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p-Adic Valuation of $(1^2 + 21)...(n^2 + 21)$ and Applications *

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Abstract. Define $P_n(a) := \prod_{k=1}^{n}(k^2 + a)$, where n and a are positive integers. Yang et al. proved that when $1 \leq a \leq 20$, there are only finite n, such that $P_n(a)$ is a square. In this paper, we study the p-adic valuation of $P_n(21)$ for all primes p. We give explicit expression and bound of the p-adic valuation of $P_n(21)$. Then as an application, we prove that $P_n(21)$ is never a square for any positive integer n.

Keywords: p-adic valuation; Quadratic reciprocity law; Square.

1. Introduction

The study of integer matrices and polynomials are common topics in number theory (see, for example [1, 4, 8]). We here mainly concentrate on the problem that representing powers by the product of consecutive terms in a sequence of

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integer quadratic polynomial. In 2010, Hong and Liu [5] studied the p-adic valuation of the product $\prod_{k=2}^{n}(k^2-1)$ and proved that there exists infinite positive integer n, such that $\prod_{k=2}^{n}(k^2-1)$ is a square. Yang et al [9] discussed the p-adic valuation of $P_n(a) := \prod_{k=1}^{n}(k^2+a)$ for positive integer a with $1 \le a \le 20$ and proved that for those a, there exists only finite positive integer n, such that $P_n(a)$ is a square. In general, it has been [?] proved that for all positive integers a, there exists a positive integer N_a which only depends on a, such that $P_n(a)$ is never a square when $n > N_a$. Recently, Chen, Wang and Hu [2] proved that $P_n(23)$ is never a square for all integers $n \ge 4$.

In this paper, we study p-adic valuation of the product $P_n := P_n(21)$. As usual, for any positive integer n, we let $v_p(n)$ denote the *p-adic valuation* of n, i.e., $v_p(n) = r$ if $p^r \parallel n$. Let $\left(\frac{\cdot}{p}\right)$ stand for the Legendre symbol (see, for example, [6]). We will give expression and bound of $v_p(P_n)$, and then using this bound, we show that P_n is always not a square. Let us state the first main result of this paper as follows.

Theorem 1.1. *Let n be a positive integer. Then $v_2(P_n) = \left\lceil \frac{n}{2} \right\rceil$, $v_p(P_n) = \left\lfloor \frac{n}{p} \right\rfloor$ for $p \in \{3,7\}$ and $v_p(P_n) = 0$ for any prime $p > 7$ with $\left(\frac{-21}{p}\right) = -1$. For $p = 5$ or any prime $p > 7$ with $\left(\frac{-21}{p}\right) = 1$, one has*

$$2\sum_{l=1}^{\lfloor \log_p n \rfloor} \left\lfloor \frac{n}{p^l} \right\rfloor \le v_p(P_n) \le 2\sum_{l=1}^{\lfloor \log_p (n^2+21) \rfloor} \left\lceil \frac{n}{p^l} \right\rceil.$$

Using these formulas, we can get the following interesting result which is the second main result of this paper.

Theorem 1.2. *For any positive integer n, the product $\prod_{k=1}^{n}(k^2+21)$ is never a square.*

The paper is organized as follows. In Section 2, we provide several preliminary lemmas. Consequently, we prove Theorems 1.1 and 1.2. Throughout the paper, p denotes a rational prime. For any nonnegative real number x, we let $\pi(x)$ denote the function $\pi(x) := \sum_{p \le x} 1$.

2. Preliminary Lemmas

In this section, we present some lemmas which will be used in the proof of Theorems 1.1 and 1.2. Write $P_n = \prod_{k=1}^{n}(k^2+21)$.

Lemma 2.1. *Let $n \geq 3$ be an integer. If P_n is a square and p is a prime factor of P_n, then $p < 2n$.*

Proof. Let P_n be a square and let p be a prime factor of P_n. Then we must have $p^2|P_n$. Consider the following two cases:

Case 1. $p^2|(k^2+21)$ for some integer k with $1 \leq k \leq n$. Then $p \leq \sqrt{k^2+21} \leq \sqrt{(n^2+21)} < 2n$ since $n \geq 3$. Lemma 2.1 is proved in this case.

