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A Short Note on the Free Implication Algebra over a Poset

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Abstract

In this short note, the free implication algebra over a poset is studied. Also, the cardinality of this free algebra is computed for special posets. In addition, examples of other algebraic structures are presented. Our work is done using a general method for constructing the free algebra over a poset in finitely generated varieties given in [8]. In the literature, there are different constructions of these free algebras (for the classes of algebras presented here). The aim of this paper is to show that this construction can be used in all these classes of algebras.

Keywords: Free algebras, Free algebras over a poset, implication algebras, Tarski algebras, De Morgan algebras, bounded distributive lattices

1 Introduction

In 1945, R. Dilworth ([6]) introduced the notion of free lattice over a poset. Later, this notion was adapted to different classes of algebras that arise from non-classical logics, these classes constitute varieties of algebras which have an underlying order structure definable by means of certain equations $p_i(x, y) = q_i(x, y)$, $1 \leq i \leq n$, in terms of the algebra's operations and some positive integer n. Constructions of this particular free algebra have been exhibited for different kind of algebras such as bounded distributive lattices, De Morgan algebras, Hilbert algebras (see [7, 8]) and, more recently, Łukasiewicz-Moisil algebras ([9]).

Consider, now, a set Ω of operations of type τ and the set E of identities. We shall note $\mathbf{Alg}_{\{\Omega,E,\leq\}}$ the category whose objects are $\{\Omega, E\}$-algebras which have an order structure definable from the operations of Ω and the arrows are the respective $\{\leq, \Omega\}$-morphisms where $\leq$ is the order from the operations of Ω.

The notion of free algebra over a poset relative to $\mathbf{Alg}_{\{\Omega,E,\leq\}}$ can be defined as follows:

Let $(X,\leq) = X_{\leq}$ be a poset. We shall say that $\mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq})$ is the free $\{\Omega, E\}$-algebra over $X_{\leq}$ if the following conditions are satisfied:

(F1) there is an one-to-one order-preserving function $g : X_{\leq} \to \mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq})$,

(F2) for each $A \in \mathbf{Alg}_{\{\Omega,E,\leq\}}$ and each one-to-one order-preserving function $f : X \to A$, there is a unique morphism $h : \mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq}) \to A$ such that $h \circ g = f$.

It is clear that from (F1), we can assert that the algebra $\mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq})$ contains a sub-poset which is isomorphic to $X_{\leq}$. If $X_{\leq}$ is an anti–chain, then the algebra $\mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq})$ is the usual free algebra (see [3]). This make us believe that may be cases where this new free algebra is not an object of the category $\mathbf{Alg}_{\{\Omega,E,\leq\}}$. Indeed, let $\mathbf{Alg}_{\{\odot,E_1,\preceq\}}$ be the category whose objects are algebras $(A,\odot)$ of type 2 characterized by the set of equations $E_1 = \{x \odot y \approx y\}$. The order relation $\preceq$ is given by $x \preceq y$ iff $x \odot y = x$. Then, it is easy to see that every object $\mathbf{Alg}_{\{\odot,E_1,\preceq\}}$ has an underlying order structure of anti–chain. However, if we take the poset $(I_2,\leq)$ where I_2 is the two–element chain, then we have that $\mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(I_2,\leq)$ is not an object of $\mathbf{Alg}_{\{\odot,E_1,\preceq\}}$.

The construction of this free algebra can be presented using techniques of the universal algebra (see [3]) or using the well–known Freyd's Adjoint Functor theorem of category theory (see [11]).

