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TAUBERIAN THEOREMS FOR ABEL SUMMABILITY*

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I. INTRODUCTION.

First we consider Abel's summability method. Assume that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ and for each $|x| < 1$ define f by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Assume further that $\lim_{x\to 1^-} f(x)$ exists and let

$$\lim_{x\to 1^-} f(x) = a.$$

The number a is the A-sum of the series $\sum_{n=0}^{\infty} a_n$. We say in this case that $\sum_{n=0}^{\infty} a_n$ is A-sununable and we write:

$$(A)\sum_{n=0}^{\infty} a_n = a.$$

The following properties are obvious:

(i) If $(A)\sum_{n=0}^{\infty} a_n = a$ and $(A)\sum_{n=0}^{\infty} b_n = b$, then

$$(A)\sum_{n=0}^{\infty} (a_n + b_n) = a + b.$$

(ii) If $(A)\sum_{n=0}^{\infty} a_n = a$, then for any c we have

$$(A)\sum_{n=0}^{\infty} ca_n = ca.$$

(iii) If $(A)\sum_{n=0}^{\infty} a_n = a$, $(A)\sum_{n=0}^{\infty} b_n = b$ and $a_n \leq b_n$ for all n, then $a \leq b$.

Thus the A-sum of an A-summable series has the most important properties of the sum of a convergent series.

We show next that the A-sum of a series $\sum_{n=0}^{\infty} a_n$ coincides with the ordinary sum of the series $\sum_{n=0}^{\infty} a_n$ whenever this series is convergent. On the other hand it is easy to show that there exists series which are not convergent, but which are A-summable. We have for example:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{and so} \quad (A)\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}, \quad \text{while} \quad \sum_{n=0}^{\infty} (-1)^n$$

does not converge. From these results and the following theorem it follows that the concept of the A-sum of a series actually extends the concept of the usual sum in a consistent way.

THEOREM 1 (Abel's Theorem). If $\sum_{n=0}^{\infty} a_n = a$, then $(A)\sum_{n=0}^{\infty} a_n = a$.

This theorem has various interpretations. One of these interpretations is the following continuity theorem for power series near the circle of convergence:

If $\sum_{n=0}^{\infty} a_n$ converges, and if the function f is defined by:

$$f(x) = \begin{cases} \sum_{n=0}^{\infty} a_n x^n, & |x| < 1 \\ \sum_{n=0}^{\infty} a_n, & x = 1 \end{cases}$$

then f is left continuous at 1, i.e.,

$$\lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n = f(1)$$

Proof of THEOREM 1. If $\sum_{n=0}^{\infty} a_n$ is convergent, then obviously $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < 1$ and we have only to show that $\lim_{x\to 1^-} f(x) = a$.

Let $s_n \to a$. For $|x| < 1$ we have $\frac{f(x)}{1-x} = \frac{1}{1-x}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} s_n x^n$, and so $f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n$. Since $a = (1-x)\sum_{n=0}^{\infty} a x^n$ for $|x| < 1$, we have $f(x) - a = -(1-x)\sum_{n=0}^{\infty}(a - s_n)x^n$. Thus

$$|f(x) - a| \le (1-x)\sum_{n=0}^{m}|a - s_n||x|^n + \sup_{\nu>m}|a - s_\nu|(1-x)\sum_{n=m+1}^{\infty} x^n$$

$$\le (1-x)\sum_{n=0}^{m}|a - s_n| + \sup_{\nu>m}|a - s_\nu| \frac{(1-x)x^{m+1}}{1-|x|}.$$

Thus for all $0 < x < 1$ we have

$$|f(x) - a| \le (1-x)\sum_{n=0}^{m}|a - s_n| + \sup_{\nu>m}|a - s_\nu|.$$

It follows that $\lim_{x\to 1^-}|f(x) - a| \le \sup_{\nu>m}|a - s_\nu|$. Since $s_n \to a$ as $n \to \infty$, the result follows by choosing m large enough.

99 The concept of A-limit and A-convergence of sequences can be introduced similarly. The number s is the A-limit of the sequence $\{s_n\}$ if and only if:

$$(A)s_0 + \sum_{n=1}^{\infty}(s_n - s_{n-1}) = s.$$

We write in this case simply $(A)\lim_{n\to\infty} s_n = s$. Such a sequence is said to be A-convergent.

THEOREM 2. $(A)\lim_{n\to\infty} s_n = s$ if and only if

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} s_n x^n = s.$$

Proof. If $(A)\lim_{n\to\infty} s_n = s$, then letting $a_0 = s_0$ and $a_n = s_n - s_{n-1}$ we have $(A)\sum_{n=0}^{\infty} a_n = s$, i.e.,

$$\lim_{x\to 1^-}\sum_{n=0}^{\infty} a_n x^n = s.$$

But $\sum_{n=0}^{\infty} a_n x^n = s_0 + \sum_{n=1}^{\infty}(s_n - s_{n-1})x^n = (1-x)\sum_{n=0}^{\infty} s_n x^n$ and the conclusion is obvious.

conversely if

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} s_n x^n = s,$$

from

$$(1-x)\sum_{n=0}^{\infty} s_n x^n = s_0 + \sum_{n=1}^{\infty}(s_n - s_{n-1})x^n$$

we have

$$\lim_{x\to 1^-}[s_0 + \sum_{n=0}^{\infty}(s_n - s_{n-1})x^n] = s,$$

i.e.,

$$(A)s_0 + \sum_{n=1}^{\infty}(s_n - s_{n-1}) = s$$

and so

$$(A)\lim_{n\to\infty} s_n = s.$$

The analog of THEOREM 1 can be stated as follows:

THEOREM 3. If $\lim_{n\to\infty} s_n = s$, then $(A)\lim_{n\to\infty} s_n = s$.

Proof. If $\lim_{n\to\infty} s_n = s$, then $s_0 + \sum_{n=1}^{\infty}(s_n - s_{n-1}) = s$ and by THEOREM 1 100

$\lim_{x\to 1^-}[s_0 + \sum_{n=1}^{\infty}(s_n - s_{n-1})x^n] = s$. And so $(A)s_0 + \sum_{n=1}^{\infty}(s_n - s_{n-1}) = s$, i.e., $(A)\lim_{n\to\infty} s_n = s$.

II. SPECIAL TAUBERIAN THEOREMS.

As we have already pointed out, an A-sumable series is not necessarily convergent. This problem of determining which A-summable series are convergent is known as a Tauberian problem in the theory of A-summability. In a Tauberian theorem we conclude from the summability of a series $\sum_{n=0}^{\infty} a_n$ and an additional hypothesis about $\{a_n\}$, that $\sum_{n=0}^{\infty} a_n$ converges. Tauberian theorems gain their name from the following theorem published by A. Tauber [1] in 1897 in which he gave the simplest converse of Abel's theorem.

THEOREM 4 (Tauber's First Theorem). If $(A)\sum_{n=0}^{\infty} a_n = s$ and $a_n = o(1/n)$ as $n \to \infty$, then $\sum_{n=0}^{\infty} a_n = s$.

Proof. Given $\epsilon > 0$, choose n_0 such that for all $n \ge n_0$ we have:

(1) $|na_n| < \epsilon/3$

(2) $\frac{|a_1| + 2|a_2| + \cdots + n|a_n|}{n} < \frac{\epsilon}{3}$

(3) $|f(1 - \frac{1}{n}) - s| = |\sum_{k=0}^{\infty} a_k (1 - \frac{1}{n})^k - s| < \frac{\epsilon}{3}$

We have for $n \ge n_0$:

$$s_n - s = f(x) - s + \sum_{i=0}^{n} a_i(1 - x^i) - \sum_{i=n+1}^{\infty} a_i x^i.$$

Since $0 \le x \le 1$, we have

$$(1 - x^i) = (1 - x)(1 + x + \cdots + x^{i-1}) \le i(1 - x)$$

and

$$|a_i| = \frac{|ia_i|}{i} < \frac{\epsilon}{3n} \quad \text{for } i > n \ge n_0.$$

Thus

$$|s_n - s| \le |f(x) - s| + (1 - x)\sum_{i=1}^{n}|ia_i| + \frac{\epsilon}{3n}\cdot\sum_{i=n+1}^{\infty} x^i$$

$$\le |f(x) - s| + (1 - x)\sum_{i=1}^{n}|ia_i| + \frac{\epsilon}{3n(1 - x)}.$$

Choosing $x = 1 - \frac{1}{n}$ we get by (2) and (3) that

$$|s_n - s| \le |f(1 - \frac{1}{n}) - s| + \frac{1}{n}\sum_{i=1}^{n}|ia_i| + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

Thus

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n = s.$$

In the proof of this theorem we have used the fact if $c_n = na_n$, then 101
$\lim_{n\to\infty} c_n = 0$ implies

$$\lim_{n\to\infty}\frac{c_1 + c_2 + \cdots + c_n}{n} = 0.$$

Consequently the condition $na_n \to 0$ as $n \to \infty$ could be weakened by assuming only that

$$\frac{a_1 + 2a_2 + \cdots + na_n}{n} \to 0 \text{ as } n \to \infty.$$

This is done in the following theorem.

THEOREM 5 (**Tauber's Second Theorem**). If (A) $\sum_{n=0}^{\infty} a_n = s$ and $\sum_{i=0}^{n} i a_i = o(n)$ as $n \to \infty$, then $\sum_{n=0}^{\infty} a_n = s$.

Proof. Define $w_n = \sum_{i=1}^{n} i a_i$ for n = 1, 2, ..., and $w_0 = 0$. Then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \frac{w_n - w_{n-1}}{n} x^n = a_0 + \sum_{n=1}^{\infty} w_n\left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} w_n\left[\frac{x^n - x^{n+1}}{n+1} - \frac{x^n}{n(n+1)}\right],$$

since

$$\frac{x^n}{n} - \frac{x^{n+1}}{n+1} = \frac{(n+1)x^n - nx^{n+1}}{n(n+1)} = \frac{x^n - x^{n+1}}{n+1} - \frac{x^n}{n(n+1)}.$$

We have

(4) $$f(x) = a_0 + (1-x)\sum_{n=1}^{\infty} \frac{w_n x^n}{n+1} + \sum_{n=1}^{\infty} \frac{w_n x^n}{n(n+1)}.$$

By hypothesis $w_n = o(n)$ as $n \to \infty$. Thus $w_n/(n+1) = o(1)$ as $n \to \infty$ and by THEOREM 2 $(1-x)\sum_{n=1}^{\infty} \frac{w_n x^n}{n+1} = o(1)$ as $x \to 1^-$. Since $f(x) \to s$ as $x \to 1^-$ we obtain from (4)

$$\lim_{x\to1^-} \sum_{n=1}^{\infty} \frac{w_n x^n}{n(n+1)} = s - a_0.$$

since $\frac{w_n}{n(n+1)} = o(1/n)$, as $n \to \infty$, by Tauber's First Theorem we obtain $\sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} = s - a_0$, i.e.,

$$s - a_0 = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{w_n}{n(n+1)} = \lim_{N\to\infty}\left[\sum_{n=1}^{N} \frac{w_n - w_{n-1}}{n} - \frac{w_N}{N+1}\right]$$

$$= \sum_{n=1}^{\infty} a_n \quad \text{as } w_N = o(N), \text{ as } N \to \infty.$$

Therefore

$$\sum_{n=0}^{\infty} a_n = s.$$

In 1910 J. E. Littlewood [2] replaced the condition $a_n = o(1/n)$ by the more general $a_n = O(1/n)$. Littlewood's proof was complex. Other proofs remained complex in spite of the number of researches [3], [4], [5], [6] devoted to it. In 1931 Karamata [7] essentially simplified the proof of Littlewood's theorem by means of the following theorem.

