xa_y - b_xb_y + a\alpha_y - b\beta_y) \ .$$

As mentionned in the introduction, this system was obtained heuristically by setting

$$(a,b,\alpha,\beta,\gamma,\delta,u)=(q+\overline{q},q-\overline{q},(q+\overline{q})_x,(q-\overline{q})_x,(q+\overline{q})_y,(q-\overline{q})_y,|q|^2+2A),$$

in order to apply Theorem 2.2.21.
By choosing $a_o \in X^m(r_o)$ and $b_o \in X^m(r_o)$ such that

$$a_o=\phi_x,\phi\in X^m(r_o) \text{ and } \frac{1}{4}(a_o^2-b_o^2)-\phi_y=\psi_x,\psi_x\in X^m(r_o)$$

with

$$Q_o=(a_o,b_o,a_{ox},b_{ox},a_{oy},b_{oy},(a_o^2-b_o^2)-\phi_y),$$

the system (2.56) has a solution in the product space $\mathbf{B^w_m}(T)=(B^w_m(T))^2\times(B^w_{m-1}(T))^4$, as in theorem 2.2.21.
We now prove that

$$V(t)=(\alpha-a_x,\beta-b_x,\gamma-a_y,\gamma-b_y,u_x-\frac{1}{4}(a^2-b^2)_x+a_y)=0 \text{ for all } t\in[-T;T].$$

By construction, by taking $U=(v_1,v_2,v_3,v_4)$, it is easy to see that $U\in B^w_{m-1}(T)$ satisfies a system of the form

$$U_t+L_1(U)+R_1(Q,\nabla Q)R_2(U,\nabla U)=0 \tag{2.57}$$

where R_j are polynomials, $R_2(0,0)=0$, and Q is a fixed function (the solution of (2.56) with initial data Q_o).
Since $Q\in B^w_{m-1}(T)$, the computation presented in Remark 2.2.15 remains valid for example in $B^w_{m-1}(T)$ and therefore, for all times, $U(t)=0$.
We can now prove by the same method that $v_5=0$, and Theorem 2.2.7 holds for

$$q(x,t)=\frac{1}{2}(a(x,t)+b(x,t)) \text{ and } A(x,t)=-\frac{1}{2}(u(x,t)-|q(x,t)|^2)$$

by looking at the first two lines of the system (2.56).

Conclusion

We were able to prove the well-posedness of system (2.38) for analytic initial data. It remains an interesting and important open problem to prove the existence of solutions for less regular initial data, say $H^s(\mathbb{R}^2)$, for some $s>0$.

2.3 Remarques sur la propagation oblique

Lorsque l'on considère le cas de la propagation oblique au champ magnétique extérieur $\mathbf{B_o}$, il est naturel de choisir une condition de frontière non nulle à l'infini pour le champ transverse [36] :

$$q \to_\infty q_o = a + ib \neq 0$$

Le modèle (1.3) ("SNLD-multidimensionnel") restant valable pour la propagation quasi-parallèle, on va choisir ici

$$q = Q + q_o \text{ avec } Q \to_\infty 0.$$

Le système (1.3) devient alors :

$$\begin{cases} Q_t + [(Q+q_o)(|Q+q_o|^2 + 2A)]_x - (|Q+q_o|^2 + 2A)_y + iQ_{xx} = 0 \\ A_x + \frac{1}{2}(Q+\overline{Q})_y = 0, \end{cases}$$

ou encore, en posant

$$\mathcal{A} = A + \frac{1}{2}|q_o|^2,$$

$$\begin{cases} Q_t + [(Q+q_o)(|Q|^2 + a(Q+\overline{Q}) + ib(\overline{Q}-Q) + 2\mathcal{A})]_x \\ -(|Q|^2 + a(Q+\overline{Q}) + ib(\overline{Q}-Q) + 2\mathcal{A})_y + iQ_{xx} = 0 \\ \mathcal{A}_x + \frac{1}{2}(Q+\overline{Q})_y = 0. \end{cases}$$

En posant

$$u = |Q|^2 + a(Q+\overline{Q}) + ib(\overline{Q}-Q) + 2\mathcal{A} :$$

$$\begin{cases} Q_t + [(Q+q_o)u]_x - u_y + iQ_{xx} = 0 \\ u_x = |Q|_x^2 + a(Q+\overline{Q})_x + ib(\overline{Q}-Q)_x - (Q+\overline{Q})_y. \end{cases}$$

Un calcul simple montre alors que, formellement, on a

$$u_t = \partial_x^{-1} u_{yy} + P(Q, \nabla Q, u, \nabla u, Q_{xx})$$

où P est un polynôme vérifiant $P(0,0,0,0,0) = 0$.
On peut par conséquent utiliser encore une fois la méthode précédente pour prouver que ce système est bien posé dans les espaces de Sobolev analytiques.

2.4 Linéarisation autour d'une onde solitaire

On linéarise l'équation

$$\begin{cases} q_t + (q(|q|^2+2A))_x - (|q|^2+2A)_y + iq_{xx} = 0 \\ A_x + \frac{1}{2}(q+\overline{q})_y = 0 \end{cases}$$

"autour" d'une onde solitaire $f(x,t)$ de l'équation DNLS-1D

$$q_t + iq_{xx} + (q|q|^2)_x = 0,$$

f de la forme :

$$f(x,t) = e^{-i\omega t} e^{i\psi(x-vt)} a(x-vt) \tag{2.58}$$

avec

$$\psi'(X) = \frac{v}{2} + \frac{3}{4}a^2(X)$$

$$a^2(X) = \frac{1}{D_1 + D_2 cosh(D_3 X)}$$

oú les D_j sont des constantes qui se calculent en fonction de ω et v.
En injectant dans (2.58) la quantité

$$q(x,y,t) = f(x,t) + r(x,y,t)$$

on obtient le problème linéaire non-autonome :

$$\begin{aligned} r_t &+ ir_{xx} + \partial_x^{-1}(r+\overline{r})_{yy} + (|f|_x^2 + f\overline{f}_x)r + ff_x\overline{r} + 2|f|^2 r_x + f^2\overline{r}_x - 2f\overline{r}_y \\ &- (f+\overline{f})r_y - f_x\partial_x^{-1}(r+\overline{r})_y = 0. \end{aligned} \tag{2.59}$$

En posant

$$R(x,y,t) = \begin{pmatrix} a(x,y,t) = Re(r) \\ b(x,y,t) = Im(r) \end{pmatrix},$$

on obtient le systéme linéaire suivant :

$$R_t + L_1(x,t)R + L_2(x,t)R + L_3(x,t)R = 0 \tag{2.60}$$

où $L_1(x,t)$ est l'opérateur anti-adjoint donné par

$$L_1(x,t) = \begin{pmatrix} 2\partial_x^{-1}\partial_y^2 + A_3(x,t)\partial_x + A_5(x,t)\partial_y & -\partial_x^2 + A_4(x,t)\partial_x + A_6(x,t)\partial_y \\ \partial_x^2 + A_4(x,t)\partial_x + A_6(x,t)\partial_y & B_2(x,t) + B_3(x,t))\partial_x \end{pmatrix},$$

L_2 est l'opérateur d'ordre 0

$$L_2(x,t) = \begin{pmatrix} A_1(x,t) & A_2(x,t) \\ B_1(x,t) & B_2(x,t) \end{pmatrix},$$

et finalement

$$L_3(x,t) = \begin{pmatrix} A_7(x,t)\partial_x^{-1}\partial_y & 0 \\ B_5(x,t)\partial_x^{-1}\partial_y & 0 \end{pmatrix},$$

avec :

$$\begin{cases} A_1(x,t) = Re(|f|_x^2 + f\overline{f}_x + ff_x) \\ A_2(x,t) = Im(f\overline{f}_x - ff_x) \\ A_3(x,t) = Re(2|f|^2 + f^2) \\ A_4(x,t) = -Im(f^2) \\ A_5(x,t) = -Re(2f + f + \overline{f}) \\ A_6(x,t) = Im(2f) \\ A_7(x,t) = -Re(f_x) \\ B_1(x,t) = Im(f\overline{f}_x - ff_x) \\ B_2(x,t) = Re(f\overline{f}_x - ff_x) \\ B_3(x,t) = -Re(f^2) \\ B_4(x,t) = -Im(f_x) \end{cases}$$

On remarque que $L_3(x,t)$, provenant lors de la linéarisation du terme

$$2Aq_x \text{ (de (2.58))}$$

peut éventuellement empêcher le problème d'être bien posé dans $L^2(\mathbb{R}^2)$.

2.4.1 Le problème à coefficients constants

On commence l'étude en "fixant" les A_j et B_j :

En prenant $A_j = A_j(x_o,t_o)$ et $B_j = B_j(x_o,t_o)$, on obtient le système à coefficients constants :

$$\{ \; R_t + L_1^o R + L_2^o R + L_3^o R = 0 R(0,x,y) = R_o(x,y) \tag{2.61}$$

avec $L_j^o = L_j(x_o,t_o)$.
On montre dans cette partie :
Theorème 2.4.1 *Le problème linéaire (2.61) est mal posé dans* $L^2(\mathbb{R}^2)$.

Preuve : Les coefficients de (2.61) étant constants, on obtient facilement ses solutions sous la forme :

$$\hat{a}(\xi,\eta,t) = F_1(\hat{a}_o,\hat{b}_o)e^{[-(A+D)-\sqrt{(A-D)^2-4BC}]t_o} + F_2(\hat{a}_o,\hat{b}_o)e^{[-(A+D)+\sqrt{(A-D)^2-4BC}]t_o} \quad (2.62)$$

$$\hat{b}(\xi,\eta,t) = F_3(\hat{a}_o,\hat{b}_o)e^{[-(A+D)-\sqrt{(A-D)^2-4BC}]t_o} + F_4(\hat{a}_o,\hat{b}_o)e^{[-(A+D)+\sqrt{(A-D)^2-4BC}]t_o} \quad (2.63)$$

où les F_j sont des fonctions de la condition intiale R_o et
et

$$\begin{cases} A = A(\xi,\eta) = A_1 + iA_3\xi + iA_5\eta + i\frac{\eta^2}{\xi} + A_7\frac{\eta}{\xi} \\ B = B(\xi,\eta) = A_2 + \xi^2 + iA_4\xi + iA_6\eta \\ C = C(\xi,\eta) = -\xi^2 + B_1 + iA_4\xi + iA_6\eta + B_4\frac{eta}{\xi} \\ D = D(\xi,\eta) = B_2 + iB_3\xi \end{cases}$$

On se ramène ainsi à l'étude des exponentielles

$$E(\xi,\eta,t_o) = e^{[-(A+D)\pm\sqrt{(A-D)^2-4BC}]t_o} \quad (2.64)$$

où l'on a noté indistinctement

$$Z = \pm\sqrt{z} \text{ les deux nombres complexes tels que } Z^2 = z$$

On prouve le lemme suivant :
Lemme 2.4.2 *A $t_o > 0$ fixé, les exponentielles $E(\xi,\eta,t_o)$ restent bornées si et seulement si la quantité $|\frac{\eta}{\xi}|$ reste bornée.*
On commence par montrer une condition nécéssaire pour que $E(\xi,\eta,t_o)$ reste bornée.

Fixons $\eta = \eta_o \in \mathbb{R}$ et faisons tendre $\xi \to 0$:
Un calcul élémentaire montre que

$$\pm\sqrt{(A-D)^2-4BC} = \pm\sqrt{(A_7\frac{\eta_o}{\xi} + \frac{i\eta_o^2}{\xi})^2(1+R(\eta_o,\xi))} = \pm(A_7\frac{\eta_o}{\xi} + \frac{i\eta_o^2}{\xi})\sqrt{1+R(\eta_o,\xi)}$$

où

$$R(\eta_o,\xi) \to 0 \text{ lorsque } \xi \to 0.$$

Ainsi,

$$E(\xi,\eta_o,t_o) = e^{[A_1+i(A_3\xi+A_5\eta+\frac{\eta^2}{\xi})+A_7\frac{\eta_o}{\xi}\pm(A_7\frac{\eta_o}{\xi}+\frac{i\eta_o^2}{\xi^2})\sqrt{1+R(\eta_o,\xi)}]t_o},$$

Et pour que $E(\xi,\eta,t_o)$ reste bornée, il faut la condition nécessaire

$$|\frac{\eta}{\xi}| \leq M \ , \ M > 0. \quad (2.65)$$

On montre maintenant que $E(\xi, \eta, t_o)$ reste bornée dans le cône

$$C_M = \{(\xi, \eta) \in \mathbb{R}^2, \ |\frac{\eta}{\xi}| \leq M\}$$

Pour $(\xi, \eta) \in C_M$,

$$\begin{cases} A = C_1(\xi, \eta) + iA_3\xi + i\eta C_2(\xi, \eta) \\ B = A_2 + \xi^2 + iA_4\xi + iA_6\eta \\ C = C_3(\xi, \eta) - \xi^2 + iA_4\xi + iA_6\eta \\ D = B_2 + iB_3\xi \end{cases}$$

où les C_j restent bornées.

Un calcul simple permet de montrer que

$$f(\xi, \eta) = (A - D)^2 - 4BC = -\xi^4 - [(C_2)^2 + A_6^2]\eta^2 + P_1(\xi^2, \xi, \eta, \eta\xi) + iP_2(\xi, \eta)$$

où les P_j sont des polynômes du premier degré à coefficients réels, variables mais bornés.

Il s'agit désormais de montrer que $Re(\pm\sqrt{f(\xi, \eta)})$ reste borné.
Et en effet

$$Re(\pm\sqrt{f(\xi, \eta)}) = \pm\frac{1}{\sqrt{2}}\sqrt{|f(\xi, \eta)| + Re(f(\xi, \eta))}$$

Pour $|(\xi, \eta)|$ assez grand, $Re(f(\xi, \eta)) < 0$.
D'où, pour $|(\xi, \eta)| \to \infty$,

$$\sqrt{|f(\xi, \eta)| + Re(f(\xi, \eta))} \sim Re(f(\xi, \eta))[1 - \sqrt{1 + (\frac{Im(f(\xi, \eta))}{Re(f(\xi, \eta))})^2} \sim \frac{1}{2}\frac{Im(f(\xi, \eta))^2}{Re(f(\xi, \eta))}$$

ce qui est une quantité bornée.
Il est maintenant aisé de montrer la proposition 2.5.1.
Il suffit pour celà de choisir une condition initiale R_o telle que, par exemple,

$$F_3 = 0 \text{ et } F_1 \sim \frac{1}{\xi^2 + \eta^2}.$$

On a alors, pour ξ fixé,

$$lim_{\eta\to\infty} F_1(x, \eta)e^{[-(A+D)-\sqrt{(A-D)^2-4BC}]=\infty}$$

et

$$a(., t_o)L^2.$$

Chapitre 3

Ondes Alfvén dans un plasma chaud sans collisions

3.1 Introduction

Dans ce chapitre, on s'intéresse à l'équation

$$q_t + iq_{xx} + (|q|^2 q)_x = \sigma(qH|q|^2)_x \tag{3.1}$$

où H désigne la transformée de Hilbert, donnée par

$$Hf(x) = v.p.(\int_{-\infty}^{+\infty} \frac{f(x')}{x - x'} dx') = v.p.(\frac{1}{x}) * f(x)$$

ou encore, en Fourier,

$$\hat{Hf}(\xi) = isgn(\xi)\hat{f}(\xi).$$

Cette équation modélise la propagation d'ondes Alfvén dans un plasma chaud sans collisions, le terme non local de droite représentant l'effet des résonances des particules sur les modulations de l'onde([60]).

Dans [11], le cas où la distribution des vitesses σ est positive est traité par une méthode d'énergie, et les auteurs obtiennent alors le théorème suivant :

Theorème 3.1.1 *Soit $T > 0$, $\alpha, \beta \in \mathbb{R}, \beta > 0$, et $s \in \mathbb{N}$, $s \geq 2$.*
Soit $q_o \in H^s(\mathbb{R})$. Alors il existe une constante $C = C(\alpha, \beta) > 0$ telle que si

$$||q_o||_{L^2} < C,$$

le problème

$$\begin{cases} q_t + iq_{xx} = \alpha(|q|^2 q)_x + \beta(H|q|^2 q)_x \\ q(0,x) = q_o(x) \end{cases}$$

admet une unique solution u telle que

$$\partial_t^h q \in L^\infty(0,T; H^{s-2h})$$

pour tout $h \in \mathbb{N}$.

A noter que la méthode de régularisation parabolique utilisée dans cet article ne tient pas compte des effets dispersifs de l'opérateur de Schrödinger.
Ainsi, on se propose d'adapter la bien connue méthode de Kenig-Ponce-Vega ([28],[29],[30]) afin de montrer des résultats (locaux) pour des données initiales moins régulières. On prouvera le résultat suivant :

Theorème 3.1.2 *Soit $\alpha, \beta \in \mathbb{C}$ et $\epsilon > 0$ fixé.*
Il existe $\delta_o > 0$ tel que si

$$||q_o||_{H^{1+\epsilon}} \leq \delta_o,$$

il existe $T = T(||q_o||_{H^{1+\epsilon}}) > 0$ tel que le problème

$$\left\{ \begin{array}{l} q_t + iq_{xx} = \alpha(|q|^2 q)_x + \beta(H|q|^2 q)_x \\ q(0,x) = q_o(x) \end{array} \right. \tag{3.2}$$

admet une unique solution forte

$$q \in C([0;T]; H^{1+\epsilon}(\mathbb{R}))$$

telle que

$$Sup_{\theta \in [0;1]} \left(\int_{\mathbb{R}_x} \left(\int_0^T |D^{\theta(\frac{3}{2}+\epsilon)} q(x,t)|^{\frac{2}{\theta}} dt \right)^{\frac{\theta}{1-\theta}} dx \right)^{\frac{1-\theta}{2}} < +\infty.$$

On remarquera que le terme de droite de (3.1) est dissipatif si et seulement si $\beta > 0$. Cette condition est vérifiée tant que la fonction de distribution de vitesse décroit avec la vitesse parallele au voisinage de la vitesse de l'onde. Il semble par conséquent difficile de montrer des résultats globaux lorsque cette condition n'est pas remplie.
De plus, la méthode présentée ici se généralise facilement à l'équation

$$q_t + iq_{xx} + \alpha_1 |q|^2 q_x + \alpha_2 q^2 \overline{q_x} = \beta_1 q H|q|^2_x + \beta_2 q_x H|q|^2 \ , \ \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$$

et, notamment, la conclusion du Théorème 3.1.2 s'applique également à l'équation de Schrödinger Dérivée :

$$q_t + iq_{xx} + \lambda |q|^2 q_x + \mu q^2 \overline{q_x} = 0$$

pour λ et μ des complexes quelconques, alors que les études précédentes concernant cette équation (voir par exemple [46], [22],[46]) nécessitent de la condition $\lambda \in \mathbb{R}$.

3.2 Propriétés de la transformée de Hilbert

Lemme 3.2.1 *Soit $1 < p < +\infty$, $1 < r < +\infty$ et $T > 0$.*
Alors il existe une constante $C = C(p,r,T) > 0$ telle que

$$\left(\int_{\mathbb{R}_x} \left[\int_0^T |H_x q(x,t)|^r dt \right]^{\frac{p}{r}} \right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}_x} \left[\int_0^T |q(x,t)|^r dt \right]^{\frac{p}{r}} \right)^{\frac{1}{p}}. \tag{3.3}$$

La démonstration de ce lemme repose sur le Théorème 3.1 de [12] :

Theorème 3.2.2 *Francia-Ruiz-Torrea(1986)*
Soient A et B deux espaces de Banach.
Pour tout $p \in [1;+\infty]$, on appelle $L^p_A(\mathbb{R}_x)$ l'espace de Bochner-Lebesgue formé des fonctions mesurables $f : \mathbb{R} \to A$ telles que

$$||f||_{L^p_A(\mathbb{R}_x)} = (\int_{-\infty}^{+\infty} ||f(x)||^p_A dx)^{\frac{1}{p}} < +\infty \text{ pour } p < +\infty, \text{ et}$$

$$||f||_{L^\infty_A(\mathbb{R}_x)} = Sup_x ||f(x)||_A < \infty.$$

Supposons que pour un certain $p_o \in [1;+\infty]$, l'opérateur H_x est continu de $L^{p_o}_A(\mathbb{R}_x)$ dans $L^{p_o}_B(\mathbb{R}_x)$. Alors H_x s'étend à $L^p_A(\mathbb{R}_x)$ pour tout $p \in [1;+\infty[$, avec,
pour tout $p \in]1;+\infty[$, l'existence d'une constante $C_p > 0$, telle que

$$||H_x f||_{L^p_B(\mathbb{R}_x)} \leq C_p ||f||_{L^p_A(\mathbb{R}_x)}.$$

Ici, on prendra $A = B = L^r_t([0,T])$:
Soit $r_o \in]1;+\infty[$.
En choisissant $p_o = r_o$,

$$\begin{aligned}
(\int_{\mathbb{R}_x} [\int_0^T |H_x q(x,t)|^{r_o} dt]^{\frac{p_o}{r_o}})^{\frac{1}{p_o}} &= (\int_0^T dt (\int_{\mathbb{R}_x} |H_x q(x,t)|^{r_o} dx))^{\frac{1}{r_o}} \\
&\leq C(\int_0^T dt (\int_{\mathbb{R}_x} |q(x,t)|^{r_o} dx))^{\frac{1}{r_o}} \text{ (H continu } L^{r_o}(\mathbb{R}_x) \to L^{r_o}(\mathbb{R}_x)) \\
&\leq C(\int_{\mathbb{R}_x} [\int_0^T |q(x,t)|^{r_o} dt]^{\frac{p_o}{r_o}} dx)^{\frac{1}{p_o}}
\end{aligned}$$

3.3 Estimations de Strichartz pour l'opérateur de Schrödinger

Dans cette partie, on déduira les estimations de Strichartz avec effet régularisant qui nous seront utiles par las suite. On interpolera les inégalités connues pour le groupe de Schrödinger, dûes à Kenig-Ponce-Vega ([28],[29]), grâce au théorème de Stein sur les familles analytiques d'opérateurs.

