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THE C.C. MACDUFFEE AWARD FOR DISTINGUISHED SERVICE

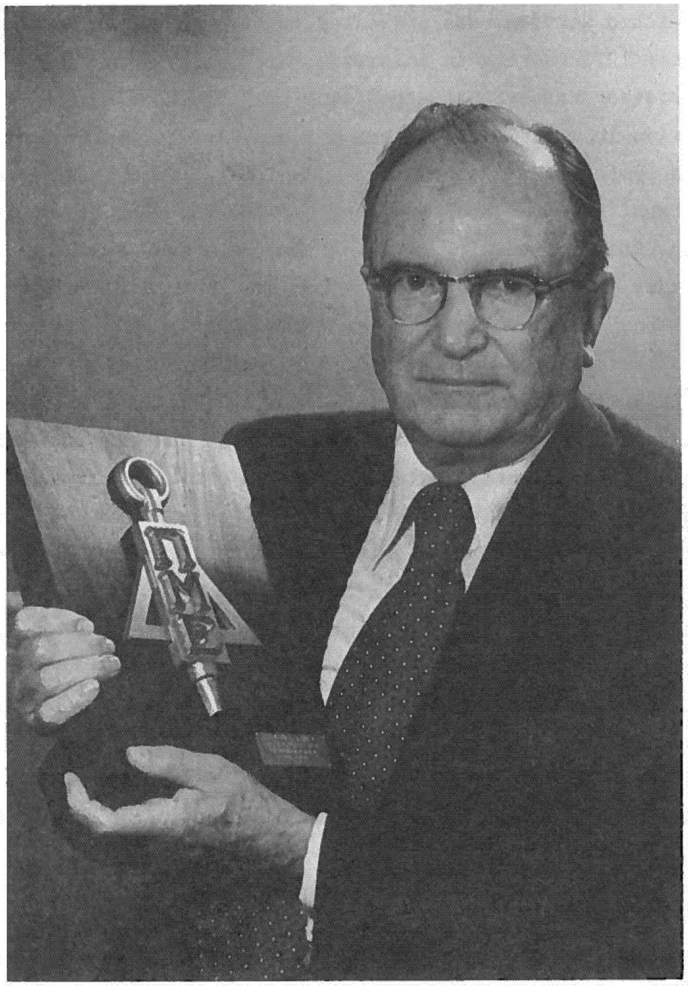
Pi Mu Epsilon's highest award, the C.C. MacDuffee Award for Distinguished Service, was presented to Dr. Houston T. Karnes at the summer 1975 meeting in Kalamazoo, Michigan.

Professor Karnes, past-president of Pi Mu Epsilon, has long been active in mathematical circles. His service included appointments as professor of Mathematics and Biology, department head, Dean of Men, public school teacher, before 1938 when he went to Louisiana State University to start the cycle over again as instructor, assistant professor, associate professor, full professor, Dean of Men, and Director of several NSF programs and institutes. He has also served as President of a Board of Trustees, Consultant and Lecturer both here in the USA and in Alahabad. As if all these projects were not enough for one soft-spoken, always courteous gentleman, he also participated in various research projects in the mathematics of genetics and in mathematics education as well as very ably promoting the interests of Pi Mu Epsilon, the national college mathematics honor society.

It is rare to find so much ability and diligence in one person. When that person is also a man of excellent taste and thoughtful consideration, we have a Houston T. Karnes. He is a man we all admire, respect and try to emulate.

It is with great pride that we add Dr. Karnes' name to those of the earlier recipients of the C.C. MacDuffee Distinguished Service Award.

1964, Dr. J. Sutherland Frame
1966, Dr. Richard V. Andree
1967, Dr. John S. Gold
1970, Dr. Francis Regan
1972, Dr. J.C. Eaves
1975, Dr. Houston T. Kames



Houston T. Karnes

MATRIX FUNCTIONS: A POWERFUL TOOL[1]

by J. S. Frame
Michigan State University

1. Introduction

Mathematical problems, both pure and applied, that involve linear relationships among several or even large numbers of variables can often be modeled most efficiently in terms of matrices. The solution of many of these problems is greatly simplified by the use of matrix functions. In this context the matrix functions considered are restricted to polynomials or convergent infinite series in an $n \times n$ matrix A, although we shall employ such scalar functions of A as its trace and determinant, which are the sum and product of certain numbers λ_j called its eigenvalues.

2. Atomic Transition Probabilities

An example of matrix modeling was related to me by Werner Heisenberg in a dinner conversation at the 1950 International Congress when I inquired what led him to describe certain aspects of atomic structure in terms of matrices. "Electrons cannot be directly observed when they remain in orbits assigned by the Bohr theory," he replied, "but only by energy changes displayed in spectral lines of certain specific frequencies when the electrons jump from one orbit to another and the atom changes its state." Observed in a spectrogram by the relative densities of spectral lines are the probabilities p_{ij} of transition from state j to state i in a certain unit time interval of exposure of the spectrogram. If the components $x_j(t)$ of a column vector $X(t)$ represent the population of atoms in state j out of the total observed population, and $P = [p_{ij}]$ is the transition probability matrix, then

$$X(t + 1) = PX(t) \tag{2.1}$$

After k such time units the state distribution vector should be given by

[1]This article is the text of the first lecture in the J. Sutherland Frame Lecture Series which was presented at the 1975 summer meeting of Pi *Mu Epsilon* at Western Michigan State University. The lecture series is named in honor of a past president and a most loyal supporter of *Pi Mu Epsilon*.

$$X(t + k) = P \cdot P \cdots PX(t) = P^k X(t) \ . \qquad (2.2)$$

So the question arises: "How does the matrix function P^k behave after a long time, when k becomes infinite?"

A two state example is illustrated by the formula

$$P^k = \begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix}^k = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix} + (1/6)^k \begin{bmatrix} 2/5 & -3/5 \\ -2/5 & 3/5 \end{bmatrix} \qquad (2.3)$$

Since $(1/6)^k$ becomes small for large k it is clear that

$$X_\infty = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .6(x_1 + x_2) \\ .4(x_1 + x_2) \end{bmatrix} \qquad (2.4)$$

We conclude that in the long run the relative population of the two states is about 0.6 and 0.4. In general, the limit of P^k as k becomes infinite is the product

$$S_1[1,1,\cdots,1] \qquad (2.5)$$

of a column vector S_1 whose entries represent the long term relative average population of the states and a row vector of 1's. Using matrix functions, we shall see why.

3. *Expansion of Matrix Functions*

It is relatively easy to compute powers and polynomial functions of an $n \times n$ diagonal matrix A

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & & \cdots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad (3.1)$$

whose diagonal entries are called its eigenvalues. If $f(\lambda)$ is any polynomial or infinite series which converges for each of the eigenvalues A_3 of A we have

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}, \quad f(\Lambda) = \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f(\lambda_n) \end{bmatrix}. \qquad (3.2)$$

In particular, the interpolating polynomial

$$q_j(\lambda) = \prod_{k \neq j} \frac{\lambda - \lambda_k}{\lambda_j - \lambda_k} \qquad (3.3)$$

which vanishes for all A except A_j and assumes the value 1 at A_j is such a polynomial of degree less than n and we have

$$E_j = q_j(\Lambda) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} \leftarrow \text{row } j \qquad (3.4)$$

where E_3 has all entries 0 except for 1's where A_j appears in A. The matrices E_3 are called constituent *idempotents* of A and satisfy the equations

$$E_j^2 = E_j \ , \quad E_i E_j = 0 \ \text{ if } i \neq j \qquad (3.5)$$

$$\sum_j E_j = I \quad \text{(Identity Matrix)}$$

The matrix $f(\Lambda)$ is a linear combination of the matrices E_j, with coefficients $f(A_3)$, namely

$$f(\Lambda) = \sum_j f(\lambda_j) E_j = \sum_j f(\lambda_j) q_j(\Lambda) \ , \qquad (3.6)$$

so $f(\Lambda)$ is a polynomial in A of degree less than n. This fundamental functional expansion formula for diagonal matrices can be extended to a much larger class of matrices, called *diagonable* matrices, which have the form

$$A = S\Lambda S^{-1} \qquad (3.7)$$

where S is an invertible matrix and A is a diagonal matrix.

Since

$$A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}, \quad \text{and} \quad A^k = S\Lambda^k 5 \qquad (3.8)$$

for any positive integral exponent k, it follows that for polynomial functions f

$$f(A) = Sf(\Lambda)S^{-1} = \Sigma f(\lambda_j) A_j \qquad (3.9)$$

where

$$A_j = SE_3 S^{-1} = Sq_j(\Lambda)S^{-1} = q_3(S\Lambda S^{-1}) = q_3(A) \ . \qquad (3.10)$$

The diagonal entries A_3 and $f(\lambda_3)$ of A and of $f(\Lambda)$ are called the *eigenvalues* of A *and* $f(A)$ respectively, and the idempotent matrices $A_j = q_j(A)$

$= SE_jS^{-1}$ are their common *constituent idempotent matrices*.

Theorem. Any polynomial or convergent series $f(A)$ in a diagonable matrix A is equal to a polynomial in A of degree less than n.

In particular, A^n is such a polynomial, so A must satisfy a polynomial equation of degree n.

Let us now reexamine the Heisenberg atomic state probability problem in the light of the expansion formula for $f(A)$. Taking P for A and λ^k for $f(\lambda)$ we have

$$P^k = \sum_j \lambda_j^k P_j \qquad \text{where} \qquad P_j = SE_jS^{-1} \tag{3.11}$$

for some modal matrix S. Since $P_jP_i = 0$ for $j \neq i$ and $RP = R$ for the row vector $R = [1,1,\cdots,1]$, it follows that

$$RP_i = RP^kP_i = \sum \lambda_j^k RP_jP_i = \lambda_i^k RP_i \qquad \text{for all k.} \tag{3.12}$$

Hence either $\lambda_i = 1$ or $RP_i = 0$. Since

$$R = RP = \sum_j RP_j \tag{3.13}$$

not all RP_i are 0. If $RP_1 \neq 0$, then $\lambda_1 = 1$ and

$$P^k = P_1 + \sum_{j>1} \lambda_j^k P_j \tag{3.14}$$

Since P^k has non-negative entries with column sums 1, the entries remain ≤ 1 as k becomes infinite, so no λ_j can exceed 1 in absolute value. Unless the states cannot be separated into two or more sets without transitions between members of two different sets, only one eigenvalue will be 1, and the rest of the λ_j^k will approach zero as k increases without limit. Thus

$$\lim P^k = P_1 = (SE_1)(E_1S^{-1}) = S_1R_1 \tag{3.15}$$

where R_1 is the first row of S^{-1}. Since $RP_1 = R$, the row R_1 is proportional to R and can be made equal to R by appropriate scaling of the column eigenvector S_1.

4. *Eigenvectors and the Characteristic Polynomial*

Given an $n \times n$ matrix A with real or complex numbers as entries, when and how is it possible to factor it so that

$$A = S\Lambda S^{-1} \qquad \text{or} \qquad AS = S\Lambda \tag{4.1}$$

for some invertible "modal" matrix S and some diagonal "spectral" matrix Λ? Equating columns in $AS = S\Lambda$ yields

$$AS_j = S_j\lambda_j \qquad \text{or} \qquad (\lambda_jI - A)S_j = 0 \tag{4.2}$$

These equations will have a non-zero solution S_j if and only if the determinant of the matrix $\lambda_jI - A$ is 0. Hence λ_j must be a root of the characteristic equation of A defined by

$$D(\lambda) = |\lambda I - A| = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \cdots + d_{n-1}\lambda + d_n = 0. \tag{4.3}$$

The sum and product of these roots, or eigenvalues, are

$$-d_1 = \Sigma\lambda_j = \Sigma a_{ii}$$

$$(-1)^n d_n = \lambda_1\lambda_2 \cdots \lambda_n = \det(A). \tag{4.4}$$

It is easily shown that similar matrices, such as A and Λ have the same characteristic polynomial $D(\lambda)$.

In the case A is the 2×2 probability matrix P mentioned above we have

$$D(\lambda) = \begin{bmatrix} \lambda - 2/3 & -1/2 \\ -1/3 & \lambda - 1/2 \end{bmatrix}$$

$$= \lambda^2 - (7/6)\lambda + 1/6 = (\lambda - 1)(\lambda - 1/6) \tag{4.5}$$

so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1/6$. Also,

$$q_1(\lambda) = (\lambda - \lambda_2)/(\lambda_1 - \lambda_2) = (6\lambda - 1)/5,$$

so

$$A_1 = (6A - I)/5 = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}; \tag{4.6}$$

$$q_2(\lambda) = (\lambda - \lambda_1)/(\lambda_2 - \lambda_1) = (6 - 6\lambda)/5,$$

so

$$A_2 = (I - A)6/5 = \begin{bmatrix} .4 & -.6 \\ -.4 & .6 \end{bmatrix}. \tag{4.6}$$

Note that the constituent matrices A_j have product 0 and sum I. Each is equal to its square, has trace 1 and determinant 0. Non-zero columns of each are eigenvectors S_j of A so a modal matrix S for A is

$$S = \begin{bmatrix} .6 & .4 \\ .4 & -.4 \end{bmatrix}. \tag{4.1}$$

5. *Longitudinal Vibrations of a Weighted Spring*

Let a spring of length $n + 1$ be stretched horizontally between points $n + 1$ units apart and let n equal masses m be attached to the spring at distances $1, 2, \cdots, n$ from one end when the spring is at rest. Consider the resulting motion if the masses are displaced horizontally, and the i'th component x_i of an n-vector X denotes the horizontal displacement from equilibrium of the i'th mass. Since the net force on the i'th mass is proportional to the difference between the stretchings $x_{i+1} - x_i$ and $x_i - x_{i-1}$ on the right and left, the equations of motion are

$$m \frac{d^2x_i}{dt^2} = mk^2(x_{i-1} - 2x_i + x_{i+1}),$$

$$x_0 = x_{n+1} = 0$$

where mk^2 denotes the spring constant. This equation may be written in matrix form as

$$\frac{d^2x}{dt^2} + B^2X = 0, \quad B^2 = k^2(2I - A) \tag{5.1}$$

where A is the tridiagonal matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \cdots\cdots & 0 \\ 1 & 0 & 1 \cdots\cdots & 0 \\ 0 & 1 & 0 \cdots\cdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\ \cdots\cdots & \cdots\cdots & 0 & 1 \\ \cdots\cdots & \cdots\cdots & 1 & 0 \end{bmatrix} \tag{5.2}$$

By analogy with the scalar equation $d^2x/dt + b^2x = 0$ this differential equation with initial conditions

$$X(0) = C, \qquad (dX/dt)_{t=0} = V \tag{5.3}$$

is satisfied by setting

$$X(t) = (\cos Bt)C + (\sin Bt/B)V \tag{5.4}$$

where the matrix functions $\cos Bt$ and $(\sin Bt/B)$ are given by convergent power series in the matrix $B^2 = k^2(2I - A)$. But these functions can be expressed in closed finite form as polynomials in A of degree less than n by using the functional expansion theorem.

In this case the matrix S with (i,j)-entry $\sin(ij\theta)$, where $\theta = \pi/(n+1)$ serves as a modal matrix for A, since the (i,j)-entry of AS is

$$\sin(i - 1)j\theta + \sin(i + 1)j\theta = \sin(ij\theta)\cdot 2\cos j\theta \tag{5.5}$$

and $\sin 0(j\theta) = \sin(n + 1)j\theta = 0$ for $i = 1$ and $i = n$. Hence $AS = S\Lambda$ where the eigenvalues λ_j of A are

$$\lambda_j = 2\cos j\theta, \quad \theta = \pi/(n + 1) . \tag{5.6}$$

The matrix S also has the convenient property that

$$S^2 = (n + 1)I/2, \qquad S^{-1} = (2/(n + 1))I \tag{5.7}$$

so the constituent matrices for A have the simple explicit form

$$A_j = SE_jS^{-1} = (2/(n + 1))S_jS_j^T \tag{5.8}$$

where the row vector S_j^T is the transpose of S_j. These A_j are also constituent matrices for $B^2 = k^2(2I - A)$, whose eignevalues are

$$\phi_j = k^2(2 - 2\cos j\theta) = 4k^2 \sin^2(j\pi/2(n + 1)) \tag{5.9}$$

Hence the solution to the vibrating spring problem may be expressed explicitly in the form

$$X = \sum_{j=1}^{n} (a_j \cos \phi_j + b_j \sin \phi_j) \tag{5.10}$$

where

$$a_j = 2S_j^TC/(n + 1), \quad b_j = 2S_j^TV/(n + 1)\phi_j,$$

$$\phi_j = 2k \sin(j\pi/2(n + 1)) \tag{5.11}$$

The columns S_j of the modal matrix S, with i'th entry $\sin(ij\pi/(n+1))$ describe the possible simple periodic modes of vibration of which the motion is composed, and the eigenvalues ϕ_j of B are proportional to the frequencies of vibration in these modes.

6. *Networks and the Exponential Function*

Many problems in electrical network theory are modeled by a system of first order linear differential equations of the form

$$\frac{dx}{dt} = AX(t) + BU(t), \quad X(0) = C \tag{6.1}$$

where X is an n-vector of states, U an m-vector of inputs or controls and A and B are constant matrices. Here a solution is expressible in terms of the exponential function of At and may be written

$$X = e^{At}C + \int_0^t e^{A(t - \tau)}BU(\tau)\ A \tag{6.2}$$

To evaluate this we compute the eigenvalues λ_j and constituent idempotents A_j of A and write

$$e^{A(t - \tau)} = \sum_j e^{\lambda_j(t - \tau)} A_j \tag{6.3}$$

whenever A is a diagonable matrix. This is discussed in the third of a series of articles on matrix functions [1]. The more complicated expansion valid for non-diagonable matrices is described in the fourth article of the series.