THEOREM 6. If $a_n \geq 0$, for n = 0, 1, ..., and

(5) $$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x} \quad \text{as } x \to 1^-$$

then

$$s_n = \sum_{i=0}^{n} a_i \sim n \quad \text{as } n \to \infty.$$

Proof. By the Weierstrass approximation theorem if g is continuous on $[0,1]$ then for any $\epsilon > 0$, there exists a polynomial Q such that $\max_{0\le x\le 1} |g(x) - Q(x)| < \frac{1}{2}\epsilon$. For all $x \in [0,1]$ we have $p(x) = Q(x) - \frac{1}{2}\epsilon < g(x) < Q(x) + \frac{1}{2}\epsilon = P(x)$. Thus we have constructed polynomials p and P such that

(6) $p(x) < g(x) < P(x)$ for all $x \in [0,1]$, and

(7) $\int_0^1 [g(x) - p(x)]dx < \epsilon$ and $\int_0^1 [P(x) - g(x)]dx < \epsilon$.

Next suppose that g is continuous on $[0,1]$ except at $c \in (0,1)$ where $g(c^-) < g(c^+)$. We can still construct polynomials p and P satisfying (6) and (7) above. Let $\delta < \min\{c, c-1\}$ and define:

$$\Phi(x) = \begin{cases} g(x) + \frac{1}{4}\epsilon, & x < c - \delta \\ \max\{\mathcal{L}(x), g(x) + \frac{1}{8}\epsilon\}, & c - \delta \le x \le c \\ g(x) + \frac{1}{4}\epsilon, & x > c \end{cases}$$

$$\varphi(x) = \begin{cases} g(x) - \frac{1}{4}\epsilon, & x < c \\ \min\{\ell(x), g(x) - \frac{1}{8}\epsilon\}, & c \le x \le c + \delta \\ g(x) - \frac{1}{4}\epsilon, & x > c + \delta \end{cases}$$

where $\mathcal{L}$ and ℓ are linear functions such that:

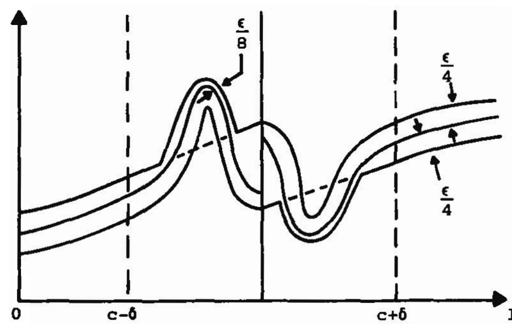
$$\mathcal{L}(c - \delta) = g(c - \delta) + \frac{1}{4}\epsilon, \quad \mathcal{L}(c) = g(c^+) + \frac{1}{4}\epsilon,$$

$$\ell(c) = g(c^-) - \frac{1}{4}\epsilon, \quad \text{and} \quad \ell(c + \delta) = g(c + \delta) - \frac{1}{4}\epsilon.$$

Clearly φ and Φ are continuous and $\varphi(x) < g(x) < \Phi(x)$. We can find polynomials r and R such that

$$|\Phi(x) - R(x)| < \frac{1}{4}\epsilon, \quad \text{and} \quad |\varphi(x) - r(x)| < \frac{1}{4}\epsilon.$$

Figure 1



We have

$$p(x) = r(x) - \frac{1}{4}\epsilon < \varphi(x) < g(x) < \Phi(x) < R(x) + \frac{1}{4}\epsilon = P(x)$$

and (6) is satisfied. Let N be the larger of the two values $\max_{0\le x\le 1}\{\Phi(x) - g(x)\}$, and $\max_{0\le x\le 1}\{g(x) - \varphi(x)\}$. We let $\delta = \epsilon/4N < \min\{c, c-1\}$.

Then as

$$P(x) - g(x) = \frac{1}{4}\epsilon + R(x) - \Phi(x) + \Phi(x) - g(x),$$

we have

$$\int_0^1 [P(x) - g(x)]dx < \frac{1}{2}\epsilon + \left[\int_0^{c-\delta} + \int_{c-\delta}^{c} + \int_c^1\right][\Phi(x) - g(x)]dx$$

$$< \frac{1}{2}\epsilon + \frac{1}{4}\epsilon + \frac{1}{4}\epsilon = \epsilon.$$

Similarly $g(x) - p(x) = \frac{1}{4}\epsilon + Q(x) - r(x) + g(x) - \varphi(x)$ and so

$$\int_0^1 [g(x) - p(x)]dx < \frac{1}{2}\epsilon + \left[\int_0^{c} + \int_{c}^{c+\delta} + \int_{c+\delta}^1\right][\Phi(x) - g(x)]dx$$

$$< \frac{1}{2}\epsilon + \frac{1}{4}\epsilon + \frac{1}{4}\epsilon = \epsilon,$$

and thus

$$\int_0^1 [g(x) - p(x)]dx < \epsilon.$$

Thus (6) and (7) have been satisfied.

Next we show that the hypothesis (5) implies that

(8) $$\lim_{x\to1^-} \sum_{n=0}^{\infty} a_n x^n P(x^n) = \int_0^1 P(t)dt$$

for any polynomial P. It is sufficient to consider the case $P(x) = x^k$.

Thus we have to show

$$\lim_{x\to1^-}\sum_{n=0}^{\infty}a_n x^{n+kn} = \frac{1}{k+1} = \int_0^1 x^k\,dx.$$

Let $\eta(x) = (1-x)\sum_{n=0}^{\infty}a_n x^n - 1$. Then $\eta(x)\to 0$ as $x\to1^-$. We have

$$\left|(1-x)\sum_{n=0}^{\infty}a_n x^{n+kn} - \frac{1}{k+1}\right|$$

$$= \left|\frac{1-x^{k+1}}{1+x+\cdots+x^k}\sum_{n=0}^{\infty}a_n x^{n+kn} - \frac{1}{k+1}\right|$$

$$= \left|\frac{\eta(x^{k+1})}{1+x+\cdots+x^k} + \frac{1}{1+x+\cdots+x^k} - \frac{1}{k+1}\right|$$

$$\le |\eta(x^{k+1})| + \left|\frac{1}{1+x+\cdots+x^k} - \frac{1}{k+1}\right| \to 0 \quad\text{as}\quad x\to1^-.$$

Next we show

$$\text{(9)}\qquad \lim_{x\to1^-}(1-x)\sum_{n=0}^{\infty}a_n x^n g(x^n) = \int_0^1 g(t)\,dt,$$

for any g which is continuous everywhere except at $c\in[0,1]$ where $g(c^-) < g(c^+)$. Let p and P be the polynomials having properties (6) and (7). Since $a_n \ge 0$ for $n = 0,1,\ldots,$ and $g(x) < P(x)$ for $x\in[0,1]$ we have

$$\lim_{x\to1^-}(1-x)\sum_{n=0}^{\infty}a_n x^n g(x^n) \le \overline{\lim_{x\to1^-}}(1-x)\sum_{n=0}^{\infty}a_n x^n P(x^n)$$

$$\le \int_0^1 P(t)\,dt < \int_0^1 g(t)\,dt + \epsilon.$$

Since ϵ can be chosen arbitrarily small we have

$$\overline{\lim_{x\to1^-}}(1-x)\sum_{n=0}^{\infty}a_n x^n g(x^n) \le \int_0^1 g(t)\,dt.$$

By a similar argument we obtain

$$\underline{\lim_{x\to1^-}}(1-x)\sum_{n=0}^{\infty}a_n x^n g(x^n) \ge \int_0^1 g(t)\,dt,$$

and (9) follows.

Finally, define g as follows:

$$g(t) = \begin{cases} 0, & t\in[0,e^{-1}) \\ 1/t, & t\in[e^{-1},1] \end{cases}.$$

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Then $\int_0^1 g(t)\,dt = 1$. By (9), given $\epsilon > 0$ we can choose δ such that $0 < \delta < x < 1$ implies

$$\left|(1-x)\sum_{n=0}^{\infty}a_n x^n g(x^n) - 1\right| < \epsilon.$$

Then for $n \ge \dfrac{1}{\log\frac{1}{\delta}}$, we have $\left|(1-e^{-1/n})\sum_{i=0}^{n}a_i - 1\right| < \epsilon$, i.e.,

$\lim_{n\to\infty}(1-e^{-1/n})s_n = 1$. Since $\dfrac{s_n}{n} = \dfrac{(1-e^{-1/n})s_n}{(1-e^{-1/n})n}$ and $\lim_{n\to\infty}\dfrac{1}{n(1-e^{-1/n})}$

$= 1$ we have $\lim_{n\to\infty}\dfrac{s_n}{n} = 1$, i.e., $s_n \sim n$, as $n\to\infty$.

Now we can give a simple proof of Littlewood's theorem.

THEOREM 7 (**Littlewood's** Theorem). If (A) $\sum_{n=0}^{\infty}a_n = s$ and $a_n = O(1/n)$ as $n\to\infty$, then

$$\sum a_n = s.$$

Proof. Since $a_n = O(1/n)$ we have $|na_n| \le c$ for $n = 1,2,\ldots,$ and so

$$|f''(x)| \le \sum_{n=0}^{\infty}n(n-1)|a_n|x^{n-2} \le c\sum_{n=0}^{\infty}(n-1)x^{n-2} = \frac{c}{(1-x)^2}.$$

First we show that $f'(x) = o(1/(1-x))$ as $x\to1^-$. Define $x' = \delta(1-x)+x$ for $0<\delta<\frac{1}{2}$. Then by Taylor's formula

$$f(x') = f(x) + \delta(1-x)f'(x) + \frac{1}{2}\delta^2(1-x)^2 f''(\zeta)$$

for $x < \zeta < x'$.