On pose $S(t) = e^{it\partial_{xx}}$ l'opérateur de Schrödinger engendrant les solutions du problème libre

$$\begin{cases} q_t + iq_{xx} = 0 \\ q(0,x) = q_o(x). \end{cases}$$

Dans tout ce qui suit, on notera

$$||q||_{L_x^p L_t^r} = \left(\int_{\mathbb{R}_x} \left(\int_{\mathbb{R}_t} |q(x,t)|^r dt\right)^{\frac{p}{r}}\right)^{\frac{1}{p}}$$

et

$$||q||_{L_x^p L_T^r} = \left(\int_{\mathbb{R}_x} \left(\int_0^T |q(x,t)|^r dt\right)^{\frac{p}{r}}\right)^{\frac{1}{p}}.$$

3.3.1 Estimation homogène

Lemme 3.3.1 *Il existe une constante $C > 0$ telle que pour tout $q_o \in L^2(\mathbb{R})$,*

$$||D_x^{\frac{1}{2}} S(t) q_o(x)||_{L_x^\infty L_t^2} \leq C ||q_o||_{L_x^2} \tag{3.4}$$

ou encore, en forme duale, pour tout $g \in L_x^1 L_t^2$,

$$|| \int D_x^{\frac{1}{2}} S(t) g(.,\tau) d\tau ||_{L^2} \leq C ||g(x,t)||_{L_x^1 L_t^2}.$$

Pour la preuve, voir [28], Théorème 4.1.
On a également l'estimation sur la fonction maximale $Sup_{t\in[0;T]} S(t) q_o(x)$:
Lemme 3.3.2 *Soit $T > 0$ et $\epsilon > 0$.*
Alors, il existe une constante $C > 0$ telle que pour tout $q_o \in H^{\frac{1}{2}+\epsilon}(\mathbb{R})$,

$$\begin{aligned} ||S(t) q_o(x)||_{L_x^2 L_t^\infty} &\leq C(1+T) ||q_o||_{H^{\frac{1}{2}+\epsilon}} \\ &\leq C(1+T)(||D_x^{\frac{1}{2}+\epsilon} q_o||_{L_x^2} + ||q_o||_{L_x^2}) \end{aligned} \tag{3.5}$$

Pour la preuve, voir [55].
Finalement, une méthode standard d'interpolation complexe donne
Lemme 3.3.3 *(Estimation homogène)*
Soit $\epsilon > 0$ fixé. Soit $T > 0$. Alors, pour tout $\theta \in [0;1]$,

$$||D_x^{\frac{\theta}{2}} S(t) q_o||_{L_x^{\frac{2}{1-\theta}} L_t^{\frac{2}{\theta}}} \leq C(T)(||D_x^{(1-\theta)(\frac{1}{2}+\epsilon)} q_o||_{L_x^2} + ||q_o||_{L_x^2}) \tag{3.6}$$

où $C(T)$ est une constante qui croit avec le temps T.

3.3.2 Estimation inhomogène

Lemme 3.3.4 *Soit $T > 0$.*
Il existe une constante $C(T) > 0$ telle que pour tout $0 \leq t \leq T$

$$\begin{aligned} \|\int_0^t S(t-\tau)f(x,\tau)d\tau\|_{L_x^2 L_T^\infty} &\leq C(T)(\|D_x^{\frac{1}{2}^+} f(x,t)\|_{L_x^2 L_T^2} + \|f\|_{L_x^2 L_T^2}) \\ &\leq C(T)(\|HD_x^{\frac{1}{2}^+} f(x,t)\|_{L_x^2 L_T^2} + \|f\|_{L_x^2 L_T^2}) \end{aligned} \quad (3.7)$$

Preuve :

$$\begin{aligned} \|\int_0^t S(t-\tau)f(x,\tau)d\tau\|_{L_x^2 L_T^\infty} &= \int_{\mathbb{R}_x} (Sup_{0\leq t\leq T}|\int_0^t S(t-\tau)f(x,\tau)d\tau|)^2 dx \\ &\leq \int_{\mathbb{R}_x} (\int_0^T Sup_{0\leq t\leq T}|S(t)f(x,\tau)|d\tau)^2 dx \\ &\leq T^2 \int_{\mathbb{R}_x}\int_0^T Sup_{0\leq t\leq T}|S(t)f(x,\tau)|^2 d\tau dx \text{ (Hölder)} \\ &\leq CT^2(1+T)^2 \int_0^T d\tau \int_{\mathbb{R}_x} (|D_x^{\frac{1}{2}^+} f(x,\tau)|^2 + |f(x,\tau)|^2) dx \text{ (par (3.4))} \\ &\leq C(T)(\|D_x^{\frac{1}{2}^+} f(x,t)\|_{L_x^2 L_T^2} + \|f(x,t)\|_{L_x^2 L_T^2}) \\ &\leq C(T)(\|HD_x^{\frac{1}{2}^+} f(x,t)\|_{L_x^2 L_T^2} + \|f(x,t)\|_{L_x^2 L_T^2}) \end{aligned}$$

Lemme 3.3.5 *Soit $T > 0$.*
Il existe une constante $C(T) > 0$ telle que pour tout $0 \leq t \leq T$,

$$\|\partial_x \int_0^t S(t-\tau)f(x,\tau)d\tau\|_{L_x^\infty L_T^2} \leq C(T)\|f(x,t)\|_{L_x^1 L_T^2}$$

ou encore

$$\|D_x^{\frac{3}{2}^+} \int_0^t S(t-\tau)f(x,\tau)d\tau\|_{L_x^\infty L_T^2} \leq C(T)\|HD_x^{\frac{1}{2}^+} f(x,t)\|_{L_x^1 L_T^2}. \quad (3.8)$$

Preuve : voir [29], Theorème 2.1.
Une fois de plus, en utilisant le lemme d'interpolation complexe de Stein , on obtient grâce aux deux lemmes précédents :

Lemme 3.3.6 *Estimation inhomogène*
Soit $\epsilon > 0$ fixé.
Soit $T > 0$. Pour tout $t \in [0;T]$ et pour tout $\theta \in [0;1]$,

$$\|\int_0^t D_x^{\theta(\frac{3}{2}+\epsilon)} S(t-\tau)f(x,\tau)d\tau\|_{L_x^{\frac{2}{1-\theta}} L_T^{\frac{2}{\theta}}} \leq C(T)(\|HD_x^{\frac{1}{2}+\epsilon} f\|_{L_x^{\frac{2}{1+\theta}} L_T^2} + \|f(x,t)\|_{L_x^2 L_T^2}) \quad (3.9)$$

où $C(T)$ est une constante qui croit avec le temps.

On termine ce chapitre par le lemme suivant :
Lemme 3.3.7 *Soit* $1 < p, r < +\infty$ *et* $0 < \alpha < 1$.
Alors, pour tout $T > 0$,

$$||D_x^\alpha(fg) - fD_x^\alpha(g) - D_x^\alpha(g)||_{L_x^p L_T^r} \leq C||D_x^{\alpha_1} f||_{L_x^{p_1} L_T^{r_1}} ||D_x^{\alpha_2} g||_{L_x^{p_2} L_T^{r_2}} \tag{3.10}$$

où $1 < p_i, r_i < +\infty$ *avec*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\ ,\ \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

et $\alpha_i \in [0;1]$ *avec* $\alpha_1 + \alpha_2 = \alpha$.
Cette inégalité reste vraie si $(p,r) = (1,2)$ *et si l'on remplace* D_x^α *par* HD_x^α.
Cette inégalité, utilisée par Kenig, Ponce et Vega ([29]), est une version vectorielle des inégalités scalaires dûes à Kato et Ponce ([26]), Christ et Weinstein ([10]) et Taylor ([54]).

3.4 Preuve du Théorème 3.1.2

Dans ce qui suit, on fixe $\epsilon > 0$.
Pour $q_o \in H^{1+\epsilon}(\mathbb{R}), ||q_o||_{H^{1+\epsilon}} \leq \delta_o$($\delta_o$ à fixer ultérieurement), on considère les solutions $\tilde{q} = \psi(q)$ du problème linéaire

$$\begin{cases} \tilde{q}_t + i\tilde{q}_{xx} = N(q) \\ \tilde{q}(0,x) = q_o(x) \end{cases} \tag{3.11}$$

où

$$N(q) = \alpha(q|q|^2)_x + \beta(qH_x|q|^2)_x\ ,\ \alpha, \beta \in \mathbb{C},$$

$$q \in X_T^a = \{q : \mathbb{R} \times [0;T]/\eta_T(q) \leq a\ ,\ \lambda_{T,\theta}(q) + \lambda_{T,\theta}(Hq) \leq a, \theta \in [0;1]\}, \text{ et}$$

$$\eta_T(q) = Sup_{t\in[0;T]}||q(t)||_{H^{1+\epsilon}}\ ,\ \lambda_{T,\theta}(q) = ||D_x^{\theta(\frac{3}{2}+\epsilon)} q||_{L_x^{\frac{2}{1-\theta}} L_t^{\frac{2}{\theta}}}.$$

Notre but ici est de montrer que pour a et T bien choisis (en fonction de $||q_o||_{H^{1+\epsilon}}$), si $q \in X_T^a$, alors $\tilde{q} \in X_T^a$, et

$$\psi = \psi_{q_o} : q \to \tilde{q}$$

est une contraction de X_T^a.

3.4.1 Estimation de $\lambda_{T,\theta}(\tilde{q})$ et $\lambda_{T,\theta}(H\tilde{q})$

On a

$$\tilde{q}(t) = S(t)q_o + \int_o^t S(t-\tau)N(q)(\tau)d\tau. \tag{3.12}$$

Ainsi, par (3.6) et (3.9),

$$\lambda_{T,\theta}(\tilde{q}) \leq c||q_o||_{H^{1+\epsilon}} + C(T)||HD^{\frac{1}{2}+\epsilon}N(q)(x,t)||_{\frac{2}{1+\theta},2} + c||N(q)(x,t)||_{2,2}$$

On pose $p = \frac{2}{1+\theta} \in [1;2]$:

$$\begin{aligned}||HD^{\frac{1}{2}+\epsilon}N(q)(x,t)||_{p,2} \leq\ & |\alpha|.||HD^{\frac{1}{2}+\epsilon}[H(|q|^2)q]_x||_{p,2} + |\alpha|||H(|q|^2q)_x||_{2,2} \\ & +||HD^{\frac{1}{2}+\epsilon}(|q|^2q)_x||_{p,2} + ||(|q|^2q)_x||_{2,2} \\ \leq\ & ||HD^{\frac{1}{2}+\epsilon}[H(q_x\overline{q})q]||_{p,2} + ||HD^{\frac{1}{2}+\epsilon}[H(q\overline{q_x})q]||_{p,2} \\ & +||HD^{\frac{1}{2}+\epsilon}[H(q\overline{q})q_x]||_{p,2} + ||[H(q_x\overline{q})q]||_{2,2} \\ & +||[H(q\overline{q_x})q]||_{2,2} + [H(q\overline{q})q_x]||_{2,2} + ||HD^{\frac{1}{2}+\epsilon}[q_x\overline{q}q]||_{p,2} \\ & +2||HD^{\frac{1}{2}+\epsilon}[q^2\overline{q_x}]||_{p,2} \\ & +2||q_x\overline{q}q||_{2,2} + ||q^2\overline{q_x}||_{2,2}.\end{aligned}$$

On estime uniquement le premier terme (les autres étant analogues) :

$$\begin{aligned}||HD^{\frac{1}{2}+\epsilon}[H(q_x\overline{q})q]||_{p,2} \leq\ & ||D^{\frac{1}{2}+\epsilon}(q_x\overline{q})q||_{p,2} + ||HD^{\frac{1}{2}+\epsilon}qH(q_x\overline{q})||_{p,2} \\ & +||HD^{\frac{1}{2}+\epsilon}[H(q_x\overline{q})q] - D^{\frac{1}{2}+\epsilon}qH(q_x\overline{q}) - HD^{\frac{1}{2}+\epsilon}H(q_x\overline{q})q||_{p,2} \\ \leq\ & ||qq_xD^{\frac{1}{2}+\epsilon}\overline{q}||_{p,2} + ||q\overline{q}HD^{\frac{3}{2}+\epsilon}q||_{p,2} \\ & +||HD^{\frac{1}{2}+\epsilon}qH(q_x\overline{q})||_{p,2} \\ & +||q[D^{\frac{1}{2}+\epsilon}[(q_x\overline{q})] - D^{\frac{1}{2}+\epsilon}\overline{q}q_x - \overline{q}HD^{\frac{3}{2}+\epsilon}q]||_{p,2} \\ & +||HD^{\frac{1}{2}+\epsilon}[H(q_x\overline{q})q] - D^{\frac{1}{2}+\epsilon}qH(q_x\overline{q}) - HD^{\frac{1}{2}+\epsilon}H(q_x\overline{q})q||_{p,2}\end{aligned} \tag{3.13}$$

Aussi,

$$||qq_xD^{\frac{1}{2}+\epsilon}\overline{q}||_{p,2} \leq ||D^{\frac{1}{2}+\epsilon}\overline{q}||_{\frac{2}{1-\theta_o},\frac{2}{\theta_o}}||\overline{q}q_x||_{\frac{2p}{2-p+p\theta_o},\frac{2}{1-\theta_o}}$$

par Hölder, et où

$$\theta_o = \frac{1+2\epsilon}{3+2\epsilon}.$$

De plus,

$$||\overline{q}q_x||_{\frac{2p}{2-p+p\theta_o},\frac{2}{1-\theta_o}} \leq ||HD^1q||_{\frac{2}{1-\theta_1},\frac{2}{\theta_1}}||q||_{a,b}$$

où

$$\theta_1 = \frac{2}{3+2\epsilon}\,,\ \frac{1}{a} + \frac{1-\theta_1}{2} = \frac{2+p(\theta_o-1)}{2p}\,,\ \text{et}\ \frac{1}{b} + \frac{\theta_1}{2} = \frac{1-\theta_o}{2},$$

i.e.

$$a = \frac{2p}{2-p} \geq 2 \text{ et } b = +\infty$$

Finalement,

$$||qq_xD^{\frac{1}{2}+\epsilon}\overline{q}||_{p,2} \leq \lambda_{T,\theta_o}(q)\lambda_{T,\theta_1}(Hq)(\lambda_{T,0}(q))^{\frac{2-p}{p}}(\eta_T(q))^{\frac{4(p-1)}{2p}} \tag{3.14}$$

De plus,

$$\begin{aligned} ||HD^{\frac{1}{2}+\epsilon}qH(q_x\overline{q})||_{p,2} &\leq ||HD^{\frac{1}{2}+\epsilon}q||_{\frac{2}{1-\theta_o}}||H(\overline{q}q_x)||_{\frac{2p}{2-p+p\theta_o},\frac{2}{1-\theta_o}} \\ &\leq \lambda_{T,\theta_0}(q)||\overline{q}q_x||_{\frac{2p}{2-p+p\theta_o},\frac{2}{1-\theta_o}}(\text{d'après (3.3)}) \\ &\leq \lambda_{T,\theta_0}(q)\lambda_{T,\theta_1}(Hq)||q||_{\frac{2p}{2-p},\infty} \end{aligned}$$

et

$$||HD^{\frac{1}{2}+\epsilon}qH(q_x\overline{q})||_{p,2} \leq \lambda_{T,\theta_0}(q)\lambda_{T,\theta_1}(Hq)(\lambda_{T,0}(q))^{\frac{2-p}{p}}(\eta_T(q))^{\frac{2(p-1)}{p}} \tag{3.15}$$

Aussi, par Hölder,

$$\begin{aligned} ||q\overline{q}HD^{\frac{3}{2}+\epsilon}q||_{p,2} &\leq ||q^2||_{p,2}||HD^{\frac{3}{2}+\epsilon}q||_{\infty,2} \\ &\leq ||q||^2_{2p,2}||HD^{\frac{3}{2}+\epsilon}q||_{\infty,2} \\ &\leq T||q||^2_{2p,\infty}||HD^{\frac{3}{2}+\epsilon}q||_{\infty,2} \\ &\leq T||q||^{\frac{2p-2}{p}}_{\infty,\infty}(\lambda_{T,0}(q))^{\frac{2}{p}}\lambda_{T,1}(Hq) \end{aligned}$$

et, par Sobolev,

$$||q\overline{q}HD^{\frac{3}{2}+\epsilon}q||_{p,2} \leq CT(\eta_T(q))^{\frac{2p-2}{p}}(\lambda_{T,0}(q))^{\frac{2}{p}}\lambda_{T,1}(Hq) \tag{3.16}$$

Finalement, d'après (3.10) et Hölder,

$$\begin{aligned} ||q[D^{\frac{1}{2}+\epsilon}[(q_x\overline{q})]-HD^{\frac{1}{2}+\epsilon}\overline{q}q_x-\overline{q}D^{\frac{3}{2}+\epsilon}q])||_{p,2} &\leq ||D^{\frac{1}{2}+\epsilon}[(q_x\overline{q})]-HD^{\frac{1}{2}+\epsilon}\overline{q}q_x-\overline{q}D^{\frac{3}{2}+\epsilon}q])||_{2,2}||q||_{\frac{2p}{2-p},\infty} \\ &\leq C||D^{\frac{1}{2}+\epsilon}\overline{q}||_{\frac{2}{1-\theta_o},\frac{2}{\theta_o}}||q_x||_{\frac{2}{1-\theta_1},\frac{2}{\theta_1}}||q||_{\frac{2p}{2-p},\infty} \end{aligned}$$

d'où

$$||q[D^{\frac{1}{2}+\epsilon}[(q_x\overline{q})] - D^{\frac{1}{2}+\epsilon}\overline{q}q_x - \overline{q}HD^{\frac{3}{2}+\epsilon}q]||_{p,2} \tag{3.17}$$

$$\leq C\lambda_{T,\theta_o}(q)\lambda_{T,\theta_1}(Hq)(\eta_T(q))^{\frac{2(p-1)}{p}}(\lambda_{T,0}(q))^{\frac{2-p}{p}}$$

De même,

$$||HD^{\frac{1}{2}+\epsilon}[H(q_x\overline{q})q] - D^{\frac{1}{2}+\epsilon}qH(q_x\overline{q}) - HD^{\frac{1}{2}+\epsilon}H(q_x\overline{q})q||_{p,2}$$
$$\leq C||H(\overline{q}q_x)||_{\frac{2p}{2-p+p\theta_o},\frac{2}{1-\theta_o}}||D^{\frac{1}{2}+\epsilon}q||_{\frac{2}{1-\theta_o},\frac{2}{\theta_o}}$$
$$\leq C\lambda_{T,\theta_0}(q)\lambda_{T,\theta_1}(Hq)(\lambda_{T,0}(q))^{\frac{2-p}{p}}(\eta_T(q))^{\frac{2(p-1)}{p}}$$

Ainsi, en combinant cette dérnière inégalité avec (3.13),(3.14) et (3.17) et en posant

$$\Lambda_T(q) = Max_{0\leq\theta\leq 1}\{\lambda_{T,\theta}(q), \lambda_{T,\theta}(Hq), \eta_T(q)\},$$

on obtient, pour tout $p \in [1;2]$,

$$||N(q)(x,t)||_{p,2} \leq C(T)(\Lambda_T(\tilde{q}))^3 \tag{3.18}$$

ou encore

$$\lambda_{T,\theta}(\tilde{q}) \leq C(T)(\Lambda_T(\tilde{q}))^3 + c||q_o||_{H^{1+\epsilon}}. \tag{3.19}$$

Les mêmes arguments montrent que

$$\lambda_{T,\theta}(Hq) \leq C(T)(\Lambda_T(q))^3 + c||q_o||_{H^{1+\epsilon}} \tag{3.20}$$

où $C(T)$ désigne une constante strictement positive, qui croit avec T.