7. *Determinants of Binomial Circulant Matrices*

Several years ago I was asked to evaluate and factor the determinants D_n of certain binomial circulant matrices M_n such as

$$M_5 = \begin{bmatrix} 1 & 5 & 10 & 10 & 5 \\ 5 & 1 & 5 & 10 & 10 \\ 10 & 5 & 1 & 5 & 10 \\ 10 & 10 & 5 & 1 & 5 \\ 5 & 10 & 10 & 5 & 1 \end{bmatrix} \tag{7.1}$$

where for general n the (i,j)-entry of M_n is the binomial coefficient $\binom{n}{|i - j|}$. Although I could evaluate these by brute force for $n \le 8$, no general pattern was obvious to me from these numerical examples. Since D_{19} is about 4.3×10^{50}, it would be hard to factor even if it could be accurately computed in non-factored form.

Matrix functions provided the key to the discovery of many patterns and the actual factorization for $n \le 30$, including the discovery of the large prime factor 969,323,029 of D_{43} which is the integral part of $\left(\frac{1 + \sqrt{5}}{2}\right)^{43}$. We write M_n in the form

$$M_n = (I_n + P_n)^n - I_n \tag{7.2}$$

where P_n denotes the circulant permutation matrix of order n with eigenvalues r^k where r is a primitive n'th root of unity. For example

$$P_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad P_5^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{7.3}$$

Since D_n is the product of the eigenvalues of M_n which are $(1 + r^k)^n - 1$, D_n can be factored in the forms

$$\begin{aligned} D_n &= \prod_{k=1}^{n} [(1 + r^k)^n - 1] \\ &= \prod_{j=1}^{n} \prod_{k=1}^{n} [r^j + r^k - 1] \end{aligned} \tag{7.4}$$

Factors with $j = n$ or $k = n$ have product 1 and can be omitted. Those with $j = k$ have product $(-1)^{n-1}(2^n - 1)$, and the remaining factors are equal in pairs, so

$$D_n = (-1)^{n-1}(2^n - 1)F_n^2 \ . \tag{7.5}$$

Since $r^j + r^k - 1$ is 0 if and only if $r^k = r^{-j}$ is a sixth root of unity, $D_n = 0$ if and only if 6 divides n. The problem for even n is simplified by showing that

$$D_{2n} = -3D_n^3 \left(\frac{2^n + 1}{3}\right)^3 K_n^6 \tag{7.6}$$

where K_n is a rational integer.

When n is an odd prime $p > 3$, we can replace the index j by jk for fixed k and write F_n^2 in the form

$$F_p^2 = \prod_{j=2}^{p-1} q_j(p) \tag{7.7}$$

where $q_j(p)$ is a symmetric function of the r^k given by

$$q_j(p) = \prod_{k=1}^{p-1} (r^{jk} + r^k - 1)(r^{-jk} + r^{-k} - 1), \quad p' = (p - 1)/2. \tag{7.8}$$

In particular

$$q_{p-1}(p) = \prod_{k=1}^{p-1} (r^{-k} + r^k - 1) = 1 \tag{7.9}$$

and $q_{j'}(p) = q_j(p)$ if $jj' \equiv 1 (\mathrm{mod}\ p)$ so

$$F_p = \prod_{1<j<j'} q_j(p) \tag{7.10}$$

where $q_j(p)$ are positive integral factors of D_p each congruent to $1 \pmod p$.

We now use functions of a matrix again to evaluate in simple form the integer

$$q_2(p) = \prod_{k=1}^{p'} (r^{2k} + \text{[illegible]} - 1)(r^{-2k} + \text{[illegible]} 1) . \tag{7.11}$$

Multiplying the pairs of complex conjugate factors yields

$$q_2(p) = \prod_{k=1} (3 - r^{2k} - r^{-2k}) . \tag{7.12}$$

Now

$$r^{2k} + r^{-2k} = 2 \cos 2k\theta, \quad \theta = 2\pi/p \tag{7.13}$$

and the $p' \times p'$ matrices having eigenvalues $2 \cos 2k\theta$ and $3 - 2 \cos 2k\theta$ are $T_{p'}$ and $3I_{p'} - T_{p'}$, where

$$T_{p'} = \begin{bmatrix} 0 & 1 & & & \cdots \\ 1 & 0 & 1 & & \\ \cdot & \cdots & \cdots & \cdots & \cdot \\ & & 1 & 0 & 1 \\ \cdots & & & 1 & -1 \end{bmatrix}, \qquad 3I_{p'} - T_{p'} = \begin{bmatrix} 3 & -1 & & & \cdots \\ -1 & 3 & -1 & & \\ \cdot & \cdots & \cdots & \cdots & \cdot \\ & & -1 & 3 & -1 \\ \cdots & & & -1 & 4 \end{bmatrix} \tag{7.14}$$

Hence $q_2(p)$ is the determinant $d_{p'}$ of $3I_{p'} - T_{p'}$, which satisfies the recurrence relation

$$d_j = 3d_{j-1} - d_{j-2}, \qquad d_1 = 4, \qquad d_2 = 11 . \tag{7.15}$$

The same relations are satisfied by $f_{2j} + f_{2j+2}$, where f_k is the kth number in the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \cdots \tag{7.16}$$

and

$$f_k = (\tau^k - \tau^{-k})/\sqrt{5}, \qquad \tau = (1 + \sqrt{5})/2 . \tag{7.17}$$

Hence for $j = p' = (p - 1)/2$,

$$q_2(p) = d_{p'} = f_{p-1} + f_{p+1} = [\tau^p(\tau^{-1} + \tau) - \tau^{-p}(\tau + \tau^{-1})]/\sqrt{5}$$

$$q_2(p) = \tau^p - \tau^{-p}, \qquad \tau = (1 + \sqrt{5})/2. \tag{7.18}$$

Since τ^{-p} is less than 1, the factor $q_2(p)$ of D_p is the greatest integer in τ^p. In particular

$$D_5 = (2^5 - 1)q_2^2(5) = 31 \cdot 11^2 \tag{7.19}$$

and $q_2(41) = 370248451$, $q_2(43) = 969{,}323{,}029$ are large prime factors of D_{41} and D_{43} respectively. Other factors satisfy more complicated recurrence relations which are a challenge to find.

REFERENCE

1. Frame, J. S., Matrix Functions and Applications (a five part series) I.E.E.E. Spectrum, 1 (1964): No. 3, pp. 208–220; No. 4, pp. 102–108; No. 5, pp. 100–109; No. 6, pp. 123–131; No. 7, pp. 103–109.

REFEREES FOR THIS ISSUE

The *Journal* recognizes with appreciation the following persons who willingly devoted their time to evaluate papers submitted for publication prior to this issue, and to serve as judges in the Undergraduate Manuscript Contest: J. Sutherland Frame, Michigan State University; Leroy J. Dickey, University of Waterloo; Ezra Brown, Virginia Polytechnic Institute and State University; A. Duane Porter, University of Wyoming; David R. Anderson, Central Washington State University; David P. Roselle, Louisiana State University; Irving Reiner, University of Illinois, E. M. Beesley, E. F. Wishart, and R. C. Hooper, University of Nevada; Ward Bouwsma, Southern Illinois University; and members of the Mathematics Department at the University of Oklahoma, Bradford Crain, Pat Lang, Bernard McDonald, Robert St. John (Physics), Albert B. Schwarzkopf, and Kirby C. Smith.

We also appreciate the competent work of the typists, Detia Roe and Claudia Embry.

REFLECTIONS OF A PROBLEM EDITOR

by Leon *Bankoff*
Los Angeles, *California*

Introduction.

Telling is not teaching and listening is not learning. This terse truism summarizes the difficulties in communication so often encountered in mathematical education. Nevertheless properly directed telling and intelligently oriented listening are essential components of successful communication. The most effective way to measure the degree of such success is by appropriate testing of the student's problem solving ability.

Volumes can be and have been written on the importance of problem solving in the learning process and in the growth and development of mathematics. History is replete with instances where entire new branches of the art and science of mathematics have sprung up as a consequence of the search for the solution of some challenging problem. A noteworthy example is the successful attack on the brachistochrone problem by the Bernoulli brothers and the role played by this solution in the birth of the Calculus of Variations. Another familiar example is the emergence of the mathematical theory of probability as an offshoot of problems considered by Pacioli, Cardan and Tartaglia and the arousal of interest by the discussions between Pascal and Fermat. Even to this day, mathematicians continue to indulge in the age-old pleasurable activity of milking one another's brains through conversation or correspondence --- exchanging ideas --- collaborating on the solution of difficult and perplexing problems --- hurling and accepting challenges emanating either from their own gnawing inquisitiveness or from the frustrated curiosity of others --- building, forging, developing and inventing new tools and ingenious devices in the never-ending struggle for the establishment of mathematical order out of chaos.

In addition to the influence of private communication in the advancement of mathematical knowledge, it is important to recognize the tremendous impetus occasioned by the dissemination of provocative, non-routine problems by way of mathematical journals. For the last three centuries, readers of periodicals that contained problem sections have been invited to submit solutions to proposed problems with the objective of competing with other solvers for the publication of what the editors later judged to be the "best" solution. First came the reader's pride in his successful bout with the challenging problem; then came his natural desire to display the results of his cerebration; and finally his curiosity as to how his solution stacked up against those submitted by other solvers. It has always been the function of the editor to solicit and select proposals suitable for the particular vehicle concerned and to use his best judgment in choosing solutions for publication. This often becomes a soul-searing problem for the editor, as will be discussed later.

One of the earliest periodicals to feature a section on problems was the Ladies' *Diary*, which first appeared in London in 1704. In 1841 the Ladies' *Diary* and the Gentleman's *Diary*, which made its debut in 1741, were united and published under the title of The Lady's *and* Gentleman's *Diary*, which came to an end in 1871. For some unaccountable reason, the title of the Ladies' *Diary* was changed to the singular form when it combined with the Gentleman's Diary. The treatment of proposals and their solutions in these and in several other British publications of that era became a model for the *Mathematical* Questions from the Educa*tional* Times, which had its inception in 1863 and continued uninterruptedly until 1918. The spirit of the problem departments of the British journals was picked up by various French publications such as *L'Enseignement Mathématique and Mathesis* (Belgium) and also by the early American journals, notably the Mathematical' Visitor, which was launched at Erie, Pennsylvania in 1878.

In his introductory editorial to Volume I, Number 1 of the *Mathematical* Visitor, Artemas Martin, editor and publisher, had this to say:

> In England and Europe, periodical publications have contributed much to the diffusion of mathematical learning, and some of the greatest scientific characters of those countries commenced their mathematical career by solving the problems proposed in such works.
>
> It was stated nearly three-quarters of a century ago that the learned Dr. Hutton declared that the Ladies' *Diary* had produced more mathematicians in England than all the mathematical authors of that kingdom.

> Similar publications have produced like results in this country. Not a few of our ablest teachers and mathematicians were first inspired with a love of mathematical science by the problems and solutions published in the mathematical department of some unpretending periodical.

A world-renowned periodical that can certainly be considered "unpretending" despite its high level of sophistication is The *American* Mathematical *Monthly*, which was founded originally as a show case for proposed and solved problems. An exhaustive historical and statistical treatment of the problem departments of this journal from 1894 to 1954 appeared in the Otto *Dunkel Memorial* Problem Book, published by the *Mathematical* Association of America in August 1957 in commemoration of that Journal's fiftieth anniversary. The author of that survey, Mr. Charles W. Trigg, Dean Emeritus of Los Angeles City College, and one of the better known and most prolific problemists of our day, has put together a most informative, interesting and entertaining article well worth the attention of all mathematicians, whether active problemists or not.

One of the striking characteristics of most problem departments is the high incidence of participation by eminent mathematicians as well as by the "man on the street" lover of mental gymnastics. As one browses through the pages of the *Lady's* and *Gentleman's* Diary, the *Mathematical* Questions from *the* Educational Times or the American Mathematical Monthly, to name a few, one is impressed to discover what an attraction problems have held for so many who have achieved great prominence in mathematics. It comes as a surprise, for example, to learn that W. G. Horner, of Horner's Method fame, solved what is now known as the Butterfly Problem in the 1815 volume of the Gentleman's Diary. The list of problemists who participated in the problem department of the Educational Times reads like a veritable Who's Who in British Mathematics from 1863 to 1918. Among the active solvers may be found the names of Cayley, Cremona, Clifford, Sylvester, Whitworth, Todhunter, Hadamard, Hardy, Salmon, Beltrami and countless others far too numerous to list.

Currently the names of numerous prominent mathematicians may be found in the problem departments of the American Mathematical Monthly, the Mathematics Magazine, the SIAM Review, the Pi Mu Epsilon *Journal*, Pentagon, School Science and Mathematics, the *Journal* of *Recreational* Mathematics, the *Fibonacci* Quarterly, the *Technology* Review, the Two-Year College Mathematics *Journal*, *Elemente der Mathematik* (Switzerland), and the Mathematics Student *Journal*. It is hard to estimate how many high schools and two-year colleges publish "newsletters" primarily for their own students. Examples are the Indiana School Mathematics *Journal* and the Oklahoma University *Mathematics* Letter. Others are listed in a booklet issued by the National Council of Teachers of Mathematics, authored by William L. Schaaf and entitled "The High School Mathematics Library".

On a less formal basis, practically every issue of Martin Gardner's Mathematical Games Department in the Scientific *American* offers several intriguing problems for the entertainment and enlightenment of its readers, with solutions revealed in the following issue. Some of these problems have been known to generate heated controversy and discussion, all to the betterment of mathematical science.

In addition to its noteworthy expository articles, the *Mathematical* Gazette, while not containing a problem department, does nevertheless publish short provocative notes that frequently set off a chain-reaction of readership discussion and development. Furthermore, the Gazette maintains a Problem Bureau which offers assistance in the solution of problems whose sources are known. From those standpoints, the publication is a problemist's delight.

Of course, there are many specialized journals that do not maintain problem sections but most of the well-known ones do. It is hard to imagine the dismal change in character that would descend on a journal if its problem department were suddenly to be abandoned.

Problems of a problem Editor.

After the foregoing prelude, let us now come home to our own *Pi Mu Epsilon Journal* and dwell a bit on what goes on behind the scenes in the conduct of the Problem Department. Let us also consider what can be done to improve the department and to provide more enthusiasm and enjoyment among our problem devotees and the readers in general.

Problems in a great variety of categories have appeared in the *Pi Mu* Epsilon *Journal* since the time of its first appearance in April 1949. The Fraternity, which started at the University of Syracuse in 1903 as a mathematics club, achieved the status of a full-fledged chartered organization shortly after the academic year 1914-15, but it was not until 1949 that the Pi Mu *Epsilon Journal* blossomed forth. In the first issue Editor Ruth W. Stokes got the problem department off to a good start by publishing eleven proposals, five of which were her own and the other six solicited from accommodating friends. With the exception of the Fall 1957 issue, the problem section has appeared regularly in each issue and it has been only on rare occasions that the editor was faced with a shortage of suitable proposals to the point where he was compelled to raid his own files to maintain an acceptable balance and variety in the proposal department.

Considering the relatively small circulation of the *Pi Mu Epsilon Journal* compared to some of the larger periodicals, the ratio of participants in the problem department is rather high. However, it is quite likely that many of the readers solve the problems, file them away and never get around to submitting the solutions. Readers are urged to try their hand at problem composition and to offer their solutions for possible publication. One never knows when the presence of an unusual gimmick or a clever solution device might in itself warrant the publication of the solution.

This could be interpreted as a cry for help. The most difficult task for the problem editor is not the selection of solutions for publication but rather the selection of proposals of a type that elicits reader response. By soliciting contributions from a wider cross-section of the membership and from other interested readers, the editor hopes to achieve a diversity of high-quality proposals in geometry, analysis, number theory, inequalities, mathematical logic, game theory, set theory, group theory, probability, paradoxes, fallacies and cryptarithms, to name a few. In general, problems should rise above the level of unimaginative text-book exercises and should strive to give solvers an opportunity to demonstrate ingenuity and inventiveness.

One of the essential attributes of a suitable proposal is the hard-to-define quality of elegance. This characteristic is usually associated more with solutions than with proposals but is nevertheless an important element in attracting the attention of would-be solvers. A beautiful example of an elegant proposal is the following one, due to W. J. Blundon, of the Memorial University of Newfoundland:

Let I, O, H denote respectively the incenter, the circumcenter and the orthocenter of a triangle with sides a, b, c and the inradius r. Prove that the area K of the triangle IOH is given by

$$K = |(a - b)(b - c)(c - a)|/8r.$$

This problem was proposed in the January 1967 issue of *Elemente der Mathematik* and a solution was published the following January. Opinions regarding beauty are often debatable but can anyone deny that the economy of expression in the displayed result constitutes a pure and austere elegance? One would hope that a proposal of such high artistic merit would elicit a solution of comparable elegance.

Not all proposals can aspire to a high level of elegance in their mere statement. Most problems are straightforward challenges to duplicate or improve upon results already found by the proposer, especially if the method of solution or the final result is significant, novel, generalized, instructive or entertaining. Ordinarily problem editors require solutions submitted along with proposals. The purpose of this is to assist the editor in the evaluation of the suitability of the proposal, the complexity of the solution or the expected readership response. On the other hand, conjectures and unsolved problems connected with related investigations or research projects are sometimes submitted with the hope that someone may successfully arrive at a satisfactory solution. When such proposals are published, the readers are alerted to the fact that solutions have not been provided.

Since the Pi Mu *Epsilon Journal* appears only twice a year, acceptable proposals are filed away for possible use some time in the future. This may entail long delays in publication, especially if other problems in like categories have priority. Unused or unusable proposals will be returned to the proposer upon request.