A general construction in varieties finitely generated in the following way can be given ([8]):

Let $\boldsymbol{\mathcal{V}}$ be a variety generated by n algebras S_i, $n < \omega$ and where the variety $\boldsymbol{\mathcal{V}}$ has an order given by the basic operations. Suppose $C = \prod_{i=1}^{n} S_i$ is not an antichain. Let I be a non-empty poset and let E be the set of all increasing functions from I to the $\boldsymbol{\mathcal{V}}-$algebra C. Besides, let $g : I \to C^E$ be defined by $g(i) = G_i$ where $G_i(f) = f(i)$, for all $f \in E$ and $i \in I$. Then, $L = [G]_{\boldsymbol{\mathcal{V}}}$ is the free $\boldsymbol{\mathcal{V}}-$algebra over I, where $G = \{G_i : i \in I\}$ and [G] is the $\boldsymbol{\mathcal{V}}-$algebra generated by G. Indeed, it follows easily that $i \leq j$ implies $G_i \leq G_j$, for all $i,j \in I$. On the other hand, let us suppose that there are $i,j \in I$ such that $G_i \leq G_j$ and $i \not\leq j$. Now, let us consider $a,b \in C$, $a < b$ and define $f^* : I \to C$ by

$$f^*(k) = \begin{cases} b & \text{if } k \geq i \\ a & \text{otherwise} \end{cases} .$$

Hence, we have that $f^* \in E$, $f^*(i) = b$ and $f^*(j) = a$. These statements imply that $G_i(f^*) \not\leq G_j(f^*)$, which is a contradiction. Thus, g is an order–embedding. Besides, by the definition of g we get that $G = g(I)$ and so, $L = [g(I)]_{\mathcal{V}}$. Therefore, (F1) holds.

Now we assume that A is a $\mathcal{V}-$algebra and $f : I \to A$ is an increasing function. Since $\mathcal{V}$ is the variety generated by C, we have that A is isomorphic to a subalgebra A^* of C^X, where X is an arbitrary set. Then, there is an isomorphism $\varphi : A \to A^*$ defined by the prescription $\varphi(a) = H_a$, where $H_a \in C^X$ for all $a \in A$ and so, let us consider the function $\varphi^* = \varphi \circ f$ where $\varphi^*(i) = \varphi(f(i)) = H_{f(i)}$. We claim that there is a homomorphism $h : L \to A^*$ such that $h \circ g = \varphi^*$. Indeed, for each $x_0 \in X$ we define $\alpha_{x_0} : I \to C$ by $\alpha_{x_0}(i) = H_{f(i)}(x_0)$. Then, we infer that $\alpha_{x_0} \in E$. This assertion allows us to consider the function $k : X \to E$, defined by $k(x) = \alpha_x$ for all $x \in X$. Hence, it is routine to check that $h : L \to C^X$ where $h(F) = \overline{F}$ being $\overline{F}(x) = F(k(x))$, is a homomorphism. Moreover, we have that $(h \circ g)(i) = h(G_i) = \overline{G_i}$. Thus, for all $x \in X$ we infer that $\overline{G_i}(x) = G_i(k(x)) = G_i(\alpha_x) = \alpha_x(i) = H_{f(i)}(x) = \varphi^*(i)(x)$, which enables us to conclude that $(h \circ g)(i) = \varphi^*(i)$, for all $i \in I$. Finally, we have that $h(L) \subseteq A^*$. Indeed, since $L' = \{F \in C^E : h(F) \in A^*\}$ is a $\mathcal{V}-$subalgebra of C^E and $G_i \in L'$ for all $i \in I$, then $L \subseteq L'$ and consequently $h(L) \subseteq A^*$. Therefore, (F2) holds.

In this work, we shall study the free algebra over a poset in the variety of implication algebras, De Morgan algebras and bounded distributive lattice. Many constructions of this free algebras to different finitely generated varieties has been given. The aim of this paper is to show all these constructions can be changed by our own. Some paper's results were presented in the preprint [7]. The paper is organized as follows. In section 2, we recall definitions and properties of these algebras to facilitate the reading of the work. In Section 3, we calculate the cardinality of the free algebra over specials finite posets. In sections 4 and 5, we show some examples.

