A unified approach to contextuality and violations of macrorealism

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Overview

1. Unified approach to nonlocality & contextuality
2. Example of experiment with non-trivial causal structure: double-slit
3. Causal measurement scenarios and causal empirical models
4. Unified description of

$$\{\text{nonlocality}, \quad \text{contextuality}, \quad \text{violations of macrorealism}\}$$

Remark

- This work extends the *sheaf-theoretic framework* for contextuality initiated by Abramsky & Brandenburger
- Key additional ingredient: causal structure
- Alternative unified approach in the *Contextuality-by-Default framework*: Kujala-Dzhafarov-Larsson [PRL 115, 150401 (2015)]

Nonlocality & contextuality recap

Empirical Data (e.g. CHSH experiment)

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
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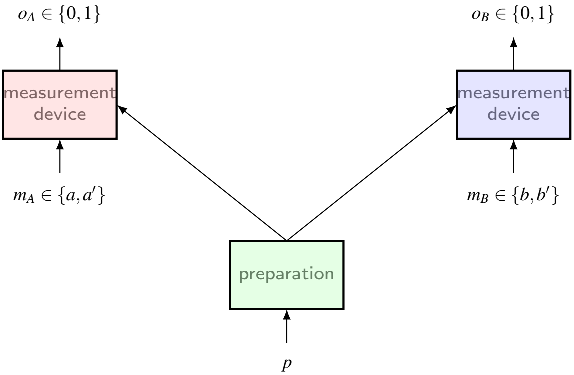

Measurement Scenarios: CHSH

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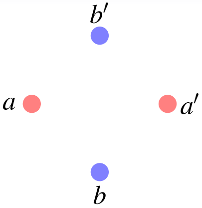

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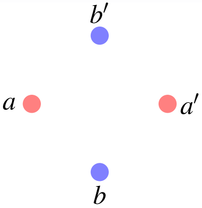

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$$X = \{a, a', b, b'\}$$

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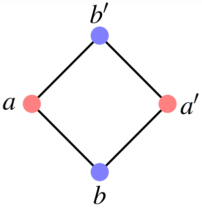

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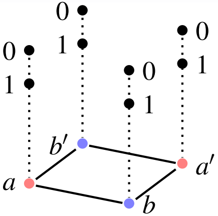

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O a finite set of outcomes — e.g.

$$O = \{0,1\}$$

Empirical Models

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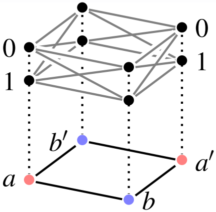

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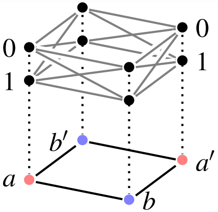

- Fix a measurement scenario $\langle X, \mathscr{M}, O \rangle$
- *Empirical model*: family $\{e_C\}_{C \in \mathscr{M}}$ where each $e_C \in \mathsf{Prob}(O^C)$
- i.e. a distribution for each context:

$$e_{\{a,b\}} = \mathsf{prob}(o_a, o_b | a, b), \quad \ldots, \quad e_{\{a',b'\}} = \mathsf{prob}(o_{a'}, o_{b'} | a', b')$$

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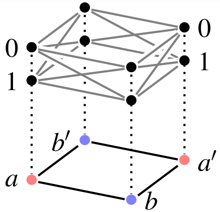

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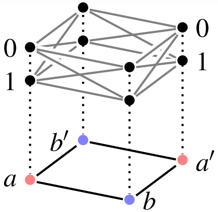

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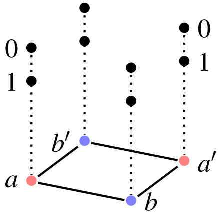
(generalised) no-signalling

Classical Correlations

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$1/2$	0	0	$1/2$
(a',b)	$1/2$	0	0	$1/2$
(a',b')	0	$1/2$	$1/2$	0

Classical data arises as a convex combination of *global assignments*:

$$
\begin{gathered}
(a,a',b,b') \mapsto (0,0,0,0), \\
(a,a',b,b') \mapsto (0,0,0,1), \\
\ldots \quad , \\
(a,a',b,b') \mapsto (1,1,1,1)
\end{gathered}
$$



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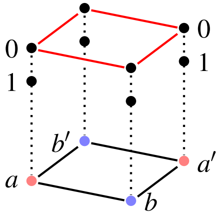

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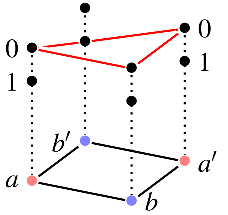