Thus

$$(1-x)f'(x) = \frac{f(x')-f(x)}{\delta} - \frac{1}{2}\delta(1-x)^2 f''(\zeta),$$

and so

$$|(1-x)f'(x)| \le \frac{|f(x')-f(x)|}{\delta} + \frac{1}{2}\delta c.$$

Choose $\delta < \epsilon/c$. By choosing x sufficiently near to 1 we obtain: $\dfrac{|f(x')-f(x)|}{\delta} < \dfrac{\epsilon}{2}$. Thus for x sufficiently close to 1, $|(1-x)f'(x)| \le \epsilon$, i.e., $f'(x) = o(1/(1-x))$ as $x\to1^-$.

Using this result we obtain

$$\sum_{n=1}^{\infty}\left[1-\frac{na_n}{c}\right]x^{n-1} = \frac{1}{1-x} - \frac{f'(x)}{c} \sim \frac{1}{1-x} \quad\text{as}\quad x\to1^-.$$

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Since $(1-na_n/c) \ge 0$ for every n, by THEOREM 6 we find that $\sum_{i=1}^{n}\frac{ia_i}{c} \sim n$, as $n\to\infty$, i.e., $\frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{ia_i}{c}\right] \to 1$ as $n\to\infty$. Therefore $\sum_{i=1}^{n}\frac{ia_i}{c} \to 0$ as $n\to\infty$, i.e., $\sum_{i=1}^{n}ia_i = o(n)$ as $n\to\infty$. Thus, the hypothesis of **Tauber's** Second Theorem is satisfied and consequently $\sum_{n=0}^{\infty}a_n = s$.

The two conditions $a_n = O\left(\frac{1}{n}\right)$ and $\sum_{i=0}^{n}ia_i = o(n)$ describe different sets of sequences. We illustrate this with the following examples. We have $(-1)^n/n = O(1/n)$ yet $\sum_{i=1}^{n}(-1)^i\frac{i}{i} \ne o(n)$. Also

$$\frac{(-1)^n\log n}{n} \ne O(1/n) \quad\text{yet}\quad \frac{\sum_{i=1}^{2n}(-1)^i\log i}{2n} = \frac{\sum_{i=1}^{n}\log\frac{2i}{2i-1}}{2n} = o(1)$$

as $\log\dfrac{2i-1+1}{2i-1} < \dfrac{2}{2i-1}$.

If we **replace** $\sum_{i=1}^{n}ia_i = o(n)$ or $a_n = O(1/n)$ by the more general $\sum_{i=1}^{n}ia_i = O(n)$, we find that $\sum_{n=0}^{\infty}a_n$ does not necessarily converge. For example, let $a_n = (-1)^n$. We have already seen that $\{(-1)^n\}$ is **A-summable** and that $\sum_{n=1}^{\infty}(-1)^n$ does not converge. But

$$\sum_{i=1}^{2n}i(-1)^i = \sum_{i=1}^{n}2i - \sum_{i=1}^{n}(2i-1) = n(n+1) - n^2 = n = O(2n).$$

However we can prove the following theorem.

THEOREM 8. If (A) $\sum_{n=0}^{\infty}a_n = s$ and $\sum_{i=1}^{n}ia_i = O(n)$, then $\dfrac{s_0+s_1+\cdots+s_n}{n+1} \to s$ as $n\to\infty$, where $s_n = \sum_{i=0}^{n}a_i$, for $n = 0,1,\ldots.$

<u>Proof</u>. Let $\sigma_n = \dfrac{1}{n+1}\sum_{i=0}^{n}s_i$ for $n = 0,1,\ldots.$ Then $f(x) = (1-x)\sum_{i=0}^{\infty}s_i x^i = (1-x)^2\sum_{j=0}^{\infty}(s_0+s_1+\cdots+s_j)x^j$ and thus

$$f(x) = (1-x)^2\sum_{i=0}^{\infty}(i+1)\sigma_i x^i \to s \quad\text{as}\quad x\to1^-.$$

If we divide by $(1-x)^2$ and integrate, we get

(10) $$\sum_{i=0}^{\infty} \sigma_i x^{i+1} = \int_0^x \frac{f(t)\,dt}{(1-t)^2} .$$

107 Since $f(x) \to s$ as $x \to 1^-$ we have for $0 < 1 - 6 \le t \le x < 1$

$$\int_{1-\delta}^{x} \frac{(s-\epsilon)}{(1-t)^2}\,dt \le \int_{1-\delta}^{x} \frac{f(t)}{(1-t)^2}\,dt \le \int_{1-\delta}^{x} \frac{(s+\epsilon)}{(1-t)^2}\,dt.$$

Thus

$$\int_0^x \frac{f(t)}{(1-t)^2}\,dt \ge \int_0^{1-\delta} \frac{f(t)}{(1-t)^2}\,dt + \frac{(s-\epsilon)}{(1-t)}\Bigg|_{1-\delta}^{x}$$

$$\le \int_0^{1-\delta} \frac{f(t)}{(1-t)^2}\,dt + \frac{(s+\epsilon)}{(1-t)}\Bigg|_{1-\delta}^{x}$$

Multiplying by $1 - x$ and subtracting s we get, since

$\int_0^{1-\delta} \frac{f(t)}{(1-t)^2}\,dt \le (1-\delta)\,\frac{f(1-\delta)}{\delta^2}$, that

$$\left|(1-x)\int_0^x \frac{f(t)}{(1-t)^2}\,dt - s\right| \le \epsilon + (1-x)\left[\frac{|s|+\epsilon}{\delta} + \frac{(1-\delta)|f(1-\delta)|}{\delta^2}\right],$$

i.e., $\left|(1-x)\int_0^x \frac{f(t)}{(1-t)^2}\,dt - s\right| \le \epsilon + (1-x)C \le 2\epsilon$ for $\delta_1 < x < 1,$

as f is bounded for such x. Thus

(11) $$\lim_{x\to 1^-} (1-x)\int_0^x \frac{f(t)}{(1-t)^2}\,dt = s.$$

From (10) and (11) it follows that

(12) $$(1-x)\sum_{i=0}^{\infty} \sigma_i x^i \to s \quad \text{as} \quad x \to 1^-.$$

From this and $(1-x)\sum_{i=0}^{\infty} s_i x^i \to s$ as $x \to 1^-$ follows

$$(1-x)\sum_{i=0}^{\infty} (s_i - \sigma_i)x^i \to 0 \quad \text{as} \quad x \to 1^-.$$

Since

$$s_i - \sigma_i = \sum_{k=0}^{n} \left[a_k - \frac{1}{n+1} s_k\right] = \frac{1}{n+1}\sum_{k=1}^{n} ka_k$$

and

$$-M \le \frac{1}{n+1}\sum_{k=1}^{n} ka_k \le M$$

we have $(s_i - \sigma_i + M) \ge 0$. Then from

108 $$(1-x)\sum_{i=0}^{\infty} (s_i - \sigma_i + M)x^i = (1-x)\sum_{i=0}^{\infty} (s_i - \sigma_i)x^i + M \to M$$

as $x \to 1^-$, and from THEOREM 6 we conclude

$$\frac{1}{n}\sum_{i=0}^{n} (s_i - \sigma_i + M) \to M \quad \text{as} \quad n \to \infty,$$

i.e., $$\frac{1}{n}\sum_{i=0}^{n} (s_i - \sigma_i) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $s_i - \sigma_i = (i+1)\sigma_i - i\sigma_{i-1} - \sigma_i = i(\sigma_i - \sigma_{i-1})$ we have $\sum_{i=0}^{n} i(\sigma_i - \sigma_{i-1}) = o(n)$ where $\sigma_{-1} = 0$. Finally from this condition and (12) we have by THEOREM 5 that

$$\lim_{n\to\infty} a = \sum_{i=0}^{\infty} (\sigma_i - \sigma_{i-1}) = s.$$

In 1913, Landau [8] weakened the **Tauberian** condition further, as **follows:**

THEOREM 9. If (A) $\sum_{n=0}^{\infty} a_n = s$ and

$$\omega(\delta) = \overline{\lim_{n\to\infty}} \max_{1\le k\le n\delta} \left|\sum_{n+1}^{n+k} a_i\right| \to 0 \quad \text{as} \quad \delta \to 0,$$

then

$$\sum_{n=0}^{\infty} a_n = s.$$

Proof. By hypothesis $\omega(\delta)$ must exist for some 6. If $6 \ge 2$, $\omega(2)$ exists since

$$\max_{1\le k\le 2n} \left|\sum_{n+1}^{n+k} a_i\right| \le \max_{1\le k\le \delta n} \left|\sum_{n+1}^{n+k} a_i\right| .$$

If $6 < 2$, let m be the greatest integer such that $(1+\delta)^m \le 3$. Then for $1 \le p \le 2n$ we have

$$\left|\sum_{n+1}^{n+p} a_i\right| \le \sum_{j=0}^{m-1} \left|\sum_{n(1+\delta)^j+1}^{n(1+\delta)^{j+1}} a_i\right| + \left|\sum_{n(1+\delta)^m+1}^{n+p} a_i\right|$$

$$\le \sum_{j=0}^{m} \max_{1\le k\le \delta n(1+\delta)^j} \left|\sum_{n(1+\delta)^j+1}^{n(1+\delta)^j+k} a_i\right|$$

and so

$$\omega(2) = \overline{\lim_{n\to\infty}} \max_{1\le p\le 2n} \left|\sum_{n+1}^{n+p} a_i\right| \le (m+1)(\omega(2)+\epsilon) < \infty.$$

Next we show 109

(14) $\sum_{i=1}^{n} ia_i = O(n)$ as $n \to \infty$.

Let $\rho(n,\delta) = \max_{1\le k\le \delta n} \left|\sum_{n+1}^{n+k} a_i\right|$, $n_1 = \left[\frac{n}{2^{\lambda+1}}\right] + 1$ and $n_2 = \left[\frac{n}{2^{\lambda}}\right]$.

