Finite element differential forms

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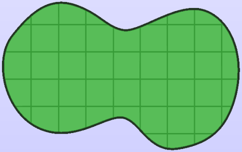
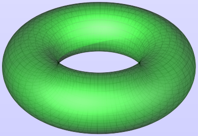
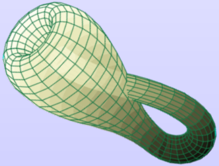
July 17, 2007

Background

- The finite element exterior calculus is a new way of looking at finite element spaces used to discretize some of the most fundamental differential operators.
- It has brought great clarity and unity to the development and analysis of mixed finite elements for a variety of problems, and has enabled major advances in finite elements for elasticity, preconditioning, a posteriori error estimates, implementation, ...
- The fundamental idea is to mimic the framework of exterior calculus by developing finite element spaces of differential forms which exactly transfer key geometrical properties (de Rham theory, Hodge theory) from the continuous to the discrete level.
- Numerical multilinear algebra? Differential forms are an important class of multilinear operator: fields of alternating multilinear forms.

Manifold concepts

n-Manifold Ω

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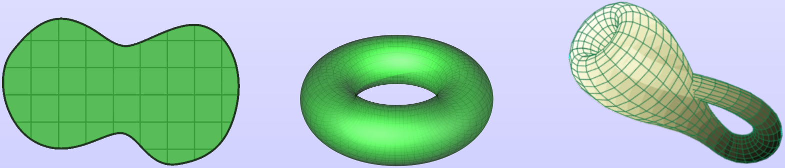
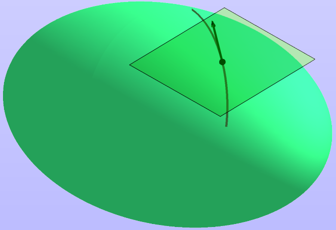


For all $x \in \Omega$, the tangent space $T_x\Omega$ is an n-dimensional vector space



If $f : \Omega \rightarrow \mathbb{R}$, its derivative $df_x : T_x\Omega \rightarrow \mathbb{R}$ is a 1-form (covector field)

Differential forms and the de Rham complex

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Case $\Omega \subset \mathbb{R}^3$:

$$0 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \to 0$$

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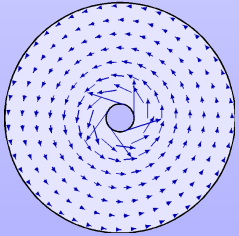

PDEs closely connected to the de Rham sequence

- $-\operatorname{div}\operatorname{grad} u = f$ or $\sigma = \operatorname{grad} u,\ -\operatorname{div} u = f$
- $(\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div})u = f$
- $\operatorname{curl}\operatorname{curl} u = f,\ \operatorname{div} u = 0$
- $\operatorname{div} u = f,\ \operatorname{curl} u = 0$
- Maxwell's equations
- dynamic problems, eigenvalue problems, lower order-terms
- variable coefficients, nonlinearities. . .

Finite element discretization

Stable discretization of such problems not easy, even in simple cases.

$$\sigma \in H(div), \quad u \in L^2 :$$

$$\langle \sigma, \tau \rangle + \langle \operatorname{div} \tau, u \rangle = 0 \qquad \forall \tau \in H(\operatorname{div})$$

$$-\langle \operatorname{div} \sigma, v \rangle = \langle f, v \rangle \qquad \forall v \in L^2$$

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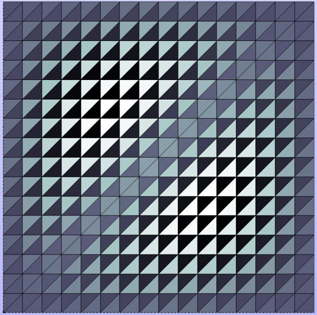

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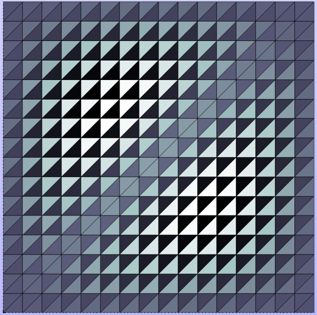