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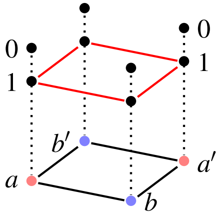

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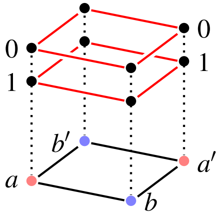

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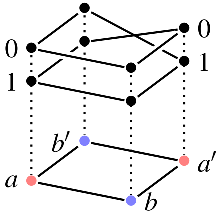


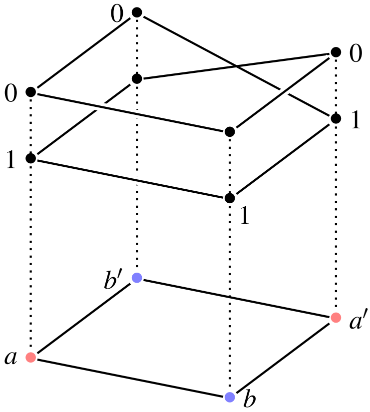
However, the above correlations *cannot* be obtained as a convex combination of global assignments!

PR box bundle diagram

Logical contextuality:
present at the level of possibilities
Strong contextuality:
no event can be extended to a global assignment

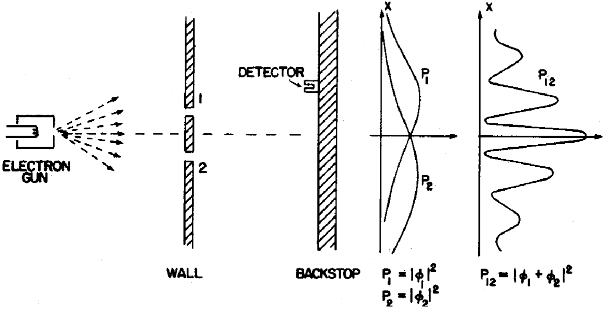
PR box possibility table:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	✓	×	×	✓
(a,b')	✓	×	×	✓
(a',b)	✓	×	×	✓
(a',b')	×	✓	✓	×



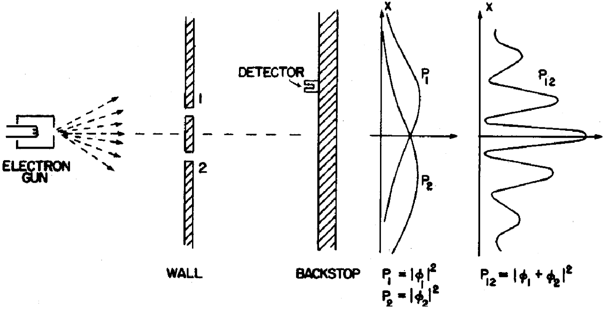
Example: the double-slit experiment

A classic experiment



- Turn down intensity to shoot one electron at a time

A classic experiment



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- Check which way electron passes: ballistic distribution $\frac{1}{2}(P_1 + P_2)$
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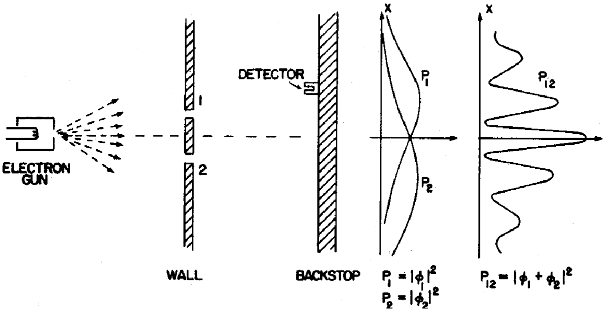

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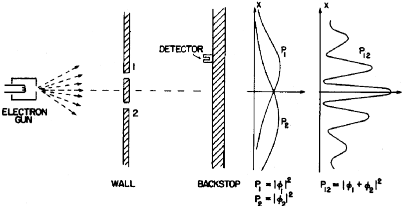
- Wave-particle duality mysticism, etc. ...

A classic experiment — empirical model?