Then

(15) $$\sum_{i=1}^{n} ia_i = \sum_{\lambda=0}^{\infty} \sum_{\left[\frac{n}{2^{\lambda+1}}\right] < i \le \left[\frac{n}{2^{\lambda}}\right]} ia_i = \sum_{\lambda=0}^{\infty} \sum_{n_1 \le i \le n_2} ia_i .$$

We have by partial summation

$$\sum_{n_1\le i\le n_2} ia_i = \sum_{n_1\le i\le n_2} i(s_i - s_{i-1})$$

$$= \sum_{n_1}^{n_2} is_i - \sum_{n_1-1}^{n_2-1} (i+1)s_i$$

$$= -\sum_{n_1}^{n_2-1} s_i + (n_2 - n_1)s_{n_2} + n_1(s_{n_2} - s_{n_1-1})$$

i.e.,

(16) $$\sum_{n_1\le i\le n_2} ia_i = \sum_{n_1}^{n_2-1} (s_{n_2} - s_i) + n_1(s_{n_2} - s_{n_1-1}).$$

Since

$$s_{n_2} - s_i = \sum_{j=n_1}^{n_1+(n_2-n_1)} a_j - \sum_{j=n_1}^{n_1+(i-n_1)} a_j ,$$

we have

$$|s_{n_2} - s_i| \le 2 \max_{1\le k\le n_2-(n_1-1)} \left|\sum_{j=n_1}^{n_1-1+k} a_j\right| .$$

Thus

$$|s_{n_2} - s_i| \le 2\rho\left(n_2, \frac{n_2-(n_1-1)}{n_1-1}\right), \quad \text{for} \quad n_1 - 1 \le i \le n_2 - 1,$$

and from (16) it follows that

$$\left|\sum_{n_1\le i\le n_2} ia_i\right| \le [2(n_2-n_1)+2n_1]\rho\left(n_2, \frac{n_2-(n_1-1)}{n_1-1}\right)$$

$$\le 2n_2\rho(n_2,2)$$

since

$$\frac{n_2-(n_1-1)}{n_1-1} \le \frac{n_2}{n_1-1} \le 2.$$

110 Using this inequality we get from (15)

$$\left|\sum_{i=1}^{n} ia_i\right| \le 2\sum_{\lambda=0}^{\infty}\left[\frac{n}{2^\lambda}\right]\rho\left(\left[\frac{n}{2^\lambda}\right],2\right) \le 2n\rho(n,2)\sum_{\lambda=0}^{\infty}\frac{1}{2^\lambda}$$

$$\le 4n\rho(n,2)$$

i.e.,

$$\frac{1}{n}\left|\sum_{i=1}^{n} ia_i\right| \le 4\rho(n,2) .$$

Consequently

$$\overline{\lim_{n\to\infty}}\ \frac{1}{n}\left|\sum_{i=1}^{n} ia_i\right| \le 4\ \overline{\lim_{n\to\infty}}\ \rho(n,2) \le 4\omega(2) < \infty.$$

Thus the relation (14) is proved.

By THEOREM 8, from (A) $\sum_{n=0}^{\infty} a = s$ and $\sum_{i=1}^{n} ia_i = O(n)$ we have

(17) $$\sigma_n \to s \quad \text{as} \quad n\to\infty.$$

It remains to be shown that this and (13) imply $\lim_{n\to\infty} s_n = s$.

From the identity

$$-\frac{(n+i+1)}{(i+1)}\sigma_{n+i} = \frac{-1}{i+1}\left[\frac{n}{n}\sum_{k=0}^{n-1} s_k + (1+i-i)s_n + \sum_{\lambda=1}^{i} s_{n+\lambda}\right]$$

we get

$$s_n - \sigma_{n+i} = \frac{n}{i+1}\left[\sigma_{n+i}-\sigma_{n-1}\right] - \frac{1}{i+1}\sum_{\lambda=1}^{i}(s_{n+\lambda}-s_n) .$$

Letting $i = [n\delta]$ we have

$$|s_n-\sigma_{n+i}| \le \frac{n}{n\delta}|\sigma_{n+i}-\sigma_{n-1}| + \frac{i}{i+1}\max_{1\le\lambda\le n\delta}|s_{n+\lambda}-s_n| .$$

Thus,

$$|s_n - s| \le |s_n-\sigma_{n+i}| + |\sigma_{n+i}-s| ,$$

$$\le \frac{1}{\delta}|\sigma_{n+i}-\sigma_{n-1}| + \max_{1\le\lambda\le n\delta}|s_{n+\lambda}-s_n| + |\sigma_{n+i}-s| .$$

Given $\epsilon > 0$. Since $\omega(\delta)\to 0$ as $\delta\to 0$, choose $\delta < 2$ such that $\omega(\delta) < \epsilon$. By (17) for any $\epsilon > 0$ and for any fixed $\delta > 0$, we can choose an N such that for $n \ge N$ we have

111 $$\frac{1}{\delta}|\sigma_{n+i}-\sigma_{n-1}| \le \frac{1}{\delta}|\sigma_{n+i}-s| + \frac{1}{\delta}|\sigma_{n-1}-s| \le \frac{\delta\epsilon}{2\delta}+\frac{\delta\epsilon}{2\delta} = \epsilon.$$

Thus, for $n \ge N$ we have

$$|s_n - s| \le \epsilon + \max_{1\le\lambda\le n\delta}|s_{n+\lambda}-s_n| + \frac{\delta\epsilon}{2} ,$$

and so

$$\overline{\lim_{n\to\infty}}\ |s_n-s| \le \epsilon+\omega(\delta)+\epsilon \le 3\epsilon.$$

Thus

$$\lim_{n\to\infty} s_n = s$$

i.e.,

$$\sum_{n=0}^{\infty} a_n = s.$$

In conclusion we note, that in these few pages we have proven four Tauberian theorems that took a number of mathematicians 34 years to prove. Only with hard work does mathematics progress.

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* This article is not original, but expository. Its results are known to those versed in series and summability, but the entire sequence of theorems presented can not be found in any one book.

FOOTNOTES

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THE GALOIS GROUP OF A NORMAL SUBFIELD

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Introduction. In the theory developed by Galois an interesting relationship exists between fields and their Galois groups. Every normal subgroup of the Galois group of a finite extension field K corresponds to a unique normal subfield L of the field K. It is this correspondence which I shall demonstrate in the following example. All the theory used in this paper can be found in [1], [2], [4], and [5].

Cyclotomic Field. The polynomial $f(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$ irreducible over the field Q of rational numbers, is called the cyclotomic polynomial of index 15, since the zeros of $f(x)$ are the primitive fifteenth roots of unity. If a primitive fifteenth root, $e(1/15)$, where $e(k/n)$ denotes $e^{2\pi ik/n}$, is adjoined to Q, a normal extension field $K_{15} = Q(e(1/15))$ is formed. The field K_{15} is called a cyclotomic field.

Subfields of K_{15}. It follows from the fact that a primitive n-th root of unity generates all the n-th roots of unity that the field K_{15} contains all the fifteenth roots of unity. The cube roots of unity, 1, $e(5/15)$, $e(10/15)$, and the fifth roots of unity, 1, $e(3/15)$, $e(6/15)$, $e(9/15)$, $e(12/15)$, are among the fifteenth roots.

The cube roots of unity are the zeros of the irreducible polynomial $g(x) = x^2 + x + 1$ and the primitive fifth roots are the zeros of $h(x) = x^4 + x^3 + x^2 + x + 1$. The extension fields $K_3 = Q(e(1/3))$ and $K_5 = Q(e(1/5))$ include all the zeros of these polynomials and are normal extension fields. Since these zeros belong to the field K_{15} the fields K_3 and K_5 are normal subfields of K_{15}.

~~The Galois~~ Group. Since K_{15} is a normal extension of Q, we consider the automorphisms of the field K_{15} which leave all elements of the field Q fixed. This set of automorphisms is called the Galois group of the field K_{15} over the field Q and is denoted by $G(K_{15},Q)$.

An arbitrary automorphism from the Galois group $G(K_{15},Q)$ carries every zero of the polynomial $f(x)$ into a zero of the same polynomial, that is, this Galois group carries a primitive fifteenth root of unity into another primitive fifteenth root of unity. There are eight of these roots so there are eight automorphisms belonging to the Galois group corresponding to the cyclotomic field K_{15}. These automorphisms may be denoted as follows:

I :	$e(1/15) \to e(1/15)$	A_8 :	$e(1/15) \to e(8/15)$
A_2 :	$e(1/15) \to e(2/15)$	A_{11} :	$e(1/15) \to e(11/15)$
A_4 :	$e(1/15) \to e(4/15)$	A_{13} :	$e(1/15) \to e(13/15)$
A_7 :	$e(1/15) \to e(7/15)$	A_{14} :	$e(1/15) \to e(14/15)$

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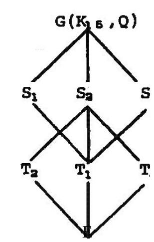
An examination of certain subgroups of the Galois group $G(K_{15},Q)$ will demonstrate the relation that exists between them and the subfields K_3 and K_5 of K_{15}. Let us look at the multiplication table of $G(K_{15},Q)$.

	I	A_2	A_4	A_7	A_8	A_{11}	A_{13}	A_{14}
I	I	A_2	A_4	A_7	A_8	A_{11}	A_{13}	A_{14}
A_2	A_2	A_4	A_8	A_{14}	I	A_7	A_{11}	A_{13}
A_4	A_4	A_8	I	A_{13}	A_2	A_{14}	A_7	A_{11}
A_7	A_7	A_{14}	A_{13}	A_4	A_{11}	A_2	I	A_8
A_8	A_8	I	A_2	A_{11}	A_4	A_{13}	A_{14}	A_7
A_{11}	A_{11}	A_7	A_{14}	A_2	A_{13}	I	A_8	A_4
A_{13}	A_{13}	A_{11}	A_7	I	A_{14}	A_8	A_4	A_2
A_{14}	A_{14}	A_{13}	A_{11}	A_8	A_7	A_4	A_2	I

This is a commutative group but not a cyclic group. Since every subgroup of a commutative group is commutative all subgroups of $G(K_{15},Q)$ are normal.

The Subgroups of $G(K_{15},Q)$. The subgroups of $G(K_{15},Q)$, besides the group itself, are:

$S_1 = \{I, A_4, A_7, A_{13}\}$ $\quad T_1 = \{I, A_{11}\}$
$S_2 = \{I, A_4, A_{11}, A_{14}\}$ $\quad T_2 = \{I, A_4\}$
$S_3 = \{I, A_2, A_4, A_8\}$ $\quad T_3 = \{I, A_{14}\}$
$E = \{I\}$.



The diagram at the right shows the inclusion relationship existing among these subgroups.

Now we shall make a study of the subfields corresponding to these subgroups. The elements of K_{15} which remain unchanged by the automorphisms of a subgroup S of $G(K_{15},Q)$ form a normal subfield of K_{15}. The degree of this field is equal to the index of S in $G(K_{15},Q)$.

~~A Basis for~~ K_{15}. A basis for the field K_{15} is formed by raising a primitive fifteenth root of unity to the 0, 1, ..., 7 powers. In general a basis is formed by raising a root of the splitting polynomial to the 0, 1, ..., n-1 powers where n is the degree of the polynomial.

The basis formed by using $e(1/15)$ is $\{1, e(1/15), e(2/15), e(3/15), e(4/15), e(5/15), e(6/15), e(7/15)\}$. Every element of K_{15} has a unique representation as a linear combination of this basis. A typical element

β of K_{15} has the form: $\beta = a_1 + a_2 e(1/15) + a_3 e(2/15) + a_4 e(3/15) + a_5 e(4/15) + a_6 e(5/15) + a_7 e(6/15) + a_8 e(7/15)$, where a_i belongs to Q.