After an issue of the *Journal* comes off the presses and is sent to the subscribers, solutions begin to trickle in. In due course the contributions are acknowledged, the solutions are filed away and the envelopes in which they were mailed are discarded. That is why solvers who would like to receive credit for their labors should be sure to identify their solutions with their names and addresses. Solutions to more than one problem should be sent on separate sheets and, to facilitate filing, should not be co-mingled with extraneous correspondence.

This saves the editor the inconvenience of photocopying portions for separate filing.

With the approach of deadlines for submitting the copy to the *Journal* Editor your problem editor examines all solutions received and is often confronted with difficult decisions as to which solution to publish. He is reminded of what motivates problemists to submit solutions in the first place. Why do they not simply solve it, file it and forget it? One incentive, of course, is the altruistic desire to share with others a well-thought out and well-expressed solution; another is to gratify one's ego in a most acceptable way by seeing his creation appreciated and published. Some problemists are so well versed in so many diverse branches of mathematics that they breeze through most of the proposals with ease and take a delight in making a marathon game of their knack for prolixity. These are individuals who generally combine quality with quantity and in many cases are legitimate candidates for inclusion in the *Guinness* **Book** of **Records.** The frequency with which their solutions are published may lead other solvers to suspect favoritism on the part of the editor, but readers are hereby assured that every effort is made to select solutions objectively on the basis of merit.

When submitting a solution, the solver should try to present it in the format adopted by the problem department. This saves the editor time and trouble in re-typing it for the printer. Most problem editors are their own secretaries -- unsung heroes who make a labor of love out of serving as intermediaries between proposers and solvers. Consequently when they are confronted with a difficult choice between two otherwise excellent solutions they may just tip the scales in favor of the solution that permits them to follow the path of least resistance. On the other hand, neatness and good form cannot in themselves supersede content; while they are qualities that are greatly appreciated, editors are often grieved to have to turn down a solution despite the evidence of painstaking care in presentation.

On occasion, excellent solutions with widely separated approaches are found to be too good to be lost to posterity. In those cases a diligent editor will attempt to do justice by concocting an amalgam of the solutions or, if space permits, publishing multiple solutions. Here again, the best mathematical and literary expression is considered

along with the quality of the solution.

It may be hard to believe, but your problem editor occasionally receives an answer to a problem instead of a solution. Participants in this arena are not really concerned with answers; their primacy"-interest is in the *way* the solution was found -- the train of thought that led to the solution, the transparency of the solver's heuristic approach to the problem, essentially, the solver's ability to take the reader by the hand and literally lead him over the various steps of the proof. One of the tests of elegance is finding a way of doing this adroitly without insulting the reader's intelligence by spelling out procedures that should be evident to him. At the same time, the solution should avoid the sins of omission -- skipping steps that are necessary for a full understanding of the solution, proof or construction, as the case may be.

To achieve this ability, the solver should be familiar with the criteria for elegance -- what we call the *ABCD*'s of Elegance. They are A for Accuracy, *B* for Brevity, *C* for Clarity, and D for the Display of Insight, Ingenuity, Imagination, Originality and, where possible, Generalization. It always helps to be able to instill a dramatic sense of awe, wonder and surprise. These are the intangible qualities that elevate mathematical creations to the realm of high art, whether they be proposals, solutions, short notes, expository essays or chapters in some impressive tome.

In conclusion, it is hoped that the enunciation of these high ideals will inspire readers to make efforts to achieve them without detering them from their most welcome participation in the Problem Department of this *Journal.*

LOCAL AWARDS

If your chapter has presented or will present awards to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the *Journal.*

A CONFORMAL GROUP ON AN n-DIMENSIONAL EUCLIDEAN SPACE

by Lonnie J. Kuss
Texas Tech University

In this paper, we study inversions and homothetic transformations on an n-dimensional Euclidean space. We shall employ the algebra and calculus of a vector space over the field of real numbers.

1. Notation

We shall denote vectors by capital letters and the scalars by smaller letters. The inner product of X and Y will be indicated by (X,Y) and $\|X\| = (X,X)^{1/2}$ means the norm of X. Whenever a basis is considered, it will be orthonormal. Let E be an n-dimensional Euclidean space and f a transformation on E. Then for A ● E, we denote the image of A under f by Af instead of $f(A)$.

2. Inversions

Let $A \neq \vec{0}$, $k \neq 0$, and $A \in E$. Then we can define a transformation f on E which satisfies for each A

(1) The set $\{A, Af\}$ is linearly dependent,

(2) $\|Af\|\,\|A\| = k \neq 0$.

This implies that $Af \neq \vec{0}$. The transformation f is called an inversion on E with pole (center) 0 and power k. (See for example [1, p. 38] and [5, p. 77].)

We shall obtain a formula for Af = B in terms of A. We observe that (1) implies that there exists a scalar m, depending on A, such that B $= mA$. Note that

$$\|B\| = \frac{k}{\|A\|}$$

Therefore,

$$|m|\;\|A\| = \frac{k}{\|A\|}$$

which implied that

$$m = \frac{\pm k}{\|A\|^2} .$$

We shall study the case $m > 0$ since the other case is quite similar. The transformation f given by the equation of the inversion is then

$$Af = B = \frac{kA}{\|A\|^2}, \qquad A \neq \vec{0} .$$

3. The Inverse Function

One can easily show that

$$A = \frac{kB}{\|B\|^2} = f^{-1}(B), \quad B \neq 0 .$$

It is interesting to see that $f^2 = e$, the identity, which means that f is the inverse of itself (see [3, p. 193]). To that end, observe that

$$Af^2 = (Af)f = \left[\frac{kA}{\|A\|^2}\right]f = \frac{k\left[\frac{kA}{\|A\|^2}\right]}{\left\|\frac{kA}{\|A\|^2}\right\|^2} = A .$$

4. The Derivative of a Vector

Let the vector $X \in E$ be a function of the real variable t; that is, $X = X(t)$. Suppose C is the curve described by the endpoint of X. Let ΔX be defined by the rule $X(t + \Delta t) = X + \Delta X$. Then, the vector ΔX is parallel to the chord connecting the endpoint of X and $X + \Delta X$, as shown in Figure 1.

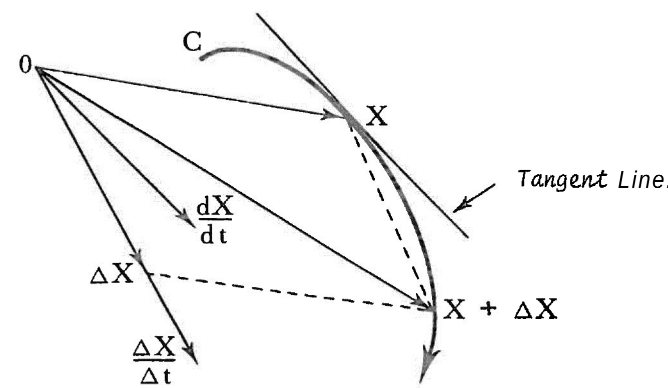


FIGURE 1

It is clear that $\frac{\Delta X}{\Delta t} = \frac{1}{\Delta t}\cdot \Delta X$ is also a vector parallel to the chord. We define the *derivative* of X with respect to t to be $\lim_{\Delta t \to 0} \frac{\Delta X}{\Delta t} = \frac{dX}{dt}$, if the limit exists. The line parallel to the vector $\frac{dX}{dt}$ at the endpoint of X is called the tangent *line* at X.

5. The Derivative of an Inner Product

Let $X = X(t)$ and $Y = Y(t)$ and also let $\frac{dX}{dt}$ and $\frac{dY}{dt}$ exist. Then $\frac{d}{dt}(X,Y)$ exists and

$$\frac{d}{dt}(X,Y) = \left(\frac{dX}{dt},Y\right) + \left(X,\frac{dY}{dt}\right).$$

This can easily be shown. First we let $X = (x_1,x_2,\cdots,x_n)$ and $Y = (y_1,y_2,\cdots,y_n)$. Then $(X,Y) = (x_1y_1 + x_2y_2 + \cdots + x_ny_n)$ and

$$\frac{d}{dt}(X,Y) = \left(\frac{dx_1}{dt}y_1 + x_1\frac{dy_1}{dt} + \cdots + \frac{dx_n}{dt}y_n + x_n\frac{dy_n}{dt}\right)$$

Therefore

$$\frac{d}{dt}(X,Y) = \left(\frac{dX}{dt},Y\right) + \left(X,\frac{dY}{dt}\right).$$

6. Conformal Property of Inversions

If X describes a curve C and Y describes the image curve $\mathcal{D}$ under the inversion f, then the following theorem states that C and $\mathcal{D}$ make equal angles with the line OX (see Figure 2). The proof for the two dimensional case is given in [1, p. 471.

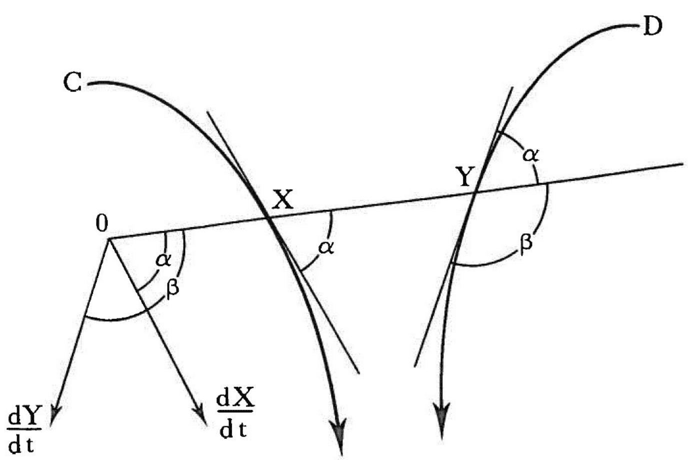


FIGURE 2

Theorem. Let $X = X(t) \in E$ such that $\frac{dX}{dt}$ in a neighborhood of $t = t_0$ exists. Let f be an inversion on E, with the equation $Xf = Y = \frac{kX}{\|X\|^2}$. Then

$$\frac{\left(Y,\frac{dY}{dt}\right)}{\|Y\|\left\|\frac{dY}{dt}\right\|} = -\frac{\left(X,\frac{dX}{dt}\right)}{\|X\|\left\|\frac{dX}{dt}\right\|}.$$

Proof. In section 5 we have shown that $\frac{dX}{dt}$ and $\frac{dY}{dt}$ are vectors parallel to the tangent lines of C and $\mathcal{D}$. Let a be the angle between X and $\frac{dX}{dt}$, and β be the angle between Y and $\frac{dY}{dt}$. Then since $Y = Xf = kX/\|X\|^2$,

$$\cos\beta = \frac{\left(Y,\frac{dY}{dt}\right)}{\|Y\|\left\|\frac{dY}{dt}\right\|} = \frac{\left(X,\frac{dY}{dt}\right)}{\|X\|\left\|\frac{dY}{dt}\right\|} \tag{1}$$

Differentiating $Y = \frac{kX}{\|X\|^2}$ gives

$$\frac{dY}{dt} = k\,\frac{\|X\|^2\frac{dX}{dt} - 2\left(X,\frac{dX}{dt}\right)X}{\|X\|^4}$$

Now we compute $\left\|\frac{dY}{dt}\right\|$. One observes that

$$\left\|\frac{dY}{dt}\right\|^2 = \frac{k^2\left\|\frac{dX}{dt}\right\|^2}{\|X\|^4} - \frac{4k^2\left(X,\frac{dX}{dt}\right)}{\|X\|^6}\left(X,\frac{dX}{dt}\right) + \frac{4k^2\left(X,\frac{dX}{dt}\right)^2}{\|X\|^8}\|X\|^2 = \frac{k^2\left\|\frac{dX}{dt}\right\|^2}{\|X\|^4}$$

Therefore

$$\left\|\frac{dY}{dt}\right\| = \frac{k\left\|\frac{dX}{dt}\right\|}{\|X\|^2}. \tag{2}$$

Next we see that

$$\left(X,\frac{dY}{dt}\right) = \frac{k}{\|X\|^2}\left(X,\frac{dX}{dt}\right) - \frac{2k\left(X,\frac{dX}{dt}\right)}{\|X\|^2} = \frac{-k\left(X,\frac{dX}{dt}\right)}{\|X\|^2}. \tag{3}$$

Substituting (2) and (3) into (1) gives

$$\cos\beta = \frac{-k\left(\|X\|^2\frac{dX}{dt}\right)}{\|X\|^2} \Big/ \|X\|\,\frac{k\left\|\frac{dX}{dt}\right\|}{\|X\|^2}$$

$$= -\frac{\left(X, \frac{dX}{dt}\right)}{\|X\| \left\|\frac{dX}{dt}\right\|} = -\cos\alpha$$

This proves the theorem and shows that $\beta = \pi - \alpha$.

Theorem. Let $X_1 = X_1(t)$, $X_2 = X_2(t) \in E$ have derivatives in a neighborhood of $t = t_0$. Let f be an inversion on E such that

$$X_1 f = Y_1 = \frac{kX_1}{\|X_1\|^2}, \qquad X_2 f = Y_2 = \frac{kX_2}{\|X_2\|^2}$$

Then at $X_1 = X_2 = X$, i.e., the point of intersection on $X_1 = X_1(t)$ and $X_2 = X_2(t)$, we have

$$\frac{\left(\frac{dY_1}{dt}, \frac{dY_2}{dt}\right)}{\left\|\frac{dY_1}{dt}\right\| \left\|\frac{dY_2}{dt}\right\|} = \frac{\left(\frac{dX_1}{dt}, \frac{dX_2}{dt}\right)}{\left\|\frac{dX_1}{dt}\right\| \left\|\frac{dX_2}{dt}\right\|}$$

(This means that f is a *conformal* or angle-preserving mapping.)

Proof. Note that

$$\frac{dY_1}{dt} = k\,\frac{\|X_1\|^2 \frac{dX_1}{dt} - 2\left(\frac{dX_1}{dt}, X_1\right)X_1}{\|X_1\|^4},$$

and

$$\frac{dY_2}{dt} = k\,\frac{\|X_2\|^2 \frac{dX_2}{dt} - 2\left(\frac{dX_2}{dt}, X_2\right)X_2}{\|X_2\|^4}$$

One obtains, as in the proof of the previous theorem,

$$\left\|\frac{dY_j}{dt}\right\| = k\,\frac{\left\|\frac{dX_j}{dt}\right\|}{\|X_j\|^2}, \qquad j = 1,2.$$

Therefore,

$$\left\|\frac{dY_1}{dt}\right\| \left\|\frac{dY_2}{dt}\right\| = k^2\,\frac{\left\|\frac{dX_1}{dt}\right\| \left\|\frac{dX_2}{dt}\right\|}{\|X_1\|^2\,\|X_2\|^2}$$

$$= k^2\,\frac{\left\|\frac{dX_1}{dt}\right\| \left\|\frac{dX_2}{dt}\right\|}{\|X\|^4}$$

Now we see that

$$\left(\frac{dY_1}{dt}, \frac{dY_2}{dt}\right) = k^2 \left[\frac{1}{\|X_1\|^2}\frac{dX_1}{dt} - 2\,\frac{\left(\frac{dX_1}{dt}, X_1\right)X_1}{\|X_1\|^4},\ \frac{1}{\|X_2\|^2}\frac{dX_2}{dt} - 2\,\frac{\left(\frac{dX_2}{dt}, X_2\right)X_2}{\|X_2\|^4}\right]$$

$$= k^2 \left[\frac{1}{\|X\|^2}\frac{dX_1}{dt} - 2\,\frac{\left(\frac{dX_1}{dt}, X\right)X}{\|X\|^4},\ \frac{1}{\|X\|^2}\frac{dX_2}{dt} - 2\,\frac{\left(\frac{dX_2}{dt}, X\right)X}{\|X\|^4}\right]$$

$$= k^2 \left[\frac{\left(\frac{dX_1}{dt}, \frac{dX_2}{dt}\right)}{\|X\|^4} - \frac{4\left(\frac{dX_1}{dt}, X\right)\left(\frac{dX_2}{dt}, X\right)}{\|X\|^6} + \frac{4\left(\frac{dX_1}{dt}, X\right)\left(\frac{dX_2}{dt}, X\right)\|X\|^2}{\|X\|^8}\right].$$

Therefore,

$$\left(\frac{dY_1}{dt}, \frac{dY_2}{dt}\right) = k^2\,\frac{\left(\frac{dX_1}{dt}, \frac{dX_2}{dt}\right)}{\|X\|^4}$$

Consequently,

$$\frac{\left(\frac{dY_1}{dt}, \frac{dY_2}{dt}\right)}{\left\|\frac{dY_1}{dt}\right\| \left\|\frac{dY_2}{dt}\right\|} = k^2\,\frac{\left(\frac{dX_1}{dt}, \frac{dX_2}{dt}\right)}{\|X\|^4} \Bigg/ k^2\,\frac{\left\|\frac{dX_1}{dt}\right\| \left\|\frac{dX_2}{dt}\right\|}{\|X\|^4}$$

$$= \frac{\left(\frac{dX_1}{dt}, \frac{dX_2}{dt}\right)}{\left\|\frac{dX_1}{dt}\right\| \left\|\frac{dX_2}{dt}\right\|}.$$

Thus the transformation is conformal.

7. *Product of Inversions*

Let f and g be two inversions on E such that

$$Xf = \frac{k_1 X}{X} \equiv Y \quad \text{and} \quad Yg = \frac{k_2 Y}{Y} \equiv Z .$$

Then

$$Xfg = Yg = \frac{k_2 \dfrac{k_1 X}{\|X\|^2}}{\left\| \dfrac{k_1 X}{\|X\|^2} \right\|^2} = \frac{\dfrac{k_2 k_1}{\|X\|^2} X}{\dfrac{k_1{}^2}{\|X\|^2}} = \frac{k_2}{k_1} X .$$

We observe that fg is not an inversion since the ratio k_2/k_1 is independent of X. We also note that in general $fg \# gf$ since

$$Xgf = (Xg)f = \frac{k_1 \dfrac{k_2 X}{\|X\|^2}}{\left\| \dfrac{k_2 X}{\|X\|^2} \right\|^2} = \frac{k_1}{k_2} X .$$

Therefore, the set of inversions having a common pole on E is not closed under multiplication. In order to obtain a group of transformations we shall turn to another class of transformations.