Position detector in dark fringe



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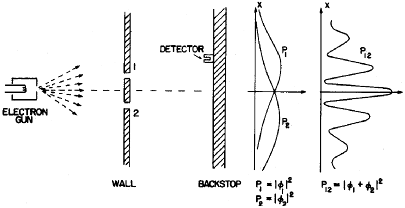

Measurements $X = \{x_e, x_w, x_d\}$,

1. x_e — *emission*, values in $\{0, 1\}$
2. x_w — *which way* (non-demolition), values in $\{0, 1, 2\}$
3. x_d — *detection*, values in $\{0, 1\}$

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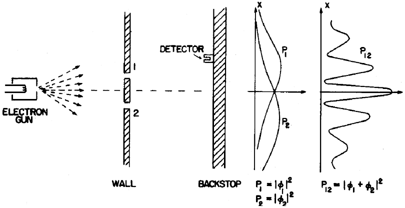

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Contexts,

$$\mathscr{M} = \quad \{\{x_e, x_w, x_d\}, \quad \{x_e, x_d\}\}$$

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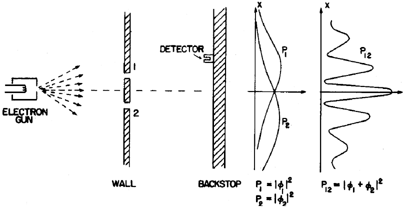

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Outcome set

$$O = \{0, 1, 2\}$$

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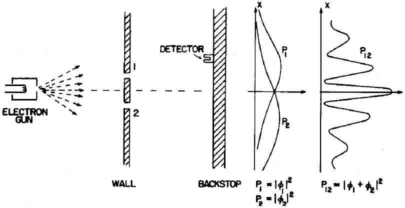

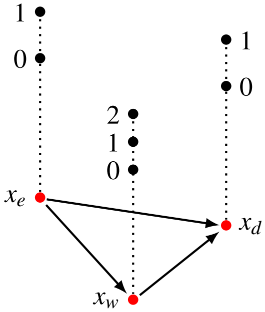

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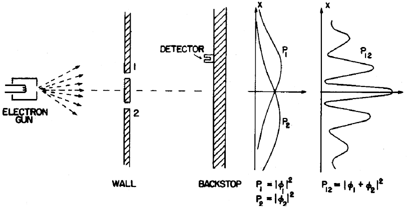

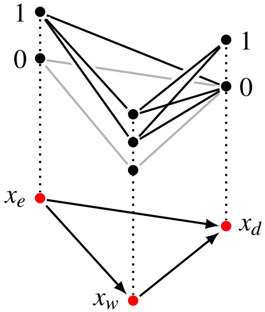

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Trouble with marginals

- Empirical models satisfy 'local' consistency:

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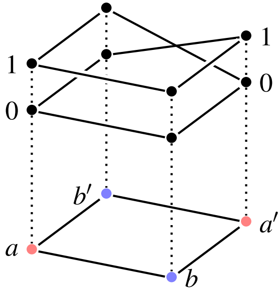
(generalised) no-signalling



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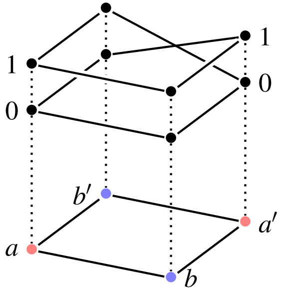
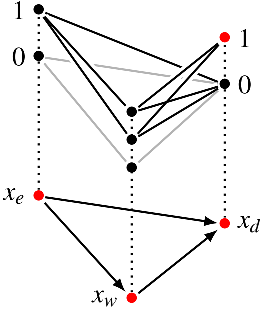
(generalised) no-signalling

- Not the case here:

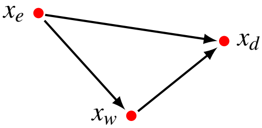
$$\text{prob}(1_d \mid x_d, x_e) \neq \text{prob}(1_d \mid x_d, x_w, x_e)$$





Solution: causal structure

- Marginals only make sense if they respect causal structure



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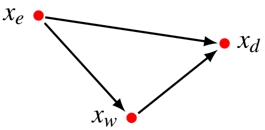

- Given a poset $(X, \leq)$ define a partial order $\leq$ on $\mathscr{P}(X)$ by

$$A \leq B \quad \textit{iff} \quad \downarrow A \cap B = A$$

i.e. B reduces to A via successive removals of maximal elements

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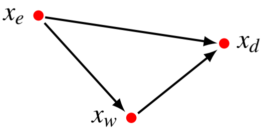

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- e.g. $\{x_e, x_w\} \leq \{x_e, x_w, x_d\}$ but $\{x_e, x_d\} \not\leq \{x_e, x_w, x_d\}$

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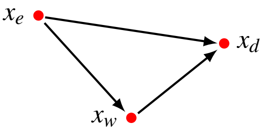

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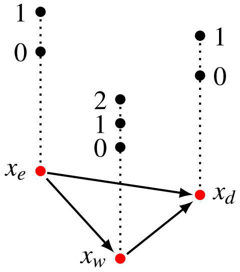
  - Intuition: *later* measurements cannot affect *earlier* outcomes
    Earlier measurements can affect *later* outcomes