3.4.2 Estimation de $\eta_T(\tilde{q})$

Pour conclure, on estime $\eta_T(\tilde{q})$:

$$\eta_T(\tilde{q}) \leq CSup_{t\in[0;T]}||q||_{L^2} + CSup_{t\in[0;T]}||D^{1+\epsilon}q||_{L^2}$$

Par (3.12),

$$\begin{aligned}\eta_T(q) \quad &\leq \quad C||q_o||_{H^{1+\epsilon}} + Sup_{t\in[0;T]}||\int_0^t S(t-\tau)N(q)(x,\tau)d\tau||_{L^2}\\ &+Sup_{t\in[0;T]}||\int_0^t D_x^{1+\epsilon}S(t-\tau)N(q)(x,\tau)d\tau||_{L^2}\end{aligned}$$

et

$$\begin{aligned}Sup_{t\in[0;T]}||\int_0^t D_x^{1+\epsilon}S(t-\tau)N(q)(x,\tau)d\tau||_{L^2} \quad &\leq \quad C||\int_0^T Sup_{t\in[0;T]}D_x^{1+\epsilon}S(t-\tau)N(q)(x,\tau)d\tau||_{L^2}\\ &\leq \quad C(T)||HD_x^{\frac{1}{2}+\epsilon}N(q)(x,t)||_{1,2} \leq C(T)(\Lambda_T(q))^3\end{aligned}$$

(par (3.9) et (3.18)).
De plus,

$$\begin{aligned}Sup_{t\in[0;T]}||\int_0^t S(t-\tau)N(u)(x,\tau)d\tau||_{L^2} \quad &\leq \quad C\int_0^T(\int_{\mathbb{R}_x}|N(q)(x,t)|^2dx)^{\frac{1}{2}}dt\\ &\leq \quad CT^{\frac{1}{2}}(\int_{\mathbb{R}_x}\int_0^T|N(q)(x,t)|^2dtdx)^{\frac{1}{2}}\\ &\leq \quad CT^{\frac{1}{2}}||N(q)(x,t)||_{2,2}\\ &\leq \quad CT^{\frac{1}{2}}(\Lambda_T(q))^3\end{aligned}$$

Finalement,

$$\eta_T(\tilde{q}) \leq C(T)(\Lambda_T(q))^3 + ||q_o||_{H^{1+\epsilon}}. \tag{3.21}$$

3.4.3 Fin de la démonstration

On a ainsi obtenu

$$\Lambda_T(\tilde{q}) \leq c||q_o||_{H^{1+\epsilon}} + C(T)(\Lambda_T(q))^3$$

où $C(T) > 0$ croit avec T.
En posant $a = 2c||q_o||_{H^{1+\epsilon}} \leq 2c\delta_o$,

$$\psi(X_T^a) \subset X_T^a$$

si $2C(T)c^2\delta_0^2 \leq \frac{1}{2}$.
Avec ces conditions sur a, T et $||q_o||_{H^{1+\epsilon}}(\mathbb{R})$, les arguments précédents permettent de montrer que

$$\Lambda(\psi(u) - \psi(v)) \leq 2C(T)c^2\delta_o^2\Lambda(u - v) \leq \frac{1}{2}\Lambda(u - v)$$

i.e. ψ est une contraction, d'où l'existence d'un point fixe encore noté q pour l'application ψ.
Ainsi, il existe $q \in X_T^a$ vérifiant l'équation intégrale

$$q(t) = S(t)q_o + \int_o^t S(t - \tau)N(q)(\tau)d\tau.$$

La technique habituelle (voir entre autres [29]) permet de montrer que $q \in C([0, T]; H^{1+\epsilon}(\mathbb{R}))$ et qu'il est unique.

Chapitre 4

L'équation de Zakharov-Rubenchik uni-dimensionnelle

Stability of the Solitons for the one-dimensional Zakharov-Rubenchik Equation

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Résumé

We prove the global well-posedness of the one-dimensional Zakharov-Rubenchik equation

$$\begin{cases} iB_t + \omega B_{xx} - k(u - \frac{v}{2}\rho + q|B|^2)B = 0 \\ \theta\rho_t + (u - v\rho)_x = -k|B|^2_x \\ \theta u_t + (\beta\rho - vu)_x = \frac{k}{2}v|B|^2_x, \end{cases} \tag{4.1}$$

in the space $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$. We also prove the existence and the orbital stability of solitary wave solutions to (4.1). [1]

4.1 Introduction

In order to describe the dynamics of dispersive Alfvén waves propagating in a plasma, we consider the Hall-MHD equations([9])

$$\begin{cases} \rho_{M_t} + \nabla.(\rho_M U) = 0 \\ \rho_M(U_t + u.\nabla U) = -\frac{\beta}{\gamma}\nabla(\rho_M^\gamma) + (\nabla \times b) \times b \\ b_t = \nabla \times (U \times b) - \frac{1}{R_i}\nabla \times (\frac{1}{\rho_M}(\nabla \times b) \times b) \\ \nabla.b = 0, \end{cases}$$

where dissipative processes have been neglected.
Here, b denotes the magnetic field, ρ_M the density of mass, U the fluid speed, R_i the normalized ion-cyclotron frequency and γ the polytropic gas constant. Assuming the existence of a strong ambient magnetic field along the x-axis and variations of the fields in this direction only, we get([9])

$$\begin{cases} \rho_{M_t} + \partial_x(\rho_M \tilde{u}) = 0 \\ \rho_M(\tilde{u}_t + u\partial_x\tilde{u}) = -\partial_x(\frac{\beta}{\gamma}\rho_M^\gamma + \frac{1}{2}|b|^2) \\ \rho_M(\partial_t v + \tilde{u}\partial_x v) = \partial_x b \\ \partial_t b + \partial_x(\tilde{u}b - v) = -\frac{i}{R_i}\partial_x(\frac{1}{\rho_M}b), \end{cases} \tag{4.2}$$

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where $\tilde{u} = U_x$ and where we have used the complex notation

$$b = b_y + ib_z \text{ and } v = U_y + iU_z$$

to describe the transverse directions of the fields.
The system (4.2) possesses the exact solutions

$$b = -\frac{\omega}{k}v = B_o e^{i(kx-\omega t)} \ , u = 0 \ , \rho_M = 1,$$

where the frequency ω and the wave number k are related by the dispersion equation

$$\omega = \frac{k^2}{2R_i} + k\sqrt{1 + \frac{k^2}{4R_i^2}}.$$

This solutions correspond to a monochromatic circulary polarized Alfvén wave propagating along the x-axis.

Again in ([9]), by a classical multi-scale analysis, the authors look for small amplitude solutions to (4.2) which are slow modulations (in time and space) of the exact solutions described above:
For a small parameter $\theta > 0$, we define the slow variables $X = \theta x$, $T = \theta t$, and expand

$$\left\{ \begin{array}{l} b = \theta e^{i(kx-\omega t)}(B_1 + \theta B_2 + \dots) \\ v = e^{i(kx-\omega t)}(\theta v_1 + \theta^2 v_2 + \dots) \\ \rho_M = 1 + \theta^2 \rho_1 + \theta^3 \rho_2 + \dots \\ \tilde{u} = \theta^2 u_1 + \theta^3 u_2 + \dots. \end{array} \right. \tag{4.3}$$

By inserting these quantities in (4.2), equating the coefficients of $\{\theta^n\}_{n\in\mathbb{N}}$ to 0, and putting $B = B_1 + \theta B_2$, $\rho = \rho_1$ and $u = u_1$, one gets the system

$$\left\{ \begin{array}{l} i(\partial_T + v_g\partial_X)B + \theta\omega\partial_{XX}B - \theta k(u - \frac{v_g}{2}\rho)B = 0 \\ \rho_T + \partial_X u = 0 \\ \partial_T u + \partial_X(\beta\rho + \frac{1}{2}|B|^2) = 0, \end{array} \right. \tag{4.4}$$

where the group velocity v_g is given by

$$v_g = \frac{2\omega^3}{k(k^2 + \omega^2)}.$$

Then, changing the system of coordinates to a frame moving at speed v_g, (4.4) is reduced to

$$\left\{ \begin{array}{l} iB_\tau + \omega B_{\xi\xi} - k(u - \frac{v_g}{2}\rho)B = 0 \\ \theta\rho_\tau + \partial_\xi(u - v_g\rho) = 0 \\ \theta u_\tau + \partial_\xi(\beta\rho - v_g u + \frac{1}{2}|B|^2) = 0, \end{array} \right. \tag{4.5}$$

where

$$(\xi, \tau) = (X - v_g T, \theta T).$$

Note that by neglecting the terms in θ, one gets the Nonlinear Schrödinger Equation with cubic nonlinearity:

$$iB_T + \omega' B_{\xi\xi} + \frac{k'v}{4(\beta - v_g^2)} |B|^2 B = 0.$$

This model clearly breaks down at the resonance

$$\beta - v^2 = 0.$$

In this case, by setting

$$\begin{cases} \beta^{\frac{1}{2}} - v_g = \theta^{\frac{2}{3}} \lambda \\ \xi = \theta^{\frac{2}{3}}(x - v_g t) \\ \tau = \theta^{\frac{4}{3}} t \\ u - \frac{1}{2} v_g (\rho - 1) = \theta^{\frac{4}{3}} \phi, \end{cases}$$

(4.4) becomes the Benney equation:

$$\begin{cases} iB_\tau + \omega B_{\xi\xi} - k\phi B = 0 \\ \phi_\tau + \lambda \phi_\xi = -\frac{1}{8} |B|^2_\xi, \end{cases} \tag{4.6}$$

This model has been extensively studied by many authors.
In [56], the local well-posedness of (4.6) is obtained in the Sobolev space $H^{\frac{1}{2}}(\mathbb{R})$.
Also, in [57], and in the case of the presence of a cubic non-linearity of the form

$$\begin{cases} iB_\tau + \omega B_{\xi\xi} - k\phi B = |B|^2 B \\ \phi_\tau + \lambda \phi_\xi = -\frac{1}{8} |B|^2_\xi, \end{cases} \tag{4.7}$$

global well-posedness has been established in $H^{\frac{3}{2}}(\mathbb{R})$.
Finally, in [34], P. Laurençot considers global weak solutions in $H^1(\mathbb{R})$ and the stability of solitary waves to (4.6).

Far from the resonance, one can get another formulation for (4.4).
By some elementary computations([9]), one obtains the Zakharov-Rubenchik equation ([62]):

$$\begin{cases} iB_t + \omega B_{xx} - k(u - \frac{v}{2}\rho + q|B|^2)B = 0 \text{ (a)} \\ \theta \rho_t + (u - v\rho)_x = -k|B|^2_x \text{ (b)} \\ \theta u_t + (\beta\rho - vu)_x = \frac{k}{2} v |B|^2_x \text{ (c)}, \end{cases} \tag{4.8}$$

where q is given by

$$q = k + \frac{v(kv - 1)}{2(\beta - v^2)}.$$

We have in fact obtained the canonical model describing the interaction between long and short waves.
Here, we will study (4.8) in the case where

$$\begin{cases} 0 < \theta << 1, \\ k > 0 \\ w > 0 \text{ (right-hand polarization)} \\ \beta - v^2 > 0. \end{cases}$$

One can find in [48] a study of the Cauchy problem for (4.8) in dimension 2 and 3.
The rest of this paper is organized as follows:
In the second section, by using "$L^p - L^q$" and Strichartz-type estimates for the Schrödinger operator, we prove the local well-posedness of (4.8) in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ via a fixed-point technique.
In the third section after deriving a few conservation laws and a priori estimates to (4.8), we prove that the local solutions described in the previous section can in fact be continued globally.
Finally, in the last section, we look for the existence of solitary wave solutions to (4.8). We also prove their orbital stability by building some appropriate conservation laws to (4.8).

4.2 Local solutions

This section is concerned with the local well-posedness of the Cauchy problem (4.8). We prove the following:
Theorème 4.2.1 *Let $B_o \in H^2(\mathbb{R})$ and $\rho_o, u_o \in H^1(\mathbb{R})$.*
Then there exists $T > 0$, and a unique solution (B, ρ, u) of (4.8), with

$$(B,\rho,u) \in \cap_{j=0,1} C^j([0,T]; H^{2-2j}(\mathbb{R})) \times \cap_{j=0,1} C^j([0,T]; H^{1-j}(\mathbb{R})) \times \cap_{j=0}^{1} C([0,T]; H^{1-j}(\mathbb{R})).$$

Moreover,

$$B_t \in L^6(0,T; L^6(\mathbb{R})).$$

4.2.1 Preliminary Remarks

The system (4.8) can be put in the form

$$\begin{cases} iB_t + B_{xx} - k(u - \frac{v}{2}\rho + q|B|^2)B = 0 \\ \rho_{tt} - 2v\rho_{xt} + (v^2 - \beta)\rho_{xx} = -k|B|^2_{xt} + \frac{k}{2}v|B|^2_{xx} \\ u_{tt} - 2vu_{xt} + (v^2 - \beta)u_{xx} = \frac{k}{2}v|B|^2_{xt} + k(\beta - \frac{v^2}{2})|B|^2_{xx}. \end{cases}$$

By setting $(x', t') = (x + vt, t)$ and

$$\tilde{B}(x', t') = e^{i(-\frac{v}{2}x' - \frac{v^2}{4}t')} B(x,t),$$

we get

$$\begin{cases} i\tilde{B}_{t'} + \tilde{B}_{x'x'} - k(u - \frac{v}{2}\rho + q|\tilde{B}|^2\tilde{B}) = 0 \\ \rho_{t't'} - \beta\rho_{x'x'} = -k|\tilde{B}|^2_{x't'} - \frac{k}{2}v|\tilde{B}|^2_{x'x'} \\ u_{t't'} - \beta u_{x'x'} = \frac{k}{2}v|\tilde{B}|^2_{x't'} + \beta k|\tilde{B}|^2_{x'x'}. \end{cases} \tag{4.9}$$

Finally, if

$$\psi = u - \frac{v}{2}\rho,$$

we obtain

$$\begin{cases} i\tilde{B}_{t'} + \tilde{B}_{x'x'} - k(\psi + q|\tilde{B}|^2\tilde{B}) = 0 \\ \psi_{t't'} - \beta\psi_{x'x'} = kv|\tilde{B}|^2_{x't'} + (\beta + \frac{v^2}{4})|\tilde{B}|^2_{x'x'}. \end{cases}$$

Therefore, when $v = 0$, the system (4.8) is reduced to the Zakharov system([47],[61]).
Also, when $\beta = \frac{v^2}{4}$, an elementary computation yields

$$\begin{cases} i\tilde{B}_{t'} + \tilde{B}_{x'x'} - k(\psi + q|\tilde{B}|^2\tilde{B}) = 0 \\ \psi_{t'} - \frac{v}{2}\psi_{x'} = v\tilde{B}_{x'}, \end{cases}$$

i.e. the Benney's equation (4.6).
In fact, the Zakharov-Rubenchik equation (4.8) contains these two models.
In what follows, we will always assume that

$$v(\beta - \frac{v^2}{4}) \neq 0.$$

Finally, setting

$$\begin{cases} \psi_1 = \frac{v}{2}\rho + u \\ \psi_2 = (\frac{1}{\sqrt{\beta}} - \frac{v}{2})(\sqrt{\beta}\rho + u), \end{cases} \tag{4.10}$$

we transform (4.9) to

$$\begin{cases} i\tilde{B}_{t'} + \tilde{B}_{x'x'} - k(\psi_1 + v\psi_2) - kq|\tilde{B}|^2\tilde{B} = 0 \\ \psi_{1t't'} - \beta\psi_{1x'x'} = (\beta - \frac{v^2}{4})|\tilde{B}|^2_{x'x'} = 0 \\ \psi_{2t'} - \beta\psi_{2x'} = (\frac{v}{2} - \sqrt{\beta})\tilde{B}|^2_{x'} = 0. \end{cases}$$

Hence, we will discuss the IVP

$$\begin{cases} iB_t + B_{xx} + \psi_1 B + \psi_2 B + |B|^2 B = 0 \\ \psi_{1tt} - \psi_{1xx} = |B|^2_{xx} \\ \psi_{2t} - \psi_{2x} = |B|^2_x \\ B(0,x) = B_o(x) \\ \psi_1(0,x) = \psi_{1_0}(x)\ ,\ \psi_{1t}(0,x) = \psi_{1_1}(x) \\ \psi_2(0,x) = \psi_{2_0}(x). \end{cases} \tag{4.11}$$

We prove :
Theorème 4.2.2 *Let $B_o \in H^2(\mathbb{R})$, $\psi_{1_0}, \psi_{2_0} \in H^1(\mathbb{R})$ and $\psi_{1_1} \in L^2(\mathbb{R})$.
Then there exists $T > 0$ and a unique strong solution to (4.11) such that*

$(B, \psi_1, \psi_2) \in \cap_{j=0,1} C^j([0,T]; H^{2-2j}(\mathbb{R})) \times \cap_{j=0,1} C^j([0,T]; H^{1-j}(\mathbb{R})) \times \cap^1_{j=0} C([0,T]; H^{1-j}(\mathbb{R}))$

Theorem 4.2.1 follows easily from Theorem 4.2.2 by inverting (4.10).

4.2.2 Proof of Theorem 4.2.2

The difficulty of (4.11) consists in a derivative-loss occuring in the nonlinear term. We will use here a technique introduced in [52] to solve the fully nonlinear wave equation : by introducing some auxiliary functions, (4.11) can be re-written without the derivative-loss (see also [47]). Then, we will be able to use the "$L^p - L^q$" and Strichartz estimates for the free Schrödinger group in order to apply a fixed-point technique.

By differenciating in respect to t the first equation of (4.11), and setting $F = B_t$, we get

$$iF_t + F_{xx} + (\psi_1 + \psi_2 + 2|B|^2)F + (\psi_{1t} + \psi_{2t} + \overline{F}B)B = 0.$$

Therefore, we consider the system :

$$\begin{cases} iF_t + F_{xx} + (\psi_1 + \psi_2 + 2|\tilde{B}|^2)F + (\psi_{1t} + \psi_{2t} + \overline{F}\tilde{B})\tilde{B} = 0 \\ \psi_{1tt} - \psi_{1xx} = |B|^2_{xx} \\ \psi_{2t} - \psi_{2x} = |B|^2_x \\ F(0,x) = F_o(x) = iB_o(\psi_{1_0} + \psi_{1_1} + |B_o|^2) + i\Delta B_o \\ \psi_1(0,x) = \psi_{1_0}(x)\ ,\ \psi_{1t}(0,x) = \psi_{1_1}(x) \\ \psi_2(0,x) = \psi_{2_0}(x) \end{cases} \tag{4.12}$$

where B and $\tilde{B}$ are given in terms of F by :

$$\begin{cases} \tilde{B}(x,t) = B_o(x) + \int_0^t F(x,s)ds \\ B(x,t) = (\Delta - 1)^{-1}[-iF(x,t) - (1 + \psi_1(x,t) + \psi_2(x,t))\tilde{B}(x,t) - |\tilde{B}(x,t)|^2\tilde{B}(x,t)]. \end{cases} \tag{4.13}$$

We have :

Lemme 4.2.3 *Let $B_o \in H^2(\mathbb{R})$, $\psi_{1_0}, \psi_{2_0} \in H^1(\mathbb{R})$ and $\psi_{1_1} \in L^2(\mathbb{R})$.*
Then there exists $T > 0$ such that (4.12)has a unique solution (F, ψ_1, ψ_2), with

$$F \in \cap_{j=0,1} C^j([0,T]; H^{-2j} \cap_{j=0,1} L^6(0,T,L^6) \text{ and } \psi_{1,2} \in \cap C^j([0,T]; H^{1-j}).$$

Moreover

$$B_t = F\ ,\ B \in C([0,T]; H^2),$$

and (B, ψ_1, ψ_2) satisfies Theorem 4.2.1.