Case 2. $p^2 \nmid (k^2+21)$ for all integers k with $1 \leq k \leq n$. Then there are integers j and k with $1 \leq k < j \leq n$, such that $p|(k^2+21)$ and $p|(j^2+21)$. It then follows that $p|(j+k)(j-k)$, which implies that $p|(j+k)$ or $p|(j-k)$. It is obvious that $j+k < 2n$ and $j-k < 2n$ since $1 \leq k < j \leq n$. So we have that $p \leq \max(j+k, j-k) < 2n$. Lemma 2.1 is true in this case.

This completes the proof of Lemma 2.1. ∎

Lemma 2.2. [7] *Let n be a positive integer. Then we have*

$$v_p(n!) = \sum_{j \leq \log n/\log p} \left\lfloor \frac{n}{p^j} \right\rfloor.$$

Lemma 2.3. [3] *Let n be a positive integer. Then $\sum_{n<p<2n} \log p \leq n \log 4$.*

Lemma 2.4. [3] *Let n be a positive integer. Then $\pi(n) \leq 2\log 4 \frac{n}{\log n} + \sqrt{n}$.*

Lemma 2.5. *Let k be a positive integer such that k^2+21 is a prime. If m is the smallest positive integer satisfying that k^2+21 divides m^2+21 and $m \neq k$, then $m = k^2-k+21$.*

Proof. Write $q = k^2+21$. Then q is a prime. Suppose that m is the smallest positive integer satisfying that $(k^2+21) \mid (m^2+21)$ and $m \neq k$. Then we derive that $q \mid (k+m)(k-m)$. It implies that $q \mid (k+m)$ or $q \mid (m-k)$. If $q \mid (k+m)$, then one deduces that $m \geq q-k$. If $q \mid (m-k)$, then one can derive that $m \geq q+k$. Note that $(k^2+21) \mid ((q-k)^2+21)$. So we have $m = q-k$ as desired. Lemma 2.5 is proved. ∎

Lemma 2.6. *Let k be a positive integer such that k^2+21 is a prime. Then for all integers n with $k \leq n \leq k^2-k+20$, P_n is not a square.*

Proof. Let $q = k^2+21$. It then follows from the fact $k \leq n \leq k^2-k+20$ and Lemma 2.5 that $q \nmid (m^2+21)$ for all integers m with $k < m \leq n$. On the other hand, $q \nmid (m^2+21)$ for integers m with $1 \leq m < k$. Thus we deduce that $v_q(P_n) = 1$. This infers that P_n is not a square. This completes the proof of Lemma 2.6. ∎

3. Proof of Theorems 1.1 and 1.2.

This section is devoted to the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By unique factorization we can write

$$P_n = \prod_p p^{\alpha_p}. \tag{1}$$

We first compute the value of $\alpha_p := v_p(P_n)$.

(i) $p = 2$. Let k be an integer with $1 \le k \le n$. Suppose that $2 \mid (k^2 + 21)$. Then k is odd. It follows that $k^2 + 21 \equiv 2 \pmod 4$, which implies that $v_2(k^2 + 21) = 1$. Thus $v_2(k^2 + 21)$ equals 1 if $2 \nmid k$, is equal to 0 if $2 \mid k$. Hence

$$\alpha_2 = \sum_{k=1}^{n} v_2(k^2 + 21) = \left\lceil \frac{n}{2} \right\rceil. \tag{2}$$

(ii) $p = 3$. Let k be an integer with $1 \le k \le n$. Suppose that $3 \mid (k^2 + 21)$. This implies that $3 \mid k$. Then $3^2 \nmid (k^2 + 21)$ since $3^2 \nmid 21$. Thus $v_3(k^2 + 21)$ is equal to 1 if $3 \mid k$, and equals 0 if $3 \nmid k$. This infers that

$$\alpha_3 = \sum_{k=1}^{n} v_3(k^2 + 21) = \left\lfloor \frac{n}{3} \right\rfloor. \tag{3}$$

(iii) $p = 7$. Using the same argument as in (ii), we deduce that $v_7(k^2 + 21)$ equals 1 if $7 \mid k$, and is equal to 0 if $7 \nmid k$. So

$$\alpha_7 = \sum_{k=1}^{n} v_7(k^2 + 21) = \left\lfloor \frac{n}{7} \right\rfloor. \tag{4}$$

(iv) $p \ne 2, 3, 7$. Then it is well known that $p \mid (k^2 + 21)$ for some integer k iff $\left(\frac{-21}{p}\right) = 1$. We consider the following two cases.