2 Implication algebras

J. Abbott ([1]) and A. Monteiro (in the 60's) studied independently and almost simultaneously implication algebras. The second author called them *Tarski algebras* in lectures given at Universidad Nacional del Sur (see [10]). More recently, several authors have been interested in these algebras (see for example [4]).

Recall that these algebras can be defined as algebras $(A, \to, 1)$ of type $(2, 1)$ which satisfy the following identities:

(I1) $1 \to p = p$,

(I2) $p \to p = 1$,

(I3) $p \to (q \to r) = (p \to q) \to (p \to r)$,

(I4) $(p \to q) \to q = (q \to p) \to p$,

The class of implication algebras will be denoted by $\boldsymbol{I}$ and we shall say that each object A of $\boldsymbol{I}$ is an I-algebra.

We state here, without proof, some results of the theory of implication algebras that are necessary in the sequel:

Theorem 2.1 [1, 10]

(I5) $p \to p = 1$,

(I6) $p \to (q \to p) = 1$,

(I7) $p \to (q \to r) \to (p \to q) \to (p \to r) = 1$,

(I8) $((p \to q) \to p) \to p) = 1$,

(I9) *the relation* $\leq$ *defined by* $x \leq y$ *if only if* $x \to y = 1$ *is a partial order on* A*, and* $x \leq 1$ *for all* $x \in A$*,*

(I10) $(A \leq)$ *is a join-semillattice, where the supremum of the elements* x, y *is* $x \vee y = (x \to y) \to y$*,*

(I11) *if* $\boldsymbol{A} = (A, \to, 1)$ *is an* I*-algebra with firts element* 0*, and the opertations* $\vee, \sim, \wedge$ *are defined by* $x \vee y = (x \to y) \to y$*,* $\sim x = x \to 0$*,* $x \wedge y = \sim (\sim x \vee \sim y)$*, then* $B(\boldsymbol{A}) = (A, \vee, \wedge, \sim, 0, 1)$ *is a Boolean algebra. If* $\boldsymbol{A} = (A, \vee, \wedge, \sim, 0, 1)$ *is a Boolean algebra and* $\to$ *is defined by the formula* $x \to y = \sim x \vee y$*, then* $T(\boldsymbol{A}) = (A, \to, 1)$ *is a* I*-algebra.*

(I12) *Let* $C_2 = \{0, 1\}$ *and* $\to$ *be defined by* $x \to y = 1$ *if* $x \leq y$ *and* $x \to y = 0$ *otherwise, then* $(C_2, \to, 1)$ *is an* I*-algebra, moreover it is a Hilbert algebra* (*see* [5]).

A subset D of an I-algebra A is said to be a deductive systems (d.s.) iff $1 \in D$ and $x, x \to y \in D$ implies $y \in D$. It is well–known that there exists a lattice isomorphism between the lattice of congruence relations of A and the set all deductive systems of A. Then, we have the next theorem.

Theorem 2.2 [10] *For every* I*-algebra* A *the following conditions hold:*

(i) *The variety* $\boldsymbol{I}$ *is* **semi-simple**, *and* A *is simple iff* $A \simeq C_2$.

(ii) *If* A *is finite and it has first element, then* $A \simeq C_2^r$ *where* r *is a positive integer.*

(iii) *The variety* $\boldsymbol{I}$ *has the* **congruence extension property**.

(iv) *The variety* $\boldsymbol{I}$ *is* **locally finite**.

3 Free algebras over a poset

Let us consider a finite poset $X_{\leq} = (X, \leq)$ and let E be the set

$$\{f : X \longrightarrow C_2;\ f \textit{ is a order-preserving function }\}.$$

Then, the set $G = \{G_j\}_{j \in X}$ where $G_j \in C_2^E$ defined by $G_j(f) = f(j)$ is the set of free-generators of $\mathbf{Free}_{\boldsymbol{I}}(X_{\leq})$, where $\mathbf{Free}_{\boldsymbol{I}}(X_{\leq})$ is the free I-algebra over the poset $X_{\leq}$. Therefore, we have the following lemma.