Every automorphism of $G(K_{15},Q)$ carries β into another element of K_{15}. The following table shows what happens to the coefficients of the basis elements when the automorphisms of $G(K_{15},Q)$ operate on β.

Automorphisms	1	e(1/15)	e(2/15)	e(3/15)	e(4/15)	e(5/15)	e(6/15)	e(7/15)
I	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
A_2	$a_1 - a_5 - a_6 + a_8$	a_6	$a_2 - a_7 - a_8$	$-a_5 + a_8$	$a_3 + a_5 - a_8$	$-a_5 - a_6$	$a_4 + a_8$	$a_6 - a_7 - a_8$
A_4	$a_1 - a_3 - a_7 + a_8$	$a_3 + a_5 - a_8$	$-a_4 + a_7$	$-a_3 - a_7$	$a_2 + a_3 - a_8$	a_5 [illegible] a_8	$-a_7$	$a_3 - a_4 + a_7 - a_8$
A_7	$a_1 + a_3 + a_5$	$-a_6$	$-a_3 - a_7$	a_3	$-a_3 - a_6 + a_8$	a_5 [illegible]	$a_3 + a_4$	$a_2 - a_3 - a_6 - a_7$
A_8	$a_1 - a_2 - a_4 - a_6$	$a_2 + a_3 - a_8$	$a_6 + a_4$	$-a_2 - a_4 + a_7$	a_2	$-a_2$ [illegible]	$-a_4 - a_8$	$a_2 + a_4$
A_{11}	$a_1 + a_5 - a_6$	$-a_2$	$-a_5 + a_8$	$a_4 + a_5$	$-a_5$	$-a_6$	$-a_2 + a_5 + a_7$	$a_3 - a_5$
A_{13}	$a_1 + a_2 - a_4$	$-a_2 - a_3 + a_8$	a_4	$-a_4 + a_7$	a_2	$a_2 + a_6$	$-a_3 - a_4$	$-a_2 + a_4 + a_5$
A_{14}	$a_1 + a_2 + a_3 - a_6 - a_7 - a_8$	$-a_3 - a_5 + a_8$	$-a_2 - a_4 + a_7$	$a_2 - a_7 - a_8$	$-a_2 - a_3 + a_8$	$a_3 - a_6 - a_8$	$a_2 - a_5 - a_7$	$-a_2 - a_3 - a_4 + a_7 + a_8$

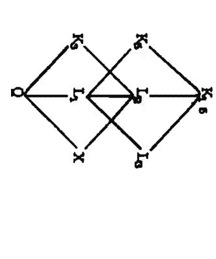
~~The~~ **~~Subfield~~** Corresponding **to** S_1. An examination of the above table shows that if the automorphisms of S_1 are to leave β unchanged the necessary condition is that $a_2 = a_3 = a_4 = a_5 = a_7 = a_8 = 0$. The cube roots of unity 1, **e(5/15)**, and **e(10/15)** are fifteenth roots of unity. By examining the elements of S_1 we see that under each of them $1 \to 1$, **e(5/15) → e(5/15)**, and **e(10/15) → e(10/15)**. The group S_1 leaves all the elements of K_3 fixed, that is, S_1 is the **Galois** group of $G(K_{15},K_3)$.

~~The~~ **~~Subfield~~** Corresponding **to** T_1. An examination of our table shows that if the automorphisms of T_1 are to leave β unchanged the necessary condition is that $a_2 = a_3 = a_5 = 0$ and $a_3 = a_8$. The automorphisms of T_1 leave elements of the form $a_1 + a_4 e(3/15) + a_7 e(6/15) - a_8 e(12/15)$ fixed. {**e(3/15)**, e(6/15), **e(12/15)**} is a basis for K_5; therefore T_1 leaves K_5 fixed and is the **Galois** group $G(K_{15},K_5)$.

~~The~~ **~~Subfield~~** Corresponding **to** S_2. An examination of the table shows that if the automorphisms of S_2 are to leave β unchanged the necessary condition is that $a_2 = a_5 = a_6 = a_7 = 0$ and $a_3 = -a_4 = a_8$. The sun of the primitive fifth root of unity **e(3/15)** and its inverse **e(12/15)** is equal to 2 cos $2\pi/5 = (-1+\sqrt{5})/2$. This sum remains unchanged when **e(3/15) → e(3/15)** and **e(12/15) → e(12/15)** and also when **e(3/15) → e(12/15)** and **e(12/15) → e(3/15)**. The extension field $L_1 = Q((-1+\sqrt{5})/2)$ is unchanged by S_2, that is; L_1 corresponds to S_2.

~~The~~ **~~Subfield~~ Corresponding to** T_2. For β to remain unchanged by the elements of T_2 it is necessary that $a_2 = a_3$, $a_4 = a_5 = -a_8$ and $a_7 = 0$. Thus elements of the form $a_1 + a_6 e(5/15) - a_4 (e(3/15) + e(12/15)) + a_2 (e(1/15) + e(4/15))$ remain fixed **under** T_2. The composite **subfield** $L_2 = Q(e(5/15), (-1+\sqrt{5})/2)$ of degree four corresponds to T_2.

The Subgroup Corresponding **to** T_3. The sun of **e(1/15)**, a primitive fifteenth root of unity, and its inverse **e(14/15)** equals 2 cos $2\pi/15$. The sum is unchanged by the elements of T_3, that is, $L_3 = Q(2 \cos 2\pi/15)$ corresponds to T_3. The degree of L_3 is four since the order of T_3 is two.



The **Subfield Corresponding to** S_3. The elements of K_{15} which remain unchanged under the automorphisms of S_3 form a field X. The author is making a further study of the nature of these elements.

Relationship *Of* Subfields **in** K_{15}. The inclusion relation among the subfields of K_{15} has the same structure as that of the subgroups of $G(K_{15},Q)$. The **relation**ship is, however, in the inverse order to that of the corresponding subgroups.

Summary.

Subgroups of $G(K_{15},Q)$	Terms left fixed	Basis of the corresponding Subfield	Subfield thus formed	Irreducible polynomial of Q thus solved
S_1	$a_1 + a_6 e(5/15)$	$\{1, e(5/15)\}$	K_3	x^2+x+1
S_2	$a_1 - a_3 (e(3/15) + e(12/15))$	$\{1, (-1+\sqrt{5})/2\}$	$L_1 = Q((-1+\sqrt{5})/2)$	x^2+x-1
S_3				
T_1	$a_1 + a_4 e(3/15) + a_7 e(6/15) - a_8 e(12/15)$	$\{1, e(3/15), e(6/15), e(12/15)\}$	K_5	$x^4+x^3+x^2+x+1$
T_2	$a_1 + a_6 e(5/15) - a_3 (e(3/15)+e(12/15)) + a_9 (e(1/15)+e(4/15))$	$\{1, e(5/15), (-1+\sqrt{5})/2, (1+\sqrt{5}+i(\sqrt{3}+\sqrt{15}))/4\}$	$L_2 = Q(e(5/15), (-1+\sqrt{5})/2)$	
T_3			$L_3 = Q(\cos 2\pi/15)$	

This paper was written while the author was a participant in a National Science Foundation Undergraduate Research Participation Program.

1. I. **Adamson,** Introduction to Field Theory, 1st ed., Interscience Publishers Inc., New York, 1964.

2. G. **Birkhoff,** S. **MacLane,** A Survey of Modern Algebra, revised edition, The **MacMillan** Co., New York, 1953.

3. W. Ledermann, Introduction to the Theory of Finite Groups, 3rd ed., R. and R. Clark, Ltd., Edinburg, England, 1957.

4. M. **Postnikov,** Fundamentals of Galois Theory, translated from 1st Russian ed. by L. Boron, P. **Noordhoff** Ltd., Groningen, The Netherlands, 1962.

5. B. van der Waerden, Modern Algebra, Vol. I, translated by F. **Blum,** Frederick **Ungar** Publishing Co., New York, 1953.

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TWO IDENTITIES INVOLVING POLYGONAL-NUMBER EXPONENTS

Myron S. Kaplan, Temple University

In this note we generalize two identities due to Euler involving square [1; 2771 and triangular [1; 2841 numbers, to polygonal numbers of arbitrary order. The development parallels that used by **Euler** [1].

In 1636, **Fermat** gave the general form of the u-th polygonal number of order $m+2$, denoted here by $P(u,m)$, as

(1) $$P(u,m) = \frac{1}{2} m(u^2 - u) + u.$$

We obtain immediately from (1) the following result:

LEMMA. For any integral value of m,

(2) $$P(n+1,m) - P(n,m) = 1 + mn, \quad n = 0,1,2,\ldots.$$

1. GENERALIZATION OF THEOREM 345 [1; 277]. Let the differences in (2) be used as the powers of x in forming the infinite product

$$Q_m(x) = (1 + x)(1 + x^{1+m})(1 + x^{1+2m}) \cdots.$$

We now use **Euler's** device of introducing a second parameter a.

Let

$$K(a) = K(a,x) = (1 + ax)(1 + ax^{1+m})(1 + ax^{1+2m}) \cdots$$
$$= 1 + c_1 a + c_2 a^2 + \cdots,$$

where $c_2 = c_2(x)$ is independent of a. Clearly

$$K(a) = (1 + ax)K(ax^m),$$

or

$$1 + c_1 a + c_2 a^2 + \ldots = (1 + ax)(1 + c_1 ax^m + c_2 a^2 x^{2m} + \cdots).$$

Hence, equating coefficients of a, we obtain

$$c_1 = x + c_1 x^m, \quad c_2 = c_1 x^{m+1} + c_2 x^{2m}, \ \cdots,$$
$$c_s = c_{s-1} x^{(s-1)m+1} + c_s x^{sm}, \ \cdots,$$

so that

$$(3) \qquad c_s = \frac{x^{(s-1)m+1}}{1-x^{sm}}\, c_{s-1}$$

$$= \frac{x^{1+(m+1)+(2m+1)+\cdots+[(s-1)m+1]}}{(1-x^m)(1-x^{2m})\cdots(1-x^{sm})}$$

But

$$1 + (m+1) + (2m+1) + \cdots + [(s-1)m+1] \stackrel{d}{=} P(s,m) .$$

Thus (3) may be written as

$$(3') \qquad c_s = \frac{x^{P(s,m)}}{(1-x^m)(1-x^{2m})\cdots(1-x^{sm})} .$$

It follows that

$$(4) \qquad (1+ax)(1+ax^{1+m})(1+ax^{1+2m})\cdots = 1 + \frac{ax}{1-x^m} + \frac{a^2x^{m+2}}{(1-x^m)(1-x^{2m})} + \frac{a^3x^{3m+3}}{(1-x^m)(1-x^{2m})(1-x^{3m})} + \cdots$$

For the special case of $a = 1$, (4) becomes

$$(5) \qquad (1+x)(1+x^{1+m})(1+x^{1+2m})\cdots = 1 + \frac{x}{1-x^m} + \frac{x^{m+2}}{(1-x^m)(1-x^{2m})} + \cdots$$

In another form, (5) is

$$(5') \qquad \prod_{j=0}^{\infty}(1+x^{1+jm}) = 1 + \sum_{s=1}^{\infty}\frac{x^{P(s,m)}}{\prod_{j=1}^{s}(1-x^{jm})} .$$

These infinite series and products are all absolutely convergent for $|x| < 1$. For the special case of $m = 2$, (5') becomes Euler's identity for self-conjugate partitions [1, 277, 279].