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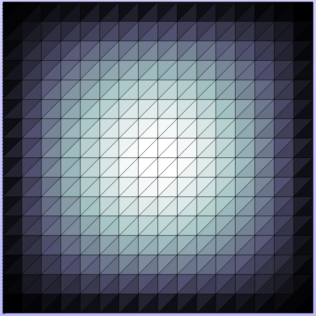

Raviart–Thomas - $\mathcal{P}_0$

Bounded cochain projections

An important observation of FEEC is that when discretizing $H\Lambda^k$ by a subspace Λ_h^k, the key property is $d\Lambda_h^k \subset \Lambda_h^{k+1}$ and there exist a bounded cochain projection, i.e., $\pi_h^k : H\Lambda^k(\Omega) \to \Lambda_h^k$ such that:

- π_h^k bounded
- π_h^k a projection
- $\pi_h^k d^{k-1} = d^{k-1}\pi_h^{k-1}$

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H\Lambda^{k-1}(\Omega) & \xrightarrow{d^{k-1}} & H\Lambda^k(\Omega) & \longrightarrow \cdots \\
 & \big\downarrow \pi_h^{k-1} & & \big\downarrow \pi_h^k & \\
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$$

Implies preservation of cohomology, discrete Poincaré lemma, stability and convergence of Galerkin's method, ...

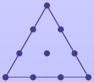
Finite element differential forms

To construct a finite element space of differential forms, we have to specify for a given simplex $T \subset \mathbb{R}^n$:

- a finite dimensional space of polynomial forms on the simplex
- a decomposition of its dual space into subspaces associated to the subsimplices (degrees of freedom)

Prototypical case: Lagrange finite elements. $V(T) = \mathcal{P}_r(T)$

$$V(T)^* = \bigoplus_{f \in \Delta(T)} W(T, f)$$



$$W(T, f) = \{\, u \mapsto \int_f \operatorname{tr}_{T,f} uv\, dx \;:\; v \in \mathcal{P}_{r-1-\dim f}(f) \,\}$$

The assembled space is then precisely

$$\{\, u \in H^1(\Omega) \;:\; u|_T \in V(T) \forall T \,\}$$

The spaces $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$

Major take-away message of this talk: For general form degree k there are *two* families of spaces of polynomial differential forms, $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$, which, when assembled lead to *the* natural finite element subspaces of $H\Lambda^k(\Omega)$.

They can be assembled into complexes with bounded cochain projections (in numerous ways).

$\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ are affine invariant subspaces of k-forms, and are almost uniquely characterized as such.

The two families are inter-related.

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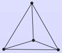
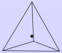
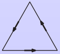
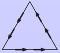
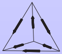
Special cases:

- $\mathcal{P}_r\Lambda^0 = \mathcal{P}_r^-\Lambda^0$, the Lagrange finite elements
- $\mathcal{P}_r\Lambda^n(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^n$ consists of all piecewise polynomials of degree r
- $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ is the space of Whitney k-forms (1 DOF per k-face)

Finite element differential forms and classical mixed FEM

- $\mathcal{P}_r^-\Lambda^0(\mathcal{T}) = \mathcal{P}_r\Lambda^0(\mathcal{T}) \subset H^1$ Lagrange elts 
- $\mathcal{P}_r^-\Lambda^n(\mathcal{T}) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}) \subset L^2$ discontinuous elts 
- $n = 2$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\text{curl})$ Raviart–Thomas elts 

  $\mathcal{P}_r\,\Lambda^1(\mathcal{T}) \subset H(\text{curl})$ Brezzi–Douglas–Marini elts 
- $n = 3$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\text{curl})$ Nedelec 1st kind edge elts 

  $\mathcal{P}_r\,\Lambda^1(\mathcal{T}) \subset H(\text{curl})$ Nedelec 2nd kind edge elts 

  $\mathcal{P}_r^-\Lambda^2(\mathcal{T}) \subset H(\text{div})$ Nedelec 1st kind face elts 

  $\mathcal{P}_r\,\Lambda^2(\mathcal{T}) \subset H(\text{div})$ Nedelec 2nd kind face elts 

The Koszul complex

The key to the construction is the Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$:

$$(\kappa\omega)_x(v^1, \ldots, v^{k-1}) = \omega_x(X, v^1, \ldots, v^{k-1}), \qquad X = x - x_0$$