Lemma 3.1 *Let* $F \in \mathbf{Free}_{\boldsymbol{I}}(X_{\leq})$ *and* $D \in \Pi(\mathbf{2}^E)$. *Then,* $F \vee D \in \mathbf{Free}_{\boldsymbol{I}}(X_{\leq})$, *where* $\Pi(\mathbf{2}^E)$ *is the set of atoms of* $\mathbf{2}^E$.

Proof. Let $E = \{h_1, \ldots, h_s\}$ with $s < \omega$. If $D \in \Pi(\mathbf{2}^E)$, then there is a unique $h_j \in E$, $1 \leq j \leq s$ such that $D(h_j) = 1$ and $D(h_k) = 0$ for all $k \neq j$, with $1 \leq k \leq s$. Then, it is verified that $D = \bigwedge_{i \in X} G_i{}'$ where

$$G'{}_i = \begin{cases} G_i, & \text{if } G_i(h_j) = 1 \\ -G_i, & \text{if } G_i(h_j) = 0 \end{cases}.$$

It is clear that there is $H_i \in L_{\boldsymbol{I}}(X_{\leq})$ such that $F \vee G_i{}' = H_i \to F$, from which: $F \vee D = F \vee \bigwedge_{i \in X} G_i{}' = \bigwedge_{i \in X} (F \vee G_i{}') = \bigwedge_{i \in X} (H_i \to F) = \bigwedge_{i \in X} (-H_i \vee F) = -(\bigvee_{i \in X} H_i) \vee F = (\bigvee_{i \in X} H_i) \to F \in L_{\boldsymbol{I}}(X_{\leq})$. ■

On the other hand, we can see that the following lemma is verified for any finite poset X.

Lemma 3.2 $\mathbf{Free}_{\boldsymbol{I}}(X_{\leq}) = \bigcup_{i \in X} [G_i)$ *where* $[G_i) = \{x \in \mathbf{2}^E : G_i \leq x\}$.

Proof.

Since $G_i \in [G_i)$ for all $i \in I$ and $\bigcup_{i \in X} [G_i)$ is a subalgebra of $\mathbf{2}^E$, we have that $\mathbf{Free}_{\boldsymbol{I}}(X_{\leq}) \subseteq \bigcup_{i \in X} [G_i)$.

Reciprocally, if $F \in \bigcup_{i\in X}[G_i)$, then there is $i_0 \in X$ such that $F \in [G_{i_0})$ and therefore $G_{i_0} \leq F$. Besides, as X is a finite set we have that $\mathbf{2}^E$ is a finite Boolean algebra. Since $F \in \mathbf{2}^E$ we may assert that $F = \bigvee\{D_F : D_F \in \Pi(\mathbf{2}^E), D_F \leq F\} = G_{i_0} \vee \bigvee\{D_F : D_F \in \Pi(\mathbf{2}^E), D_F \leq F\} = \bigvee\{(G_{i_0} \vee D_F) : D_F \in \Pi(\mathbf{2}^E), D_F \leq F\}$.

Then, by Lemma 3.1, we have that $F \in \mathbf{Free}_{\boldsymbol{I}}(X_{\leq})$. ■

In order to determine the number of elements of $\mathbf{Free}_{\boldsymbol{I}}(X_{\leq})$ when X is an antichain with n elements, we can write the following equation:

$$|\boldsymbol{Free}_I(X_{\leq})| = |\bigcup_{i=1}^{n}[G_i)| = \sum_{k=1}^{n}(-1)^{k+1}\alpha_k(n),$$

where $\alpha_k(n) = \sum_{1\leq i_1<\cdots<i_k\leq n} |[G_{i_1})\cap[G_{i_2})\cap\cdots\cap[G_{i_k})|$ and for every set Y we denoted the cardinalty of Y by $|Y|$.