We denote by $\{S(t)\}_{t\in\mathbb{R}}$ the one parameter Schrödinger group. We start by quoting a few well known results :

Lemme 4.2.4 *($L^p - L^q$ estimates [7])*
Let p and q such that $2 \leq p \leq \infty$ et $\frac{1}{p} + \frac{1}{q} = 1$.
Then there exists $C > 0$ such that for all $f \in L^q(\mathbb{R})$,

$$||S(t)f||_{L^p} \leq C|t|^{\frac{1}{p}-\frac{1}{2}}||f||_{L^q}\ ,\ t \neq 0$$

Lemme 4.2.5 *(Strichartz estimate [7])*
Let (r,q) such that $2 \leq q < \infty$ and $(\frac{1}{2}-\frac{1}{q})r = 2$.
Then there exists $K = K(q) > 0$ such that for all $f \in L^2(\mathbb{R})$,

$$||S(.)f||_{L^r(\mathbb{R},L^q(\mathbb{R}))} \leq K||f||_{L^2}$$

Lemme 4.2.6 *[7]*
Let (r,q) as in the previous lemma. Let I be an interval of $\mathbb{R}$.
Then there exists $A = A(q) > 0$ such that for all $f \in L^{r'}(I, L^{q'}(\mathbb{R}))$,

$$||\int_0^t S(t-s)f(s)ds||_{L^a(I,L^b(\mathbb{R}))} \leq A||f||_{L^{r'}(I,L^{q'}(\mathbb{R}))}$$

where $\frac{1}{q}+\frac{1}{q'} = 1$, $\frac{1}{r}+\frac{1}{r'} = 1$, $2 \leq b < \infty$ and $(\frac{1}{2}-\frac{1}{b})a = 2$.

Let $T > 0$ and $X(T)$ the Banach space equipped with the norm

$$\begin{aligned} ||(F,\psi_1,\psi_2)||_X &= ||F||_{L^\infty(0,T,L^2)} + ||F||_{L^6(0,T,L^6)} + ||\psi_1||_{L^\infty(0,T,H^1)} \\ &+||\psi_2||_{L^\infty(0,T,H^1)} + ||\psi_{1t}||_{L^\infty(0,T,L^2)} + ||\psi_{2t}||_{L^\infty(0,T,L^2)}. \end{aligned}$$

We set

$$a = Sup\{||F_o||_{L^2}, ||B_o||_{H^2}, ||\psi_{1_0}||_{H^1}, ||\psi_{2_0}||_{H^1}, ||\psi_{1_1}||_{L^2}\}.$$

We define the map

$$\Theta : (F,\psi_1,\psi_2) \to (\theta_1,\theta_2,\theta_3)$$

with

$$\begin{cases} \theta_1(x,t) = S(t)F_o - \int_0^t S(t-s)[(\psi_1(s)+\psi_2(s)+2|\tilde{B}(s)|^2)F + (\psi_{1t}(s)+\psi_{2t}(s)+\tilde{B}(s)^2\overline{F})\tilde{B}(s)]ds \\ \theta_2(x,t) = cos(Dt)\psi_{1_o} + \frac{1}{D}sin(Dt)\psi_{1_1} + \int_0^t \frac{1}{D}sin(D(t-s))|B(s)|^2_{xx}ds \\ \theta_3(x,t) = e^{\partial_x t}\psi_{2_o} + \int_O^t e^{\partial_x(t-s)}|B(s)|^2_x ds, \end{cases}$$

where $W(t)$ is the well-known linear operator which generates the solutions of the free wave equation, given by

$$W(t)f(x) = \frac{1}{2\pi}\int e^{ix\xi}\frac{sin(t|\xi|)}{|\xi|}\hat{f}(\xi)d\xi,$$

and B,$\tilde{B}$ are given by (4.13).
Moreover, we consider

$$\begin{aligned} Y(T) = \{(F,\psi_1,\psi_2) \in X/ &||F||_{L^\infty(0,T,L^2)} \leq 2a, ||F||_{L^6(0,T,L^6)} \leq 2Ka, ||\psi_1||_{L^\infty(0,T,H^1)} \leq 2a, \\ &||\psi_2||_{L^\infty(0,T,H^1)} \leq 2a, ||\psi_{1t}||_{L^\infty(0,T,L^2)} \leq 2a\}, ||\psi_{2t}||_{L^\infty(0,T,L^2)} \leq 2a\} \end{aligned}$$

(where K=K(6) is the constant given by Lemma 4.2.5).

Lemme 4.2.7 *For T small enough, Θ is a contraction from $Y(T)$ into itself.*

Proof:
We first prove that Θ maps Y into itself.
Let $(F, \psi_1, \psi_2) \in Y$.
Then,

$$\begin{aligned}||\theta_1(t)||_{L^\infty(0,T,L^2)} &\leq ||F_o||_{L^2} + ||\int_0^t S(t-s)([\psi_1(s)+\psi_2(s)]F(s))ds||_{L^\infty(0,T,L^2)} \\ &+||\int_0^t S(t-s)([\psi_{1t}(s)+\psi_{2t}(s)]\tilde{B}(s))ds||_{L^\infty(0,T,L^2)} \\ &+||\int_0^t S(t-s)(2|\tilde{B}(s)|^2F(s)+\tilde{B}(s)^2\overline{F}(s))ds||_{L^\infty(0,T,L^2)}.\end{aligned}$$

By Lemma 4.2.6, with $(q,r)=(3,12)$,

$$\begin{aligned}||\int_0^t S(t-s)(\psi_1(s)F(s))ds||_{L^\infty(0,T,L^2)} &\leq C||F\psi_1||_{L^{\frac{12}{11}}(0,T,L^{\frac{3}{2}})} \\ &\leq C||F||_{L^{\frac{12}{11}}(0,T,L^6)}||\psi_1||_{L^\infty(0,T,L^2)} \\ &\leq CT^{\frac{1}{4}}||F||_{L^6(0,T,L^6)}||\psi_1||_{L^\infty(0,T,L^2)}.\end{aligned}$$

Also,

$$||\int_0^t S(t-s)(\psi_2(s)F(s))ds||_{L^\infty(0,T,L^2)} \leq CT^{\frac{1}{4}}||F||_{L^6(0,T,L^6)}||\psi_2||_{L^\infty(0,T,L^2)}.$$

Moreover,

$$\begin{aligned}||\int_0^t S(t-s)(\psi_{1t}(s)\tilde{B}(s))ds||_{L^\infty(0,T,L^2)} &\leq T^{\frac{1}{2}}||\psi_{1t}B_o||_{L^\infty(0,T,L^2)} + ||\psi_{1t}\int_0^t F(\tau)d\tau||_{L^{\frac{12}{11}}(0,T,L^{\frac{3}{2}})} \\ &\leq T^{\frac{1}{2}}||\psi_{1t}||_{L^\infty(0,T,L^2)}||B_o||_{L^\infty(0,T,L^\infty)} \\ &\quad +||\psi_{1t}||_{L^\infty(0,T,L^2)}||\int_0^t F(\tau)d\tau||_{L^{\frac{12}{11}}(0,T,L^6)} \\ &\leq ||\psi_{1t}||_{L^\infty(0,T,L^2)}(CT^{\frac{1}{2}}||B_o||_{L^\infty(0,T,H^2)} + T^{\frac{15}{12}}||F||_{L^6(0,T,L^6)}),\end{aligned}$$

and

$$||\int_0^t S(t-s)(\psi_{2t}(s)\tilde{B}(s))ds||_{L^\infty(0,T,L^2)} \leq ||\psi_{2t}||_{L^\infty(0,T,L^2)}(CT^{\frac{1}{2}}||B_o||_{L^\infty(0,T,H^2)}+T^{\frac{15}{12}}||F||_{L^6(0,T,L^6)}).$$

Finally,

$$\begin{aligned}||\int_0^t S(t-s)(\tilde{B}^2(s)F(s))ds||_{L^\infty(0,T,L^2)} &\leq ||\int_0^t S(t-s)(B_o^2(s)F(s))ds||_{L^\infty(0,T,L^2)} \\ &+2||\int_0^t S(t-s)(\tilde{B}(s)F(s)(\int_O^s F(\tau)d\tau))ds||_{L^\infty(0,T,L^2)} \\ &+||\int_0^t S(t-s)(F(s)(\int_O^s F(\tau)d\tau))ds||_{L^\infty(0,T,L^2)}.\end{aligned}$$

by Lemma 4.2.5 for the pairs $(3,12)$ and $(6,6)$,

$$\begin{aligned}
\|\int_0^t S(t-s)(\tilde{B}^2(s)F(s))ds\|_{L^\infty(0,T,L^2)} &\leq \|B_o^2F\|_{L^{\frac{12}{11}}(0,T,L^{\frac{3}{2}})} + \|B_oF(\int_O^t F(s)ds)\|_{L^{\frac{12}{11}}(0,T,L^{\frac{3}{2}})} \\
&+\|F(\int_O^t F(s)ds)^2\|_{L^{\frac{6}{5}}(0,T,L^{\frac{6}{5}})} \\
&\leq C\|F\|_{L^\infty(0,T,L^2)}\|B_o\|^2_{L^\infty(0,T,H^2)} \\
&+\|B_o\|_{L^\infty(0,T,L^\infty)}\|F\|_{L^\infty(0,T,L^2)}\|\int_0^t F(s)ds\|_{L^{\frac{12}{11}}(0,T,L^6)} \\
&+\|F\|_{L^\infty(0,T,L^2)}\|(\int_0^t F(s)ds)^2\|_{L^{\frac{6}{5}}(0,T,L^3)} \\
&\leq CT^{\frac{21}{12}}\|F\|_{L^6(0,T,L^6)}\|B_o\|^2_{L^\infty(0,T,H^2)} \\
&+T^{\frac{21}{12}}C\|B_o\|_{L^\infty(0,T,H^2)}\|F\|_{L^\infty(0,T,L^2)}\|F\|_{L^6(0,T,L^6)} \\
&+T^{\frac{10}{3}}\|F\|^2_{L^6(0,T,L^6)}\|F\|_{L^\infty(0,T,L^2)},
\end{aligned}$$

and

$$\|\theta_1(t)\|_{L^\infty(0,T,L^2)} \leq a + aC(aKT^{\frac{1}{4}} + aT^{\frac{1}{2}} + aKT^{\frac{15}{12}} + (K+1)a^2T^{\frac{21}{12}} + K^2a^2T^{\frac{10}{3}}).$$

By choosing T small enough, in order to have

$$C(aKT^{\frac{1}{4}} + aT^{\frac{1}{2}} + aKT^{\frac{15}{12}} + (K+1)a^2T^{\frac{21}{12}} + K^2a^2T^{\frac{10}{3}}) \leq 1,$$

we get

$$\|\theta_1(t)\|_{L^\infty(0,T,L^2)} \leq 2a.$$

Next, we estimate $\|\theta_1(t)\|_{L^6(0,T,L^6)}$:
We begin by taking the L^6-norm in space. By lemma 4.2.5 :

$$\begin{aligned}
\|\theta_1(t)\|_{L^6_x} &\leq \|S(t)F_o\|_{L^6} + C\int_O^t |t-s|^{-\frac{1}{3}}\|F(s)(\psi_1(s)+\psi_2(s)) \\
&+\tilde{B}(s)(\psi_{1t}(s)+\psi_{2t}(s)) + F(s)|\tilde{B}(s)|^2 + \overline{F}(s)\tilde{B}(s)^2\|_{L^{\frac{6}{5}}}ds \\
&\leq \|S(t)F_o\|_{L^6} + C\int_O^t |t-s|^{-\frac{1}{3}}[\|F(s)\|_{L^2}\|\psi_1(s)+\psi_2(s)\|_{L^3} \\
&+\|B_o\|_{L^3}\|\psi_{1t}(s)+\psi_{2t}(s)\|_{L^2} \\
&+\int_0^s \|F(\tau)\|_{L^3}d\tau\|\psi_{1t}(s)+\psi_{2t}(s)\|_{L^2} + \|B_o^2F(s)\|_{L^{\frac{6}{5}}} \\
&+2\|B_oF(s)\int_0^s F(\tau)d\tau\|_{L^{\frac{6}{5}}} + \|F(s)(\int_0^s F(\tau)d\tau)^2\|_{L^{\frac{6}{5}}}]ds
\end{aligned} \quad (4.14)$$

$$\begin{aligned}
\le\;& ||S(t)F_o||_{L^6} + C\int_O^t |t-s||^{-\frac{1}{3}}[||F(s)||_{L^2}||\psi_1(s)+\psi_2(s)||_{L^3} \\
&+||B_o||_{L^3}||\psi_{1t}(s)+\psi_{2t}(s)||_{L^2} \\
&+\int_0^s ||F(\tau)||_{L^2}^{\frac{1}{2}}||F(\tau)||_{L^6}^{\frac{1}{2}}d\tau + ||B_o||_{L^6}^2||F(s)||_{L^2} \\
&+2||B_o||_{L^6}||F(s)||_{L^2}\int_0^s ||F(\tau)||_{L^6}d\tau + ||F(s)||_{L^2}(\int_0^s ||F(\tau)||_{L^6}d\tau)^2]ds.
\end{aligned}$$

Finally, by Lemma 4.2.4,

$$\begin{aligned}
||\theta_1(t)||_{L^6(0,T,L^6)} \le\;& Ka + \frac{3}{2}T^{\frac{5}{6}}[C||F||_{L^\infty(0,T,L^2)}||\psi_1+\psi_2||_{L^\infty(0,T,H^1)} + C||B_o||_{H^2}||\psi_{1t}+\psi_{2t}||_{L^\infty(0,T,L^2)} \\
&+T^{\frac{11}{12}}||\psi_{1t}+\psi_{2t}||_{L^\infty(0,T,L^2)}||F||_{L^\infty(0,T,L^2)}^{\frac{1}{2}}||F||_{L^6(0,T,L^6)}^{\frac{1}{2}} + C||F||_{L^\infty(0,T,L^2)}||B_o||_{H^2}^2 \\
&+CT^{\frac{5}{6}}||B_o||_{H^2}||F||_{L^\infty(0,T,L^2)}||F||_{L^6(0,T,L^6)} + T^{\frac{5}{3}}||F||_{L^\infty(0,T,L^2)}||F||_{L^6(0,T,L^6)}^2]
\end{aligned}$$

i.e.

$$||\theta_1(t)||_{L^6(0,T,L^6)} \le aK + aCT^{\frac{5}{6}}(a(1+T^{\frac{11}{12}}) + a^2T^{\frac{5}{6}} + a^2T^{\frac{5}{3}}).$$

By choosing T small enough, in order to have

$$CT^{\frac{5}{6}}(a(1+T^{\frac{11}{12}}) + a^2T^{\frac{5}{6}} + a^2T^{\frac{5}{3}}) \le K,$$

we get

$$||\theta_1(t)||_{L^6(0,T,L^6)} \le 2aK.$$

We now derive an estimate for $||B||_{L^\infty(0,T,H^2)}$, which will be useful for the estimates concerning $\theta_{2,3}$:

Lemme 4.2.8 *For* $0 \le t \le T$,

$$||B||_{L^\infty(0,T,H^2)} \le C(a + a^2 + a^3(1 + T + T^{\frac{5}{3}} + T^{\frac{5}{2}})).$$

Proof:

$$\begin{aligned}
||B(t)||_{H^2} \le\;& ||(\psi_1(t)+\psi_2(t)-1+\tilde{B}(t)^2)\tilde{B}(t)||_{L_2} + ||F||_{L^\infty(0,T,L^2)} \\
\le\;& ||\psi_1(t)+\psi_2(t)||_{L^\infty}(||B_o||_{L^2} + ||\int_0^t F(s)ds||_{L^2}) + ||B_o||_{L^2}||\int_0^t F(s)ds||_{L^2} \\
&+||B_o + (\int_0^t F(s)ds)^3||_{L^2} + ||F||_{L^\infty(0,T,L^2)} \\
\le\;& C(1+||\psi_1(t)||_{H^1}+||\psi_2(t)||_{H^1})||B_o||_{L^2} + ||B_o||_{L^6}^3 + 3CT||B_o||_{H^2}||F||_{L^\infty(0,T,L^2)} \\
&+T^{\frac{5}{3}}||B_o||_{L^6}||F||_{L^6(0,T,L^6)}^2 + T^{\frac{5}{2}}||F||_{L^6(0,T,L^6)}^3 + ||F||_{L^\infty(0,T,L^2)},
\end{aligned}$$

i.e.

$$||B||_{L^\infty(0,T,H^2)} \leq C(a + a^2 + a^3(1 + T + T^{\frac{5}{3}} + T^{\frac{5}{2}})).$$

Therefore,

$$\begin{aligned} ||\theta_2(t)||_{H^1} &= ||(1+D^2)^{\frac{1}{2}}\theta_2(t)||_{L^2} \\ &\leq (1+t)a + \int_0^t (1+s)||B_{xx}(s)^2||_L^2 ds \\ &\leq (1+T)a + T(1+T)(||B||_{L^\infty(0,T,L^\infty)} + ||B_x||_{L^\infty(0,T,L^\infty)})||B||_{L^\infty(0,T,H^2)}, \end{aligned}$$

and

$$||\theta_2(t)||_{L^\infty(0,T,H^1)} \leq a + CT(a + a||B||_{L^\infty(0,T,H^2)}) \leq 2a,$$

for $T > 0$ small enough.
Also,

$$\begin{aligned} \frac{\partial}{\partial t}\theta_2(t) &= -\sin Dt\frac{\partial}{\partial x}\psi_{1_0} + \cos Dt\psi_{1_1} + \frac{1}{D}\sin Dt|B_o|^2_{xx} \\ &\quad + \int_0^t \cos D(t-s)|B(s)|^2_{xx} ds \end{aligned} \tag{4.15}$$

and we get as before

$$||\theta_{2t}(t)||_{L^\infty(0,T,L^2)} \leq 2a$$

for $T > 0$ small enough.

We can prove by analogous computations that if $T > 0$ is small enough,

$$||\theta_3||_{L^\infty(0,T,H^1)} \leq 2a \text{ and } ||\theta_{3t}||_{L^\infty(0,T,L^2)} \leq 2a.$$

Yet, we have shown that

$$\Theta(Y) \subset Y.$$

By the same computations, we can prove that for T small enough , Θ is a contraction of Y, which completes the proof of the Lemma 4.2.7.

□

Therefore we get the existence of a unique fixed-point (F, ψ_1, ψ_2), solution to (4.12). A standard argument then yields :

$$F \in C([0,T], L^2) \ , \psi_{1,2} \in C^j([0,T]; H^{1-j}) \ , j = 0,1 \text{ et } B \in C([0,T]; H^2).$$

To complete the proof of Lemma 4.2.3, we still have to prove that $B(0,.) = B_o$ and that $F = B_t$.
From the first line of (4.12), one gets

$$(\Delta - 1)F = -iF_t - (\psi_1 + \psi_2 + 1 + 2|\tilde{B}|^2)F + (\psi_{1t} + \psi_{2t} + \overline{F}\tilde{B})\tilde{B}$$

Also, by differentiating (4.13) with respect to time:

$$(\Delta - 1)B_t = -iF_t - (\psi_1 + \psi_2 + 1 + 2|\tilde{B}|^2)F + (\psi_{1t} + \psi_{2t} + \overline{F}\tilde{B})\tilde{B},$$

so that

$$B_t = F \text{ in } H^{-2}(\mathbb{R}).$$

Also,

$$(\Delta - 1)B(0,.) = -iF_o - B_o(\psi_{1_O} + \psi_{1_O} + 1) - B_o|B_o|^2 = (\Delta - 1)B_o$$

by the choice of F_o.

4.3 Global solutions

4.3.1 The formal invariants

Lemme 4.3.1 *The system (4.8) possesses the following (formal) invariants:*

$$I_1(t) = \int_{\mathbb{R}} |B|^2,$$

$$I_2(t) = \frac{\omega}{2}\int_{\mathbb{R}} |B_x|^2 + \frac{kq}{4}\int_{\mathbb{R}} |B|^4 + \frac{k}{2}\int_{\mathbb{R}} (u - \frac{v}{2}\rho)|B|^2 + \frac{\beta}{4}\int_{\mathbb{R}} |\rho|^2 + \frac{1}{4}\int_{\mathbb{R}} |u|^2 - \frac{v}{2}\int_{\mathbb{R}} u\rho,$$

and

$$I_3(t) = \theta\int_{\mathbb{R}} u\rho + \frac{i}{2}\int_{\mathbb{R}} (B\overline{B_x} - B_x\overline{B}).$$

Remarque 4.3.2 *By combining I_2 and I_3, we obtain the following fourth invariant:*

$$I_4(t) = \frac{\omega}{2}\int_{\mathbb{R}} |B_x|^2 + \frac{kq}{4}\int_{\mathbb{R}} |B|^4 + \frac{k}{2}\int_{\mathbb{R}} (u - \frac{v}{2}\rho)|B|^2 + \frac{\beta}{4}\int_{\mathbb{R}} |\rho|^2 + \frac{1}{4}\int_{\mathbb{R}} |u|^2 + \frac{iv}{4\theta}\int_{\mathbb{R}} (B\overline{B_x} - B_x\overline{B}).$$

Proof:

By multiplying (4.8,a) by $\overline{B}$ and integrating the imaginary part, we get the conservation of I_1.