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Key relation: $(d\kappa + \kappa d)\omega = (r + k)\omega \quad \forall \omega \in \mathcal{H}_r\Lambda^k$ (homogeneous polys)

κ is a contracting chain homotopy

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$$\mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$$

Definition of $\mathcal{P}_r^- \Lambda^k$

Using the Koszul differential, we define $\mathcal{P}_r^- \Lambda^k$ contained between $\mathcal{P}_r \Lambda^k$ and $\mathcal{P}_{r-1} \Lambda^k$:

$$\mathcal{P}_r \Lambda^k := \mathcal{P}_{r-1} \Lambda^k + \kappa \mathcal{H}_{r-1} \Lambda^{k+1} + d\mathcal{H}_{r+1} \Lambda^{k-1}$$

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Note

$$\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$$

$$\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$$

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God made $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$,
all the rest is the work of man.

Degrees of freedom

To define the finite element spaces, we must specify *degrees of freedom*, i.e., a decomposition of the dual spaces $(\mathcal{P}_r\Lambda^k(T))^*$ and $(\mathcal{P}_r^-\Lambda^k(T))^*$, into subspaces associated to subsimplices f of T.

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$$\omega \mapsto \int_f \operatorname{Tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f) \qquad \text{Hiptmair}$$

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Degrees of freedom

To define the finite element spaces, we must specify *degrees of freedom*, i.e., a decomposition of the dual spaces $(\mathcal{P}_r\Lambda^k(T))^*$ and $(\mathcal{P}_r^-\Lambda^k(T))^*$, into subspaces associated to subsimplices f of T.

DOF for $\mathcal{P}_r^-\Lambda^k(T)$: to a subsimplex f of dim. $d \geq k$ we associate

$$\omega \mapsto \int_f \operatorname{Tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f)$$ Hiptmair

DOF for $\mathcal{P}_r\Lambda^k(T)$:

$$\omega \mapsto \int_f \operatorname{Tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d}^-\Lambda^{d-k}(f)$$

The resulting FE spaces have exactly the continuity required by $H\Lambda^k$:

Theorem. $\mathcal{P}_r\Lambda^k(\mathcal{T}) = \{\,\omega \in H\Lambda^k(\Omega) \;:\; \omega|_T \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T}\,\}$.

Similarly for $\mathcal{P}_r^-$.

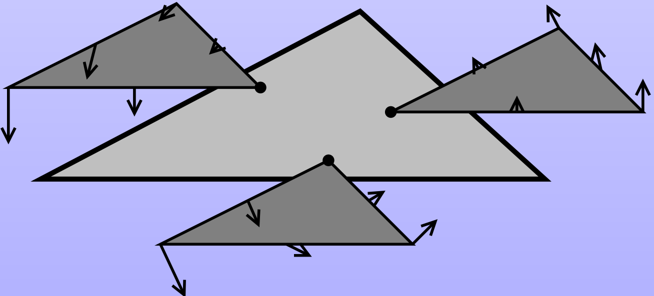
Dual bases

As a basis for $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ we may take the dual basis to the degrees of freedom.

For $k = 0$ this is the standard Lagrange basis.

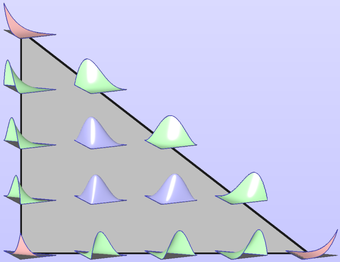
For $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ there is one basis element for each k-simplex, the Whitney form

$$\phi_{\sigma_0\cdots\sigma_k} := \sum_{i=0}^{k}(-1)^i\lambda_{\sigma_i}d\lambda_{\sigma_0}\wedge\cdots\wedge\widehat{d\lambda_{\sigma_i}}\wedge\cdots\wedge d\lambda_{\sigma_k}$$



Geometric bases

The *Bernstein basis*, given by monomials in the barycentric coords, is an explicit alternative to the Lagrange basis for the Lagrange finite elts.