We can rewrite the above equation as:

$$|\bigcap_{j=1}^{k}[G_{i_j})| = |\bigcap_{t=1}^{k}[G_t)|.$$

Let $S_k = \bigcap_{t=1}^{k}[G_t)$ and $G_k^* = \bigvee_{i=1}^{k} G_i$. Then, we have that $S_k = [G_k^*)$ and therefore S_k is an I-subalgebra of $\boldsymbol{Free}_I(X_{\leq})$ which has first element. Besides, it is clear that for each maximal d.s. D of S_k there is (by Theorem 2.2 (iii)) a maximal deductive system M of $\boldsymbol{Free}_I(X_{\leq})$ such that $D = M \cap S_k$. Now, let $\boldsymbol{M}_k$ be the set of all maximal deductive system of $\boldsymbol{Free}_I(X_{\leq})$ such that $M \in \boldsymbol{M}_k$ implies $S_k \nsubseteq M$. Hence, according to Theorem 2.2 (ii), we have that $S_k \simeq \prod_{D\in\boldsymbol{M}_k} S_k/D \simeq {C_2}^{\alpha_k}$, where α_k is the number of maximal deductive systems of S_k. Therefore,

$$|\mathbf{Free}_{\boldsymbol{I}}(X_{\leq})| = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}|C_2|^{\alpha_k}$$

On the other hand, for each $M \in \boldsymbol{M}_k$ there exists a unique I-epimorphism $h : \boldsymbol{Free}_I(X_{\leq}) \to C_2$ such that $M = Ker(h)$ and $S_k \nsubseteq Ker(h)$. Also, for each I-epimorphism h (in the same conditions above) there exists a funtion $f : \{G_i\}_{i\in X} \to C_2$ such that $h|\{G_i\}_{i\in X} = f$. It is clear that $f(G^k) = 0$ where $G^k = \{G_1, G_2, \cdots, G_k\}$ and therefore $\alpha_k = 2^{m-k}$. From what was sated above, we have that the following theorem holds.

Theorem 3.3 $\sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}2^{2^{m-k}}$ *is the cardinalty of the free implication algebra with n generators.*

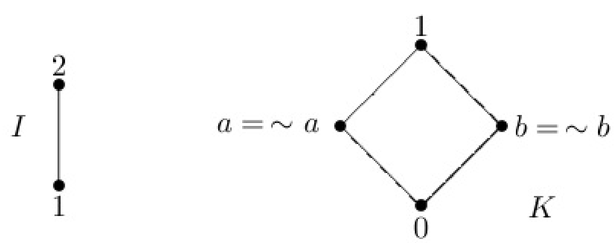
Note that the theorem 3.3 was obtined by A. Monteiro in 60's year (see [10]) using different techniques.

4 De Morgan algebras

In what follows we shall exhibit the free De Morgan algebra over the two-element chain. Firstly, we recall that an algebra $\boldsymbol{L} = \langle L, \vee, \wedge, \sim, 0, 1\rangle$ of tipe $(2, 2, 1, 0, 0)$ is said to be a De Morgan algebra if the reduct $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, and the following identities hold:

$$\sim\sim x = x, \sim (x \wedge y) =\sim x \vee \sim y.$$

The next Hasse diagrams represent the poset I and the algebra K (respectively) that generates the variety of De Morgan algebras:



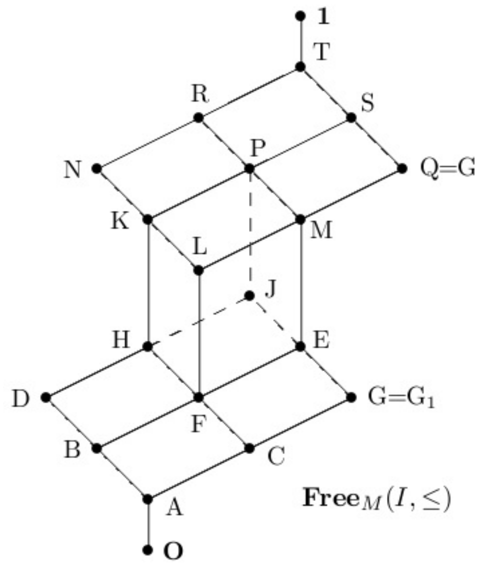
Let E be the set of all order-preserving functions from I to K. Let us consider the function $g : I \to K^E$ defined by $g(i) = G_i$ where $G_i(f) = f(i)$, for all $f \in E$ and all $i \in I$.

Therefore, the set of E has 9 elements as we indicate in the following table:

I	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
1	0	0	0	0	a	a	b	b	1
2	0	a	b	1	a	1	b	1	1

Then, the free De Morgan algebra $\mathbf{Free}_M(I, \leq)$ over I has the following diagram:

Where, we obtain A, B, C, D, F, G, H, E and J from of G_1 and $\sim G_2$ (free generators), in the following way:



A=$G_1 \wedge \sim G_2$, B= $G_2 \wedge \sim G_2$, C= $G_1 \wedge \sim G_1$, F=B$\vee$C, H=$\sim G_2 \vee$C, E=B$\vee G_1$, J=$\sim G_2 \vee G_1$.

O	G=G_1	D=$\sim G_2$	A	B	C	F	H	E	J
0	0	1	0	0	0	0	1	0	1
0	0	a	0	a	0	a	a	a	a
0	0	b	0	b	0	b	b	b	b
0	0	0	0	0	0	0	0	0	0
0	a	a	a	a	a	a	a	a	a
0	a	0	0	0	a	a	a	a	a
0	b	b	b	b	b	b	b	b	b
0	b	0	0	0	b	b	b	b	b
0	1	0	0	0	0	0	0	1	1

Then, the rest of elements are obtained as follows:

X	**O**	A	B	C	D	F	G	H	E	J
$\sim$X	**1**	T	S	R	Q	P	N	M	K	L

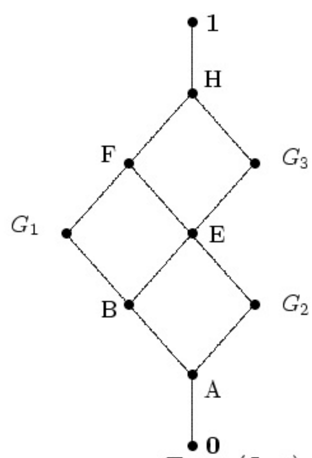
5 Bounded distributive lattices

Now, we shall build the free bounded distributive lattice (for short, l-algebras) over a poset $I_{\leq}$ with 3 elements which is represented by the following diagram:

where L is the algebra which generates the variety of l-algebras. In a similar way to the one of section 4, we can define the set of all order-preserving functions from I into L as is indicated in the next table:

I	f_1	f_2	f_3	f_4	f_5	f_6
a	0	0	0	1	1	1
c	0	1	1	0	1	1
b	0	0	1	0	0	1

Therefore, the free l-algebras over the poset I has the following diagram:



$\mathbf{Free}_L(I, \leq)$

where $G_1 = (0,0,0,1,1,1)$, $G_2 = (0,1,1,0,1,1)$ and $G_3 = (0,0,1,0,0,1)$ are the generators of $\mathbf{Free}_L(I, \leq)$. Besides, the other elements are obtained in the following way: $A = G_1 \wedge G_2$, $B = G_1 \wedge G_3$, $F = G_1 \vee G_2$, $H = G_1 \vee G_3$, $E = (G_1 \vee G_2) \wedge G_3$ and the constants $\mathbf{0} = (0,0,0,0,0,0)$ and $\mathbf{1} = (1,1,1,1,1,1)$.

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