. *Dilatations.*

Let A $\neq \vec{0}$, k $\neq$ 0, and A $\in$ E. Then we define the transformation f on E by the rule Af = B iff $B = kA$. The transformation f is called a dilatation on E with center 0 and ratio k. (See [1, p. 311 and [5, 68].)

One can easily show that a dilatation is a conformal transformation. Since the proof is very simple, we omit it.

Theorem (A conformal group). Let $\{f\}$ be the set of all inversions and dilatations on E. Then $\{f, \cdot\}$ is a group.

Proof. We have already proved that the product of two inversions is a dilatation. (See section 7.) We shall study other products,

Let f and g be two dilatations given by Xf = $k_1 X$ and Xg = $k_2 X$. Then we observe that $Xf \cdot g = (k_1 k_2) X$. Thus the product of two dilatations is a dilatation.

Let f be the inversion given by Xf = $kX/\|X\|^2$ and g the dilatation given by Xg = hX. Then $X(fg) = (Xf)g = hkX/\|X\|^2$. Similarly,

$$X(gf) = (Xg)f = \frac{k(hX)}{\|hX\|^2} = \frac{kX}{|h|\ \|X\|^2} = \frac{\frac{k}{|h|} X}{\|X\|^2}$$

Thus the product of an inversion and a dilatation is an inversion..

Note that we may obtain a function which looks like an inversion with a negative power. Thus we shall study

$$Xf = \frac{-kX}{\|X\|^2}, \qquad k > 0.$$

If we let $Xf_1 = kX/\|X\|^2$ and $Xf_2 = (-1)X$, then $[Xf_1]f_2 = (-1)Xf_1 = -kX/\|X\|^2 =$ Xf. Thus f $= f_1 f_2$ where f_1 is an inversion and f_2 is a dilatation of ratio -1. Consequently $\{f\}$ is closed under products.

Let f $\in \{f\}$. We shall show that f^{-1} exists and $f^{-1} \in \{f\}$. Let Xf = Y. Then there are two cases:

(i) In section 3 we have shown that the inverse of an inversion is itself.

(ii) Let f be a dilatation. Then Xf $= hX$, and $Xf^{-1} = \frac{1}{h}$ X. Thus f^{-1} exists and $f^{-1} \in \{f\}$.

Consequently, $\{f\}$ is a group under multiplication [4, p. 115, 127].

9. *The Transform of a Hyperplane.*

A hyperplane in E has an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c \qquad (1)$$

Let $(a_1, \cdots, a_n) = A$ and $(x_1, \cdots, x_n) =$ X. Then (1) becomes $(A, X) = c$. Consider the inversion

$$X = \frac{kY}{\|Y\|^2}, \qquad Y \neq \vec{0} .$$

Then the transform of (1) by f will be

$$\left(A, \frac{kY}{\|Y\|^2}\right) = c .$$

This will become $(kA, Y) = c\|Y\|^2$.

Two cases may be considered:

(i) If e = 0, we get $k(A, Y) =$ 0 which is the same hyperplane as $(A, X) = 0$.

(ii) If c # 0, then we obtain the **hypersphere** $k(A, Y) = c\|Y\|^2$ with center $kA/2c$ and radius $|k/2c|\ \|A\|$ and which contains $\vec{0}$.

10. The Transform of a Hypersphere.

The equation of a hypersphere in E is $\|X - C\| = r$. We can write this equation as $(X - C, X - C) = r^2$. Expanding this we obtain

$$\|X\|^2 - 2(C,X) + \|C\|^2 - r^2 = 0 . \tag{1}$$

Now consider the inversion $X = kY/\|Y\|^2$, $Y \neq 0$. Then this inversion transforms (1) into

$$\left\|\frac{kY}{\|Y\|^2}\right\|^2 - 2\left(C, \frac{kY}{\|Y\|^2}\right) + \|C\|^2 - r^2 = 0 . \tag{2}$$

Two cases may be considered:

(i) If $\|C\| = r$, then (2) will become $(C,Y) = \frac{k}{2}$, which is a hyperplane.

(ii) If $\|C\| \neq r$, then (2) will become the hypersphere

$$[\|C\|^2 - r^2]\|Y\|^2 - 2k(C,Y) + k^2 = 0 .$$

There is one special case which is important. Let the hypersphere be $\|X\| = \sqrt{k}$. Then the inversion preserves this particular hypersphere; that is, applying $X = kY/\|Y\|^2$, $Y \neq \vec{0}$, we obtain $\|Y\| = \sqrt{k}$. Thus this hypersphere is invariant under the inversion.

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HYPERPERFECT NUMBERS

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In this paper we generalize the concept of perfect number; some basic properties are given, along with a few conjectures.

Definition. Let e be a divisor of m; e is said to be a proper divisor if $1 < e < m$.

Definition. An integer m is n-hyperperfect if

$$m = 1 + n\ \Sigma d_i$$

where the summation is taken over all proper divisors of m.

It is clear that 1-hyperperfect and perfect are equivalent concepts. An n-hyperperfect number is highly deficient.

Theorem 1. If $j|m$ with $1 < j \leq n$, then m is not n-hyperperfect $(n > 1)$.

Proof. Say $j = m$. Then $m \leq n$, so that $m \neq 1 + n\ \Sigma d_i$. Say now that j is a proper divisor of m. Then m/j is also a proper divisor of m. Consider

$$s = 1 + n\ \Sigma d_i = 1 + n\left(\frac{m}{j} + \cdots\right)$$

where the dots indicate other divisors of m. Since $j \leq n$, $s > m$ and so the definition is not satisfied. In conclusion, m is not n-hyperperfect, as was to be proved.

Corollary 1. If m is even, then m is not n-hyperperfect since it has $n > 1$.

Proof. If m is even, $2|m$. By Theorem 1, m cannot be n-hyperperfect for $n \geq 2$.

A prime number p is obviously not n-hyperperfect since it has no proper divisors and

$$p > 1 + n \cdot 0$$

Similarly, if $m \le n$ then m is not n-hyperperfect. This follows since

$$m < 1 + m\,\Sigma d_i < 1 + n\,\Sigma d_i\,. \quad (m \text{ non prime})$$

It should be observed that if m is n-hyperperfect, then $m \equiv 1 \bmod n$.

<u>*Theorem 2.*</u> An n-hyperperfect number $(n > 1)$ cannot be a power of a prime.

Proof. Let $m = p^k$ be n-hyperperfect. Then $p, p^2, \cdots, p^{k-1}$ are proper divisors. We know that

$$1 + p + p^2 + \cdots + p^{k-1} = \frac{p^k - 1}{p - 1}$$

so that

$$p + p^2 + \cdots + p^{k-1} = \frac{p^k - p}{p - 1}\,.$$

This implies

$$1 + n(p + p^2 + p^3 + \cdots + p^{k-1}) = 1 + n\,\frac{p^k - p}{p - 1}$$

<u>Case 1</u>. $n \ge p$. Impossible since then m has a divisor p with $p \le n$ (Theorem 1).

<u>Case 2</u>. Hence $n = 2, 3, \cdots, p - 1$; by hypothesis

$$m = p^k = 1 + n(p + p^2 + \cdots + p^{k-1}) = 1 + n\,\frac{p - p^k}{1 - p} = 1 + n\,\frac{p^k - p}{p - 1}\,.$$

This implies

$$p^k \le 1 + (p - 1)\,\frac{p - p^k}{1 - p} = 1 + p^k - p\,.$$

This implies $0 \le 1 - p$. Since $p > 1$, this is a contradiction. Therefore, $m \ne p^k$, completing the proof.

<u>*Theorem 3.*</u> Let $m = pq$ with p, q primes, be n-hyperperfect. Then the following relation must hold: $n < p < 2n < q$.

Proof. Let $p < q$. Thus

$$pq = 1 + n(p + q) = 1 + np + nq < 1 + 2nq$$

so

$$pq \le 2nq$$

from which follows that $p \le 2n$. Now $p \ne 2n$, otherwise $2|m$, which is a contradiction of Corollary 1. Finally $p < 2n$. Similarly, $pq > 1 + 2np$, so that

$$pq \ge 2np$$

and thus

$$q \ge 2n\,;$$

as above, $q \# 2n$ so that $q > 2n$. Also $(p - n)q = 1 + np > 0$, implying $p - n > 0$ or $p > n$, as was to be proved.

Of course this theorem says nothing about more general n-hyperperfect numbers of the form $p_1^{\alpha_1}\,p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. In view of the constraints imposed by Theorem 1, limiting the number of candidates for **n-hyperperfection, it** is natural to ask if for any given n there exists at least one **hyperperfect** number. For example, let M be some positive integer; $\frac{1}{2}M - P$ numbers are candidates for 2-hyperperfection (P = number of primes between 2 and M); only $\frac{1}{2}M - \frac{1}{6}M - P$ numbers are candidates for 3-hyperperfection (see Theorem 1). Also, having fixed M, the number of candidates for **n-hyper**perfection decreases to zero as n approaches M.

The search conducted by the authors, in which the integers between 2 and 25,000 were investigated, gives the following results (table exhaustive to 25,000).

n	m
2	21
6	301
3	325
12	697
18	1333
18	1909
12	2041
2	2133
30	3901
11	10693
6	16513
2	19521
60	24601

An astonishing relation exists between n-hyperperfect numbers and generalized Mersenne numbers — for which we discuss here only the second degree extension.

Definition. Let t be an integer. If $3^t - 2$ is prime, then $3^{t-1}(3^t - 2)$ is called a second degree Mersenne number and 3 - 2 a Mersenne *prime*.

Observe the similarity with the regular Mersenne numbers defined as $2^{t-1}(2^t - 1)$, for $2^t - 1$ prime.

Theorem 4. If for some t, $3^{t-1}(3^t - 2)$ is a second degree Mersenne number, then $m = 3^{t-1}(3^t - 2)$ is 2-hyperperfect.

Proof. Let $m = 3^{t-1}(3^t - 2)$; we want to determine Σd_i. It is known from number theory that if $(a,b) = 1$ then $\sigma(a,b) = \sigma(a)\sigma(b)$.

By assumption on 3 - 2, $(3^{t-1}$, 3 - 2) = 1. Then

$$\sigma(m) = \sigma[(3^{t-1})(3^t - 2)] = \sigma(3^{t-1})\sigma(3^t - 2)$$
$$= \sigma(3^{t-1})[(3^t - 2) + 1] ,$$

since $3^t - 2$ is prime. Now,

$$\sigma(3^{t-1}) = (1 + 3 + \cdots + 3^{t-1}) = \frac{1 - 3^t}{1 - 3} = \frac{1}{2}(3^t - 1)$$

so that $\sigma(m) = \frac{1}{2}(3^t - 1)^2$. Now

$$1 + 2\,\Sigma d_i = 1 + 2[\sigma(m) - m - 1]$$
$$= 1 + 2[\tfrac{1}{2}(3^t - 1)^2 - 3^{t-1}(3^t - 2) - 1]$$
$$= (3^t - 1)^2 - 2\cdot 3^{t-1}(3^t - 2) - 1$$
$$= 3^{2t} - 2\cdot 3^t - 2\cdot 3^{2t-1} + 4\cdot 3^{t-1} .$$

Factoring,

$$1 + 2\,\Sigma d_i = 3^{2t-1}(3 - 2) + 3^{t-1}(4 - 2\cdot 3)$$
$$= 3^{2t-1} - 2\cdot 3^{t-1} = m$$

Hence m is 2-hyperperfect, ending the proof.

Whether the above condition is also necessary has not been established. The table below shows how the 2-hyperperfect numbers can be generated; the question whether these are the only 2-hyperperfect numbers between

2 and 4,782,963 × 4,348,905 is contingent upon settling the necessary part of the theorem.

t	3^{t-1}	$3^t - 2$	Prime?	$M^* = 3^{t-1}(3^t - 2)$
1	1	1	No	
2	3	7	Yes	21
3	9	25	No	
4	27	79	Yes	2,133
5	81	241	Yes	19,521
6	243	727	Yes	176,661
7	729	2,185	No	
8	2,187	6,559	No	
9	6,561	19,681	Yes	129,127,041
10	19,683	59,047	No	
11	59,049	177,145	No	
12	177,147	531,439	No	
13	531,441	1,594,321	No	
14	1,594,323	4,782,967	No	
15	4,782,969	14,348,905	No	

We caution the reader from inferring that

$$(1 + k)^{t-1}[(1 + k)^t - k], \qquad k = 1,2,3,\cdots$$

will always be sufficient to generate k-hyperperfect numbers; in fact, it is not, though it will work in certain situations. The process of deriving appropriate Mersenne generalizations is an intriguing and difficult task, and is discussed at length by the authors elsewhere. We conclude with two conjectures:

Conjecture 1. For all n, there exists at least one n-hyperperfect number.

Conjecture 2. The converse to Theorem 4 is true.

THE HEIGHT OF THE LATTICE OF FINITE TOPOLOGIES

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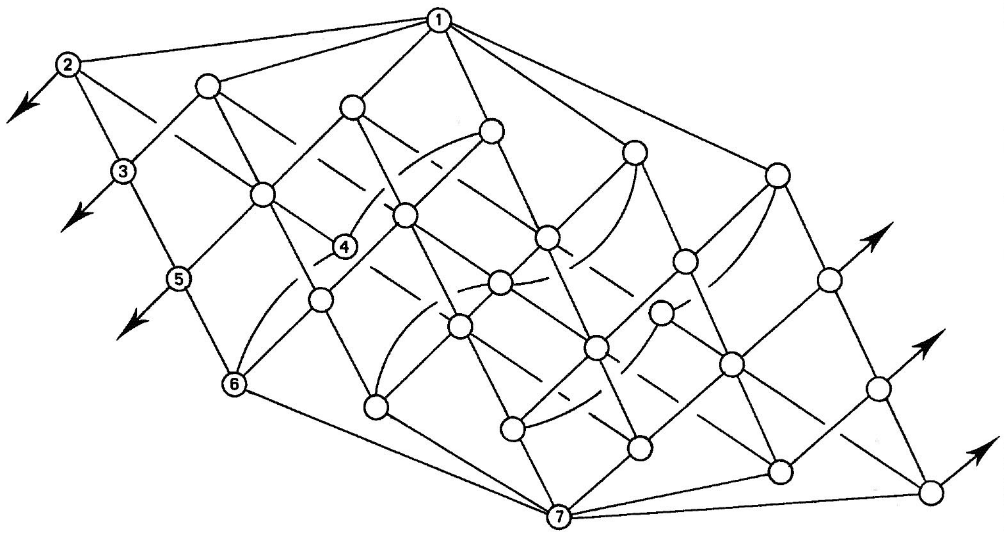
Several articles have appeared concerning the lattice of topologies on a finite set. Many of these papers deal with counting procedures for different types of topologies definable on n elements. For example, the lattice of topologies on a set containing n elements has $2^n - 2$ atoms and $n^2 - n$ anti-atoms.

Example 1. If n is equal to three, the following diagram taken from (9) illustrates that both of these formulas yield six.



As indicated by the diagram, there are 29 topologies definable on a finite set of three elements. If the finite set consists of the elements a, b, and c, then the topologies labeled 1 through 7 in the diagram are as follows:

(1) The discrete topology.
$\{\emptyset, [a], [b],$ [el, $[a,b], [a,c], [b,c], [a,b,c]\}$

(2) One of the six anti-atoms.
$\{\emptyset, [a], [b], [a,b], [b,c], [a,b,c]\}$

(3) $\{\emptyset, [a], [b], [a,b], [a,b,c]\}$

(4) $\{\emptyset, [a], [b,c], [a,b,c]\}$

(5) $\{\emptyset, [a], [a,b], [a,b,c]\}$

(6) *One of the six atoms.*
$\{\emptyset, [a], [a,b,c]\}$

(7) *The indiscrete topology.*
$\{\emptyset, [a,b,c]\}$

The two chains

$$7 \subseteq 6 \subseteq 5 \subseteq 3 \subseteq 2 \subseteq 1$$

and

$$7 \subseteq 6 \subseteq 4 \subseteq 2 \subseteq 1$$

are both maximal in the sense that no other topologies may be included in either of the two chains. However, the first is a chain of maximal length, in this case 6, while the second is not.

We can observe from the diagram that the maximum chain length in this lattice is also six. This maximumlength is called the ***height*** of the lattice. We will show in this paper that the height of the lattice of topologies on n elements is $(n^2 + n)/2$.

1. *Notation and Definitions*

For a definition of a lattice, we refer the reader to (1) or any of several other standard texts on lattices.

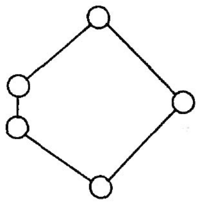
Definition. If L is a lattice and C is a subset of L, then C is a chain if for every x, y $\in$ C either $x \le y$ or $y \le x$. The length of the *chain* C is simply its cardinality.

Finite lattices can be represented by diagrams, as seen in Example 1. In these diagrams, the lattice ordering of $x \le y$ is indicated by placing x lower than y and connecting them with a line.

Definition. A chain C in a lattice L is called *maximal* if by adding any other elements of L to C, C is no longer a chain.

Definition. If L is a lattice, then the height of L is the maximum length of a chain in L.

Example 2. Note that it is possible to have maximal chains of different lengths. A simple example is illustrated in the diagram below:



This lattice has a height of four, but along with a maximal chain of length four, it also has a maximal chain of length three.