Moreover, by multiplying (4.8,a) by $\overline{B_t}$, and integrating the real part,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} (\frac{\omega}{2}|B_x|^2 + \frac{kq}{4}|B|^4) = \int_{\mathbb{R}} \frac{1}{2}k(\frac{v}{2}\rho|B|_t^2 - u|B|_t^2),$$

i.e.

$$\frac{\partial}{\partial t}\int_{\mathbb{R}}(\omega|B_x|^2+\frac{kq}{2}|B|^4+ku|B|^2-\frac{1}{2}kv\rho|B|^2)=\int_{\mathbb{R}}ku_t|B|^2-\frac{1}{2}\int_{\mathbb{R}}kv\rho_t|B|^2. \qquad (4.16)$$

Also, by (4.8,b) and (4.8,c),

$$\begin{aligned}\theta(\int_{\mathbb{R}}\frac{1}{2}ku_t|B|^2-\frac{1}{4}\int_{\mathbb{R}}kv\rho_t|B|^2) &= \frac{k}{2}\int_{\mathbb{R}}(\beta\rho-vu)|B|_x^2+(v\rho-u)(\frac{v}{2}|B|_x^2)\\ &= \frac{1}{2}\int_{\mathbb{R}}(\beta\rho-vu)[(v\rho-u)_x-\theta\rho_t]\\ &\quad+\frac{1}{2}\int_{\mathbb{R}}(v\rho-u)[\theta u_t+(\beta\rho-vu)_x]\\ &= -\theta\frac{\partial}{4\partial t}(\int_{\mathbb{R}}(|\rho|^2+|u|^2-2v(u\rho)_t)),\end{aligned}$$

and we obtain the second invariant.

Finally, by multiplying (4.8,b) by ρ, and (4.8,c) by u, we easily get

$$\theta\frac{\partial}{\partial t}\int_{\mathbb{R}}(u\rho+i(B_x\overline{B}-B\overline{B_x}))=0.$$

4.3.2 A priori estimates

In this section we derive a priori estimates for a solution $(B,\rho,u)\in H^2(\mathbb{R})\times H^1(\mathbb{R})\times H^1(\mathbb{R})$ to (4.8).

Lemme 4.3.3 *There exists $C>0$ such that*

$$||u(t)||^2_{L^2(\mathbb{R})}+||\rho(t)||^2_{L^2(\mathbb{R})}+||B_x(t)||^2_{L^2(\mathbb{R})}\leq C, \ , t\in\mathbb{R}. \qquad (4.17)$$

Proof:

We begin by noticing that

$$\frac{\omega}{4}\int_{\mathbb{R}}|B_x|^2+\frac{iv}{4\theta}\int_{\mathbb{R}}(B\overline{B_x}-B_x\overline{B})\geq -CI_1(0). \qquad (4.18)$$

We have for any fixed $\eta>0$,

$$|\frac{iv}{4\theta}\int_{\mathbb{R}}(B\overline{B_x}-B_x\overline{B})|\leq 2\int_{\mathbb{R}}|\frac{v\overline{B}}{2\theta\eta}(\eta B_x)|\leq\frac{\eta^2}{2}\int_{\mathbb{R}}|B_x|^2+\frac{v^2}{8\theta^2\eta^2}\int_{\mathbb{R}}|B_o|^2$$

since

$$\int_{\mathbb{R}} |B|^2 = \int_{\mathbb{R}} |B_o|^2.$$

By choosing η such that $\frac{\omega}{2} - \eta^2 \geq 0$, we get (4.18).

Therefore,

$$\frac{w}{4}\int_{\mathbb{R}} |B_x|^2 + \int_{\mathbb{R}} \frac{kq}{4}|B|^4 + \frac{k}{2}\int_{\mathbb{R}} (u - \frac{v}{2}\rho)|B|^2 + \frac{\beta}{4}\int_{\mathbb{R}} |\rho|^2 + \frac{1}{4}\int_{\mathbb{R}} |u|^2 \leq I_4(0) + CI_1(0). \tag{4.19}$$

Observe that for all $\eta_1 > 0$ and $\eta_2 > 0$,

$$\int_{\mathbb{R}} ku|B|^2 \leq \frac{1}{2}\int_{\mathbb{R}} \eta_1^2 u^2 + \int_{\mathbb{R}} \frac{1}{2\eta_1^2}k^2|B|^4$$

and

$$\int_{\mathbb{R}} \frac{kv}{2}\rho|B|^2 \leq \frac{1}{2}\int_{\mathbb{R}} \eta_2^2\rho^2 v^2 + \int_{\mathbb{R}} \frac{1}{8\eta_2^2}k^2|B|^4.$$

By choosing η_1 and η_2 such that

$$a_1 = \frac{1}{4}(1 - \eta_1^2) > 0 \ , \ a_2 = \frac{1}{4}(\beta - \frac{\eta_2^2}{2}v^2) > 0,$$

and setting

$$a_3 = k^2(\frac{1}{4\eta_1^2} + \frac{1}{16\eta_2^2}),$$

$$a_1\int_{\mathbb{R}} u^2 + a_2\int_{\mathbb{R}} \rho^2 - a_3\int_{\mathbb{R}} |B|^4 + \frac{w}{4}\int_{\mathbb{R}} |B_x|^2 \leq CI_1(0) + I_4(0).$$

Finally, by noticing that

$$\int_{\mathbb{R}} |B|^4 \leq ||B||_\infty^2 ||B||_{L^2}^2 \leq ||B||_{L^2}^3 ||B_x||_{L^2} \leq \frac{1}{\epsilon}I_1(0)^3 + \epsilon||B_x||_{L^2}^2,$$

for any $\epsilon > 0$, we get the announced result for $\frac{\omega}{4} - \epsilon > 0$□

Lemme 4.3.4 *(Estimates for $||B||_{H^2}$,$||u||_{H^1}^2$ and $||\rho||_{H^1}^2$)*
For any $T \in \mathbb{R}^+$, there exists $D = D(T) > 0$ such that for all $0 \leq t \leq T$,

$$||B(t)||_{H^2}^2 + ||u(t)||_{H^1}^2 + ||\rho(t)||_{H^1}^2 \leq D(T).$$

Proof:
Differentiating (4.8,a) with respect to t, we get

$$iB_{tt} + \omega B_{xxt} - k[(u - \frac{v}{2}\rho + q|B|^2)B]_t = 0.$$

We multiply this expression by $\overline{B}_t$ and integrate the imaginary part :

$$\begin{aligned}\frac{\partial}{\partial t}\int |B_t|^2 &= k\int (u-\frac{v}{2}\rho)_t Im(B\overline{B}t) + q\int |B|_t^2 Im(B\overline{B}t)\\ &= k\int ((\frac{v^2}{2}-\beta)\rho_x + \frac{3}{2}vu_x)Im(B\overline{B}t) + k^2\int |B|_x^2 Im(B\overline{B}t) + q\int |B|_t^2 Im(B\overline{B}t).\end{aligned}$$

The Cauchy-Schwarz inequality then yields

$$||B||_{L^\infty} \le C||B_x||_{H^1} \le C,$$

and we obtain

$$\frac{\partial}{\partial t}\int |B_t|^2 \le C_1 + C_2\int |B_t|^2 + C_3\int |\rho_x|^2 + C_4\int |u_x|^2 \tag{4.20}$$

where the constants $C_j > 0$ are independent of time.
Moreover, by (4.8,b) and (4.8,c),

$$\begin{aligned}\frac{\partial}{\partial t}(\beta||\rho_x||_{L^2}^2 + ||u_x||_{L^2}^2) &= -k\beta\int |B|_{xx}^2\rho_x + \frac{k}{2}v\int |B|_{xx}^2 u_x\\ &\le C||B_x||_{L^\infty}\int |B_x||u_x+\rho_x| + C||B||_{L^\infty}\int |B_{xx}||u_x+\rho_x|.\end{aligned}$$

From (4.8,a), $||B_{xx}||_{L^2} \le ||B_t||_{L^2}$, and using the Cauchy-Schwarz inequality and the Sobolev imbeddings,

$$\frac{\partial}{\partial t}(\beta||\rho_x||_{L^2}^2 + ||u_x||_{L^2}^2) \le C(||B_t||_{L^2} + ||u_x||_{L^2} + ||\rho_x||_{L^2} + 1)^2 \tag{4.21}$$

Finally, setting

$$\alpha^2(t) = ||B_t||_{L^2}^2 + \beta||\rho_x||_{L^2}^2 + ||u_x||_{L^2}^2,$$

we combine (4.20) and (4.21) in order to obtain

$$\frac{\partial}{\partial t}\alpha^2(t) \le C(1+\alpha(t)+\alpha^2(t)).$$

The result follows by Gronwall's lemma.

□

4.3.3 The global solutions

We prove here that the local solutions of (4.12) found in the last section are in fact global in time.

Let $[0; T_{max}[$ be the maximal time interval on which the Cauchy problem (4.12) has a unique solution (F, ψ_1, ψ_2) with

$$F \in \cap_{j=0;1} C^j([0,T[; H^{-2j} \cap L^6(0,T,L^6) \text{ and } \psi_{1,2} \in \cap_{j=0;1} C^j([0,T[; H^{1-j}),$$

for any $T < T_{max}$. By Theorem 4.2.1, one clearly has $T_{max} > 0$. Assume that $T_{max} < +\infty$. Let $0 < T < T_{max}$ and (B, u, ρ) the associated solutions to (F, ψ_1, ψ_2), solutions of (4.8). The regularity of (B, ρ, u), allows one to validate all the computations in the previous section, hence, for any $0 \leq t \leq T$,

$$||F(t)||_{L^2} = ||B_t(t)||_{L^2} \leq M_1 \text{ and } ||\psi_1||_{H^1} + ||\psi_2||_{H^1} \leq M_2,$$

where the constants $M_j > 0$ depend only on the initial data.
Moreover, in view of (4.15), one has clearly

$$||\theta_2(t)||_{L^2} + ||\theta_3(t)||_{L^2} \leq M_3.$$

Finally, we estimate $||F||_{L^6(0,T;L^6)}$. Since $\tilde{B} = B$, by (4.14), one obtains

$$||F||_{L^6(0,T;L^6)} \leq ||F_o||_{L^6(0,T;L^6)} + C||F(\psi_1 + \psi_2) + B(\psi_{1t} + \psi_{2t}) + F|B|^2 + \overline{F}B^2||_{L^\infty(0,T;L^{\frac{6}{5}})}.$$

By the Sobolev imbeddings, one easily checks that

$$||F||_{L^6(0,T;L^6)} \leq M_4$$

Finally,

$$||(F, \psi_1, \psi_2)||_{X(T)} \leq M_1 + M_2 + M_3 + M_4,$$

which contradicts the maximality of T_{Max}.
Therefore, $T_{Max} = +\infty$, and :

Theorème 4.3.5 *Let $B_o \in H^2(\mathbb{R})$ et $\rho_o, u_o \in H^1(\mathbb{R})$.*
Then there exists a unique strong solution (B, ρ, u) for the system (4.8), and

$$(B, \rho, u) \in \cap_{j=0,1} C^j(\mathbb{R}^+; H^{2-2j}(\mathbb{R})) \times \cap_{j=0,1} C^j(\mathbb{R}^+; H^{1-j}(\mathbb{R})) \times \cap_{j=0}^{1} C(\mathbb{R}^+; H^{1-j}(\mathbb{R})).$$

Remark 4.3.6

Although the natural energy space for (4.8) is $\mathcal{E} = H^1 \times L^2 \times L^2$, the fact that a derivative-loss occurs in the nonlinear terms turns a global existence result in this space hard to get. For instance, in [56], when dealing with the particular case of the Benney equation

$$\begin{cases} iB_t + iB_{xx} = uB + |B|^2 B \\ u_t = |B|^2_x, \end{cases} \tag{4.22}$$

the authors were only able to establish the existence of global strong solutions for $B_o \in H^{\frac{3}{2}}$. Although the local problem can be solved for initial data in $H^{\frac{1}{2}} \times L^2$ (see[4]), it remains an open problem to prove the global well-posedness of (4.22) for $B_o \in H^s$, $s < \frac{3}{2}$.

4.4 Stability of the solitary waves

In this section we prove the existence and the orbital stability of the solitary wave solutions to (4.8).

4.4.1 Existence of solitary waves

Let $c > 0$.
We look for solutions to the system (4.8), of the form $Q = (\phi_1, \phi_2, \phi_3)$, with

$$\begin{cases} \phi_1(x,t) = e^{i\lambda t} A(x-ct) \\ \phi_2(x,t) = a|A(x-ct)|^2 \\ \phi_3(x,t) = b|A(x-ct)|^2, \end{cases} \tag{4.23}$$

$(a,b) \in \mathbb{R}^2$.
By (4.8,b) and (4.8,c), we get

$$\begin{aligned} b &= b(c) = \frac{k(-c\theta - \frac{v}{2})}{\beta - (c\theta + v)^2} \text{ and} \\ a &= a(c) = \frac{k(-\beta + \frac{v}{2}(c\theta + v))}{\beta - (c\theta + v)^2}. \end{aligned} \tag{4.24}$$

Moreover, by (4.8,a)

$$\omega A'' - icA' - \lambda A = k(a - \frac{v}{2}b + q)|A|^2 A. \tag{4.25}$$

Finally, setting

$$R(x) = e^{\frac{-icx}{2\omega}} A(x),$$

we obtain

$$R'' + \frac{1}{\omega}(\frac{c^2}{4\omega} - \lambda)R - \frac{k}{\omega}(a - \frac{v}{2}b + q)R^3 = 0, \tag{4.26}$$

which is the well-known stationary equation associated to the cubic NLS equation. A simple computation shows that if

$$\frac{c^2}{4\omega} - \lambda \geq 0 \text{ or } a(c) - \frac{v}{2}b(c) + q \geq 0,$$

then (4.26) has no solutions.
Otherwise, if

$$\frac{c^2}{4\omega} - \lambda < 0 \text{ and } a(c) - \frac{v}{2}b(c) + q < 0,$$

then there exists a unique positive solution to (4.26), which is even and exponentially decreasing.
We can therefore state the following result :

Lemme 4.4.1 *Let $c > 0$.*
Assume that $E := \frac{1}{\omega}(\lambda - \frac{c^2}{4\omega}) > 0$ and that $a(c) - \frac{v}{2}b(c) + q < 0$.
Then $Q_R(x,t) = (\phi_1(x,t), \phi_2(x,t), \phi_3(x,t))$, with

$$\begin{cases} \phi_1(x,t) = e^{i\lambda t} e^{\frac{icx}{2\omega}} R(x-ct) \\ \phi_2(x,t) = a(c)|R(x-ct)|^2 \\ \phi_3(x,t) = b(c)|R(x-ct)|^2, \end{cases}$$

is a solitary wave to (4.8).
Here, $R = R(., E)$ is the unique positive solution to (4.26), which is even and exponentially decreasing, and $a(c)$, $b(c)$ are given by (4.24).

Remarque 4.4.2 *For θ small enough, the condition $a(c) - \frac{v}{2}b(c) + q < 0$ is satisfied. In fact, for $\theta = 0$:*

$$a(c) - \frac{v}{2}b(c) + q = -\frac{v}{4(\beta - v^2)} < 0.$$

Remarque 4.4.3 *The expression of R is known :*

$$R(x,E) = \sqrt{\frac{2E\omega}{k(a(c) - \frac{v}{2}b(c) + q)} \frac{1}{\cosh(\sqrt{E}x)}}. \tag{4.27}$$

4.4.2 Orbital stability

In this section, we prove the orbital stability of the solitary waves described earlier. We will use the method developed in [34], [58], [5] and [59].

We begin by introducing the natural orbit

$$\mathcal{O}(B,u,\rho) = \{e^{i\alpha}B(.+x_o), u(.+x_o), \rho(.+x_o)/\alpha \in \mathbb{R}, x_o \in \mathbb{R}\}. \tag{4.28}$$

Note that if (B,u,ρ) is a solution to the system (4.8), then all the elements in its orbit $\mathcal{O}(B,u,\rho)$ remain solutions.

We denote by N_E the $H^1(\mathbb{R})$-norm given by:

$$N_E^2(f) = ||f'||^2_{L^2(\mathbb{R})} + E||f||^2_{L^2(\mathbb{R})}.$$

Finally, if $Q = (B,u,\rho)$ is a solution to (4.8) for some initial data $Q_o = (B_o, u_o, \rho_o)$, we define for all times the distance between Q and the orbit of the ground state Q_R by:

$$d_E(Q_o,t) = \inf_{\alpha, x_o}[N_E(e^{i\alpha}e^{-\frac{ic.}{2\omega}}B(.+x_o) - R)] \tag{4.29}$$

We can now state our main theorem :
Theorème 4.4.1 *Let $\omega > 0$, $v > 0$, $0 < \theta < 1$ and $\beta - v^2 > 0$. There exists $A > 0$ such that for all $(\lambda, c) \in \mathbb{R}^+ \times]0; \frac{A}{\theta}[$ satisfying*

$$E := \frac{1}{\omega}(\lambda - \frac{c^2}{4\omega}) > 0,$$

the solitary wave

$$Q_R(x,t) = (\phi_1(x,t), \phi_2(x,t), \phi_3(x,t)) = (e^{i\lambda t} e^{\frac{icx}{2\omega}} R(x-ct), a(c)R^2(x-ct), b(c)R^2(x-ct))$$

is orbitally stable, i.e. :
There exists $\epsilon_1 > 0$ such that for all $\epsilon_1 > \epsilon > 0$ and for all $(B_o, u_o, \rho_o) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$, there exists $\delta(\epsilon) > 0$ such that if

$$||B_o - e^{\frac{ic.}{2\omega}} R||_{H^1(\mathbb{R})} \leq \delta(\epsilon),$$

$$||u_o - a(c)R^2||_{L^2(\mathbb{R})} \leq \delta(\epsilon),$$

$$||\rho_o - b(c)R^2||_{L^2(\mathbb{R})} \leq \delta(\epsilon),$$

then for all $t \in \mathbb{R}^+$,

$$Inf_{\alpha, x_o} ||e^{i\alpha} B(. + x_o, t) - \phi_1(.,t)||_{H^1(\mathbb{R})} < \epsilon \quad (4.30)$$

$$Inf_{x_o} ||u(. + x_o, t) - \phi_2(.,t)||_{L^2(\mathbb{R})} < \epsilon \quad (4.31)$$

$$Inf_{x_o} ||\rho(. + x_o, t) - \phi_3(.,t)||_{L^2(\mathbb{R})} < \epsilon, \quad (4.32)$$

where (B, u, ρ) is the solution of (4.8) corresponding to the initial data (B_o, u_o, ρ_o).

Remark 4.4.4

Note that we must take $Q_o \in H^2(\mathbb{R})$, since we do not know if (4.8) is strongly well-posed in $H^1(\mathbb{R})$.

Proof of the Theorem 4.4.1 :

We begin by noticing that :
Remarque 4.4.5
One can assume, without loss of generality, that

$$||B_o||_{L^2(\mathbb{R})} = ||\phi_1(.,0)||_{L^2(\mathbb{R})}.$$

First, we construct another ground state $Q_{\tilde{R}}$, corresponding to a speed $\tilde{c}$ close to c, and satisfying $||B_o||_{L^2(\mathbb{R})} = ||R_{\tilde{c}}||_{L^2(\mathbb{R})}$.

$$||R_{\tilde{c}}||^2_{L^2(\mathbb{R})} = \frac{a(c) - \frac{v}{2}b(c) + q}{a(\tilde{c}) - \frac{v}{2}b(\tilde{c}) + q}||R_c||^2_{L^2(\mathbb{R})}.$$

We choose $\tilde{c}$ such that

$$\frac{a(\tilde{c}) - \frac{v}{2}b(\tilde{c}) + q}{a(c) - \frac{v}{2}b(c) + q} = ||R_c||^2_{L^2(\mathbb{R})}||B_o||^{-2}_{L^2(\mathbb{R})},$$

which is possible if $\delta(\epsilon)$ small enough:
By setting $f(c) = a(c) - \frac{v}{2}b(c) + q$, we have

$$f'(c) = \theta\frac{2\beta(2\theta c + v) - v^3 - 6\theta vc^2 - 3v^2\theta c}{2(\beta - (v+\theta c)^2)^2} \geq \theta\frac{v^3 + 4vc^2 + 2c^3}{2(\beta - (v+\theta c)^2)^2} > 0.$$

The Remark 4.4.2 now comes from the fact that the orbits of Q_{R_c} and $Q_{R_{\tilde{c}}}$ remain close when c is close to $\tilde{c}$.