Case 1. $\left(\frac{-21}{p}\right) = -1$. Then $p \nmid (k^2 + 21)$ for all integers k, which implies that $\alpha_p = 0$ in this case.

Case 2. $\left(\frac{-21}{p}\right) = 1$. Since $\left(\frac{-21}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)\left(\frac{7}{p}\right)$, we need only to consider the following four cases.

(a) $\left(\frac{-1}{p}\right) = 1, \left(\frac{3}{p}\right) = 1, \left(\frac{7}{p}\right) = 1$. Using quadratic reciprocity law, we deduce that $p \equiv 1 \pmod 4, p \equiv 1 \pmod 3$ and $p \equiv 1, 2, 4 \pmod 7$. So applying the Chinese remainder theorem gives us that $p \equiv 1, 25, 37 \pmod{84}$.

(b) $\left(\frac{-1}{p}\right) = 1, \left(\frac{3}{p}\right) = -1, \left(\frac{7}{p}\right) = -1$. Then one can similarly deduce that $p \equiv 5, 17, 41 \pmod{84}$.

(c) $\left(\frac{-1}{p}\right) = -1, \left(\frac{3}{p}\right) = 1, \left(\frac{7}{p}\right) = -1$. We can derive that $p \equiv 11, 23, 71 \pmod{84}$.

(d) $\left(\frac{-1}{p}\right) = -1, \left(\frac{3}{p}\right) = -1, \left(\frac{7}{p}\right) = 1$. It can be deduced that $p \equiv 19, 31, 55 \pmod{84}$.

Therefore we have that $\left(\frac{-21}{p}\right) = 1$ if and only if

$$p \equiv 1, 5, 11, 17, 19, 23, 25, 31, 37, 41, 55, 71 \pmod{84}.$$

In this case we have $p|(k^2+21)$ for some integer k. Meanwhile $x^2+21 \equiv 0 \pmod{p}$ has two solutions in each interval of length p. In general, for all positive integers j, the congruence $x^2+21 \equiv 0 \pmod{p^j}$ also has two solutions in each interval of length p^j by Hensel's lemma. Let N_j denote the number of integers k with $1 \le k \le n$ satisfying $k^2+21 \equiv 0 \pmod{p^j}$. It follows that $2\left\lfloor \frac{n}{p^j} \right\rfloor \le N_j \le 2\left\lceil \frac{n}{p^j} \right\rceil$. On the other hand, it is easy to see that $\alpha_p = \sum_{j \le \log(n^2+21)/\log p} N_j$. Then

$$\sum_{j \le \log n/\log p} 2\left\lfloor \frac{n}{p^j} \right\rfloor \le \alpha_p \le \sum_{j \le \log(n^2+21)/\log p} 2\left\lceil \frac{n}{p^j} \right\rceil. \tag{5}$$

This ends the proof of Theorem 1.1. ■

For the sake of convenience, we define two sets: $\Re := \{1, 5, 11, 17, 19, 23, 25, 31, 37, 41, 55, 71\}$ and $\wp := \bigcup_{a \in \Re} \wp_a$, where $\wp_a := \{p | p \equiv a \pmod{84}\}$. Then for any prime $p \ne 2, 3, 7$, if $p \in \wp$, then (5) holds. If $p \notin \wp$, then $\alpha_p = 0$. We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. By direct computation, we obtain $P_1 = 22$, $P_2 = 550$, $P_3 = 16500$, $P_4 = 610500$, $P_5 = 28083000$, $P_6 = 1600731000$, $P_7 = 112051170000$, $P_8 = 9524349450000$, $P_9 = 971483643900000$, $P_{10} = 117549520911900000$. Hence P_n is not a square for all positive integers n with $n \le 10$. In what follows we let $n > 10$ be an integer. We assume that P_n is a square. By the unique factorization theorem and Lemma 2.1, we write

$$P_n = \prod_{p<2n} p^{\alpha_p}, \tag{6}$$

where $\alpha_p = v_p(P_n)$. Then Theorem 1.1 gives us the expression and bound of α_p for each prime p with $p < 2n$.