2. GENERALIZATION OF THEOREM 354 [1, 284]. An altered form of one of Jacobi's identities [1, 283] may be written as

$$(6) \qquad \prod_{n=0}^{\infty}[(1+x^{2kn+k-h})(1+x^{2kn+k+h})(1-x^{2kn+2k})] = \sum_{n=-\infty}^{\infty} x^{kn^2+hn} ,$$

with $0 < |x| < 1$. But (1) can be rearranged as

$$\frac{1}{2}mu^2 + \left(1 - \frac{1}{2}m\right)u.$$

Thus, by setting $k = \frac{1}{2}m$, $h = 1 - \frac{1}{2}m$, (and $n = u$) in (6), and noting from (2) that n is non-negative, we now have for the right-hand side of (6):

$$\sum_{n=0}^{\infty} x^{P(n,m)} ;$$

this is the infinite series with polygonal-number exponents, which is equivalent to an infinite product--the left-hand side of (6). For the case $m = 1$ ($k = h = 1/2$), we have Euler's identity [1, 284] involving the triangular numbers.

REFERENCE

1. G. H. Hardy and E. M. Wright, Theory of Numbers, 4th ed., reprinted (with corrections), Oxford University Press, 1962, pp. 273-281.

RESEARCH PROBLEMS

This section is devoted to suggestions of topics and problems for Undergraduate Research Programs. Address all correspondence to the Editor.

Proposed by LEO MOSER.

If two numbers are expressible as sum of two squares then their product is so expressible. If $n-1$, n, $n+1$ are each so expressible then so are n^2-1, n^2, n^2+1 since the first is $(n-1)(n+1)$, the second is n^2+0^2 and the third n^2+1^2. Since $8 = 2^2+2^2$, $9 = 3^2+0^2$ and $10 = 3^2+1^2$ it follows that there are infinitely many triples of consecutive numbers each expressible as sum of two squares. On the other hand, since no number leaving remainder of 3 on division by 4 is so expressible, no 4 consecutive numbers are so expressible. Perhaps one could prove, however, that apart from every 4th number, longer blocks are expressible. For examples is it true that there exist infinitely many blocks of 7 consecutive numbers, 6 of which can be represented as sum of two squares?

Proposed by PAUL C. ROSENBLOOM.

Differential actuations -- Differential Geometry.

A space curve is determined, to within a rigid motion, by the curvature and torsion as functions of arc length, $\kappa(s)$ and $\tau(s)$. The curve can be constructed by solving the Riccati equation

$$2\frac{d\theta}{ds} = -i\tau\left(1 + \frac{2\kappa\theta}{\tau} - \theta^2\right),$$

or an equivalent linear differential equation of the second order.

In principle, therefore, theorems on such differential equations can be interpreted in terms of the geometry of space curves. Investigate such interpretations and find the geometric implications of the theorems on differential equations.

References: Struik, Differential Geometry
Eisenhart, Differential Geometry
Coddington and Levinson, Theory of Ordinary Differential Equations

PROBLEM DEPARTMENT

Edited by
M. S. Klamkin, Ford Scientific Laboratory

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the **Master's** Degree. Solutions of these problems should be submitted on separate, signed sheets within **four months** after publication.

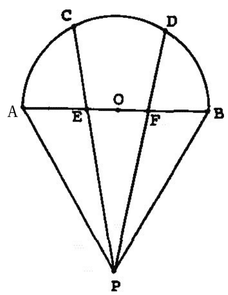
An asterisk (*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to Mr. M. S. Klamkin, Ford Scientific Laboratory, P. 0. Box 2053, Dearborn, Michigan 48121.

PROBLEMS FOR SOLUTION

172. Proposed by John **Baudhuin,** Sparta High School, Sparta, Wisconsin (student).

Given: Semi-circle 0 with diameter AB and equilateral triangle PAB; C and D are trisection points of $\widehat{AB}$ (i.e., $AC = \widehat{CD} = \widehat{DB}$).



Prove: E and **F** are trisection points **of** $\overline{AB}$.

Note: A synthetic proof is desired.

173. Proposed by **K. S.** Murray, New York City.

If
$$D^n\Phi(x)/x = \Psi_n(x)/x^{n+1},$$
show that
$$D\Psi_n(x) = x^n D^{n+1}\Phi(x).$$

174. Proposed by **C.** S. Venkataraman, Sree **Kerzala** Vanna College, Trichur, South India.

Find the locus of a point which moves such that the squares of the lengths of the tangents from **it** to three coplanar circles are in arithmetic progression.

175. Proposed by **R.** C. Gebhart, Parsippany, New Jersey.

The twenty-one dominoes of a set may be denoted by (1,1), (1,2), ..., (1,6), (2,6), ..., (6,6).

(a) Is there any arrangement of these, end-to-end with adjacent ends matching, such as ... (3,1)(1,1)(1,6)(6,4) ..., such that all twenty-one dominoes may be involved?

(b) What conditions must a general set of dominoes satisfy in order that such an arrangement in (a) exists?

Editorial Note: A related problem would be to find the largest and the smallest chain which can be formed with a given set of general dominoes.

176. Proposed by M. **S.** Klamkin, Ford Scientific Laboratory.

Determine all continuous functions $F(x)$ in $[0,1]$, if possible, such that $F(x^2) = F(x)^2$ and

(a) $F(0) = F(1) = 0$,

(b) $F(0) = F(1) = 1$,

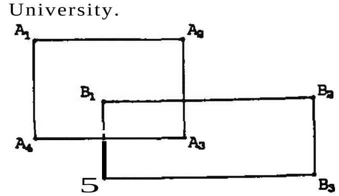
(c) $F(0) = 0$, $F(1) = 1$

(d) $F(0) = 1$, $F(0) = 0$.

SOLUTIONS

150. Proposed by **D. J. Newman,** Yeshiva University.

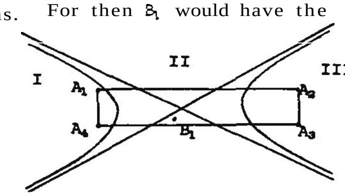
Given two overlapping parallel rectangles $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ and a quadratic polynomial $Q(x,y)$.



Show that Q cannot be > 0 at A_1, A_2, A_3, A^* and < 0 at B_1, B_2, B_3, B_4.

Solution by M. **S.** Klamkin, Ford Scientific Laboratory.

The result is also valid for any function $Q(x,y)$ whose graph divides the plane into only two regions. For then B_1 would have the same sign as A_1, A_2, A_3 and A_4. Consequently, we only have to consider the case when $Q(x,y) = 0$ is a hyperbola (or the various degenerate cases of straight lines).



A line through $B_1 \parallel A_1A_2$ must intersect both branches of the hyperbola exactly once each. Whence By lies in region **I** or **III** and has the same sign as A_1.

Also solved by **H.** Kaye, D. Smith and the proposer.

159.* Proposed by David L. Silverman, Beverly Hills, California.

If A_n denotes the largest integer divisible by all the integers less than its n^{th} root, show that $A_2 = 24$ and $A_3 = 420$. Find a general formula for A_n.

Editorial **Note:** A partial answer is given by Mathematics Review 1085, Aug., 1965:

'Ozeki, Nobuo

On the problem 1, 2, 3, ..., $[n^{1/k}] \mid n$.

J. College Arts Sci. Chiba Univ. 3 **(1961/62), 427-431.**

It is proved that 720720 is the largest integer which is divisible by all the positive integers that do not exceed the 5^{th} root of n. Similar results for k^{th} roots are proved in the cases $6 \le k \le 10$. The results for k = 2, 3, 4 are known."

160. Proposed by Sidney **Kravitz, Dover,** New Jersey.

I have here," said the editor, "a cryptarithm which shows a two digit number being multiplied by itself. You will note that the subproducts are not shown, only the number being squared and the final product."

"Well," said the reader, "I've tried to solve this cryptarithm but the solution is not unique. It is possible that I might be able to give you the answer if you told me whether the number being squared is odd or even."

"The number being squared is odd," said the editor.

"Good," said the reader. "I was hoping you would say that. I now know the answer."

What is the solution to this unique cryptarithm?

Solution by Charles W. Trigg, San **Diego,** California.

Consider all the following possible patterns with their solutions:

1. AS" = CDEA; 42^2 = 1764, 48^2 = 2304, 93^2 = 8649.
2. AB^2 = CDEF; AB = 53, 57, 59, 79, or 54, 72, 84.
3. AS" = BCDE; AB = 52 or 87.
4. AS" = **ACDB;** AB = 95 or 96.
5. AS" = **CDDB;** AB = 35, 65, 85, or 46.
6. AS" = **CDCB;** AB = 45, 81, 91, or 56.

125 7. AH" = CDEB; AB = 36, 86, or 51, 61, 71.

8. AH" = CCDE; AB = **34,** 58, or **47,** 67.
9. AH" = **CAAB;** AB = 76.
10. AB^2 = BCDB; AB = 41 **or 75.**
11. AB^2 = **CDBE;** AB = 32, 78, or 82.
12. AB^2 = CBDE; AB = **73 or 89.**
13. AB^2 = BCDA; AB = **64.**
14. AB^2 = CBAD; AB = **74.**
15. AH" = **BCAC;** **AB = 63.**
16. AB^2 = ACDE; AB = 98.
17. AB^2 = CADE; AB = **37** or **49.**
18. AB^2 = CDAE; AB = **43** or 69.
19. AB = CADC; AB = 68.
20. AB = CAAD; AB = 83.
21. **AB = ACDA;** **AB = 97.**
22. AB = CDEC; AB = 39.
23. **AB = CDED;** **AB = 92.**
24. **AA = CDEF;** **AB = 33 or 44.**
25. AA = CDEA; AB = 55 or **66.**
26. AA = ACDE; AB = 99.
27. AA = CDED; AB = 77.

The three digit possibilities are given by

AB^2: **CDB(16,31); ACD(13,14); CDE(17,29,18,24);**
CCB(15,21); CAB(25); BCB(26); BAC(27);
CBD(28); CAD(23); CDA(19).