Definition. If L is a lattice with least element 0, then a is called an atom of L if $0 < c \leq a$ implies $c = a$. The dual notion of an atom is called an anti-atom.

If X is a set, then we use $\Sigma(X)$ to denote the lattice of topologies on X under the ordering of set inclusion. In this paper, we will assume that X is finite.

In the proof of our main theorem, we will rely heavily upon Fröhlich's ultratopologies [5] which form the anti-atoms in $\Sigma(X)$. If X is finite, the anti-atoms in $\Sigma(X)$ are precisely the ultratopologies of the form:

$$T(a,b) = [G \subseteq X \mid a \notin G \text{ or } b \in G].$$

Fröhlich has shown that $T(a,b) = T(c,d)$ iff $a = c$ and $b = d$, and that $\Sigma(X)$ is anti-atomic; that is, any topology in $\Sigma(X)$ may be written as the intersection of anti-atoms.

Maximal chains in $\Sigma(X)$ may be nicely represented by sequences of ultratopologies in $\Sigma(X)$ as follows. Assume we have a maximal chain in $\Sigma(X)$ of length k: $[\emptyset,X] = T_1 \subseteq T_2 \cdots \subseteq T_k = P(X)$. T_{k-1} must be an ultratopology, say $T(a_1,b_1)$ and T_{k-2} must be the intersection of this ultratopology with another, say $T_{k-2} = T(a_1,b_1) \cap T(a_2,b_2)$. We may continue this process to obtain a sequence of ultratopologies such that $T_{k-m} = T(a_1,b_1) \cap T(a_2,b_2) \cap \cdots \cap T(a_m,b_m)$. There will be exactly $k - 1$ ultratopologies in this sequence since the discrete topology $P(X)$ is greater than any ultratopology.

Definition. A sequence $[T_1,T_2,\cdots,T_n]$ of ultratopologies in $\Sigma(X)$ is *independent* if for every $k \leq n$, $\cap[T_i \mid i < k] \neq \cap[T_i \mid i \leq k]$. A sequence which is not independent is called dependent.

Example 3. The sequence $[T(a,b), T(a,c), T(a,d)]$ is independent as long as a, b, c, and d are distinct.

Example 4. The sequence $[T(a,c), T(c,b), T(a,b)]$ is dependent since $T(a,c) \cap T(c,b) = T(a,c) \cap T(c,b) \cap T(a,b)$. This example does not depend upon the set X as long as X has at least three elements. Note that the order of a sequence of ultratopologies may be critical to its dependence or independence. In this example, if we reorder the sequence to produce $[T(a,c), T(a,b), T(c,b)]$, we obtain an independent sequence.

Definition. If S is a sequence of ultratopologies in $\Sigma(X)$, then by the *basic* set of the sequence we mean the set of all a, b $\in$ X such that $T(a,b) \in S$.

2. Height of $\Sigma(X)$

Theorem 1. If X is finite and if an independent sequence of distinct ultratopologies in $\Sigma(X)$ possesses a basic set containing n elements, then the sequence contains at most $[(n^2 + n)/2] - 1$ ultratopologies.

Proof. The proof is by induction on n. If $n = 2$, the only possible ultratopologies in the sequence would be of the form $T(a,b)$ and $T(b,a)$. Since for $n = 2$, $[(n^2 + n)/2] - 1 = 2$, the result is obvious. If $n = 3$, $[(n^2 + n)/2] - 1 = 5$ and since there are at most six distinct ultratopologies in a sequence whose basic set has three elements, it is clear that any sequence of more than five distinct ultratopologies is dependent since it would contain a subsequence like that of Example 4.

We will assume the theorem to be true for $n - 1$. That is, any independent sequence having a basic set of $n - 1$ elements cannot have $[(n - 1)^2 + (n - 1)]/2$ distinct ultratopologies.

Assume $S = [T(a_1,b_1), T(a_2,b_2), \cdots, T(a_k,b_k)]$ is a sequence of distinct ultratopologies in $\Sigma(X)$ where $k = (n^2 + n)/2$, and assume that the basic set of S contains n elements.

Case 1. If there exists an element m such that m appears n or less times as either the first or second element in the representation of the ultratopologies, then there would be at least $[(n^2 + n)/2] - n$ ultratopologies in S whose representations involved only the remaining $n - 1$ elements other than m. Since $[(n^2 + n)/2] - n = [(n - 1)^2 + (n - 1)]/2$, we may apply the induction assumption to this subsequence involving the $n - 1$ elements other than m to conclude that the subsequence is dependent. It is easily seen that any subsequence of an independent sequence must be independent; therefore, we conclude that S is dependent.

Case 2. If every element of the basic set of S appears exactly $n+1$ times in the representation of the ultratopologies in S, then we will focus upon the ultratopology $T(a_k,b_k)$. Note that no element in the basic set can appear more than $n + 1$ times in the representation of the ultratopologies. We leave this verification for the reader. Since a_k could only have appeared in the second position in at most $n - 1$ ultratopologies in S, there must exist some $c \in X$ such that $c \neq b_k$ and $T(a_k,c) \in S$. If $T(c,b_k) \in S$, then $\cap[T(a_i,b_i) \mid i < k] = \cap[T(a_i,b_i) \mid i \leq k]$ and the proof would be complete. Therefore, assume $T(c,b_k) \notin S$. We also assume that the subsequence $[T(x,y) \in S \mid x,y \neq a_k]$ is independent, since if it were dependent, then S would be dependent. However, by adjoining $T(c,b_k)$ as the final term of this subsequence, we obtain a sequence on $[(n - 1)^2 + (n - 1)]/2$ elements whose basic set has only $n - 1$ elements. As in Case 1, we can apply our induction assumption to conclude that this new sequence is dependent. Since the subsequence without $T(c,b_k)$ was independent but with $T(c,b_k)$ is dependent, the following intersections must be equal:

$$\cap[T(x,y) \in S \mid x,y \neq a_k] = \cap[T(x,y) \in S \mid x,y \neq a_k] \cap T(c,b_k).$$

Call this topology T_1 and let $T_2 = \cap[T(a_i,b_i) \mid i < k]$. We claim that $T_2 \cap T(a_k,b_k) = T_2$. To see this, choose $G \in T_2$. By the inclusion $T_2 \subseteq T_1 \subseteq T(c,b_k)$ we know that $G \in T(c,b_k)$. Also, we know $T_2 \subseteq T(a_k,c)$ which implies that $G \in T(a_k,c)$. As in Example 4, this implies $G \in T(a_k,b_k)$ and our proof is complete.

Corollary. If X has n elements, then every chain in $\Sigma(X)$ has a length of at most $(n^2 + n)/2$.

Theorem 2. If $X = [1,2,\cdots,n]$ there exists a chain of length $(n^2 + n)/2$ in $\Sigma(X)$.

Proof. We claim that the following sequence of ultratopologies is independent:

$$[T(1,2), \cdots, T(1,n), T(2,3), \cdots, T(2,n), T(2,1), \cdots,$$
$$T(n - 2, n - 1), T(n - 2, n), T(n - 2, 1), T(n - 1, n),$$
$$T(n - 1, 1), T(n,1)].$$

Verification of this claim is messy, but straightforward. We will indicate a method of proof by the following table. If the sequence is relabeled $[T_1,T_2,\cdots,T_k]$, then by G_m we mean a set $G_m \in T_1 \cap T_2 \cap \cdots \cap T_{m-1}$ such that $G_m \notin T_m$.

T_m	$T(1,2)$	$T(1,3)$	$\cdots$	$T(1,n)$	$T(2,3)$	$\cdots$
G_m	$[1]$	$[1,2]$	$\cdots$	$[1,2,\cdots,n - 1]$	$[2]$	$\cdots$

T_m	$T(2,n)$	$T(2,1)$	$T(n - 2, n - 1)$	$T(n - 2, n)$
G_m	$[2,3,\cdots,n - 1]$	$[2,3,\cdots,n]$	$[n - 2]$	$[n - 2, n - 1]$

T_m	$T(n - 2, 1)$	$T(n - 1, n)$	$T(n - 1, 1)$	$T(n,1)$
G_m	$[n - 2, n - 1, n]$	$[n - 1]$	$[n - 1, n]$	$[n]$

This sequence has $n - 1$ ultratopologies whose first term is 1, $n - 1$ whose first term is 2, $n - 2$ whose first term is 3, and so on. Therefore, we have $(n - 1) + (n - 1) + (n - 2) + \cdots + 3 + 2 + 1 = [(n^2 + n)/2] - 1$ ultratopologies in the sequence. Since $P(X)$, the discrete topology, is greater than any of these ultratopologies, we may form a chain of length $(n^2 + n)/2$.

The combination of the first two theorems brings us to our third theorem.

Theorem 3. The height of the lattice of topologies on n elements is $(n^2 + n)/2$.

The following table compares four different dimensions of $\Sigma(X)$ for $|X| \leq 8$.

n	number of atoms	number of anti-atoms	height	number of topologies [6]
2	2	2	3	4
3	6	6	6	29
4	14	12	10	355
5	30	20	15	6942
6	62	30	21	209527
7	126	42	28	9535241
8	254	56	36	646555994

One interesting side result concerning chains of maximal length in $\Sigma(X)$ is that they cannot contain a non-trivial partition topology. By glancing back at the diagram of $\Sigma(X)$ where $X = [a,b,c]$, we see that the three topologies in the center of the lattice which are not contained in a chain of maximal length are the partition topologies $\{\emptyset, [b], [a,c], X\}$, 10, $[a]$, $[b,c]$, $X\}$, and $\{\emptyset, [c], [a,b], X\}$.

Definition. A topology on X is called a partition topology if the minimal open sets of the topology form a partition on X. Partition topologies are called symmetric topologies in (7).

Theorem 4. A chain of maximal length in $\Sigma(X)$ cannot contain a non-trivial partition topology.

Proof. Assume $P(X) = T_0, T_1, \cdots, T_k = [\emptyset,X]$ is the chain and assume that it is represented by a sequence of ultratopologies $S = [T(a_1,b_1), \cdots, T(a_k,b_k)]$ where $T_3 = T(a_1,b_1)$ n $\cdots$ n $T(a_3,b_3)$. Assume that T_i, i # 0, $i \neq k$, is a partition topology. Since T_i is non-trivial, there must be at least two distinct minimal open sets in the partition and one of them must have at least two elements. Say the partition is $[P_1,P_2,\cdots,P_m]$ where $P_1 = [p_1,p_2,\cdots,p_r]$. The sequence of ultratopologies has the property that if $j \leq i$, then a_3 and b_3 are both in the same equivalence class in the partition. Since $r \leq n - 1$, we know one of the members of P_1, say p_1, appears at most $n - 1$ times as some a_j or b_j where $j \leq i$. We may also assume that if $i < j$, then $a_3 \neq p_1$ and $b_3 \neq p_1$. If either a_j or b_3 were equal to p_1, they could be replaced by p_2. Finally, the sequence $[T(x,y) \in S \mid x,y \neq p_1]$ can have at most $[(n-1)^2 + (n-1)]/2 - 1$ elements, and since p_1 appeared at most $n - 1$ times in the sequence S, there is at most $[(n-1)^2 + (n-1)]/2 - 1 + (n-1) = [(n^2+n)/2] - 2$ ultratopologies in the sequence. Therefore, this sequence could not generate a chain of maximal length and we have a contradiction. This completes the proof.

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ANOTHER PUZZLE

The day before his execution a prisoner was brought before the king. The people were demanding execution, but the high court was seeking a stay of execution in favor of a lesser penalty. The king decided on a compromise in the form of a wager. The prisoner was presented three urns, 1 black ball and 23 white balls, and was told that the next day three palace guards would be blindfolded and would independently take a ball from one of the urns, without replacing the ball. The prisoner was to be allowed to distribute the balls in the urns. If the black ball was chosen by any one of the three guards, he was to be executed, How should the prisoner arrange the balls?

INITIATION CEREMONY

The editorial staff of the Journal has prepared a special publication entitled *Initiation Ritual* for use by local chapters containing details for the recommended ceremony for initiation of new members. If you would like one, write to:

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Editor, Pi Mu Epsilon Journal
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The University of Oklahoma
Norman, Oklahoma 73069

MATRIX MULTIPLICATION AS AN APPLICATION OF THE PRINCIPLE OF COMBINATORIAL ANALYSIS

by Mary Zimmerman
Western Michigan State University

The *principle of combinatorial analysis* states that if there are two events A and B which are independent (that is, the outcome of neither event depends on the outcome of the other), there are m possible outcomes for event A, and there are n possible outcomes for event B, then there are $m \cdot n$ total outcomes for the two events A and B. For example, if a student is to take an examination consisting of two problems, one problem from a list A of three problems and one problem from a list B of five problems, then the total number of possible examinations is $3 \cdot 5 = 15$. It therefore follows that if A , A_2, $\cdots$, A_k are k events which are pair-wise independent (that is, every two events are independent) and there are a_I possible outcomes for A_i, for $i = 1,2,\cdots,k$, then the total number of outcomes for all k events is $a_1 \cdot a_2 \cdot \cdots \cdot a_k$.

As a somewhat different illustration of this principle, suppose we are given three sets X, Y, Z of cities. Suppose further that there are roads from cities in X to some cities in Y and roads from cities in Y to some cities in Z, but no road from any city in X to any city in Z (that is, to travel from a city in X to a city in Z, one must pass through a city in Y). In this context, we say that a "route" from a city $x \in X$ to a city $z \in Z$ consists of a road from x to some city $y \in Y$ followed by a road from y to a. As a more specific example, assume that $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Z = \{z_1, z_2, z_3\}$ and that the various roads are as depicted in the "graph" G of Fig. 1, where the cities are represented by 'vertices" and the roads by "edges".

The graph G of Fig. 1 is said to model the road system consisting of the ten cities and the various roads between certain pairs of cities. We now consider the number of routes from a city in X to a city in Z. The number of different routes from x_1 to z_1, for example, can be determined by calculating the number of such routes which pass through each of the cities y_1, y_2, y_3, and y_4 and adding these four numbers. To

calculate each of these four numbers, one can think of employing the principle of combinatorial analysis. There is one road from x_1 to y_1 and one road from y_1 to z_1; therefore, there is $1 \cdot 1 = 1$ route from x_1 to z_1 through y_1. In a like manner, we see that there are 0 routes from x_1 to z_1 through y_2, 0 through y_3, and 1 through y_4. Hence, the total number of routes from x_1 to z_1, is $1 + 0 + 0 + 1 = 2$. Similarly, we can calculate the number of different routes from x. to z_k, for any x_I $(i = 1,2,3)$ and any 2, $(k = 1,2,3)$.

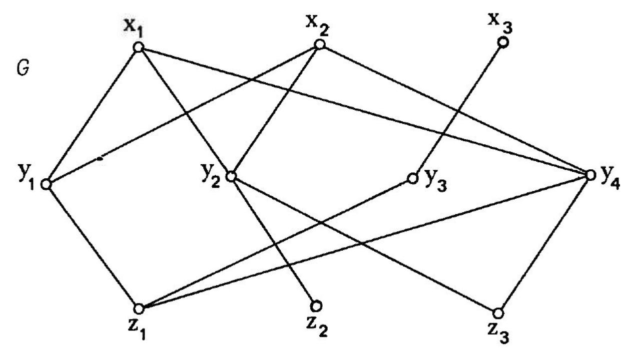

FIGURE 1

In a natural way, two matrices A and B may be associated with the graph G. The matrix A is a representation of the roads between X and Y, and, in particular, $A = [a_{ij}]$ is a 3-by-4 matrix, where a_{ij} is the number of roads from x_i to y_j. Since there is one road from x_1 to y_1, it follows that $a_{11} = 1$. Similarly, $a_{12} = 1$, $a_{13} = 0$, a = 1, and, in fact,

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$(X \text{ to } Y)$

The matrix B is a representation of the roads from Y to Z, namely, $B = [b_{ij}]$ is a 4-by-3 matrix, where b_{Ij} to the number of roads from y_i to z_3 so that

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(Y to Z)

Since a_{11} is the number of roads from x_1 to y_1 and b_{11} is the number of roads from y_1 to z_1, $a_{11} \cdot b_{11}$ is the number of roads from x_1 to z_1, through y_1. Hence, the total number of different routes from x_1 to z_1 can be written as

$$a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} + a_{14} \cdot b_{41} = 1 + 0 + 0 + 1 = 2 ,$$

which we observed earlier. This is reminiscent of matrix multiplication; in fact, this number is calculated in precisely the same manner as the entry in row one, column one of the product matrix $C = A \cdot B$. The matrix $C = [c_{ij}]$, therefore, is a 3-by-3 matrix for which c_{ij} is the number of different routes from x_i to z_j. Hence, we find that

$$C = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

(X to Z)

The foregoing discussion concerning road systems may be extended so that four, five, or more sets of cities are involved. There is a particularly interesting consequence when four sets of cities are considered. Let us suppose there is a fourth set $W = \{w_1, w_2, w_3, w_4\}$ of cities and that the road system is now represented by the graph H of Fig. 2. (We say that the graph G of Fig. 1 is a subgraph of H.)

We now investigate the number of routes from a city in X to a city in W. (By a *route* from a city $x \in X$ to a city $w \in W$, we, of course, mean a road from x to some $y \in Y$, followed by a road from that y to some $z \in Z$, which is then followed by a road from that z to w.) Suppose we wish to calculate the number of routes from x_1 to w_1. We already know that there are 2 routes from x_1 to z_1. Since there is a road from z_1 to w_1, it follows by the principle of combinatorial analysis that the number of routes from x_1 to w_1 through z_1, is $2 \cdot 1 = 2$. Similarly, we see that there are $1 \cdot 0 \cdot 0$ routes from x_1 to w_1 through z_2 and $2 \cdot 1 = 2$ routes from x_1 to w_1

through z_3. Therefore, the total number of routes from x_1 to w_1 is $2 + 0 + 2 = 4$. In a like manner, one can calculate the number of different routes from x_i to w_ℓ, for any x_i ($i = 1,2,3$) and any w_ℓ ($\ell = 1,2,3,4$).