Lemme 4.4.6 *For all $t \geq 0$, there exists $\alpha(t) \in \mathbb{R}$ et $x_o(t) \in \mathbb{R}$ minimising (4.29). Moreover,*

$$\begin{cases} \Theta : \mathbb{R}^+ \to H^1(\mathbb{R}) \\ t \to e^{i\alpha(t)}e^{-\frac{-ic.}{2\omega}}B(. + x_o(t)) \end{cases}$$

is continue.
Also,

$$\int R^3(x)\frac{\partial}{\partial x}Re(e^{i\alpha(t)}e^{-\frac{-ic.}{2\omega}}B(x + x_o(t), t) - R(x))dx = 0 \tag{4.33}$$

and

$$\int R^3(x)Im(e^{i\alpha(t)}e^{-\frac{-ic.}{2\omega}}B(x + x_o(t), t) - R(x))dx = 0. \tag{4.34}$$

This technical lemma is proved for example in [3].

We now build a Lyapunov-invariant associated to the system (4.8).
We set

$$\begin{cases} h(x,t) = e^{i\alpha(t)}e^{-\frac{-icx}{2\omega}}B(x + x_o(t)) - R(x) \\ w_1(x,t) = u(x + x_o(t)) - aR^2(x) \\ w_2(w,t) = \rho(x + x_o(t)) - bR^2(x) \\ h_1(x,t) = Re(h(x,t)) \\ h_2(x,t) = Im(h(x,t)) \end{cases} \tag{4.35}$$

First, we compute $I_j(t) := I_j(Q(t))$, where I_j, $1 \leq j \leq 3$, are the three invariants introduced in section 3.
By setting

$$r := \frac{c}{2\omega},$$

$$I_1(t) = \int R^2 + \int |h|^2 + 2\int Rh_1,$$

$$\begin{aligned}
I_2(t) &= \frac{\omega}{2}(\int R'^2 + r^2 \int R^2) + \frac{a^2+\beta b^2 + k(q+2a-vb) - 2abv}{4}\int R^4 \\
&+ \int h_1[\omega r^2 R - \omega R'' + k(q+a-\frac{v}{2}b)R^3] - 2\omega r \int R'h_2 \\
&+\frac{\omega}{2}\int |h'|^2 + r^2\frac{\omega}{2}\int |h|^2 + kq\int R^2|h_1|^2 + \frac{k}{4}(2q+2a-vb)\int R^2|h|^2 \\
&+2\omega r\int h_1h_2' + \frac{kq}{4}\int |h|^4 + kq\int R|h|^2h_1 + \int w_1R^2[\frac{k-vb+a}{2}] \\
&+\int w_2R^2[\frac{2\beta b - kv - 2va}{4}] + \frac{1}{4}\int (w_1^2+\beta w_2^2) \\
&\frac{k}{2}\int w_1|h|^2 + k\int Rh_1w_1 - \frac{kv}{2}\int Rh_1w_2 - \frac{kv}{4}\int |h|^2w_2 - \frac{v}{2}\int w_1w_2,
\end{aligned}$$

and, finally,

$$\begin{aligned}
I_3(t) &= \int \theta abR^4 + \int \theta aR^2\omega_2 + \int \theta bw_1R^2 + \int \theta w_1w_2 \\
&+r\int R^2 + 2r\int Rh_1 + r\int |h|^2 - 2\int h_1'h_2 + 2\int Rh_2'.
\end{aligned}$$

We set

$$\mathcal{I}(t) = \frac{\lambda}{\omega}I_1(t) + \frac{2}{\omega}I_2(t) - \frac{c}{\omega}I_3(t)$$

and

$$\Delta\mathcal{I}(t) = \mathcal{I}(t) - \mathcal{I}(\phi_1(0), \phi_2(0), \phi_3(0)).$$

We get

$$\begin{aligned}
\Delta\mathcal{I}(t) &= \int |h'|^2 + \int |h|^2(-\frac{c^2}{4\omega^2} + \frac{k}{2\omega}(2q+2a_vb)R^2 + \frac{\lambda}{\omega}) \\
&+\int |h_1|^2(\frac{2kq}{\omega}R^2) + \frac{1}{2\omega}\int (w_1^2+\beta w_2^2) \\
&-\frac{c\theta}{\omega}\int w_1w_2 + \frac{k}{\omega}\int w_1|h|^2 + \int 2k\omega\int Rh_1w_1 \\
&-\frac{kv}{2}\int Rh_1w_2 - \frac{kv}{4}\int |h|^2w_2.
\end{aligned}$$

Note that the linear terms in h, w_1 and w_2 dropped out, which means that the ground state (ϕ_1, ϕ_2, ϕ_3) is a critical point for the functional $\mathcal{I}$.
Moreover,

$$\begin{aligned}\Delta\mathcal{I}(t) &= < L_+h_1, h_1 >_{L^2} + < L_-h_2, h_2 >_{L^2} + \frac{1}{2\omega}[w_1 - (v+\theta c)w_2 + 2kRh_1 + k|h|^2]^2 \\ &+ \frac{1}{2\omega}[\sqrt{\beta - (v+\theta c)^2}(v+\theta c)w_2 + Rkh_1\frac{(v+2\theta c)}{\sqrt{\beta-(v+\theta c)^2}} \\ &+ k|h|^2\frac{(v+2\theta c)}{2\sqrt{\beta-(v+\theta c)^2}}]^2 + F(h),\end{aligned} \tag{4.36}$$

where

$$L_- = -\partial_x^2 + E.Id + R^2\frac{k}{\omega}(q + a - \frac{v}{2}b),$$

$$L_+ = -\partial_x^2 + E.Id + 3R^2\frac{k}{\omega}(q + a - \frac{v}{2}b)$$

and F is composed of the terms of higher order:

$$\begin{aligned}F(h) &= \int k|h|^4(\frac{q-k}{2\omega} - \frac{k}{8\omega}\frac{(v+2\theta c)}{\beta-(v+\theta c)^2}) \\ &+ \frac{k}{\omega}\int R|h|^2h_1(2q - 2k - k\frac{(v+2\theta c)}{\beta-(v+\theta c)^2}).\end{aligned}$$

Next, we estimate $\mathcal{I}(t)$. By the imbeddings

$$H^1(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$$

and

$$H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}),$$

we easily prove the existence of two positive constants A_1 et A_2 such that

$$F(h) \geq -A_1||h||^3_{H^1} - A_2||h||^4_{H^1}.$$

Therefore, we are reduced to the study of the quadratic forms $< L_+h_1, h_1 >_{L^2}$ and $< L_-h_2, h_2 >_{L^2}$, which correspond to the second derivative of $\mathcal{I}$.

Lemme 4.4.7 *On the varieties defined by the restriction conditions (4.33) et (4.34), we have the existence of* $B_1 > 0$, $C_1 > 0$, $C_2 > 0$ *and* $C_3 > 0$ *such that*

$$< L_-h_2, h_2 >_{L^2} \geq B_1||h_2||^2_{H^1}$$

and

$$< L_+h_1, h_1 >_{L^2} \geq C_1||h_1||^2_{H^1} - C_2||h||^3_{H^1} - C_3||h||^4_{H^1}.$$

Proof:
The estimate concerning L_- is easier.
Since $R > 0$ and $L_- R = 0$, L_- is non-degenerated, and therefore $< L_-.,. >$ is positive.
Moreover, one can prove (see [5]) that if $\frac{1}{||h_2||^2_{H1}} < L_- h_2, h_2 >_{L^2}$ vanishes under the constraint (4.34), thus this minimum is attained in R, which contradicts (4.34).
The proof of the second estimate is harder to obtain. Since we assumed that

$$||B_o||_{L^2(\mathbb{R})} = ||\phi_1(.,0)||_{L^2(\mathbb{R})},$$

the proof stated in [59], Proposition 3.3 can be applied here.

We can now end the proof of Theorem 4.4.1 by the standard method:

$$\mathcal{I}(t) \geq (B_1 + C_1)||h||^2_{H^1} - (C_2 + A_1)||h||^3_{H^1} - (C_3 + A_2)||h||^4_{H^1},$$

i.e.

$$\mathcal{I}(t) \geq G(N_E(h(t))),$$

where

$$G(x) = c_1 x^2 - c_2 x^3 - c_3 x^4,$$

and $c_1 = \frac{B_1+C_1}{Sup(1,E)}$, $c_2 = \frac{C_2+A_1}{Inf(1,E)}$ and $c_3 = \frac{C_2+A_1}{Inf(1,E)}$.

More, since (B, u, ρ) is a solution to 4.8,

$$\Delta\mathcal{I}(t) = \Delta\mathcal{I}(0)\ \forall t \in \mathbb{R}^+.$$

Let $\epsilon_o > 0$ such that G increases on $[0; \epsilon_o]$.
Let $0 < \epsilon < \epsilon_o$:
At $t = 0$,

$$|\Delta\mathcal{I}(0)| < G(\epsilon) \text{ for } \delta(\epsilon) \text{ small enough.}$$

Hence,

$$G(\epsilon) > G(N_E(h(t)))$$

Finally, by the continuity of $t \to N_E(h(t))$ and since G increases,

$$d_E(Q_o, t) = N_E(h(t)) < \epsilon\ \forall t \in \mathbb{R}+,$$

h being a minimizer for the problem (4.29).

Finally, we check the estimates (4.31) and (4.32):
We only need to notice that by (4.36), and the previous study of h,

$$G(\epsilon) \geq \frac{1}{2\omega}[w_1 - (v + \theta c)w_2 + 2kRh_1 + k|h|^2]^2, \tag{4.37}$$

and

$$G(\epsilon) \geq \frac{1}{2\omega}[\sqrt{\beta-(v+\theta c)^2}(v+\theta c)w_2 + Rkh_1\frac{(v+2\theta c)}{\sqrt{\beta-(v+\theta c)^2}} + k|h|^2\frac{(v+2\theta c)}{2\sqrt{\beta-(v+\theta c)^2}}]^2. \tag{4.38}$$

Chapitre 5

Approximation de l'équation DNLS par l'équation de Schrödinger cubique

A modulation equation for the DNLS model

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Résumé

We approach rigorously the Schrödinger Equation of Derivative type $q_t + iq_{xx} + \lambda|q|^2 q_x + \mu q^2 \overline{q_x} = 0$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{C}$, by the cubic Nonlinear Schrödinger Equation $A_T + iA_{XX} + ik_o(\lambda - \mu)|A|^2 A = 0$.
We also study the case of the KdV-like equation $q_t + iq_{xx} + aq_{xxx} + i|q|^2 q + \tilde{\lambda}(|q|^2 q)_x + \tilde{\mu}|q|^2 q_x = 0$, $\tilde{\lambda}, \tilde{\mu} \in \mathbb{R}$, arizing in optical physics. Finally, we illustrate these results numerically. [1]

5.1 Introduction

We consider the Nonlinear Schrödinger Equations of the form:

$$q_t + iq_{xx} + \lambda|q|^2 q_x + \mu q^2 \overline{q_x} = 0, \tag{5.1}$$

where $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}$.
These equations have been extensively studied by many authors (see for instance [53], [22],[35],[46]), and cover the case of the so-called Derivative Nonlinear Schrödinger Equation (DNLS)

$$q_t + iq_{xx} + (|q|^2 q)_x = 0,$$

describing the parallel propagation (to the ambient magnetic field) of circulary polarized Alfvén waves in a plasma. When linearized at $q \equiv 0$, (5.1) has the exact solutions:

$$q = q_o e^{i(k_o x + \omega_o t)}, \quad q_o \in \mathbb{C},$$

where the wave number k_o and the frequency ω_o are related by the dispersion equation

$$\omega_o = k_o^2. \tag{5.2}$$

In order to study how these plane waves are modulated by the nonlinear effects, we introduce for a small $\epsilon > 0$, the formal approximation ([44], [42])

$$q_A(x,t) = \epsilon A(X,T) e^{i\phi(x,t)}, \tag{5.3}$$

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where $X = \epsilon(x - v_G t)$, $T = \epsilon^2 t$ and $\phi(x,t) = k_o x + \omega_o t$,
and for an energy speed v_G given by

$$v_G = -\frac{\partial \omega_o}{\partial k_o}.$$

We expect with this approximation to describe solutions to (5.1) which are slow modulations (in time and space) of a wave train of small amplitude: the exponential will take into account the "fast" oscillation of the phase while the "slow" rescaled variables X and T will be responsible for the small variations of the complex amplitude.
In order to derive an equation for the amplitude A, we inject q_A in (5.1), and then equate the coefficients of $[\epsilon^j$, $j \in \mathbb{N}$, to 0. This formal multi-scale method was used in [44], [42].

By this method, we get the cubic Non Linear Schrödinger Equation (NLS).

$$A_T + iA_{XX} + ik_o(\lambda - \mu)|A|^2 A = 0, \qquad (5.4)$$

In this sense, the NLS equation appears here as an approximation to equation (5.1).

In this paper we will justify mathematically this approach.
This line of work is not new, and we can quote the works of P. Kirmann and al.([32]), T. Gallay and al.([14]), and G. Schneider([51]):
In [51], the NLS equation is rigorously derived as an approximation for the KdV equation:

$$u_t + u_{xxx} + uu_x = 0$$

and, in [32], for the Sine-Gordon equation:

$$u_{tt} = u_{xx} + sin(u).$$

Also, in this same paper, the Swift-Hoenberg equation

$$u_t = L_\lambda(\partial_x)u - u^3 \ , \ L_\lambda(\partial_x)u = -(1 + \partial_x^2)u + \lambda u.$$

is correctly approximated by the Ginzburg-Landau equation:

$$A_T = 4A_{XX} + A - 3|A|^2 A.$$

Finally, in [14], T. Gallay and G. Schneider describe the KP equation

$$[2A_T + (A^2)_X + A_{XXX}]_X + A_{YY} = 0.$$

as an approximation for the Boussinesq equation

$$\phi_{tt} = \Delta\phi + \Delta(\phi^2) + \Delta\phi_{tt}.$$

Here, we will combine the different arguments presented in these papers to prove our main result :

Theorème 5.1.1 *Let $s > \frac{3}{2}$.*
Let A be a solution to the NLS equation (5.4), with

$$A \in C([0, T_o]; H^{s+3}(\mathbb{R})),$$

$T_o > 0$, and let q_A be given by (5.3).
Then there exists $\epsilon_o > 0$ such that for all $0 < \epsilon \leq \epsilon_o$ and for all $q_o \in H^s(\mathbb{R})$ satisfying

$$||q_o - q_A(.,0)||_s \leq D_1\epsilon^2, \tag{5.5}$$

we have the existence of a solution $q \in C([0;T]; H^s(\mathbb{R}))$ to (5.1), with $q(.,0) = q_o$ and

$$Sup_{[0,\frac{T_o}{\epsilon^2}]}||q(.,t) - q_A(.,t)||_s \leq D_2\epsilon^{\frac{3}{2}}. \tag{5.6}$$

Here the $D_{1,2}$ are strictly positive constants depending exclusively on ϵ_o and T_o.
We can now state a few remarks :

The solution q (as well as the approximated solution q_A) has a $H^s(\mathbb{R})$-norm of order $\epsilon^{\frac{1}{2}}$, while the error $q - q_A$ is of order $\epsilon^{\frac{3}{2}}$, which is smaller .

This result proves the existence of solutions to the DNLS equation that behave approximately as NLS solitons. In fact, a part of the NLS dynamics are contained in the DNLS model.

Finally, the regularity condition $A \in H^{s+3}(\mathbb{R})$ is usual in this kind of results and is in a certain way needed for the estimate (5.6) to remain valid in the "large" time interval $[0; \epsilon^{-2}T_o]$.

The rest of this paper is organized as follows :
In the second section, and essentially for technical reasons, we build an approximation r_A, more accurate than q_A.
In the third section, we evaluate the error and prove Theorem 5.1.1.
In the fourth section, we consider the nonlinear cubic problem

$$\begin{cases} q_t + iq_{xx} + aq_{xxx} + i|q|^2q + \tilde{\lambda}(|q|^2q)_x + \tilde{\mu}|q|^2q_x = 0 \\ q(0,X) = q_o(X) \end{cases} \tag{5.7}$$

with $a \neq 0$, which we will approximate by the NLS equation

$$A_T + i(ak_o + 1)A_{xx} + i(1 + k_o(\tilde{\lambda} + \tilde{\mu}))|A|^2A = 0, \tag{5.8}$$

and by

$$A_T + aA_{xxx} + i(1 + k_o(\tilde{\lambda} + \tilde{\mu}))|A|^2A = 0 \tag{5.9}$$

in the case where the wave number k_o is close to the critical value $-\frac{1}{a}$. Finally, in the last section, we illustrate Theorem 5.1.1 by a numerical method.
We end this introduction with a few notations:

We denote by $< .,. >$ the canonical scalar product of $L^2(\mathbb{R})$, given by

$$< f,g >=< f,g >_{L^2}= \int_{\mathbb{R}} f\overline{g}.$$

We denote by Λ^s the operator $(1-\partial_x^2)^{\frac{s}{2}}$.

For $s \in \mathbb{R}$, we introduce the usual Sobolev spaces

$$H^s(\mathbb{R}) = \{f/||f||_s^2 = ||\Lambda^s f||_{L^2}^2 < \infty\}$$

and, if s is an integer,

$$W^{s,\infty}(\mathbb{R}) = \{f/||f||_{W^{s,\infty}} = \sum_{k=0}^{s} ||\partial_x^k f||_{L^\infty} < \infty\}.$$

Finally, we denote by $[\Lambda^s f, g]$ the commutator given by

$$[\Lambda^s f, g] = \Lambda^s(fg) - f\Lambda^s(g).$$

5.2 The approximated solution

Let $s > \frac{3}{2}$ and $A \in C([0,T_o]; H^{s+3}(\mathbb{R}))$ a solution to (5.4).
Let $\epsilon_o > 0$, $\epsilon \in]0,\epsilon_o]$ and $q_o \in H^s(\mathbb{R})$ such that

$$||q_o(.) - q_A(.,0)||_s \leq D_1\epsilon^2,$$

for q_A given by (5.3) and $D_1 > 0$.

We denote by $Res(q_A)$ the residual

$$Res(q_A)(\epsilon,x,t) = q_{A_t} + iq_{A_{xx}} + \lambda|q_A|^2 q_{A_x} + \mu q_A^2 \overline{q_{A_x}},$$

formed by the terms that do not drop out after inserting q_A in (5.1).
A simple computation yields:

$$\begin{aligned} Res(q_A) &= i\epsilon e^{i\phi}[\omega_o - k_o^2] \\ &+\epsilon^2 e^{i\phi} A_X[-v_G - 2k_o] \\ &+\epsilon^3 e^{i\phi}[A_T + iA_{XX} + ik_o\lambda - \mu|A|^2 A] \\ &+\epsilon^4 e^{i\phi}[\lambda|A|^2 A_X + \mu A^2 \overline{A}_X] \end{aligned}$$

By the dispersion equation (5.2), the value of the energy speed v_G and (5.4), we get

$$Res(q_A)(\epsilon, x, t) = \mathcal{O}(\epsilon^4).$$

In order to prove Theorem 5.1.1, we need to get a "smaller" residual. For this reason, we define a more accurate approximated solution, by adding one more term to q_A. Hence, we set

$$r_A(\epsilon, x, t) = q_A(\epsilon, x, t) + \epsilon^2 B(X, T) e^{i\phi}$$

where B will be chosen later.
The new residual yields:

$$\begin{aligned} Res(r_A) \quad = \quad & \epsilon^4 e^{i\phi}[B_T + iB_{XX} + \lambda|A|^2 A_X + \mu A^2 \overline{A}_X + ik_o\lambda - \mu(A^2\overline{B} + 2|A|^2 B)] \\ & + \epsilon^5 e^{i\phi}[ik_o\lambda - \mu(2|B|^2 A + B^2\overline{A}) + \lambda(|A|^2 B_X + AA_X\overline{B} + B\overline{A}A_X)] \\ & + \epsilon^5 e^{i\phi}\mu(A^2 B_X + 2AB\overline{A}_X) + \epsilon^6 e^{i\phi}\mu(B^2\overline{A}_X + 2AB\overline{B}_X) \\ & + \epsilon^6 e^{i\phi}[ik_o\lambda - \mu B|B|^2 + \lambda(|B|^2 A_X + A\overline{B}B_X + BB_X\overline{A})] \\ & + \epsilon^7 e^{i\phi}[\lambda|B|^2 B_X + \mu B^2\overline{B}_X]. \end{aligned}$$

Now, consider the linear (in B) equation:

$$\left\{ \begin{array}{l} B_T + iB_{XX} + \lambda|A|^2 A_X + \mu A^2\overline{A}_X + ik_o\lambda - \mu(A^2\overline{B} + 2|A|^2 B) = 0 \\ B(X, 0) = B_o(X). \end{array} \right. \tag{5.10}$$

Since $H^{s+2}(\mathbb{R})$ is an algebra, it is clear that the Initial Value Problem (5.10) is well-posed, and that its solutions exist and are bounded in $H^{s+2}(\mathbb{R})$ as long as A exists and is bounded in $H^{s+3}(\mathbb{R})$.
Hence, we choose $B \in C([0, T_o]; H^{s+2}(\mathbb{R}))$ the solution of (5.10) corresponding to the initial data $B_o = 0$.