AA^a: **ACA(11); CDC(22).**

From the question asked and the answer given it follows that the particular pattern must lead to only one odd value of AB and more than one even value. This corresponds to (1) and the value

$$93^2 = 8649.$$

Also solved by H. Kaye, **Paul** Meyers, K. S. Murray, M. Wagner, F. Zetto and the proposer.

162. Proposed by M. **S.** Klamkin, Ford Scientific Laboratory.

If a surface is one of revolution about two axes, show that it must be spherical.

Solution by Sidney Spital, California State Polytechnic College. 126

Denote the two axes of revolution by A and B and their intersection by 0. Consider a plane through 0 normal to A. Its intersection with the surface is a circle all of whose points are equidistant from 0. Revolve this circle about B. It sweeps out a spherical zone all of whose points are equidistant from 0. Now revolve this zone about A, thus increasing the width of the spherical zone. By continued rotations about alternating axes, the entire sphere will be covered.

Solution by the proposer.

1. Also solved similarly as above, but one has to first **prove** that the two axes intersect. Assuming the surface is bounded, it follows by symmetry that the centroid of the figure must lie on each axis and thus the axes must intersect. Also, the surface could be a spherical annulus.

2. Analytically the functional form of a surface of revolution about the axis

$$\frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n}$$

is given by

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = G(\ell x + my + nz) .$$

This is obtained by noting that the circular cross-sections of the surface to the axis can be gotten either by intersections of the surface with spheres

$$(x-a)' + (y-b)^2 + (z-c)^2 = r^2$$

centered on the **axis, or** by planes

$$\ell x + my + nz = p$$

which are ⊥ to the axis. Then there has to be some functional relationship between p and **r,** say $r^2 = G(p)$. Since the two axes intersect, we can choose a coordinate system whose origin is the point of intersection and such that the two axes (of revolution) are symmetric with respect to the z-axis and to the y-axis. Then the equation of the surface is given by both

$$x^2 + y^2 + z^2 = F(nz + \ell x) ,$$

$$x^2 + y^2 + z^2 = G(nz - \ell x) .$$

Choose x and **z** as independent variables (**y** will then be the dependent one). For all points **(x,y,z)** on the surface,

$$F(nz + \ell x) \equiv G(nz - \ell x) .$$

Since x and **z** are independent variables, so are $nz + \ell x$ and $nz - \ell x$. The only way a function of one independent variable can be equal to a function of another independent variable is for both functions to be constant. Whence,

$$x^2 + y^2 + z^2 = \text{constant},$$

which is a sphere. 127

Also solved by James Opelka (incompletely), M. Wagner and **F.** Zetto.

Editorial Note: The geometric solution suggests a new problem. Given two axes of revolution meeting at a given angle. Now starting with a given point of the figure, how many alternate rotations about the two axes successively does it take to generate the entire surface of the sphere? If the two axes are orthogonal, the number will be two if the point is on an axis (not the center) and three for any other point.

163. Proposed by Seymour Schuster, University of Minnesota.

Can any real polynomial be expressed as the difference of two real polynomials each of which having only positive roots?

Solution by the proposer.

Assume, without loss of generality, that the leading coefficient of the given polynomial $P(x)$ of degree n is unity. We can then write

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Can any real polynomial be expressed as the difference of two real polynomials each of which having only positive roots?

Solution by the proposer.

Assume, without loss of generality, that the leading coefficient of the given polynomial $P(x)$ of degree n is unity. We can then write

$$\frac{P(x)}{\prod_{i=1}^{n}(x-i)} = \left\{\lambda + \sum_{i=1}^{n_1}\frac{a_i}{x-k_i}\right\} - \left\{\lambda - 1 + \sum_{i=1}^{n_2}\frac{b_i}{x-\ell_i}\right\}$$

where $a_i, b_i > 0$ and $n_1 + n_2 = n$ (the k_i and ℓ_i are the positive integers $1,2,\ldots,n$). Let

$$f_1(x) = \lambda + \sum_{i=1}^{n_1}\frac{a_i}{x-k_i}, \qquad f_2(x) = \lambda - 1 + \sum_{i=1}^{n_2}\frac{b_i}{x-\ell_i}.$$

$f_1(x)$ has n_1 Positive poles and n_1 zeros. Now consider the graph of $f_1(x)$. By continuity, there must be a zero in between each pair of consecutive poles which accounts for $n_1 - 1$ of the zeros (which are positive). The n_1th zero is in the interval $(-\infty, \min k_i)$. Since $f_1(x)$ is negative just to the left of min k_i, this zero will be positive (by continuity) if $f_1(0) > 0$. This can be insured by taking λ sufficiently large. Similarly, $f_2(x)$ has n_2 positive zeros. Then

$$P(x) = f_1(x)\prod_{i=1}^{n}(x-i) - f_2(x)\prod_{i=1}^{n}(x-i)$$

gives an affirmative answer to the question.

Also solved by Robert J. Hursey, Jr. and K S. Murray.

164. Proposed by F. Zetto, Chicago.

Which numbers of the form 300...007 are divisible by 37?

Solution by Charles Ziegenfus, Madison College.

Let

$$N = \sum_{i=0}^{n} c_i 10^{3i},$$

where $0 \leq c_i \leq 999$. Since $10^{3n} \equiv 1 \pmod{37}$ for $n \geq 1$, we see that N is divisible by 37 if and only if $\sum_{i=0}^{n} c_i$ is divisible by 37. In the special case of 300...007, if the 3 occupies the $(3k + 2)$-th position, $k = 0,1,2,\ldots,$ then 300...007 is divisible by 37.

R. C. Gebhardt, Parsippany, New Jersey, and Robert L. Winkler, University of Chicago, in their solutions note that, equivalently, the number of zeros must be divisible by 3. Gebhardt also gives the following table:

$$\begin{aligned}
37 &= (37)(1)\\
30007 &= (37)(811)\\
30000007 &= (37)(810811)\\
30000000007 &= (37)(810810811)\\
30000000000007 &= (37)(810810810811)
\end{aligned}$$

Also solved by H. Kaye, P. Myers, D. Smith, M. Wagner and the proposer.

* * * * *

SEX IN THE MODERN MATHEMATICS CURRICULUM
-- a letter to Professor Paul C. Rosenbloom

The following is an unaltered letter received by Professor P. C. Rosenbloom from the Director of Instruction of School District No. 6, Greenfield, Wisconsin.

Dr. Paul Rosenbloom
Department of Mathematics
University of Minnesota
Minneapolis, Minnesota

Dear Dr. Rosenbloom:

During the summer Of 1964 you presented an address to the National Council of the Teachers of Mathematics, entitled "Science and the Math Curriculum." Dr. Emma Carroll heard this talk and suggested that we procure the tape and use it to help present to our teachers the ideas put forth. I asked one of our students to listen to the tape and present me with a typewritten copy. Her lack of familiarity with a mathematics vocabulary produced this inadvertent but humorous result.

"Let me start with the idea of sex, which has been considered a sort of hallmark for a simple modern mathematics curriculum. For purely mathematical purposes you need only the term sex, the idea of one to one correspondence, and the union of sex without common members in the teaching of addition of integers. For geometry the English word sex is certainly better than the Latin word, locus.

"In solving some equations in algebra, the solutions are in the intersection of the graphs of the equation in the system. You don't need the notation or the techniques of the algebra of sex. Any more extensive treatment of sex hangs in midair, since the student has nothing to do with this knowledge.

"The first place in school mathematics where you can do anything non-trivial with sex is in the theory of probability. Except for a chapter in the S.M.S.G. text and some enrichment material such as that of Glen and McCully and Professor Johnson, no one has written anything on probability for school use, below the 12th grade level.

"So long as you stick to pure mathematics curriculum, the critics of an over emphasis of sex in school mathematics are entirely justified. But, classification taxonomy is fundamental to the science curriculum. The whole system of Linnaeus for classifying plants and animals constructing sex, genus, the family, the species, which form a structure of sex and subsex.

"Some of our experimental teachers recommend that we throw out our discussion of the intersection of sex in kindergarten. We won't do this. First, our achievement tests, test those children who have mastered the concept very well. Second, the scientist used the concepts very extensively in their first grade unit on objects and their properties. Here the child classified objects with respect to several properties at the same time and the need to form the intersection of sex."

I hope, Dr. Rosenbloom, that you too share my view that this slight change produced some "interesting" results.

Yours truly,

SCHOOL DISTRICT NO. 6,
CITY OF GREENFIELD

Clyde G. Wallenfang,
Director of Instruction

BOOK REVIEWS

Edited by
Roy B. Deal, Oklahoma State University

The Elements of Real Analysis. By Robert G. Bartle. New York, John Wiley, 1964. xiv + 447 pp.

An elementary introduction to real analysis with precise definitions, rigorous proofs, biographical sketches, and a wide variety of levels of problems, some designed to give research orientation. Based on lectures to students from freshman to graduate level, often non-mathematics majors, and covers the topology, differentiation, and integration of finite-dimensional Cartesian spaces, as well as the Riemann-Stieltjes integral, infinite series, manifolds, differentials, line and surface integrals, and Green's and Stokes* Theorems.

Readings & Mathematical Psychology, Volume 11. Edited by R. D. Luce, R. R. Bush, and E. Galanter. New York, John Wiley, 1965. ix + 568 pp., $8.95.

With one exception, this volume consists of papers deemed by the authors and the editors of the Handbook of Mathematical Psychology to be especially relevant to approximately half the chapters of the Handbook. These articles partition naturally into six categories: computers, language, social interaction, sensory processes, preference and utility, and Bayesian statistics. Articles on measurement, psychophysics, reaction time, learning, and the stochastic processes that are relevant to the remaining chapters of the Handbook were included in Volume I of the Readings in Mathematical Psychology, which was published about a year earlier.

Basic Concepts of Geometry. By Walter Prenowitz and Meyer Jordan. New York, Blaisdell Publishing CO., 1965. xix + 350 pp., $7.50.

A modern treatment of the foundations of Euclidean and non-Euclidean Geometry with incidence properties for affine and projective geometry as well.

Analysis, Volume I. By Einar Hille. New York, Chelsea Publishing Co. vi + 234 pp.

A modern elementary Cours d' Analyse of functions of one variable with sufficient complex analysis to develop the theory of elementary transcendental functions, treating rigorously, with historical perspective and many examples and problems, the topics normally covered in an integrated course in calculus and analytical geometry.

~~Textbook of~~ Algebra, Volumes I, II. By G. Chrystal. New York, Chelsea Publishing Co., 1964. xxii + 584, xxiii + 628 pp., $4.70.

The young reader should perhaps know that these old classics contain a rich source of complicated classical results in classical algebra and elementary complex function theory which are often useful.

Multidimensional Gaussian Distributions. By Kenneth S. Miller. New York John Wiley, 1964. viii + 129 pp., $9.50.