$$\mathcal{P}_r(T) = \bigoplus_{f \text{ subsimplex}} \mathcal{P}_r(T, f)$$

$$\mathcal{P}_r(T, f) \xrightarrow[\text{trace}]{\cong} \mathring{\mathcal{P}}_r(f) \cong \mathcal{P}_{r - \dim f - 1}(f)$$

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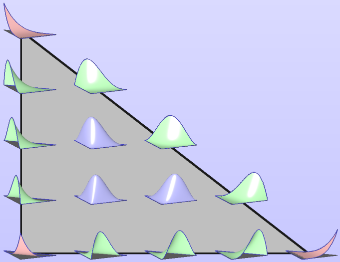

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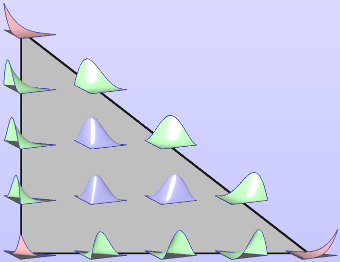

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Construction of the geometric basis

The Bernstein basis for $\mathcal{P}_r(T)$ begins with the barycentric monomials

$$\lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n}, \quad |\alpha| = r$$

Associating the monomial to the subsimplex determined by $\operatorname{supp}(\alpha)$ we get the geometric decomposition.

For $\mathcal{P}_r^- \Lambda^k(T)$ we start with the spanning set

$$\lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n} \phi_\rho, \quad |\alpha| = r - 1, \quad 0 \le \rho_0 < \cdots < \rho_k \le n$$

These are not linearly independent, but associating this form to the subsimplex determined by $\operatorname{supp}(\alpha) \cup \{\rho_0, \ldots, \rho_k\}$ gives a direct sum decomposition.

For $\mathcal{P}_r \Lambda^k(T)$, the obvious spanning set is

$$\lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n} d\lambda_{\rho_1} \wedge \cdots \wedge d\lambda_{\rho_k}, \quad |\alpha| = r, \quad 0 \le \rho_1 < \cdots < \rho_k \le n$$

but these do not give a direct sum decomposition. A modification does work, namely we substitute a more complicated expression for $d\lambda_{\rho_i}$ if $\rho_i \in \operatorname{supp}(\alpha)$

Finite element de Rham subcomplexes

- The polynomial dR complex assembles into a FEdR subcomplex

$$0 \to \mathcal{P}_r\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathcal{T}) \to 0$$

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- For $r \geq 1$, the $\mathcal{P}_r^-\Lambda^k$ spaces give another FEdR subcomplex:

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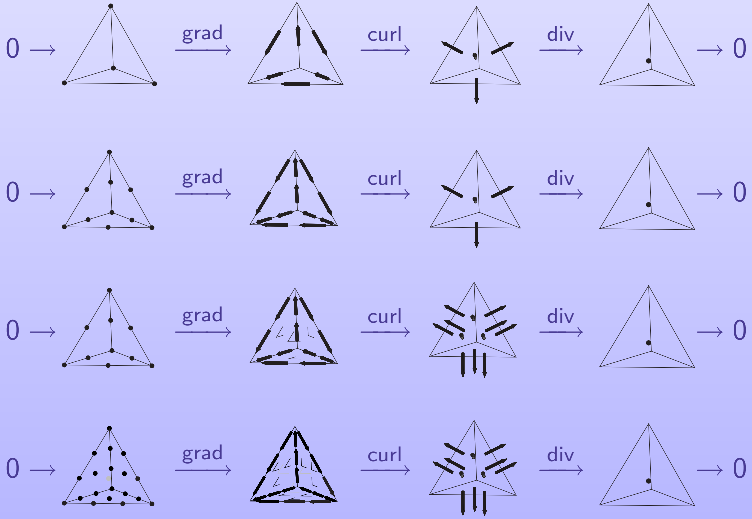
- These are extreme cases. For every r there are 2^{n-1} such FEdR subcomplexes.

The 4 FEdR subcomplexes ending with $\mathcal{P}_0\Lambda^3$ in 3D



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However the composition

$$\pi_h^k = (Q_h^k|_{\Lambda_h^k})^{-1} \circ Q_h^k$$

can be shown to be a *bounded cochain projection*.

AFW, Christiansen, Schöberl