Lemme 5.2.1 *With the above choices of B and r_A, there exists three positive constants C_j, $j = 1, 2, 3$, depending exclusively on k_o and ϵ_o, and three functions f, g, h such that:*

1) $Res(r_A) = \epsilon^{\frac{9}{2}} g(\epsilon, x, t)$, $Sup_{[0, \frac{T_0}{\epsilon^2}]}||g(\epsilon, ., t)||_{s+1} \leq C_1$

2) $r_A = \epsilon^{\frac{1}{2}} f(\epsilon, x, t)$, $Sup_{[0, \frac{T_0}{\epsilon^2}]}||f(\epsilon, ., t)||_{s+2} \leq C_2$

3) $r_A = \epsilon h(\epsilon, x, t)$, $Sup_{[0, \frac{T_0}{\epsilon^2}]}||h(\epsilon, ., t)||_{W^{s+1,\infty}} \leq C_3$

In fact,

$$\begin{aligned} Res(r_A) \quad = \quad & \epsilon^5 e^{i\phi}[ik_o\lambda - \mu(2|B|^2 A + B^2\overline{A}) + \lambda(|A|^2 B_X + AA_X\overline{B} + B\overline{A}A_X)] \\ & + \epsilon^5 e^{i\phi}\mu(A^2 B_X + 2AB\overline{A}_X) + \epsilon^6 e^{i\phi}\mu(B^2\overline{A}_X + 2AB\overline{B}_X) \\ & + \epsilon^6 e^{i\phi}[ik_o\lambda - \mu B|B|^2 + \lambda(|B|^2 A_X + A\overline{B}B_X + BB_X\overline{A})] \\ & + \epsilon^7 e^{i\phi}[\lambda|B|^2 B_X + \mu B^2\overline{B}_X]. \end{aligned}$$

One only has to notice that for $F \in H^s(\mathbb{R})$, the estimate

$$||F(\epsilon x)e^{ik_o x}||_s \leq C(k_o, \epsilon_o)\epsilon^{-\frac{1}{2}}||F||_s,$$

holds.

The Lemma then follows by using the imbedding

$$H^{s+\frac{3}{2}^+}(\mathbb{R}) \hookrightarrow W^{s+1,\infty}(\mathbb{R}).$$

5.3 The error estimate

We next consider the I.V.P.

$$\begin{cases} q_t + iq_{xx} + \lambda|q|^2 q_x + \mu q^2 \overline{q_x} = 0 \\ q(x,0) = q_o(x). \end{cases}$$

It is well known that this problem is locally well-posed in $H^s(\mathbb{R})$ (globally well-posed in $H^1(\mathbb{R})$, see ([53])).

Therefore, in order to prove Theorem 1.1, we only need to bound uniformly the quantity $\epsilon^{-\frac{3}{2}}||q(.,t) - q_A(.,t)||_s$ over the time interval $[0; \epsilon^{-2}T_o]$.

We set

$$\epsilon^2 R = q - r_A. \tag{5.11}$$

By an elementary computation,

$$\begin{aligned} R_t + iR_{xx} &= (\lambda\overline{r_A}r_{A_x} + \mu\overline{r_{A_x}}r_A)R + (\mu\overline{r_{A_x}}r_A + \lambda r_A r_{A_x})\overline{R} + \lambda|r_A|^2 R_x + \mu\overline{R}_x r_A^2 \\ &+\epsilon^2[\lambda|R|^2 r_{A_x} + \lambda R_x(\overline{R}r_A + R\overline{r_A}) + \mu R^2\overline{r_{A_x}} + 2\mu R\overline{R}_x r_A] \\ &+\epsilon^4[\lambda|R|^2 R_x + \mu R^2\overline{R}_x] - \epsilon^{-2}Res(r_A), \end{aligned}$$

i.e.

$$\begin{aligned} R_t + iR_{xx} &= \epsilon^2[(\lambda\overline{h}h_x + \mu\overline{h}_x h)R + (\mu\overline{h}_x h + \lambda hh_x)\overline{R} + \lambda|h|^2 R_x + \mu\overline{R}_x h^2] \\ &+\epsilon^{\frac{5}{2}}[\lambda|R|^2 f_x + \lambda R_x(\overline{R}f + R\overline{f}) + \mu R^2\overline{f}_x + 2\mu R\overline{R}_x f] \\ &+\epsilon^4[\lambda|R|^2 R_x + \mu R^2\overline{R}_x] - \epsilon^{\frac{5}{2}}g \\ &:= \epsilon^2\phi_1(R,h,R_x,h_x) + \epsilon^{\frac{5}{2}}\phi_2(R,R_x,f,f_x) + \epsilon^4\phi_3(R,R_x) - \epsilon^{\frac{5}{2}}g, \end{aligned}$$

where f, g et h are given by the Lemma 5.2.1.

We now estimate R:

Lemme 5.3.1 *For all $0 < \epsilon \leq \epsilon_o \leq 1$ and for all $t \in [0; \epsilon^{-2}T_o]$,*

$$\frac{1}{2}\frac{d}{dt}||R||_s^2 \leq \epsilon^2[C_1 + C_2||R||_s^2 + \epsilon^{\frac{1}{2}}C_3||R||_s^4], \tag{5.12}$$

where the C_j are positive constants.

By applying the operator Λ^s to (5.12), multiplying by $\Lambda^s\overline{R}$ and integrating the real part, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||R||_s^2 &= \epsilon^2 Re(<\Lambda^s\phi_1, \Lambda^s R>) + \epsilon^{\frac{5}{2}} Re(<\Lambda^s\phi_2, \Lambda^s R>) \\ &\quad +\epsilon^4 Re(<\Lambda^s\phi_3, \Lambda^s R>) - \epsilon^{\frac{5}{2}} Re(<\Lambda^s g, \Lambda^s R>), \qquad (5.13)\end{aligned}$$

for all $t \in [0, T_o\epsilon^{-2}]$. In what follows, we will denote indistinctly by C an arbitrary positive constant.

Plainly,

$$Re<\Lambda^s g, \Lambda^s R> \leq C||R||_s \leq C(1+||R||_s^2). \qquad (5.14)$$

Also,

$$\begin{aligned}Re(<\Lambda^s\phi_3, \Lambda^s R>) &= \lambda Re<\Lambda^s(|R|^2R_x), \Lambda^s R> + Re<(\mu\Lambda^s(R^2\overline{R}_x), \Lambda^s(R)> \\ &= \lambda Re<|R|^2\Lambda^s R_x, \Lambda^s R> + \lambda Re<[\Lambda^s, |R|^2]R_x, \Lambda^s R> \\ &\quad + Re<\mu R^2\Lambda^s\overline{R}_x, \Lambda^s R> + Re<\mu[\Lambda^s, R^2]\overline{R}_x, \Lambda^s R> \\ &\leq -\frac{\lambda}{2}\int |R^2|_x|\Lambda^s R|^2 + |\mu||\int (R^2)_x|\Lambda^s\overline{R}|^2| \\ &\quad + |\lambda|.||[\Lambda^s, |R|^2]R_x||_o||R||_s + |\mu|.||[\Lambda^s, R^2]\overline{R}_x||_o||R||_s \\ &\leq C||R||_s^4, \qquad (5.15)\end{aligned}$$

where we have used the fact that, for $s > 1$,

$$||[\Lambda^s, f]g||_o \leq C||f||_s||g||_{s-1},$$

and the Sobolev imbedding

$$H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}).$$

Note that this estimate does not hold if $\lambda \in \mathbb{C}/\mathbb{R}$.
Next,

$$\begin{aligned}Re(<\Lambda^s\phi_2, \Lambda^s R>) &= \lambda Re(<\Lambda^s(|R|^2 f_x), \Lambda^s R>) + \lambda Re(<\Lambda^s(R_x(\overline{R}f + R\overline{f})), \Lambda^s R>) \\ &\quad + Re(<\Lambda^s(\mu R^2\overline{f}_x), \Lambda^s R>) + 2Re(<\mu\Lambda^s(R\overline{R}_x f), \Lambda^s R>).\end{aligned}$$

Since $\overline{R}f + R\overline{f} \in \mathbb{R}$, the same method as presented in the previous estimate yields

$$Re(<\Lambda^s(R_x(\overline{R}f + R\overline{f})), \Lambda^s R>) \leq C||R||_s^3 \qquad (5.16)$$

and

$$Re(<\mu\Lambda^s(R\overline{R}_x f), \Lambda^s R>) \leq C||R||_s^3. \qquad (5.17)$$

The other terms do not contain x-derivatives of R, hence it is clear that

$$Re(<\Lambda^s\phi_2, \Lambda^s R>) \leq C||R||_s^3. \tag{5.18}$$

Finally, we estimate the last term:

$$\begin{aligned} Re(<\Lambda^s\phi_1, \Lambda^s R>) &= \lambda < Re\Lambda^s(\overline{h}h_x R), \Lambda^s R> + Re < \Lambda^s(\mu\overline{h}_x hR), \Lambda^s R> \\ &\quad + Re < \Lambda^s(\mu\overline{h}_x h\overline{R}), \Lambda^s R> + \lambda Re < \Lambda^s(hh_x\overline{R}), \Lambda^s(R)> \\ &\quad + Re < \Lambda^s(\lambda|h|^2 R_x + \mu\overline{R}_x h^2), \Lambda^s R> . \end{aligned}$$

We use here the following remark:
Remarque 5.3.2 *Let $s \in \mathbb{N}$, $R \in H^s(\mathbb{R})$ and $\psi \in W^{s+1,\infty}(\mathbb{R})$.*
There exists $C > 0$ such that

$$<\Lambda^s(\psi\overline{R})_x, \Lambda^s R> \leq C||\psi||_{W^{s+1,\infty}}||R||_s^2.$$

Furthermore, if ψ is real-valued,

$$Re(<\Lambda^s(\psi R)_x, \Lambda^s R>) \leq C||\psi||_{W^{s+1,\infty}}||R||_s^2$$

Proof:
Let $R \in H^s(\mathbb{R})$, $\psi \in W^{s+1,\infty}(\mathbb{R})$, real-valued, and s an integer. Then,

$$\begin{aligned} <\Lambda^s(\psi R)_x, \Lambda^s R> &= \sum_{\alpha=0}^{s} C(\alpha) < \partial_x^{\alpha+1}(H\psi), \partial_x^\alpha R> \\ &= \sum_{\alpha=0}^{s} C(\alpha) < \sum_{\beta=0}^{\alpha+1} C_{\alpha+1}^{\beta}\partial_x^\beta R\partial_x^{\alpha+1-\beta}\psi, \partial_x^\alpha R> \end{aligned}$$

Plainly, for $\beta \leq s$,

$$<\partial_x^\beta R\partial_x^{\alpha-\beta}H, \partial_x^\alpha R> \leq ||R||_s^2||f||_{W^{s+1,\infty}}.$$

Also,

$$Re < \psi\partial_x^{s+1}R, \partial_x^s R> = \frac{1}{2}\int \psi|\partial_x^s R|_x^2$$

and we obtain the desired result by integrating by parts. The proof of the first statement follows the same lines.
By using Remark 5.3.2, we obtain that

$$Re(<\Lambda^s(\phi_1), \Lambda^s(R)>) \leq C||R||_s \leq C(1 + ||R||_s^2). \tag{5.19}$$

By combining (5.14), (5.15), (5.18) and (5.19), we get Lemma 5.3.1.

End of the proof: By noticing that $r_A(0) = q_A(0)$ (since we have chosen $B_o = 0$),

$$R(\epsilon, x, O) = \epsilon^{-2}(q_o(x) - r_A(0, x)) = \epsilon^{-2}(q_o(x) - q_A(0, x)),$$

and, by the condition (5.5),

$$||R(\epsilon, x, O)||_s \leq D_1.$$

We set

$$t(\epsilon) = Sup\{t \in [0, \epsilon^{-2}T_o]/\epsilon^{\frac{1}{2}}||R(\tau)||_s^2 \leq 1 \forall \tau \in [0, t]\}.$$

If one chooses $\epsilon_0 > 0$ small enough, $t(\epsilon) > 0$, since at $t = O$,

$$\epsilon^{\frac{1}{2}}||R(0)||_s^2 \leq \epsilon_o^{\frac{1}{2}}||R(0)||_s^2 < 1.$$

Hence, for all $t \in [0; t(\epsilon)]$,

$$\frac{d}{dt}||R(t)||_s^2 \leq A\epsilon^2(1 + ||R(t)||_s^2),$$

where

$$A = Max\{2C_1; 2(C_2 + C_3)\}.$$

By Grönwall's lemma, for all $t \in [0; t(\epsilon)]$,

$$||R(t)||_s^2 \leq (1 + ||R(., 0)||_s^2)e^{A\epsilon^2 t} - 1. \tag{5.20}$$

In particular, for ϵ_o small enough,

$$\epsilon^{\frac{1}{2}}||R(t)||_s^2 < \epsilon_o^{\frac{1}{2}}(1 + ||R(., 0)||_s^2)e^{AT_o} \leq 1,$$

and

$$t(\epsilon) = \epsilon^{-2}T_o.$$

Therefore, (5.20) holds for all $t \in [0, \epsilon^{-2}T_o]$, and there exists $C > 0$, depending exclusively on ϵ_o and T_o, such that

$$||R(t)||_s \leq C \text{ , for all } t \in [0; \epsilon^{-2}T_o],$$

i.e.

$$Sup_{[0, \frac{T_o}{\epsilon^2}]}||q - r_A||_s \leq C\epsilon^2. \tag{5.21}$$

We end the proof of Theorem 5.1.1 by noticing that

$$r_A = q_A + \epsilon^2 F,$$

where $F(\epsilon, x, t) = B(X, T)e^{i(k_o x + w_o t)}$ and

$$\epsilon^{\frac{1}{2}}||F||_s \leq C,$$

hence

$$Sup_{[0, \frac{T_o}{\epsilon^2}]}||q - q_A||_s \leq D_2\epsilon^{\frac{3}{2}}.$$

It is straightforward to derive from Theorem 5.1.1 the following corollary:

Corollaire 5.3.3 *Let $T_o > 0$, $k > 0$ and $s > \frac{3}{2}$.*
Let A be a solution to the NLS equation

$$A_T + iA_{XX} + ik|A|^2 A = O$$

such that

$$A \in C([0; T_o]; H^{s+3}(\mathbb{R})).$$

Then we have a constant $C > 0$ and ϵ_o such that for all $0 < \epsilon \leq \epsilon_0$, there are solutions q to the DNLS equation

$$q_t + iq_{xx} + (|q|^2 q)_x = 0$$

such that

$$Sup_{[0;\frac{T_0}{\epsilon^2}]} ||q(.,t) - q_A(.,t)||_\infty \leq C\epsilon^2. \tag{5.22}$$

Proof:
One only needs to choose $q(.,0)$ in the ball of $H^s(\mathbb{R})$ of center q_A and radius $D_1\epsilon^2$. The result then follows from Theorem 5.1.1 and the imbedding

$$H^s(\mathbb{R}) \to L^\infty(\mathbb{R})\ ,\ s > \frac{1}{2}.$$

5.4 The case of a cubic linearity

We consider the KdV-like equation

$$\begin{cases} q_t + iq_{xx} + aq_{xxx} + i|q|^2 q + \tilde{\lambda}(|q|^2 q)_x + \tilde{\mu}|q|^2 q_x = 0 \\ q(0,x) = q_o(x), \end{cases} \tag{5.23}$$

$q \in \mathbb{C}$, and where a,$\tilde{\lambda}$ and $\tilde{\mu}$ are three fixed real parameters, $a \neq 0$.
This model arises in optical physics, when studying the propagation of localized waves along a fiber ([19]).
For the linearized equation at $q = 0$,

$$q_t + iq_{xx} + aq_{xxx} = 0,$$

we find the exact solutions

$$q(x,t) = q_o e^{i(w_o t + k_o x)}$$

with the dispersion equation

$$w_o(k_o) = k_o^2(1 + ak_o). \tag{5.24}$$

In what follows, we assume that $k_o \neq 0$.
Once again, we introduce the formal approximation

$$\begin{aligned} r_A(\epsilon,x,t) &= \epsilon A(X,T)e^{i(w_o t+k_o x)} + \epsilon^2 B(X,T)e^{i(w_o t+k_o x)} \\ &= q_A(\epsilon,x,t) + \epsilon^2 B(X,T)e^{i(w_o t+k_o x)} \end{aligned} \quad (5.25)$$

with $X = \epsilon(x - v_G t)$, $T = \epsilon^2 t$, and

$$-v_G = \frac{\partial w_o}{\partial k_o} = 2k_o + 3ak_o^2.$$

By inserting (5.25) in (5.23), we obtain the residual

$$\begin{aligned} Res(r_A)(\epsilon,x,t) &:= r_{At} + ir_{Axx} + ar_{Axxx} + i|r_A|^2 r_A + \tilde{\lambda}(|r_A|^2 r_A)_x + \tilde{\mu}|r_A|^2 r_{Ax} \\ &= \epsilon[iAe^{i\phi}(w_o - k_o^2 - ak_o^3)] + \epsilon^2[iBe^{i\phi}(w_o - k_o^2 - ak_o^3)] \\ &\quad +\epsilon^2[A_x e^{i\phi}(-v_G - 2k_o - 3ak_o^2)] + \epsilon^3[B_x e^{i\phi}(-v_G - 2k_o - 3ak_o^2)] \\ &\quad +\epsilon^3 e^{i\phi}[A_T + i(ak_o+1)A_{xx} + i(1 + k_o(\tilde{\lambda}+\tilde{\mu}))|A|^2 A] \\ &\quad +\epsilon^4 e^{i\phi}[B_T + i(ak_o+1)B_{xx} + aA_{xxx} + \tilde{\lambda}(|A|^2 A)_x + \tilde{\mu}|A|^2 A_x] \\ &\quad +\epsilon^4 e^{i\phi}[(ik_o(\tilde{\lambda}+\tilde{\mu})+1)(2|A|^2 B + \overline{B}A^2)] \\ &\quad +O(\epsilon^5). \end{aligned}$$

The amplitude A must then be a solution to the NLS equation

$$A_T + i(ak_o+1)A_{XX} + i(1 + k_o(\tilde{\lambda}+\tilde{\mu}))|A|^2 A = 0. \quad (5.26)$$

The techniques presented in the previous section allow one to derive an analogous theorem for the problem (5.23) :

Theorème 5.4.1 *Let $s > \frac{3}{2}$, $T_o > 0$ and $A \in C([O,T_o];H^{s+4})$ a solution to (5.26). Then there exists $\epsilon_o > 0$ such that for all $\epsilon \leq \epsilon_o$ and for all $q_o \in H^s(\mathbb{R})$ satisfying*

$$||q_o - q_A(.,0)||_s \leq D_1\epsilon^2,$$

we have the existence of a solution $q \in C([0;T],H^s(\mathbb{R}))$ to (5.23), with $q(.,0) = q_o$ and

$$Sup_{[0,\frac{T_o}{\epsilon^2}]}||q(.,0) - q_A(.,0)||_s \leq D_2\epsilon^{\frac{3}{2}} \quad (5.27)$$

where the D_j are strictly positive and depend exclusively on T_o and ϵ_o.

Note that for $ak_o + 1 = 0$, (5.26) degenerates in

$$A_T + i(1 + k_o(\tilde{\mu}+\tilde{\lambda}))|A|^2 A = 0. \quad (5.28)$$

However, one would like to approach (5.23) by a "good" dispersive model also in this case.
We then look for another envelope equation, valid for wave numbers k_o such that the expression $ak_o + 1$ is small or even null.