We may now introduce a 3-by-4 matrix $D = [d_{ij}]$ to represent the roads between Z and W, where d_{ij} equals the number of roads from z_i to w_j. From what we have now seen, the number of different routes from a city $x_i \in X$ to a city $w_\ell \in W$ can be determined by calculating the product $E = CD$. The matrix $E = [e_{ij}]$ is a 3-by-4 matrix where e_{ij} is the number of routes from x_i to w_j. We see that

$$CD = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 & 4 \\ 4 & 3 & 5 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = E$$

(X to Z) (Z to W) (X to W)

There is another way of determining the number of different routes from x_i to w_ℓ. If we were to evaluate the product $F = BD$, then we see that $F = [f_{ij}]$ is a 4-by-4 matrix where f_{ij} is the number of routes from y_i ($i = 1,2,3,4$) to w_j ($j = 1,2,3,4$). Therefore,

$$BD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{bmatrix} = F$$

H

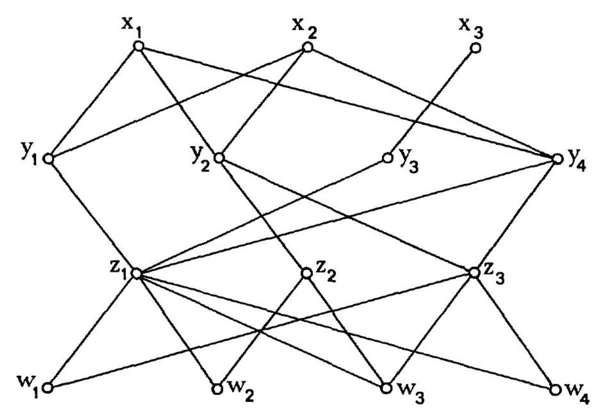

FIGURE 2

We know there is one road from x_1 to y_1 and one route from y_1 to w_1. Thus, by the principle of combinatorial analysis, there is $1 \cdot 1 = 1$ route from x_1 to w_1 through y_1. Similarly, the number of routes from x_1 to w_1 through y_2 is 1, through y is 0, and through y_4 is 2. Therefore, the total number of routes from x_1 to w_1 is $1 + 1 + 0 + 2 = 4$. In this manner, we can determine the number of routes from x_i ($i = 1,2,3$) to w_j ($j = 1,2,3,4$). This, however, is equivalent to calculating the product $M = AF$, where then $M = [m_{ij}]$ is a 3-by-4 matrix and m_{ij} is the number of routes from x_i to w_j. The calculations give us

$$AF = \underset{(X \text{ to } Y)}{\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \underset{(Y \text{ to } W)}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{bmatrix}} = \underset{(X \text{ to } W)}{\begin{bmatrix} 4 & 3 & 5 & 4 \\ 4 & 3 & 5 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = M$$

Of course, $M = E$ since M and E count the same things. This says that $M = AF = CD = E$, or since $F = BD$ and $C = AB$, we conclude that

$$A(BD) = (AB)D .$$

This verifies the associative law of multiplication for the three matrices A, B, and D. This may make it appear more reasonable why matrix multiplication, in general, is associative.

Thus far we have been discussing graphs. The above analysis can also be applied to certain kinds of networks. A *network* is a graph in which each edge is assigned a number or value. (In fact, as we shall see, a graph itself may be considered as a network in which each edge is assigned the number 1.) We consider two examples of networks.

A *multigraph* is a network in which each edge is assigned a positive integer. Often a multigraph is represented by a diagram in which a pair of vertices is joined by the number of edges equal to the value assigned the appropriate edge. For example, we might use a multigraph M to depict roads between a set $X = \{x_1, x_2\}$ of cities and a set $Y = \{y_1, y_2, y_3\}$ of cities as well as roads between Y and a set $Z = \{z_1\}$ of cities (one city in this case). Such an illustration is given in Fig. 3.

As with the graphs we considered earlier, we may associate two matrices A and B with the multigraph M. The matrix A is a representation of the roads between X and Y; namely, $A = [a_{ij}]$ is a 2-by-3 matrix, where

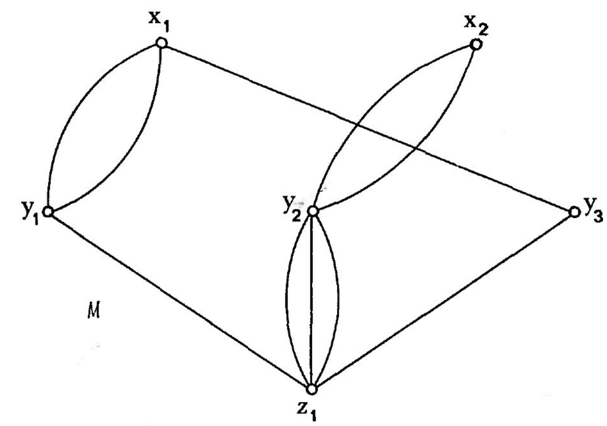

FIGURE 3

a_{ij} is the number of roads from x_i to y_j. The matrix B is a representation of the roads from Y to Z, that is, $B = [b_{ij}]$ is a 3-by-1 matrix, where b_{ij} is the number of roads from y_i to z_j. Hence, it follows that

$$A = \underset{(X \text{ to } Y)}{\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}} \quad \text{and} \quad B = \underset{(Y \text{ to } Z)}{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}} .$$

The product matrix $C = AB$ describes the number of routes from cities in X to cities in Z; in particular, $C = [c_{ij}]$ is a 2-by-1 matrix, where c_{ij} is the number of different routes from x_i to z_j. In this case, we find that

$$C = \underset{(X \text{ to } Z)}{\begin{bmatrix} 3 \\ 6 \end{bmatrix}} .$$

As in the discussion with graphs, the number of routes from cities in X to cities in Z given by the matrix C can be interpreted as an application of the principle of combinatorial analysis. Furthermore, the discussion on multigraphs may be extended to any number of cities. From this, one can deduce the validity of the associative law of multiplication of matrices whose entries are non-negative integers.

As a second example of networks we consider "probability networks". For a specific illustration, we investigate the network N of Fig. 4.

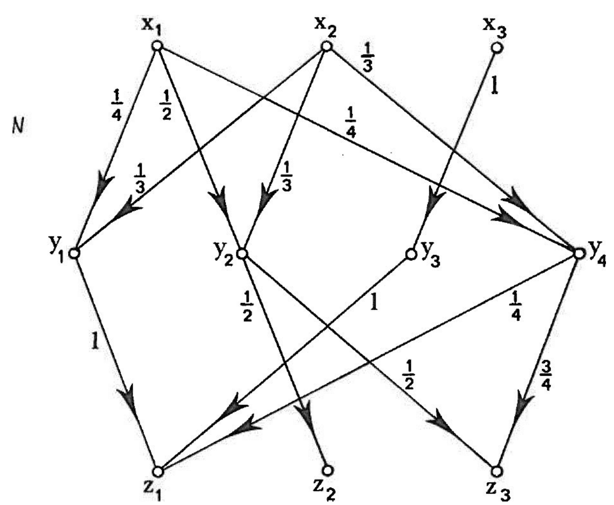

FIGURE 4

The network of Fig. 4 is directed, that is each edge is "directed". We may interpret the network N as a model of a road system; in fact, it is the same road system as in the graph G of Fig. 1. Each edge of N is assigned a number, namely a probability; the probability of going to one city given that we are at a specific city. In particular, the numbers assigned to the edges of N imply the probability of being at city x_1 and going to y_1 is 1/4, of going to y_2 is 1/2, of going to y_3 is 0, and of going to y_4 is 1/4. Hence, the probability of being at x_1 and going to a city in Y is 1/4 + 1/2 + 0 + 1/4 = 1. In the language of probability theory, the conditional probability of going to y_1 given that we are at x_1 is $P(y_1|x_1) = 1/4$. All other probabilities can be described in a like manner.

In a probability network, each probability is a number p such that $0 < p \le 1$ (although an edge with probability 0 is ordinarily omitted) and the sum of the probabilities of the edges incident from a vertex of X or a vertex of Y is one. In symbols, this says that

$$\sum_{j=1}^{4} P(y_j|x_i) = 1 \quad \text{for} \quad i = 1,2,3$$

and

$$\sum_{k=1}^{3} P(z_k|y_j) = 1 \quad \text{for} \quad j = 1,2,3,4.$$

The probability of being at x_1 and going to z_1, written $P(x_1|z_1)$, can also be computed. Again, we turn to the principle of combinatorial analysis. In probability theory notation, we have ,

$$P(z_1|x_1) = \sum_{i=1}^{4} [P(y_i|x_1)\cdot P(z_1|y_i)]$$

$$= \frac{1}{4}\cdot 1 + \frac{1}{2}\cdot 0 + 0\cdot 1 + \frac{1}{4}\cdot\frac{1}{4} = \frac{5}{16} \quad .$$

Thus, the probability of beginning at x_1 and going to z_1 is 5/16. Observe that the method of computing this is very similar to the method used to determine the number of routes from x_1 to z_1 in the graph G of Fig. 1.

The foregoing discussion suggests the introduction of two matrices, namely a 3-by-4 matrix A = $[a_{ij}]$ and a 4-by-3 matrix B = $[b_{ij}]$, where $a_{ij} = P(y_j|x_i)$ and $b_{ij} = P(z_j|y_i)$. Hence we have

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix}$$

(X to Y) (Y to Z)

The probability of going to city z_j ($j = 1,2,3$) given that one begins at city x_i can be determined with the aid of the principle of combinatorial analysis (as we have seen) or by computing the product matrix C = AB, where $C = [c_{ij}]$ is a 3-by-3 matrix with $c_{ij} = P(z_j|x_i)$. In particular,

$$AB = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{16} & \frac{1}{4} & \frac{7}{16} \\ \frac{5}{12} & \frac{1}{6} & \frac{5}{12} \\ 1 & 0 & 0 \end{bmatrix} = C \ .$$

(X to Y) (Y to Z) (X to Z)

We note that the sum of the entries in each row of the matrix C is one; that is,

$$\sum_{j=1}^{3} P(z_j|x_i) = 1 , \qquad i = 1,2,3 .$$

This, of course, states that the probability of going to some city of Z given that we begin at some city of X is one.

In a like manner, probability networks may involve a larger number of sets of cities. In a natural way, several matrices may be introduced and the associative law of matrix multiplication may be verified for matrices having rational numbers as entries.

As a further illustration of the preceding discussion on probability networks, we consider a manufacturing process. Assume that a raw part starts off with three choices of assembly lines, with certain probabilities that it will go to each assembly line. On each assembly line there are specified probabilities that the part will take various routes during the manufacturing process to eventually become one of three different finished products. The probability network of Fig. 5 models the above situation with the probabilities of a part proceeding from one step to another displayed next to the appropriate directed edge.

This manufacturing process may be described in terms of tasks. One task T_1 is to move a raw part to an assembly line, a second task T_2 to transfer the part to a manufacturing process from the assembly line, and a third task T_3 is to finally produce a finished product. We might also

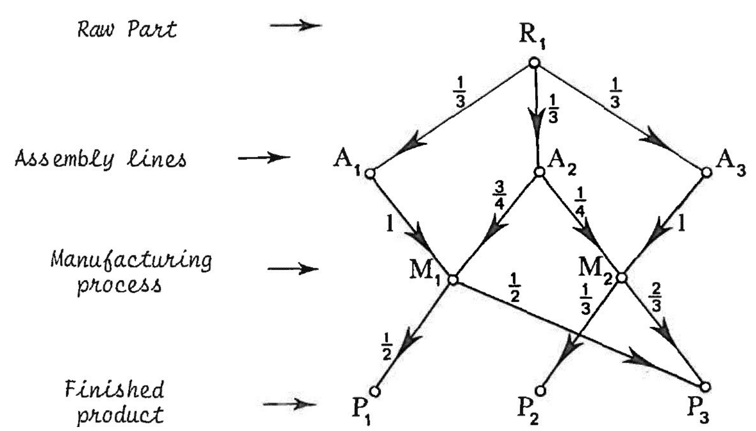

FIGURE 5

consider the "composite task" T of manufacturing a finished product from a raw part. The entire process of converting a raw part into a finished product may be described by means of three matrices, which we denote in the same manner as the corresponding tasks, T_1, T_2, and T_3. These matrices are given below:

$$T_1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad T_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(Raw part to assembly lines) (Assembly lines to manufacturing process) (manufacturing process to finished product)

The product matrix T = $(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3)$ is then a 1-by-3 matrix whose $(1,j)$ entry, $j = 1,2,3$, represents the probability that a raw part R_1 becomes the final finished product P_j. Here, we have that

$$T = \begin{bmatrix} \frac{7}{24} & \frac{5}{36} & \frac{41}{72} \end{bmatrix}$$

, in terms of percentages,

$$T = [29\% \quad 14\% \quad 57\%].$$

This analysis could as well be applied to the probabilities of defects occurring in the assembling process. Such analysis has many applications. In general, it can he used to describe the end results of a process or task that can be expressed as a succession of component tasks.

MATCHING PRIZE FUND

If your chapter presents awards for outstanding mathematical papers or student achievement in mathematics, you may apply to the National Office to match the amount spent by your chapter. For example, $30 of awards can result in the chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of mathematical films. To apply, or for more information, write to:

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1973-1974 MANUSCRIPT CONTEST WINNERS

The judging for the best expository papers submitted for the 1973-74 school year has now been completed. The winners are:

FIRST PRIZE ($200): Charles D. Keys, Louisianna State University, for his paper "Graphs Critical for Maximal Book-thickness" (this *Journal*, Vol. 6, No. 2, pp. 79-84).

SECOND PRIZE ($100): S. Brent Morris, Duke University, for his paper "The Basic Mathematics of the Faro Shuffle" (this *Journal*, Vol. 6, No. 2, pp. 85-92).

THIRD PRIZE ($50): H. Joseph Straight, Western Michigan University, for his paper "Applications of Finite Differences to the Summation of Series" (this *Journal*, Vol. 6, No. 2, pp. 93-98).

1975-1976 CONTEST

Papers for the 1974-75 contest are now being judged, and we are receiving papers for this year's contest, so be sure to send us your paper, or your chapter's papers (at least 5 entries must be received from the same chapter in order to qualify, with a $20 prize for the best paper in each chapter). For all manuscript contests, in order for authors to be eligible, *they must not have received a Master's degree at the time they submit their paper,*

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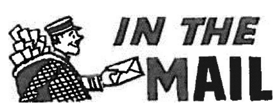

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PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems displaying novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions, if available, and by any information that will assist the editor,

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before June 15, 1976.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

350. *Proposed by R. Robinson Rowe, Sacramento, California.*

In the game of *ELDOS*, an acronym for *Each Loser Doubles Opponents Stacks*, each of n players starts with his 'bank' (B) and at any point in the play holds his 'stack' (S), which he bets on the next round. For each round there is just one loser; in paying the $n - 1$ winners, he doubles their stacks. Consider here a unique game when, after n rounds, each player has lost once and all players end with equal stacks.

(a) For $n = 5$, what was the minimum bank, B, for each player?

(b) How many players, n, were there if the least initial B was 11 cents?

(c) Find a general formula for B_m, the initial B of the m'th player to lose, as a function of m and n.

(d) Using the formula or any other appropriate method, what was the initial bank B of the 9'th of 13 players to lose?

351. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

Angle A and angle B are acute angles of a triangle ABC. If angle $A = 30°$ and h_a, the altitude issuing from A, is equal to m_b, the median issuing from B, find angles B and C.

352. *Proposed by Charles W. Trigg, San Diego, California.*

The edges of a semi-regular polyhedron are equal. The faces consist of eight equilateral triangles and six regular octagons. In terms of the edge e, find the diameters of the following spheres: (a) the sphere touching the octagonal faces, (b) the circumsphere, and (c) the sphere touching the triangular faces. (See solution to problem 198, on page 390 of this *Journal*, Vol. 4, No. 9.)

353. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

It is easy to show that if a and b are complex numbers such that $a + b = 0$ and $|a| = |b|$, then $a^2 = b^2$. Prove that if a, b and c are complex numbers such that $a + b + c = 0$ and $|a| = |b| = |c|$, then $a^3 = b^3 = c^3$. Can this result be extended to more than three numbers?

354. *Proposed by Arthur Bernhart and David C. Kay, University of Oklahoma, Norman, Oklahoma.*

In a triangle ABC with angles less than $2\pi/3$, the *Fermat* point, defined as that point which minimizes the function $f(X) = AX + BX + CX$, may be determined as the point P of concurrence of lines AD, BE and CF, where BCD, ACE and ABF are equilateral triangles constructed externally on the sides of triangle ABC. If R, S and T are the points where PD, PE, and PF meet the sides of triangle ABC, prove that PD, PE and PF are twice the arithmetic means, and that PR, PS and PT are half the harmonic means of the pairs of distances (PB, PC), (PC, PA) and (PA, PB) respectively.

355. *Proposed by John M. Howell, Littlerock, California.*

On the TV game show called "Who's Who?", four panelists try to match the occupations of four contestants with signs marking their occupations. If the first panelist matches correctly, the contestants get nothing and the game is over. If the second panelist succeeds in matching correctly, the contestants get \$25. If the second panelist fails but the third succeeds, the contestants get \$50. If the fourth panelist matches after the third fails, the contestants get \$75. If there is no match, the contestants win \$100. What is the expected value of the contestants' winnings? Assume pure guessing and that no panelist repeats a previous arrangement.