And by the unique factorization theorem, we can write

$$n! = \prod_{p \le n} p^{\beta_p}, \tag{7}$$

where $\beta_p := v_p(n!)$. It is clear that $P_n > (n!)^2$. So by (6) and (7) and taking logarithm, we get that

$$\sum_{p\le n} \beta_p \log p < \frac{1}{2} \sum_{p<2n} \alpha_p \log p.$$

This is equivalent to

$$\sum_{\substack{p\le n\\ p\notin\wp}} \beta_p \log p < \sum_{\substack{p\le n\\ p\in\wp}} (\frac{\alpha_p}{2} - \beta_p) \log p + \frac{1}{2} \sum_{\substack{p\le n\\ p\notin\wp}} \alpha_p \log p + \frac{1}{2} \sum_{n<p<2n} \alpha_p \log p. \quad (8)$$

We consider each term on the right hand side of (8): For the first term, since $p \in \wp$, by Lemma 2.2 and (5), we have

$$\frac{\alpha_p}{2} - \beta_p \le \sum_{j\le \log n/\log p} (\left\lceil \frac{n}{p^j} \right\rceil - \left\lfloor \frac{n}{p^j} \right\rfloor) + \sum_{\log n/\log p<j\le\log(n^2+21)/\log p} \left\lceil \frac{n}{p^j} \right\rceil.$$

Noting that $\left\lceil \frac{n}{p^j} \right\rceil - \left\lfloor \frac{n}{p^j} \right\rfloor \le 1$ and $\left\lceil \frac{n}{p^j} \right\rceil = 1$ if $j > \frac{\log n}{\log p}$, so the above inequality becomes

$$\frac{\alpha_p}{2} - \beta_p \le \sum_{j\le \log n/\log p} 1 + \sum_{\log n/\log p<j\le\log(n^2+21)/\log p} 1 \le \frac{\log(n^2+21)}{\log p}.$$

It then follows that

$$\sum_{\substack{p\le n\\ p\in\wp}} (\frac{\alpha_p}{2} - \beta_p) \log p \le \log(n^2+21) \sum_{\substack{p\le n\\ p\in\wp}} 1. \quad (9)$$

Consequently, we consider the second term: If $p \ne 2, 3, 7$, then $\alpha_p = 0$ because $p \notin \wp$. As we assume $n > 10$ at the beginning, we use (2), (3) and (4) to get that

$$\begin{aligned} \frac{1}{2} \sum_{\substack{p\le n\\ p\notin\wp}} \alpha_p \log p &= \frac{1}{2}(\alpha_2 \log 2 + \alpha_3 \log 3 + \alpha_7 \log 7) \\ &= \frac{1}{2}(\left\lceil \frac{n}{2} \right\rceil \log 2 + \left\lfloor \frac{n}{3} \right\rfloor \log 3 + \left\lfloor \frac{n}{7} \right\rfloor \log 7). \end{aligned} \quad (10)$$

Finally, we deal with the last term. As $n > 10$, we have $p \ge 11$ in this case. If $p \notin \wp$, then $\alpha_p = 0$. If $p \in \wp$, it then follows from $p > n$ that $p^2 \ge (n+1)^2 > n^2 + 21$. Hence $\frac{\log(n^2+21)}{\log p} < 2$, which implies that $\alpha_p \le 2$ by (5). Then by Lemma 2.3, we have

$$\frac{1}{2} \sum_{n<p<2n} \alpha_p \log p \le \sum_{n<p<2n} \log p \le n \log 4. \quad (11)$$

Putting (9), (10) and (11) into (8), we obtain that

$$\begin{aligned}&\sum_{\substack{p\le n\\ p\notin\wp}}\beta_p\log p\\ <&\frac{1}{2}\Big(\Big\lceil\frac{n}{2}\Big\rceil\log 2+\Big\lfloor\frac{n}{3}\Big\rfloor\log 3+\Big\lfloor\frac{n}{7}\Big\rfloor\log 7\Big)+\log(n^2+21)\sum_{\substack{p\le n\\ p\in\wp}}1+n\log 4.\end{aligned}\tag{12}$$