We expect to find small space-time modulations of the critical mode $e^{ik_o x}$.
In order to do so, we have to rescale the time variable and the amplitude of the modulated wave. We set, for a small parameter $\epsilon > 0$,

$$r_A(\epsilon, x, t) = \epsilon^{\frac{3}{2}} A(X,T)e^{ik_o x} + \epsilon^{\frac{5}{2}} B(X,T)e^{ik_o x} \tag{5.29}$$

with

$$X = \epsilon(x - v_c t) \ , \ T = \epsilon^3 t \ ,$$

where v_c is the critical energy speed

$$-v_c = \frac{\partial w_o}{\partial k_o}(-a^{-1}) = a^{-1}.$$

and

$$|ak_o + 1| \leq \epsilon^4.$$

We compute the new value for the residual $Res(r_A)$:

$$\begin{aligned} Res(r_A)(\epsilon, x, t) \quad = \quad & \epsilon^{\frac{3}{2}} e^{ik_o x}[-iAk_o^2(1 + ak_o)] \\ & +\epsilon^{\frac{5}{2}} e^{ik_o x}[-iBk_o^2(1 + ak_o) + A_X(-v_c - 2k_o - 3ak_o^2)] \\ & +\epsilon^{\frac{7}{2}} e^{ik_o x}[B_X(-v_c - 2k_o - 3ak_o^2) + i(ak_o + 1)A_{XX}] \\ & +\epsilon^{\frac{9}{2}} e^{ik_o x}[A_T + i(1 + k_o(\tilde{\lambda} + \tilde{\mu}))|A|^2 A + aA_{XXX} + i(ak_o + 1)B_{XX}] \\ & +\epsilon^{\frac{11}{2}} e^{ik_o x}[B_T + \tilde{\lambda}(|A|^2 A)_X + \tilde{\mu}|A|^2 A_X \\ & +i(1 + k_o(\tilde{\lambda} + \tilde{\mu}))(A^2\overline{B} + 2|A|^2 B)] \\ & +O(\epsilon^{\frac{13}{2}}) \end{aligned}$$

Since

$$|1 + ak_o| \leq \epsilon^4$$

and

$$| - v_c - 2k_o - 3ak_o^2| = |1 + ak_o||\frac{1}{a} - 3k_o|,$$

$$\begin{aligned} Res(r_A)(\epsilon, x, t) \quad = \quad & \epsilon^{\frac{9}{2}} e^{ik_o x}[A_T + aA_{XXX} + i(1 + k_o(\tilde{\lambda} + \tilde{\mu}))|A|^2 A] \\ & +\epsilon^{\frac{11}{2}} e^{ik_o x}[B_T + \tilde{\lambda}(|A|^2 A)_X + \tilde{\mu}|A|^2 A_X \\ & +i(1 + k_o(\tilde{\lambda} + \tilde{\mu}))(A^2\overline{B} + 2|A|^2 B)] \\ & +O(\epsilon^{\frac{13}{2}}). \end{aligned}$$

By choosing two convenient functions A et B , we easily get the following result, analogous to Lemma 5.2.1 :

Lemme 5.4.2 *There exists three strictly positive constants* $C_{1,2,3}$ *and three functions* f, g, h *such that :*

1) $Res(r_A) = \epsilon^6 g(\epsilon, x, t)$, $Sup_{[0,\frac{T_o}{\epsilon^3}]}||g(\epsilon,.,t)||_{s+1} \leq C_1$

2) $r_A = \epsilon f(\epsilon, x, t)$, $Sup_{[0,\frac{T_o}{\epsilon^3}]}||f(\epsilon,.,t)||_{s+2} \leq C_2$

3) $r_A = \epsilon^{\frac{3}{2}} h(\epsilon, x, t)$, $Sup_{[0,\frac{T_o}{\epsilon^3}]}||h(\epsilon,.,t)||_{W^{s+1,\infty}} \leq C_3$

Then, if $q \in C([0,T_o]; H^s(\mathbb{R}))$ is a solution to (5.23), we derive an equation for the error $R = \epsilon^{-3}(q - r_A)$:

$$\begin{aligned} R_t + iR_{xx} + aR_{xxx} = & -\epsilon^6(i|R|^2R + \tilde{\lambda}(|R|^2R)_x + \tilde{\mu}|R|^2R_x) \\ & -\epsilon^4(iR^2\overline{f} + 2i|R|^2 f + \lambda(2|R|^2 f_x + R^2\overline{f}_x + 2fR\overline{R}_x \\ & +2R_x(R\overline{f} + \overline{R}f)) \\ & -\epsilon^4\tilde{\mu}(|R|^2 f_x + R_x(R\overline{f} + \overline{R}f)) \\ & -\epsilon^3(2iR|h|^2 + i\overline{R}h^2 + \tilde{\lambda}(R|h|^2_x + 2\overline{R}hh_x + |h|^2R_x + h^2\overline{R}_x)) \\ & -\epsilon^3\tilde{\mu}(Rh_x\overline{h} + \overline{R}hh_x + |h|^2R_x) \\ & -\epsilon^3 g(\epsilon, x, t) \end{aligned}$$

As before, we get the estimate

$$\frac{d}{dt}||R||_s^2 \leq \epsilon^3(C_1 + ||R||_s^2(C_2 + \epsilon||R||_s^2)) \tag{5.30}$$

for all $t \in [0, \frac{T_o}{\epsilon^3}]$.
Therefore, we can prove by the method presented before the following result:

Theorème 5.4.3 *Let $s > \frac{3}{2}$.*
Let $A \in C([0,T_o]; H^{s+3}(\mathbb{R}))$ be a solution to the equation

$$A_T + aA_{XXX} + i(1 + k_o(\lambda + \mu))|A|^2A = 0,$$

for some $T_o > 0$, and with $a \neq 0$ and $k_o \neq 0$.
Then, there exists $\epsilon_o > 0$ and strictly positive constants $D_j(\epsilon_o, T_o)$ such that for all $0 < \epsilon < \epsilon_o$, if
1) $q_o \in H^s(\mathbb{R})$ satisfies

$$||q_o - \epsilon^{\frac{3}{2}} A(\epsilon x, 0)e^{ik_o x}||_s \leq D_1\epsilon^3$$

and
2) $|ak_o + 1| \leq \epsilon^4$,
then there exists a solution $q \in C([0,T_o]; H^s(\mathbb{R}))$ to (5.23) such that $q(.,0) = q_o$ and

$$Sup_{[0,\frac{T_o}{\epsilon^3}]}||q(x,t) - \epsilon^{\frac{3}{2}} A(X, \epsilon^3 t)e^{ik_o x}||_s \leq D_2\epsilon^2,$$

where $X = \epsilon(x - v_c t)$ and $v_c = \frac{1}{a}$.

5.5 Numerical applications

In this section, we illustrate the Corollary 5.3.3, emphasizing the correspondance between NLS and DNLS. For this, we use a finite-differences scheme in order to solve numerically DNLS and to recover solitonic structures of NLS.

Finite differences methods are very often used for the numerical resolution of dispersive equations such as Schrödinger-like equations (see for instance [13], [50], [17]). It is well-known that accurate solutions can be obtained with use of the Crank-Nicholson scheme, with discrete conservation properties corresponding to the two formal invariance laws for the NLS equation

$$\left\{ \begin{array}{l} M(t) = ||u(t)||_{L^4}^4 = M(0) \\ E(t) = \frac{1}{2}||\nabla u(t)||_{L^2}^2 - \frac{1}{4}||u(t)||_{L^4}^4 = E(0). \end{array} \right.$$

Furthermore, the Crank-Nicholson scheme is known to be unconditionaly l^2-stable.
We then describe our numerical method. We introduce the standard notations for finite-difference schemes:
For an arbitrary space-time domain $[a;b] \times [0;t_{max}]$, and for $(J,N) \in \mathbb{N}^2$, we set

$$\Delta x = \frac{b-a}{J}\,;\; x_j = a + j\Delta x \text{ for } j = 0;...;J,$$

and

$$\Delta t = \frac{t_{max}}{N}\,;\; t_n = n\Delta t \text{ for } n = 0;...;N.$$

Also, the standard difference operators are:

$$(\delta q)_j = \frac{1}{\Delta x}(q_{j+1} - q_j)$$

and

$$(\delta^2 q)_j = \frac{1}{\Delta x^2}(q_{j+1} - 2q_j + q_{j-1}).$$

The scheme can then be written as:

$$\frac{1}{\Delta t}(q_j^{n+1} - q_j^n) + \frac{i}{2}\delta^2(q_j^n + q_j^{n+1}) + \frac{1}{2}\delta(|q_j^n|^2|q_j^n| + |q_j^{n+1}|^2|q_j^{n+1}|) = 0. \tag{5.31}$$

At the boundary of the space domain, we set the Dirichlet condition

$$q_0^n = q_J^n = 0.$$

Note that the scheme (5.31) is implicit. In order to implement it, we have chosen to use a fixed-point algorithm consisting in successive linear resolutions of a released problem. The scheme is supplemented with the initial values

$$q_j^0 = \epsilon A(\epsilon x_j, 0)e^{ikx_j},$$

where

$$A(X,T) = e^{-i\lambda t} R(X) \ , \ R(X) = \sqrt{\frac{2\lambda}{k}} \frac{1}{\cosh(\sqrt(\lambda) X)}$$

are the well-known ground states of the NLS equation.

In order to measure the numerical difference between q and q_A, we introduce the quantity

$$\Delta_\epsilon = ||q_o||_{l^\infty}^{-1} ||q(t_{max}) - q_A(t_{max})||_{l^\infty}.$$

We have made several experiments with the following parameters: $T_o = 10^{-5}$, $k = 5$ and $\lambda = 10^4$, and for different values of $\epsilon = 0.1$, 0.025 , 0.01 , 0.075. In the next figures, we have displayed the corresponding plots for the quantities $|q_o|$, $|q(t_{max})|$, $|q_A(t_{max})|$, and $|q(t_{max}) - q_A(t_{max})|$.

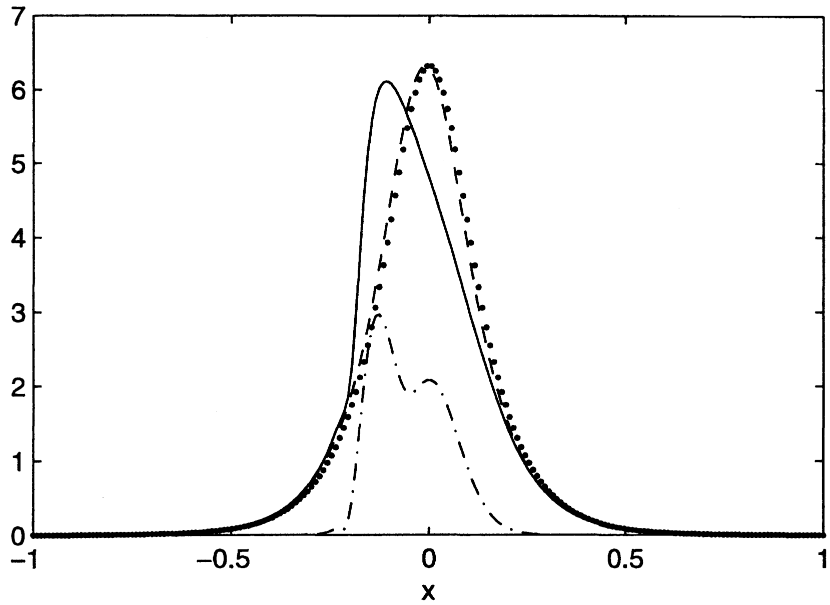

Fig. 1: $|q_o|$ (dotted ..), $|q(t_{max})|$ (continuous line -), $|q_A(t_{max})|$ (dashed –) and $|q(t_{max}) - q_A(t_{max})|$(dashed-dotted .-) for: $\epsilon = 0.1$, $t_{max} = 0.01$, $\Delta x = 0.01$, $\Delta t = 1.10^{-5}$, $[a;b] = [-1;1]$, $\Delta_\epsilon = 0.46$.

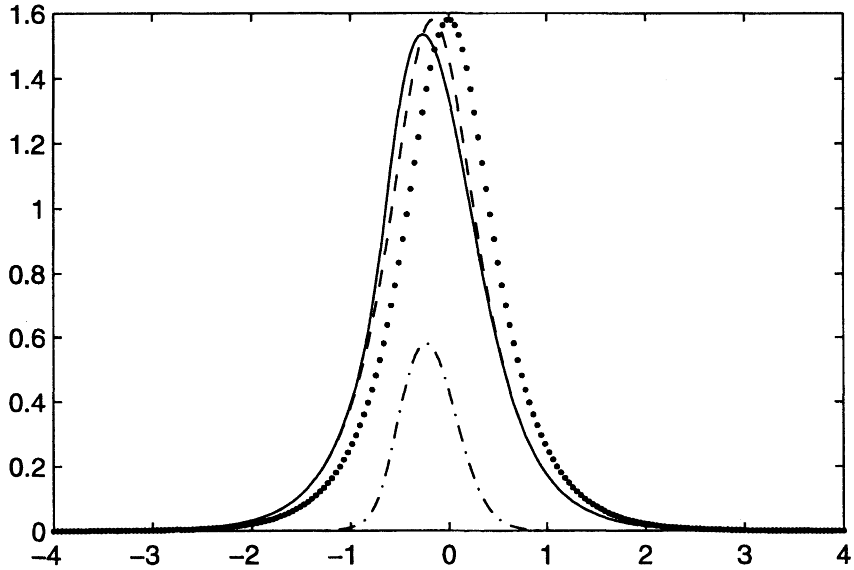

Fig. 2: $|q_o|$ (dotted ..), $|q(t_{max})|$ (continuous line -), $|q_A(t_{max})|$ (dashed –) and $|q(t_{max}) - q_A(t_{max})|$(dashed-dotted .-) for: $\epsilon = 0.025$, $t_{max} = 0.088$, $\Delta x = 0.04$, $\Delta t = 16.10^{-5}$, $[a;b] = [-4;4]$, $\Delta_\epsilon = 0.36$.

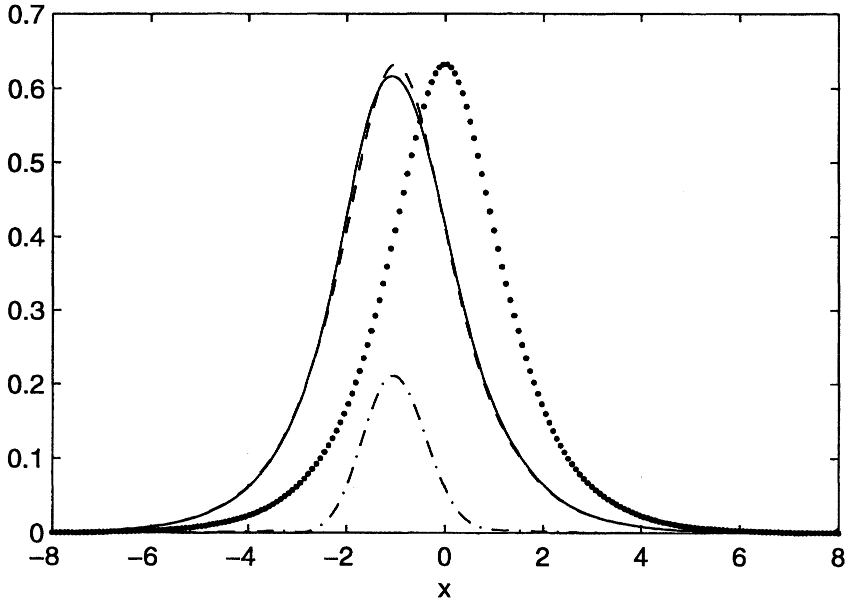

Fig. 3: $|q_o|$ (dotted ..), $|q(t_{max})|$ (continuous line -), $|q_A(t_{max})|$ (dashed –) and $|q(t_{max}) - q_A(t_{max})|$(dashed-dotted .-) for: $\epsilon = 0.01$, $t_{max} = 0.1$, $\Delta x = 0.08$, $\Delta t = 10^{-3}$, $[a; b] = [-8; 8]$, $\Delta_\epsilon = 0.33$.

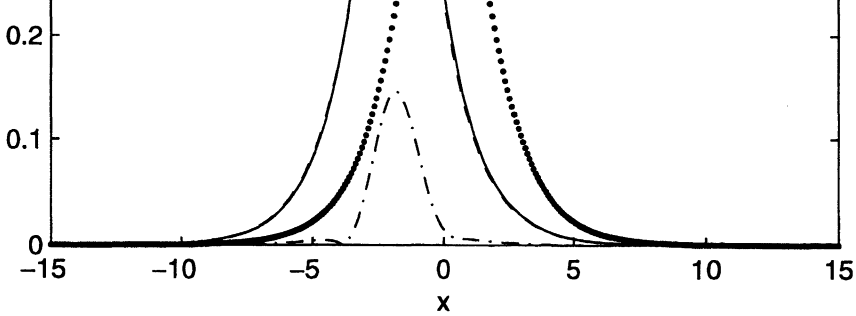

Fig. 4: $|q_o|$ (dotted ..), $|q(t_{max})|$ (continuous line -), $|q_A(t_{max})|$ (dashed –) and $|q(t_{max}) - q_A(t_{max})|$(dashed-dotted .-) for: $\epsilon = 0.0075$, $t_{max} = 0.18$, $\Delta x = 0.08$, $\Delta t = 10^{-3}$, $[a; b] = [-15; 15]$, $\Delta_\epsilon = 0.29$.

We see that as ϵ becomes small, q and q_A are in good agreement numerically, and that Δ_ϵ decreases with ϵ.
Note however that the error on the phase is considerably larger than the error on the amplitude. This is essencially due to the fact that the very fast oscillation of the phase ($\lambda = 10^4$) creates an error that our scheme is unable to cope with.
Moreover, the Corollary 3.3 suggests that $||q_A(.,t_{max}) - q(.,t_{max})||_\infty$ is of order ϵ^2.
In order to check this assumption numerically, we display on table I the value of $r_\epsilon = \epsilon^{-2}||q(.,t_{max}) - q_A(.,t_{max})||_{l^\infty}$ for different values of ϵ.

Table I:

ϵ	$[a;b]$	$(\Delta x, \Delta t)$	t_{max}	r_ϵ
0.01	$[-10;10]$	$(0.08, 1.10^{-3})$	0.10	2050
0.0095	$[-10;10]$	$(0.08, 1.10^{-3})$	0.11	2133
0.009	$[-10;10]$	$(0.08, 1.10^{-3})$	0.12	2224
0.0085	$[-13;13]$	$(0.08, 1.10^{-3})$	0.14	2316
0.008	$[-15;15]$	$(0.08, 1.10^{-3})$	0.16	2397
0.0075	$[-15;15]$	$(0.08, 1.10^{-3})$	0.18	2496
0.007	$[-16;16]$	$(0.08, 1.10^{-3})$	0.20	2544
0.0065	$[-16;16]$	$(0.08, 1.10^{-3})$	0.23	2640
0.006	$[-16;16]$	$(0.08, 1.10^{-3})$	0.28	2676
0.0055	$[-20;20]$	$(0.08, 1.10^{-3})$	0.33	2689
0.005	$[-20;20]$	$(0.08, 1.10^{-3})$	0.4	2700

Graphically:

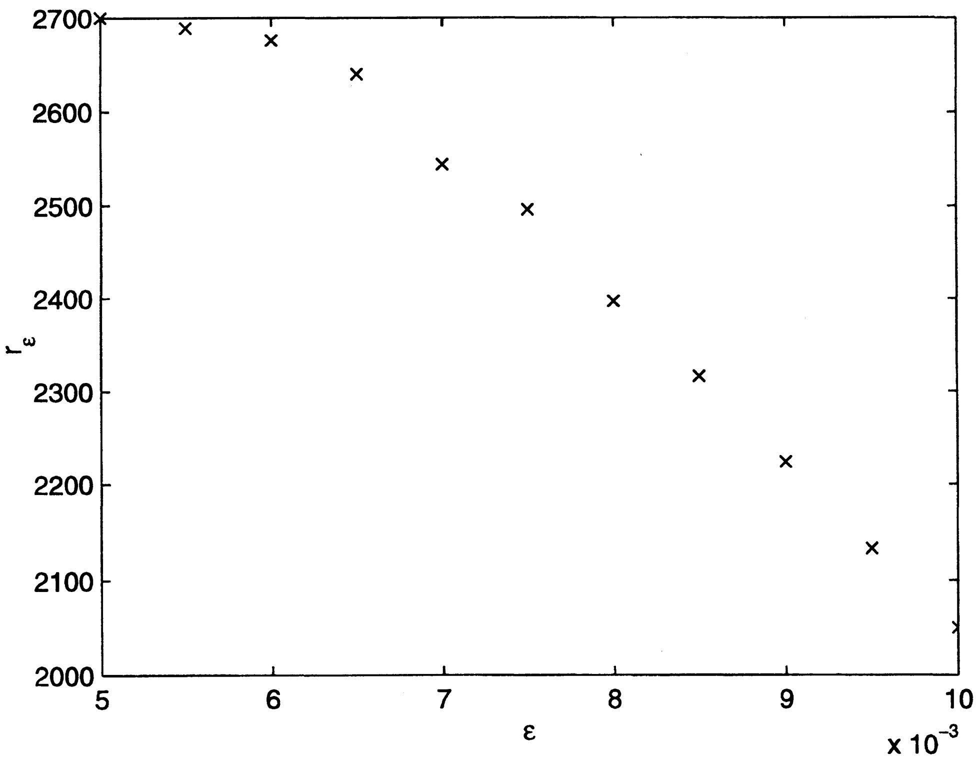

Fig.5 : $r_\epsilon = \epsilon^{-2}||q(.,t) - q_A(.,t)||_{l^\infty}$ for different values of ϵ.
This plot suggests that the estimate (5.22) in the Corollary 5.3.3 seems to be optimal, since r_ϵ stays bounded as $\epsilon \to 0$.

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