356. *Proposed by Erwin Just, Bronx Community College, Bronx, New York.*

From the set of integers contained in $[1, 2n]$ a subset K consisting of $n + 2$ integers is chosen. Prove that at least one element of K is the sum of two other distinct elements of K.

357. *Proposed by David L. Silverman, West Los Angeles, California.*

Able, Baker and Charlie, with respective speeds $a > b > c$, start at point P with Able designated "it" in a game of Tag which terminates when Able has tagged both Baker and Charlie. At time $-T$, Baker heads north and Charlie south. After a count taking time T, Able starts chasing one of the two quarries. Assuming that Baker and Charlie will maintain their speeds and directions, whom should Able chase first in order to minimize the time required to make the second and final tag?

358. *Proposed by Sidney Penner and H. Ian Whitlock, Bronx Community College, Bronx, New York.*

From a $2n + 1$ by $2n + 1$ checkerboard, in which the corner squares are black, two black squares and one white square are deleted. If the deleted white square and at least one of the deleted black squares are not edge squares, then the reduced board can be tiled with 2×1 dominoes.

359. *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Show that there is an infinitude of pairs of consecutive integers, each pair consisting of a pentagonal number $P_n^5 = \frac{n}{2}(3n - 2)$ and an hexagnumber $P_m^6 = \frac{m}{2}(4m - 2)$.

360. *Proposed by Paul Erdös and Ernst Strauss, University of California at Los Angeles.*

Denote by A_n the least common multiple of the integers $\le n$ and denote by $d(n)$ the number of divisors of n.

(a) Prove that $\sum_{n=1}^{\infty} \frac{1}{A_n}$ is irrational.

(b) Prove that $\sum_{n=1}^{\infty} \frac{d(n)}{A_n}$ is irrational.

(c) Prove that $\sum_{n=1}^{\infty} \frac{f(n)}{A_n}$ is irrational, where $f(x)$ is a polynomial with integer coefficients.

361. Proposed by Carl A. Argila, De La Salle College, Manila, Philippines.

Consider any triangle ABC such that the midpoint P of side BC is joined to the midpoint Q of side AC by the line segment PQ. Suppose R and S are the projections of P and Q respectively on AB, extended if necessary. What relationship must hold between the sides of the triangle if the figure $PQRS$ is a square.

Solutions

326. [Fall 19741 Proposed by Zazou Katx, Beverly Hills, California.

Find solutions of the equation $x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + d^2$, where each of the sets x, y, z and a, b, c, d consists of consecutive integers.

I. Solution by J. S. Frame, Michigan State University.

Writing the two arithmetic sequences in the form

$$y - 1,\ y,\ y + 1 \qquad \text{and} \qquad b - 1,\ b,\ b + 1,\ b + 2 \tag{1}$$

the equality of the two sums of squares requires

$$y^2 = 1 + (2b + 1)^2/3 \tag{2}$$

All positive integral solutions of the Diophantine equation $y^2 = 1 + 3u^2$ are given by the formula

$$\begin{pmatrix} y \\ 3u \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} r^k + r^{-k} \\ \sqrt{3}\,(r^k - r^{-k}) \end{pmatrix} \tag{3}$$

where $r = 2 + \sqrt{3}$ and $1/r = 2 - \sqrt{3}$ are the eigenvalues of the square matrix in (3). Since $3u = 2b + 1$ is odd, k must also be odd, say k = 2m + 1. If q = r^2 = 7 + $4\sqrt{3}$, then

$$\begin{pmatrix} y \\ 2b + 1 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 12 & 7 \end{pmatrix}^m \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 14 & -1 \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} 2y \\ 2b + 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (q^{m+1} + q^{-m-1}) \\ (q^m + q^{-m}) \end{pmatrix} \tag{5}$$

Using $[t]$ to denote the greatest integer $\le t$ we have

$$2y_m = [(q^{m+1} + q^m + 2)/4]$$
$$2b_m + 1 = [(q^{m+1} - q^m)/4], \qquad q = 7 + 4\sqrt{3} \tag{6}$$

The first few solutions for the midvalues (y_m, b_m + 1/21 in the two arithmetic progressions (1) are

$$(\pm 2, \pm 1.5), (\pm 26, \pm 22.5), (\pm 362, \pm 313.5), (\pm 5042, \pm 4366.5) \tag{7}$$

For example, the second positive solution for y and b yields

$$25^2 + 26^2 + 27^2 = 21^2 + 22^2 + 23^2 + 24^2 = 2030 \tag{8}$$

II. Solution by Clayton W. Dodge, University of Maine at Orono.

We have $(y - 1)^2 + y^2 + (y + 1)^2 = a^2 + (a + 1)^2 + (a$ t $2)^2 + (a + 3)^2$, which can be written in the form $3y^2 - 3 = (2a + 3)^2$, so a is divisible by 3. Let a = $3r$. The equation reduces to $y^2 - 1 = 3(2r + 1)^2$ or, letting $u = 2r + 1$, we obtain the Pell equation $y^2 - 3u^2 = 1$, where $a = 3(u - 1)/2$, which requires, of course, that u be odd.

The solutions to this equation are well known and are found by setting $y + u\sqrt{3} = (2 + \sqrt{3})^n$ where n is an odd positive integer. Thus (y,u) = (2,1), (26,151, (362, 209), $\cdots$, which produce (y,a) = (2, 0), (26, 21), (362, 312), $\cdots$; that is,

$$1^2 + 2^2 + 3^2 = 0^2 + 1^2 + 2^2 + 3^2,$$

$$25^2 + 26^2 + 27^2 = 21^2 + 22^2 + 23^2 + 24^2,$$

$$361^2 + 362^2 + 363^2 = 312^2 + 313^2 + 314^2 + 315^2, \cdots$$

Also solved by LOUIS H. CAIROLI, *Syracuse University;* VICTOR G. FESER, *St. Louis University;* R. C. GEBHARDT, *Hopatcong, N. J.;* MARK JAEGER, *Ames, Iowa;* ARTHUR M. KELLER, *Brooklyn, New York;* ROY HAGGARD, *University of Akron, Ohio;* JOHN M. HOWELL, *Littlerock, California;* J.A. HUNTER, *Toronto, Canada;* EDITH E. KISEN, *Portland State University, Oregon;* JIM METZ, *Griffin High School, Springfield, Illinois;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh;* R. ROBINSON ROWE, *Sacramento, Calif.;* CHARLES W. TRIGG, *San Diego, California;* GREGORY WULCZYN, *Bucknell University, Lewisburg, Pennsylvania; and the Proposer.*

Comment by the Problem Editor

Charles W. Trigg offered the following references:

1. H. L. Alder, "n and $n + 1$ Consecutive Integers With Equal Sums of Squares," *American Mathematical Monthly*, 69 (April 1962), 282-285.

2. Brother U. Alfred, "n and $n + 1$ Consecutive Integers With Equal Sums of Squares," Mathematics *Magazine*, 35 (May 1962), 155-164. For $x < y < z$ and $a < b < c < d$, the given equation reduces to $3(x + 1)^2 - 3 = (2a + 3)^2$. Alfred gives the first five values of (x,a) as (25,21), (361, 312), (5041, 4365), (70225, 60816), and (978121, 8470770).

Louis H. Cairoli supplied a reference to the article by Brother Alfred noted above and to an article by Brother Alfred in the September 1967 issue of the *Mathematics Magazine*.

Metz supplied a BASIC program and Keller a FORTRAN IV program used to generate solutions and to confirm their results.

The Proposer derived solutions by using convergents of the simple continued fraction expansion of $\sqrt{3}$ in connection with the Pell equation $p^2 - 3q^2 = 1$.

327. **[Fall 1974]** *Proposed by Charles W. Trigg, San Diego, California.*

On a remnant counter there are six rolls of ribbons containing 31, 19, 17, 15, 13 and 8 yards. There are two widths of ribbons, some rolls being twice as wide as the others. There are no price marks, but all the ribbons sell for the same price per square inch. If you wish to buy \$14.00 worth of each width, buying every roll but one, which roll would you leave on the counter?

Solution by Clayton W. Dodge, University of Maine at Orono.

In the five rolls we buy there must be twice as many yards of the narrow width as of the wide width. Hence the total yardage must be divisible by 3. Since the total yardage of the 6 rolls is $103 \equiv 1 \pmod 3$, we must omit the 31-, 19-, or 13-yard roll. The remaining yardages are 72, 84, and 90 respectively. Now no combination of the remaining rolls will add up to either 24 (= 72/3) or 30 (= 90/3). Since 15 + 13 = 28 = 84/3 and 31 + 17 + 8 = 56 = 2-28, the 19-yard roll must be left.

Also solved by VICTOR G. FESER, St. Louis University, Missouri; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; ROY HAGGARD, University of Akron, Ohio; ARTHUR M. KELLER, Brooklyn College, New York; EDITH E. KISEN, Portland State University, Oregon City, Oregon; STEVE LEELAND, University of South Florida; CHARLES H. LINCOLN, Raleigh, North Carolina; R. ROBINSON ROWE, Sacramento, California; and the Proposer.

328. **[Fall 1974]** *Proposed by Joe Dan Austin, Emory University, Atlanta, Georgia.*

A group of 366 people are sequentially asked their date of birth. Assuming birthdates are independent and all days are equally like@--find P_k, the probability that the first match is obtained when the k'th person is asked. As 366 people must have at least one match,

$$\sum_{k=1}^{366} P_k = 1.$$

Show this directly.

Solution by R. Robinson Rowe, Sacramento, California.

The text assumes no leap-day birthdates. In addition to the definition of P_k as the probability of a first match when the k'th person is asked, we define Q_k by

$$1 - Q_k = \sum_{k=1}^{k} P_k \qquad (1)$$

The first person asked has no one to match, so $P_1 = 0$ and $Q_1 = 1$. The next person has only one to match, so $P_2 = 1/365$ and $Q_2 = 364/365$. For the third person, there is a chance of Q_2 that there has been no previous match and 2 persons with different birthdates to match, so $P_3 = (364/365)(2/365)$ and $Q_3 = Q_2 - P_3 = (364/365)(363/365)$.

With the sequence of persons, corresponding relations will be of the form $P_n = Q_{n-1}(n - 1)/365$ and $Q_n = Q_{n-1}(366 - n)/365$.

This can be expressed generally

$$P_k = \frac{(k - 1)\, 365!}{365^k (366 - k - 1)!} \qquad (2)$$

which answers the first question, and

$$Q_k = \frac{365!}{365^k (366 - k - 1)!}\,. \qquad (3)$$

For the second question, note from (1) that with k = 366 as the upper limit,

$$\sum_{k=1}^{366} P_k = 1 - Q_{366} = 1 - \frac{365!}{365^{366}\,(-1)!} = 1 - 0 = 1$$

since factorial -1 is infinite.

Also solved by JOHN M. HOWELL, Littlerock, California; ZAZOU KATZ, Beverly Hills, California; NOSMO KING, Raleigh, N. C.; MARK JAEGER, Carlton College, Northfield, Minnesota; and the Proposer.

329. [Fall 1974] Proposed by Bernard C. Anderson, Henry Ford Community College, Dearborn, Michigan.

Show that $f(x) = 2x + \sin x$ is a strictly increasing function on $(-\infty, +\infty)$ by using only pre-calculus methods.

Solution by N. J. Kuenzi, The University of Wisconsin-Oshkosh.

More generally, it can be shown that any function of the form $f(x) = ax + \sin x$ with $a \geq 1$ is strictly increasing on $(-\infty, +\infty)$. Let u and v be any real numbers with $u < v$. Then

$$f(v) - f(u) = a(v - u) + \sin v - \sin u.$$

Recall that if $x > 0$ then $\sin x < x$. (Refer to the unit circle.) Hence,

$$\left|\sin\left(\frac{v-u}{2}\right)\right| < (v - u)/2 ,$$

and

$$2\left|\cos\left(\frac{v+u}{2}\right) \sin\left(\frac{v-u}{2}\right)\right| < v - u .$$

Using a standard trigonometric identity yields

$$|\sin v - \sin u| < v - u ,$$

or equivalently,

$$-(v - u) < \sin v - \sin u < v - u .$$

Adding $a(v - u)$ to each term yields

$$(a - 1)(v - u) < f(v) - f(u) < (a + 1)(v - u) .$$

Since $a \geq 1$, it follows that $f(u) < f(v)$ whenever $u < v$.

Also solved by JEFFREY BERGEN, Brooklyn College, Brooklyn, N. Y.; R. C. GEBHARDT, Hopatcong, N. J.; RAY HAERTEL, Central Oregon Community College; CHARLES H. LINCOLN, Raleigh, N. C.; C. B. A. PECK, State College, Pennsylvania; R. ROBINSON ROWE, Sacramento, California; CHANDER LEKHA SABHARWAL, Parks College of St. Louis University, Cahokia, Illinois; and the Proposer.

330. [Fall 1974] Proposed by R. Robinson Rowe, Sacramento, California.

Starting at zero-zero latitude and longitude at 12:00 noon on Monday, Rumline Crowe flew his plane at a constant 180 knots loxodromically North 45° West. Where was he on Tuesday at 12:00 noon, local standard time?

Solution by Zazou Katz, Beverly Hills, California.

Since the northerly and westerly components of the plane's velocity are each equal to 180 knots divided by $\sqrt{2}$, the northerly position after T hours is $180T/\sqrt{2}$ times the conversion factor of $1°$ per 60 nautical miles, or $3°T/\sqrt{2}$ N. Latitude.

One degree of longitude corresponds to 60 nautical miles times the cosine of the latitude. Hence the westerly velocity of $180/\sqrt{2}$ knots can be expressed in terms of degrees of longitude as $3° \sec(\text{latitude})/\sqrt{2}$ or $(3°/\sqrt{2})\left(\sec \frac{3°T}{\sqrt{2}}\right)$.

Then

$$\text{Longitude} = \int_0^T \frac{3°}{\sqrt{2}} \sec \frac{3°T}{\sqrt{2}}$$

$$= \frac{180°}{\pi} \ln \tan\left(45° + \frac{3°T}{2\sqrt{2}}\right) \tag{1}$$

After 24 hours, the position of the plane will be $24(3°/\sqrt{2})$, or 50.91° North Latitude and, by substitution of 24 for T in (1), 59.34° West Longitude, thereby placing Rumline four time zones west, where the local standard time is 8:00 A.M. He now has four additional hours of flying time at his disposal.

Trying T = 28 in (1), we find Rumline at Latitude 59.40° North, Longitude 74.26° West, thus reaching the fifth time zone west and allowing one extra hour of flying time.

Finally, at T = 29, Rumline's position is

Latitude 61.52° or 61°31'06" N

Longitude 78.56° or 78°33'54" W,

which is still in the fifth time zone west, with 75th-meridian time (since one time zone is equivalent to 15° of longitude).

Also solved by STEVE LEELAND, University of South Florida, Tampa, Florida; LEONARD BARR, Beverly Hills, California; and the Proposer, who notes that Rumline reaches the east shore of Hudson Bay near Kingway, Quebec at 12:00 noon on Tuesday. Two incorrect solutions were received.

331. **[Fall 1974]** *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

In a right triangle ABC, $A = 60°$ and $B = 30°$, with D, E, F the points of trisection nearest A, B, C on the sides AB, BC and CA respectively. Extend CD, AE and BF to intersect the circumcircle (O) at points P, Q, R. Show that triangle PQR is equilateral.

I. *Solution by Jeanette Bickley, Webster Groves High School, Webster Groves, Missouri.*

In triangle BCD, $DB = 4r/3$, $CB = r\sqrt{3}$, where r is the radius of the circumcircle of triangle ABC (see Fig. 1). With angle DBC = 30°, the Law of Cosines yields $CD = r\sqrt{7}/3$; from the Law of Sines, $\sin BCD = 2/\sqrt{7}$. Hence $\cos BCD = \sqrt{3/7}$.

Similarly in triangle BAE, $AB = 2r$, $BE = r\sqrt{3}/3$, angle ABE = 30°. Hence $AE = r\sqrt{7/3}$, $\sin BAE = \sqrt{7}/14$ and $\cos BAE = \sqrt{27/28}$.

Then $\cos PQR = \cos(PRB + BRQ) = \cos(BCD + BAE) = 1/2$, and angle PRQ = 60°.

In triangle BAF, $AF = 2r/3$, $AB = 2r$, angle BAF = 60°. Hence $BF = 2r\sqrt{7}/3$, $\sin ABF = \sqrt{21}/14$ and $\cos ABF = 5/7/14$.

In triangle ACD, $AC = r$, $AD = 2r/3$, angle CAD = 60°. Hence $CD = r\sqrt{7}/3$, $\sin ACD = \sqrt{3/7}$, and $\cos ACD = 2/\sqrt{7}$.

Since $\cos RQP = \cos(RQA + AQP) = \cos(ABF + ACD) = 1/2$, it follows that angle RQP = 60°.

Then angle PRQ = angle RQP = 60° implies that triangle PQR is equilateral.

Also solved similarly by the Proposer.

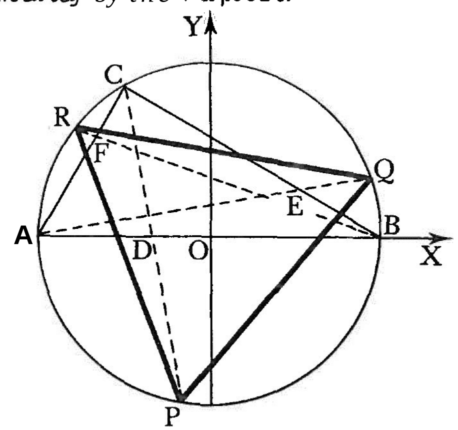

FIGURE 1

II. *Nearly identical solutions by R. Robinson Rowe, Gregory Wulczyn, Charles H. Lincoln, Mike Keith, Mark Jaeger, Jeffrey Bergen and Zazou Katz.*

On the unit circle $x^2 + y^2 = 1$, the coordinates of the vertices of triangle ABC are $A = (-1,0)$; $B = (1,0)$; $C = (-1/2,\sqrt{3}/2)$ whence $D = (-1/3,0)$; $E = (1/2,\sqrt{3}/6)$; $F = (-2/3,\sqrt{3}/3)$.