Now we treat the left hand side of the (12): By Lemma 2.2,

$$\begin{aligned}\beta_p=\sum_{j\le\log n/\log p}\Big\lfloor\frac{n}{p^j}\Big\rfloor\ge\sum_{j\le\log n/\log p}(\frac{n}{p^j}-1)&\ge\frac{n-p}{p-1}-\frac{\log n}{\log p}\\ &>\frac{n-1}{p-1}-\frac{\log(n^2+21)}{\log p}.\end{aligned}\tag{13}$$

So by (12) and (13), we deduce that

$$\begin{aligned}(n-1)\sum_{\substack{p\le n\\ p\notin\wp}}\frac{\log p}{p-1}<\frac{1}{2}(\Big\lceil\frac{n}{2}\Big\rceil\log 2+\Big\lfloor\frac{n}{3}\Big\rfloor\log 3+\Big\lfloor\frac{n}{7}\Big\rfloor\log 7)\\ +\log(n^2+21)\pi(n)+n\log 4.\end{aligned}$$

Then using Lemma 2.4 and noting that $\Big\lceil\frac{n}{2}\Big\rceil\le\frac{n+1}{2}$, $\Big\lfloor\frac{n}{3}\Big\rfloor\le\frac{n}{3}$, $\Big\lfloor\frac{n}{7}\Big\rfloor\le\frac{n}{7}$, we obtain that

$$\begin{aligned}\sum_{\substack{p\le n\\ p\notin\wp}}\frac{\log p}{p-1}<&\frac{1}{2(n-1)}(\frac{n+1}{2}\log 2+\frac{n}{3}\log 3+\frac{n}{7}\log 7+2n\log 4)\\ &+\frac{\log(n^2+21)}{n-1}((2\log 4)\frac{n}{\log n}+\sqrt{n}).\end{aligned}\tag{14}$$

With a little more effort, we can see that the limit of the right hand side of (14) is equal to $\frac{41}{4}\log 2+\frac{1}{6}\log 3+\frac{1}{14}\log 7$ (about 7.427) as n tends to ∞. By the computer we can check that the right hand side is less than 7.78 when $n\ge 4000000$, and the left hand side is bigger than 7.78 when $n\ge 4000000$. So we arrive at a contradiction. Therefore we have proved that for all integers n with $n\ge 4000000$, P_n is not a square.

Now we use Lemmas 2.5 and 2.6 to show that P_n is not a square for all integers n with $10<n<4000000$:

(i) Since $4^2+21=37$ is a prime, the smallest positive integer m satisfying $37\mid(m^2+21)$ and $m\ne 4$ is $37-4=33$ by Lemma 2.5. Then by Lemma 2.6, P_n is not a square for all integers n with $4\le n\le 32$.

(ii) Because $16^2+21=277$ is a prime, the smallest positive integer m satisfying $277\mid(m^2+21)$ and $m\ne 16$ is $277-16=261$ by Lemma 2.5. Hence by Lemma 2.6, P_n is not a square for all integers n with $16\le n\le 260$.

(iii) For $50^2 + 21 = 2521$ is a prime, the smallest positive integer m satisfying $2521 \mid (m^2 + 21)$ and $m \neq 50$ is $2521 - 50 = 2471$ by Lemma 2.5. Thus by Lemma 2.6, P_n is not a square for all integers n with $50 \leq n \leq 2470$.

(iv) Since $2026^2 + 21 = 4104697$ is a prime, the smallest positive integer m satisfying $4104697 \mid (m^2 + 21)$ and $m \neq 2026$ is $4104697 - 2026 = 4102671$ by Lemma 2.5. Then by Lemma 2.6, P_n is not a square for all integers n with $2026 \leq n \leq 4102670$. Therefore combining (i), (ii), (iii) with (iv), we obtain that P_n is not a square for all integers n with $10 < n < 4000000$.

Thus we deduce that P_n is not a square for all positive integers n. This completes the proof of Theorem 1.2. ■

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