Causal empirical models

Causal measurement scenarios



A *causal measurement scenario* is a tuple $\langle X, \leq, \mathscr{M}, O \rangle$ where:

- $(X, \leq)$ a partially ordered set of measurements — e.g.

$$X = \{x_e, x_w, x_d\}$$

- $\mathscr{M}$ the maximal contexts (wrt '$\leq$') — e.g.

$$\mathscr{M} = \{\{x_e, x_d\}, \quad \{x_e, x_w, x_d\}\}$$

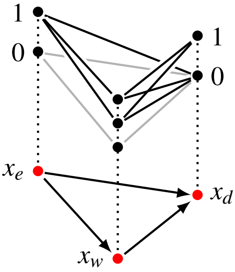
- O a finite set of outcomes — e.g.

$$O = \{0, 1, 2\}$$

Causal empirical models

Fix a causal measurement scenario $\langle X, \leq, \mathcal{M}, O \rangle$

Causal empirical model: family $\{e_C\}_{C \in \mathcal{M}}$ where each $e_C \in \mathrm{Prob}(O^C)$



- i.e. a distribution for each context:

$$e_{\{x_e, x_w, x_d\}} = \mathrm{prob}(o_e, o_w, o_d \mid x_e, x_w, x_d), \qquad e_{\{x_e, x_d\}} = \mathrm{prob}(o_e, o_d \mid x_e, x_d)$$

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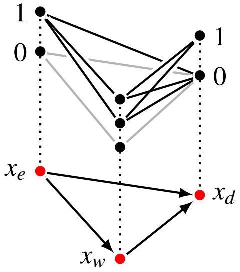

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- 'Local' consistency of causal marginals:

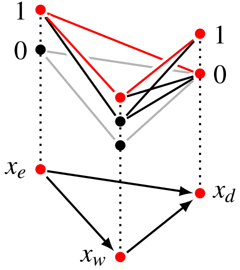
$$\mathsf{prob}(o_e \mid x_e, x_w, x_d) = \mathsf{prob}(o_e \mid x_e, x_w) = \mathsf{prob}(o_e \mid x_e), \text{ etc.}$$

generalised no signalling (outside of forward light-cone)

Contextuality and macrorealism

Double-slit model is logically contextual

i.e. cannot be obtained as convex combination of global assignments



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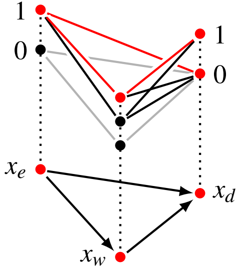

For totally ordered scenarios, *macrorealism* (Leggett-Garg) has 3 requirements (Maroney-Timpson):

1. Macrorealism *per se* — mixtures of deterministic hidden states
2. Non-invasiveness — parameter-independence (forwards in time)
3. Induction — parameter-independence (backwards in time)

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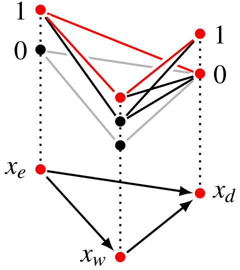

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Macrorealism $\longleftrightarrow$ non-contextuality

Special causal scenarios

1. (Standard/static) empirical models:
   '$\leq$' is trivial *or* each maximal context contains only pairwise incomparable elements

2. Leggett-Garg-type models:
   '$\leq$' is a total order

3. Temporal Bell-type models (Fritz):
   X can be partitioned into time steps as $(X_t)_{t\in\mathbb{N}}$, such that:
   - $\mathscr{M} = \prod X_t$
     (contexts contain one measurement per time step)
   - For all $x \in X_t, y \in X_{t'}$ we have $x \leq y$ iff $t \leq t'$
     (causal order is induced by the partition)

Conclusion

- Unified approach to

$$\{\text{nonlocality}, \quad \text{contextuality}, \quad \text{violations of macrorealism}\}$$

Representative examples:

$$\{\text{Bell}, \quad \text{Kochen-Specker}, \quad \text{Leggett-Garg}\}$$

Conclusion

- Unified approach to

  {nonlocality, contextuality, violations of macrorealism}

  Representative examples:

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Future directions (relating to other talks):

- Quantifiying ‘contextuality’ in causal scenarios (Rui)
- Process calculus (Rui)
- Advantages over classical causal models in informatic tasks (Samson)
- Extension to indefinite causal structures (Philip)
- Clarify relation to unified approach in CbD (Ehtibar)