A concise presentation of basic facts about multidimensional Gaussian distributions (or multivariate normal) for those with basic knowledge in linear algebra, probability theory, and advanced calculus, including some applications to Gaussian Noise.

Combinatorial Mathematics. By Herbert John Ryser. New York, Wiley, 1963. xiv + 154 pp., $4.00.

An introduction of the same fine caliber as the other Carus Monographs. Presupposes elementary modern algebra, particularly some matrix theory. Many counting arguments of an elementary, but difficult nature are used, which seems characteristic of the subject.

~~First Course in~~ Mathematical Logic. By Patrick Suppes and Shirley Hill.

An outgrowth of the famous experiments in teaching logic to selected elementary school students, which develops for utilization in the study of mathematics the sentential inference, inference with universal quantifiers, and applications, of the theory of inference developed, to the elementary theory of commutative groups. Existential quantifiers are not discussed in this volume.

~~Lectures on Modern~~ Mathematics, Volume I. Edited by T. L. Saaty. New York, John Wiley, 1963. vii + 175 pp., $5.75.

This volume contains the first six expositions in a series of 18 lectures given at George Washington University and sponsored jointly by the University and the office of Naval Research. These are excellent discussions by six of the very best mathematicians, for research mathematicians to learn what the current trends are in fields related to the specialty, but most under-graduates should not attempt to read this book.

Mathematical Discovery, Volume 11. By George Polya. New York, John Wiley, 1965. 220 pp., $5.50.

Professor Polya continues his illuminating heuristic discussions on the ways and means of discovery, and a 43 page chapter on "Learning, Teaching, and Learning Teaching."

~~First Course in~~ Functional Analysis. By Casper Goffman and George Pedrick. Englewood cliffs, New Jersey; Prentice-~all,1965. xi + 282 pp., $12.00.

A beginning graduate text which is done so thoroughly, however, that a good undergraduate student with some elementary general topology, modern algebra, and a modern advanced calculus course can gain an excellent introduction to modern analysis from it.

Distributions, An Outline. By Jean-Paul Marchand. Amsterdam, North-Holland Publishing Company, 1962. ix + 90 pp.

By confining himself to less general situations, the author is able to obtain the fundamental theorems of both Schwartz and Mikusinski for readers with an elementary knowledge of functional analysis.

Ordinary Differential Equations. By Philip Hartman. New York, John Wiley, 1964. xiv + 612 pp.

A comprehensive treatment of ordinary differential equations for those in mathematics, physics, and engineering with a knowledge of matrix theory and modern advanced calculus, with an impressive collection of classical and modern theorems and theories on the qualitative stability and asymptotic behavior of solutions.

Elements of Numerical Analysis. By Peter Henrici. New York, John Wiley, 1964. xv + 328 pp., $8.00.

Based on lecture notes for a course at U.C.L.A., and a summer institute for numerical analysis sponsored by the National Science Foundation. This book covers quite well the fundamental facets of numerical analysis with many modern algorithms, excluding linear algebra, eigenvalue problems, and machine language.

Philosophy of Mathematics. By Stephen F. Barker. Englewood Cliffs, New Jersey: Prentice-Hall, 1964. xiii + 111 pp., $1.50.

This little book in the Foundations of Philosophy Series discusses questions of truth, existence, and knowledge attained in mathematics, focusing on geometry and numbers from both literalistic and non-literalistic use with some interesting comments on axiomatized and formalized systems, the synthetic a priori, the logistic thesis, the paradoxes, constructivity, and Gödel's theorem.

Conformal Mapping. By L. Bieberbach. New York, Chelsea Publishing Co. vi + 234 pp., $1.50.

A translation of the last edition (fourth) of Bieberbach's well-known Einfuhrung in die Konfonne Abbildung, Berlin 1949, covers the fundamental facets of conformal mapping including a proof of Riemann's mapping theorem and many examples. For those with a bare introduction to the theory of complex variables, including use of the Cauchy integral theorem.

NOTE: All correspondence concerning reviews and all books for review should be sent to PROFESSOR ROY B. DEAL, DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA, 74075

BOOKS RECEIVED FOR REVIEW

L. J. Adams: Applied Calculus. New York, John Wiley, 1963. ix + 278 pp.

Shmuel Agmon: Lectures on Elliptic Boundary Value Problems. Princeton, Van Nostrand, 1965. iii + 291 pp., $3.95.

Nathan Altshiller-Court: Modern Pure Solid Geometry. New York, Chelsea Publishing Company, 1964. xiv + 353 pp.

Aaron Bakst: Mathematical Puzzles and Pastimes. Second Edition. Princeton, Van Nostrand Publishing Company, 1965. vii + 242 pp., $5.50.

Richard E. Barlow and Frank Proschan: Mathematical Theory of Reliability. New York, John Wiley, 1965. xiii + 256 pp.

Beckenbach, Drooyan, and Wooton: College Algebra. Belmont, California; Wadsworth Publishing Company, 1964. x + 438 pp.

W. G. Bickley and R. E. Gibson: Via Vector to Tensor. New York, John Wiley, 1962. xv + 152 pp.

Emile Borel: Elements of the Theory of Probability. Englewood Cliffs, New Jersey; Prentice-Hall, 1965. 179 pp., $5.75.

Edward L. Braun: Digital Computer Design. New York, Academic Press, 1963. xii + 606 pp., $16.50.

J. R. Britton, R. B. Kriegh, and L. W. Rutland: University Mathematics, Volume II. San Francisco, W. H. Freeman and Company, 1965. xii + 650 pp., $9.50.

Bryant, Graham, and Wiley: Nonroutine Problems in Algebra, Geometry and Trigonometry. New York, McGraw-Hill Book Company, Inc., 1965. 89 pp.

Robert R. Christian: A Brief Trigonometry. New York, Blaisdell, 1965. xii + 108 pp., $1.75.

Haskell Cohen and B. E. Mitchell: A New Look at Elementary Mathematics. Englewood Cliffs, New Jersey; 1965. x + 354 pp., $4.95.

Richard M. Cohna Difference Algebra. New York, Wiley Interscience, 1965. xiv + 355 pp., $12.95.

J. Cunningham: Complex Variable Methods in Science and Technology. Princeton, Van Nostrand, 1965. vii + 178 pp., $7.50.

Flora Dinkines: Abstract Mathematical Systems. New York, Appleton-Century-Crofts, Division of Meredith Publishing Company, 1961. viii + 97 pp.

Flora Dinkinesa Elementary Concepts of Modern Mathematics. New York, Appleton-Century-Crofts, Division of Meredith Publishing Company, 1961. x + 457 pp.

Flora Dinkines: Elementary Theory of Sets. New York, Appleton-Century-Crofts, Division of Meredith Publishing Company, 1961. viii + 237 pp.

Flora Dinkines: Introduction to Mathematical Logic. New York, Appleton-Century-Crofts, Division of Meredith Publishing Company, 1961. viii + 122 pp.

John R. Dixon: A Programmed Introduction to Probability. New York, John Wiley, 1964. xiv + 392 pp., $3.95.

Fobes and Smyth: Calculus and Analytic Geometry, Volume I. Englewood Cliffs, New Jersey; Prentice-Hall, 1963. xv + 660 pp.

Fobes and Smyth: Calculus and Analvtic Geometry, Volume 11. Englewood Cliffs, New Jersey; Prentice-Hall, 1963. xi + 450 pp.

J. Heading: An Introduction to Phase-Integral Methods. New York, John Wiley, 1962. 160 pp.

Francis B. Hildebrand: Methods of Applied Mathematics. Second Edition. Englewood Cliffs, New Jersey; Prentice-Hall, 1965. ix + 362 pp., $10.00.

Rufus Isaacs: Differential Games. New York, John Wiley, 1965. xxii + 384 pp., $15.00.

J. C. Jaeger: An Introduction to the Laplace Transformation. New York, John Wiley, 1962. viii + 156 pp.

N. L. Johnson and F. C. Leone: Statistics and Experimental Design: In Engineering and the Physical Sciences, Volume I. New York, John Wiley, 1964. ix + 523 pp., $10.95.

N. L. Johnson and F. C. Leone: statistics and Experimental Design: In Engineering and the Physical Sciences, Volume II. New York, John Wiley, 1964. ix + 399 pp., $11.50.

Edmund Landau: Differential and Integral Calculus. New York, Chelsea Publishing Company, 1965. 372 pp.

Serge Lang: A First Course in Calculus. Reading, Massachusetts; Addison-Wesley, 1964. xii + 258 pp., $6.75.

Howard Levi: Foundations of Geometry and Trigonometry. Englewood Cliffs, New Jersey; Prentice-Hall, 1960. xiv + 347 pp.

A. I. Lur'e: ——— Dimensional Problems of the Theory of Elasticity. New York, Wiley Interscience, 1964. xii + 493 pp.

A. I. Mal'cev: Foundations of Linear Algebra. San Francisco, W. H Freeman and Company, 1963. xi + 304 pp., $7.50.

A. I. Markushevich: Theory of Functions of a complex Variable, Volume I. Englewood Cliffs, New Jersey; Prentice-Hall, 1965. xiv + 459 pp., $12.00.

Leopoldo Nachbin: The Haar Integral. Princeton, Van Nostrand Company, 1965. xii + 156 pp., $2.50.

Louis L. Pennisi: Elements of Complex Variables. New York, Holt, Rinehart and Winston, 1963. x + 459 pp.

Edwin J. Purcell: Calculus with Analytic Geometry. New York, Appleton-Century-Crofts, Division of Meredith Publishing Company, 1965. xv + 843 pp.

L. B. Rall: Error in Digital computation, volume I. New York, John Wiley, 1965. ix + 324 pp.. $6.75.

J. B. Roberts: The Real Number System in an Algebraic Setting. San Francisco, W. H. Freeman and Company, 1962. ix + 145 pp., $1.75.

Evelyn B. Rosenthal: Understanding the New Math. New York, Hawthorn Books Inc., 1965. 240 pp., $4.95.

Shepley L. Ross: Differential Equations. New York, Blaisdell Publishing Company, 1964. xi + 594 pp., $10.00.

Frank M. Stewart; Introduction to Linear Algebra. Princeton, Van Nostrand Company, 1963. xv + 281 pp.

Hugh A. Thurston: Calculus for Students of Engineering and the Exact Sciences, Volume I. Englewood Cliffs; Prentice-Hall, 1962. ix + 193 pp.

Hugh A. Thurston: Calculus for Students of Engineering and the Exact Sciences, Volume II. Englewood Cliffs, New Jersey; Prentice-Hall, 1963. 208 pp.

Bevan K. Youse: The Number System. Belmont, California; Dickenson Publishing Company, 1965. vii + 72 pp.

C. H. Wilcox (Editor): Asymptotic Solutions of Differential Equations and their Applications. New York, John Wiley, 1964. x + 249 pp., $4.95.