After the slopes of CD, AE and BF are computed, their intersections with the circle are found to be $P = (-1/7, -4\sqrt{3}/7)$; $Q = (13/14), 3/3/14)$; $R = (-11/14, 5\sqrt{3}/14)$.

Application of the distance formula shows that $PQ = QR = RP = \sqrt{3}$. As a result, triangle PQR is equilateral.

<u>*Comment by the Problem Editor.*</u>

If CD is a Cevian to side AB of any triangle ABC, it is easily verified that $AD/DB = AC \sin ACD/CB \sin CBD$. Applying this principle to the Cevians AE and BF with respect to (AC, AB) and (AB, CB), and noting that $AC:AB:CB = 1:2:\sqrt{3}$, we find that $\tan QAB = \tan CBR = \sqrt{3}/9$, with the result that arc RC and BQ are equal, triangles BCR and RQB are congruent and $RQ = CB = \sqrt{3}$ in the circumcircle of unit radius. Hence angle RPQ = 60°.

In a similar manner, considering the Cevian AE with respect to (AB, AC) and the Cevian CD with respect to (AC, CB), we obtain tan CAE = tan PCB, so that angle CAE = angle PCB and arc CQ = arc PB. It then follows that arc RQ = arc PQ and that $RQ = PQ = CB = \sqrt{3}$ and triangle PQR is equilateral.

Note that CP is perpendicular to AQ. Why?

332. **[Fall 1974]** *Proposed by Richard Field, Santa Monica, California.*

*Several years ago I was spending the evening at the home of a friend who is a musicologist. While there, I received a call from the president of my company, who apologetically told me that he had traced me to ask a question he had to answer at the next morning's board meeting. Specifically, was our monthly average rate of sales growth (6%) compatible with his forecast that our business would double in the next year? I promised to call him back as quickly as possible with an answer. At first I thought I would have to dash home to consult my slide rule, log tables, etc.— but then in a flash it occured to me that my musicologist's library should provide the answer. And indeed it did! I called back in 5 minutes with the answer and proceeded without further disturbance to my social evening. What do you suppose gave me the answer?

Solution by R. Robinson Row, Sacramento, California.

The interval between two musical notes is the ratio of the frequency of the higher note to that of the lower. For consonance, the ratio should be expressible with small numbers—such as 3/2 for a perfect fifth (G/C) or 4/3 for a perfect fourth (F/C). On a stringed instrument, such ratios are possible in any scale, but in the scale of G, for instance, there is an F-sharp and in the scale of D-flat, there is a G-flat—with a slight difference (called a 'comma') between F-sharp and G-flat. Such differences were compromised for the piano and its predecessors by 'tempering' so that there were 12 equal intervals between any note and its octave. Now the octave interval is 2, so if each of the 12 equal intervals (called semitones) is i, then $i^{12} = 2$, whence $i = 1.059463094\cdots$. Analogously, if sales growth was 5.9463094% per month, business would double in 12 months—a year. Surely the value of i could be found in the musicologist's library; since 6% was even more, the boss could be assured of doubling in the next year.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono, Maine;* VICTOR G. FESER, *St. Louis University, St. Louis, Missouri;* R. C. GEBHARDT, *Hopatcong, N. J.;* MARK JAEGER, *Carleton College, Northfield, Minnesota; and the Proposer.*

333. [Fall 1974] *Proposed by Charles W. Trigg, San Diego, California.*

Find integers in the scale of eight whose 6-digit squares are permutations of sets of consecutive digits.

Solution by the Proposer.

If N^2 contains exactly six digits, then $266 \le N < 1000$.

If N^2 is composed of six consecutive digits then it is a permutation of

0, 1, 2, 3, 4, 5 with $N^2 \equiv 1 \pmod 7$ and $N \equiv 1$ or 6; or

1, 2, 3, 4, 5, 6 with $N^2 \equiv 0 \pmod 7$ and $N \equiv 0$; or

2, 3, 4, 5, 6, 7 with $N^2 \equiv 6 \pmod 7$, an impossible situation.

Clearly, N cannot end in 0. If N ends in 2 preceded by an odd digit, or ends in 6 preceded by an even digit, then N^2 ends in 44. If N ends in 3 preceded by a 0 or 4, or ends in 5 preceded by a 3 or 7, then N^2 ends in 11.

These conditions reduce the number of possible values of N in the established range from 142 to 97. Of these, only three have squares with distinct consecutive digits, namely: $(527)^2 = 345621$; $(627)^2 = 503421$; and $(634)^2 = 513420$.

Also solved by R. ROBINSON ROWE, *Sacramento, California.*

334. [Fall 1974] *Proposed by Richard Field, Santa Monica, California.*

What is the 37'th digit in the decimal fraction

$$\sum_{n=1}^{\infty} \frac{1}{10^n - 1} = .122324\cdots?$$

After how many digits does the first zero occur?

Solution by Bob Prielipp, The University of Wisconsin-Oshkosh.

It is easy to see that $\frac{1}{10-1} = \frac{1}{9} = .\overline{1}$, $\frac{1}{10^2-1} = \frac{1}{99} = .\overline{01}$, $\frac{1}{10^3-1} = \frac{1}{999} = .\overline{001}$, $\cdots$, and in general $\frac{1}{10^n - 1} = .\overline{\underbrace{00\cdots0}_{n-1 \text{ zeros}}1}$.

Let $\tau(k)$ denote the number of positive integer divisors of the positive integer k. If $\sum_{n=1}^{\infty} \frac{1}{10^n - 1}$ is thought of as an infinite sum of decimals, then the number of 1's which appear in the kth column to the right of the decimal point is $\tau(k)$. Using the procedures of elementary number theory one can easily determine that the smallest positive integer solution of the equation $\tau(k) = 10 = 5\cdot 2$ is $48 (2^4\cdot 3)$. Also $\tau(49) = \tau(7^2) = 3$ and $\tau(50) = \tau(2\cdot 5^2) = 2\cdot 3 = 6$. Thus the first zero in the given decimal fraction occurs in the forty-eighth place. Because there is no "carry number" involved, the 37'th digit in the given decimal fraction is $\tau(37)$, which is 2 since 37 is a prime.

Also solved by LOUIS G. CAIROLI, *Syracuse University, N. Y.;* VICTOR G. FESER, *St. Louis University, Missouri;* RICHARD A. GIBBS, *Fort Lewis College, Durango, Colorado;* MIKE KEITH, *Hazlet, N. J.;* EDITH E. KISEN, *Portland State University;* CHARLES H. LINCOLN, *Raleigh, N. C.;* SIDNEY PENNER, *Bronx Community College, Bronx, New York;* R. ROBINSON ROWE, *Sacramento, California; and the Proposer.*

335. [Fall 1974] *Proposed by Victor G. Feser, St. Louis University, St. Louis, Missouri.*

Problem 65 in this *Journal* (first presented in April 1954; re-presented in Fall 1968; solved in Fall 1969) showed that every simple non-triangular polygon has at least one interior diagonal, that is, a diagonal lying entirely inside the polygon.

(a) Show that every simple polygon of n sides, $n \geq 3$, has at least $(n - 3)$ interior diagonals.

(b) Show that for every $n \geq 3$, there exists a simple polygon having exactly $(n - 1)$ interior diagonals.

Solution by Charles ff. Lincoln, Raleigh, North Carolina.

(a) A polygon with 4 sides has at least one interior diagonal, so since $n = 4$, the statement is true for $n = 4$. (It is trivially true for $n = 3$.) Assume that the statement is true for all simple polygons up to $n - 1$ sides. Let P be a polygon with n sides. It has at least one interior diagonal. This diagonal creates two simple polygons, one having k sides, the other having $n - k + 2$ sides.

The first has $k - 3$ interior diagonals; the second has $n - k + 2 - 3$. Thus the two together have $n - 4$ interior diagonals, all of which are interior diagonals of P. They are, of course, different from the one which separated P into two polygons. Hence, there are at least $n - 3$ interior diagonals in P.

(b) This part is shown by giving a method of constructing a simple polygon with n sides and exactly $n - 3$ interior diagonals.

On a circle 0, choose points A and B so that arc AB = 60°. Let T be the point of intersection of the tangents to O at A and B. For $n \geq 3$, put $n - 3$ points on 0 between A and B. Polygon $TAP_1P_2 \cdots P_{n-3}$ has exactly $n - 3$ interior diagonals: $TP_1, TP_2, \cdots, TP_{n-3}$.

Also solved by R. ROBINSON ROWE, Sacramento, California; and the Proposer.

Murray S. Klamkin, of the University of Waterloo, Ontario, Canada., states that a solution to this problem may be found in the American **Mathematical** Monthly, *December 1970, page 1111, problem E 2274. For a more difficult unsolved related problem, see 7-25* SIAM Review.

336. [Fall **1974**] *Proposed by Zazou Katz, Beverly Hills, California.*

On the diameter AB of a semicircle (0) perpendiculars are erected at arbitrary points C and D cutting the semi-circumference at points E and F respectively. A circle (P) touches the arc of the semicircle and each of the two half-chords. Show that PQ, the distance from P to the diameter AB, is equal to the geometric mean of AC and DB.

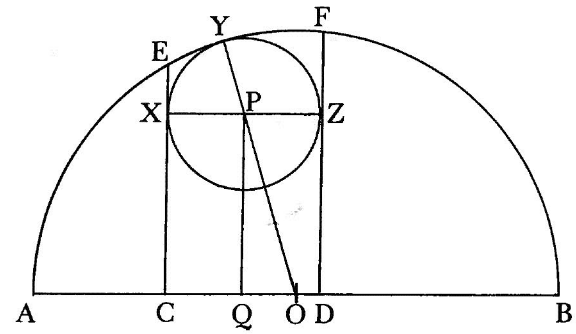

FIGURE 2

Solution by Charles W. Trigg, San Diego, California—practically identical to those submitted by Clayton W. Dodge, University of Maine at Orono, and by Charles H. Lincoln, Raleigh, North Carolina and R. Robinson Rowe, Sacramento, California.

Draw the radius of the semicircle, OPY = R = $AO + OB$, and the radii of the circle (P), XP = PZ = r = PY = CQ = QD. Then

$$AC = AO - CQ - QO = R - r - QO,$$

and

$$DB = OB - QD + QO = R - r + QO.$$

(These equations hold whether 0 falls on the segment CD or outside it.) Multiplying,

$$(AC)(DB) = (R - r)^2 - (QO)^2 = (PO)^2 - (QO)^2 = (PQ)^2.$$

Also solved by VICTOR G. FESER, St. Louis University; RICHARD A. GIBBS, Font Lewis College, Durango, Colorado; DAVID C. KAY, University of Oklahoma, Norman, Oklahoma; GUS MAVRIGIAN, Youngstown State University, Youngstown, Ohio; and the Proposer.

337. [Fall **1974**] *Proposed by the Problem Editor.*

If R, r and ϕ denote the circumradius, the **inradius** and the orthic triangle **inradius** respectively of an acute triangle ABC, show that $r^2 \geq \phi R$. (The orthic triangle is determined by the feet of the altitudes of the parent triangle).

Solution by Zazou Katz, Beverly Hills, California.

It is known that the distance between the **incenter I** and the **orthocenter** H of a triangle is given by the relation $IH^2 = 2r^2 - 4R \cos A \cos B \cos C$ and that ϕ, the **inradius** of the orthic triangle is equal to $2R \cos A \cos B \cos C$.

It follows that

$$\phi/2R \equiv \cos A \cos B \cos C = (2r^2 - IH^2)/4R^2 \leq r^2/2R^2$$

and that $\phi R \leq r^2$.

Also solved by **R. ROBINSON ROWE** *and the Proposer.*

298 [Spring 1973; Spring 1974] *Proposed by Paul Erdös, Budapest, Hungary, and Jan Mycielski, University of Colorado, Boulder, Colorado.*

Prove that

(1) $\lim_{n\to\infty} \frac{1}{n}(\sqrt{n} + \sqrt[3]{n} + \cdots + \sqrt[n]{n}) = 1$,

(2) $\lim_{n\to\infty} \frac{1}{n}(n^{1/\log 3} + n^{1/\log 4} + \cdots + n^{1/\log n}) = e$.

The solution to part (1), by Donnelly J. Johnson, was published in the Spring 1974 issue. The following solution to part (2) is by the Proposers.

First of all, observe that $n^{1/\log 3} > n^{1/\log 4} + \cdots + n^{1/\log n} \geq (n - 2)e$ since $n^{1/\log n} = e$ and the terms are decreasing. Next we give an upper bound for the sum as follows:

Put

$$\sum_{k=3}^{n} n^{1/\log k} = \Sigma_1 + \Sigma_2 + \Sigma_3$$

where in Σ_1, $3 \leq k \leq n^{1/100}$; in Σ_2, $n^{1/100} < k \leq n/\log n$ and in Σ_3, $n/\log n < k \leq n$.

We evidently have

$$\Sigma_1 < n^{1/100} n^{1/\log 3} < n^{1-(1/1000)} < \eta n \text{ for every } \eta > 0.$$

$$\Sigma_2 < e^{100}(n/\log n) < \eta n \text{ for every } \eta \text{ if } n > n_0(\eta)$$

$$\Sigma_3 < n \cdot n^{1/(\log n - \log\log n)} < en(1 + \eta) \text{ if } n > n_0(\eta).$$

Thus

$$\sum_{k=3}^{n} n^{1/\log k} < n + 5\eta n \text{ for every } \eta > 0 \text{ if } n > n_0(\eta),$$

which proves the result.

Comments

The following observation was sent to the Problem Editor by John Oman, The University of Wisconsin-Oshkosh:

"I would like to comment on problem 294. The comment by K. R. S. Sastry that Michael Goldberg's solution holds for any regular polygon is incorrect. The January 1971 issue of *Mathematics* Magazine contains some comments by J. F. Rigby (pages 45-53) which includes a counter-example to the analogous problem for hexagons. In the same comments by Rigby there is a proof for problem 294.

Two other issues of the Mathematics Magazine which contain results on analogous problems are the November 1970 and the May 1971 issues.

I personally think it would be of interest to publish as many different solutions to problem 294 as possible."

Comment by the Problem Editor.

It is the policy of the Problem Department to welcome novel and elegant solutions, including significant improvements on those previously published.

REGIONAL MEETINGS OF MAA

Many regional meetings of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send request to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

JOURNAL WELCOMES NEW OFFICERS

The Pi Mu Epsilon Fraternity elected a new slate of officers during the past year, so we congratulate them and wish them well in their new offices. For the benefit of the membership at large, we introduce them below and include a brief background sketch for each officer.

President

E. Allan Davis, Professor of Mathematics at the University of Utah, received his bachelor's and master's degree from the University of California at Berkeley and earned his Ph. D. in 1951 there also. He came to the University of Utah in 1955, and has taught at the Universities of California and Oregon. He was the Associate Program Director of the National Science Foundation Special Projects in Science Education in 1961-62 and was Program Director for the Student and Cooperative Program in Pre-College Education in Science from 1967 to 1970. He has also served as the faculty advisor of the Utah Alpha Chapter.

President-Elect

Richard V. Andree, Professor of Mathematics and Professor of Information and Computing Sciences at the University of Oklahoma, received his bachelor's degree at the University of Chicago, Ph. M. and Ph. D. degrees at the University of Wisconsin. He has been National Secretary-Treasurer since 1957. An active teacher, author and research worker both in mathematics and computing science, he has written a dozen books and more than 200 articles on mathematics and computing. He is a frequent lecturer at national and international meetings as well as a travelling lecturer for MAA, SIAM, and ACM.

Secretary-Treasurer

Richard A. Good, Professor of Mathematics at the University of Maryland, received his bachelor's degree at Ashland College and his master's and Ph. D. degrees at the University of Wisconsin. Renowned teacher, author, editor, and lecturer, he is active in many local and national mathematical organizations. He is the author and co-author of several books and numerous expository and research articles. Director of several NSF supported projects on teaching of mathematics as well as supervising undergraduate mathematics instruction at the University of Maryland, he is the originator and producer of the University of Maryland television mathematics instruction program.

Councillors

Milton D. Cox, Assistant Professor of Mathematics at Miami University, Ohio, received his bachelor's degree at Depauw University and his master's and Ph. D. degrees at Indiana University. He served as Research Analyst at Aerospace Research Labs, Wright-Patterson AAB. Has been sponsor of Ohio Delta since 1969, and has sponsored the regional Pi Mu Epsilon meeting in the spring of both 1974 and 1975.

Robert M. Woodside, Associate Professor of Mathematics at the University of East Carolina, received his bachelor's and master's degrees at North Carolina State University, and has completed post master's work at Indiana University and Harvard. He is founder and sponser of North Carolina Delta Chapter, and has sponsored student speakers at six national Pi Mu Epsilon meetings, as well as organizing a regional meeting. He has served as chairman of the Faculty Senate at the University of East Carolina, and is a member of numerous honorary and professional organizations.

We also welcome back E. *Maurice Beesley*, Professor and Chairman of the Department of Mathematics at the University of Nevada, and *Eileen L Poiani*, Assistant Professor of Mathematics at Saint Peter's College (Jersey City), who will each be serving another term as *Councillor*. Professor Beesley has been a Councillor since 1969, and Professor Poiani, since 1972. *Houston T. Karnes*, Professor of Mathematics at Louisiana State University, will serve as *Poet-President*, and *David* C. *Kay*, Associate Professor of Mathematics at the University of Oklahoma, will continue as *Journal Editor*.

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