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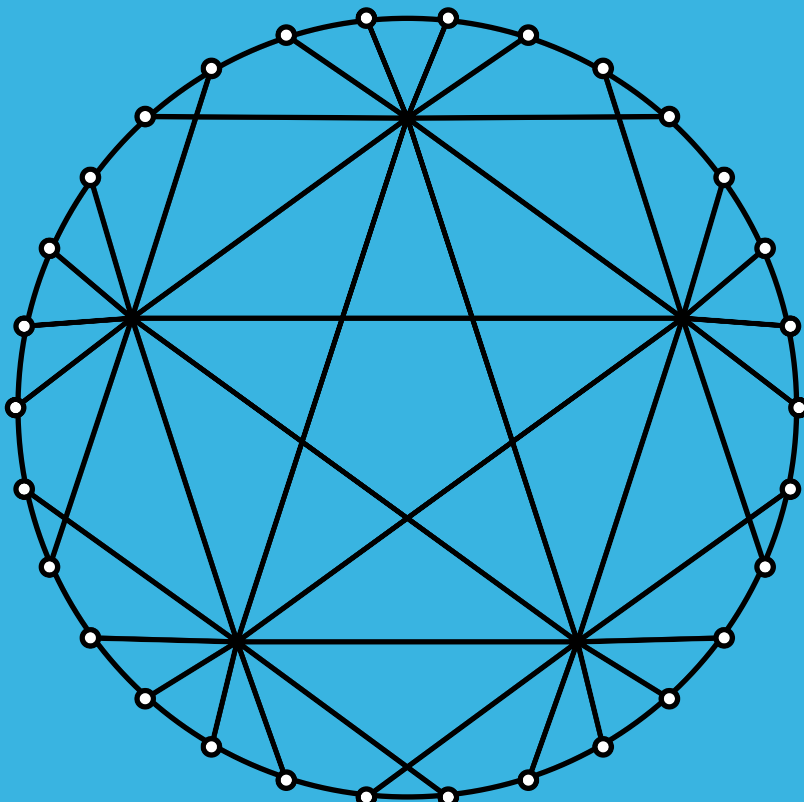

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Signed magic rectangles with three filled cells in each column

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Abstract

A *signed magic rectangle* $SMR(m, n; k, s)$ is an $m \times n$ array with entries from X, where $X = \{0, \pm 1, \pm 2, \ldots, \pm(mk-1)/2\}$ if mk is odd and $X = \{\pm 1, \pm 2, \ldots, \pm mk/2\}$ if mk is even, such that precisely k cells in every row and s cells in every column are filled, every integer from set X appears exactly once in the array and the sum of each row and of each column is zero. In this paper, we prove that a signed magic rectangle $SMR(m, n; k, 3)$ exists if and only if $3 \le m, k \le n$ and $mk = 3n$.

1 Introduction

A *magic rectangle* of order $m \times n$ (see [8]) with precisely r filled cells in each row and precisely s filled cells in each column, $MR(m, n; r, s)$, is an arrangement of the numbers from 0 to $mr - 1$ in an $m \times n$ array such that each number occurs exactly once in the array and the sum of the entries of each row is the same and the sum of entries of each column is also the same. If $r = n$ or $s = m$, then the array has no empty cells and we denote it by $MR(m, n)$.

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The following theorem (see [5, 6, 10]) settles the existence of an $MR(m, n)$.

Theorem 1. *An $m \times n$ magic rectangle exists if and only if $m \equiv n \pmod 2$, $m + n > 5$, and $m, n > 1$.*

An *integer Heffter array* $H(m, n; s, t)$ is an $m \times n$ array with entries from $X = \{\pm 1, \pm 2, \ldots, \pm ms\}$ such that each row contains s filled cells and each column contains t filled cells, the elements in every row and column sum to 0 in $\mathbb{Z}$, and for every $x \in A$, either x or $-x$ appears in the array. The notion of an integer Heffter array $H(m, n; s, t)$ was first defined by Archdeacon in [1]. Heffter arrays can be used for construction of orthogonal cycle systems or embeddings of pairs of cycle systems on surfaces.

Integer Heffter arrays $H(m, n; s, t)$ with $m = n$ represent a type of magic square where each number from the set $\{1, 2, \ldots, ms\}$ is used once up to sign. A Heffter array is *tight* if it has no empty cell; that is, $n = s$ (and necessarily $m = t$). We denote a signed magic rectangle $SMR(m; n; n; m)$ by $SMR(m; n)$.

Theorem 2. [2] *Let m, n be integers at least 3. There is a tight integer Heffter array $H(m, n)$ if and only if $mn \equiv 0, 3 \pmod 4$.*

A *square* integer Heffter array $H(n; k)$ is an integer Heffter array with $m = n$ and $s = t = k$. In [3, 4] it is proved that

Theorem 3. *There is an integer $H(n; k)$ if and only if $3 \leq k \leq n$ and $nk \equiv 0, 3 \pmod 4$.*

We now define signed magic rectangles which are similar to Heffter arrays. A *signed magic rectangle* $SMR(m, n; r, s)$ is an $m \times n$ array with entries from X, where $X = \{0, \pm 1, \pm 2, \ldots, \pm(mr - 1)/2\}$ if mr is odd and $X = \{\pm 1, \pm 2, \ldots, \pm mr/2\}$ if mr is even, such that precisely r cells in every row and s cells in every column are filled, every integer from set X appears exactly once in the array and the sum of each row and of each column is zero. By the definition, $mr = ns$, $r \leq n$ and $s \leq m$. If $r = n$ or $s = m$, then the rectangle has no empty cells. We denote by $SMR(m, n)$ a signed magic rectangle $SMR(m, n; n, m)$. In the case where $m = n$, we call the array a *signed magic square*. Signed magic squares represent a type of magic square where each number from the set X is used once. If A is a Heffter array $H(m, n; s, t)$, then the array $[A \ -A]$ is a signed magic array $SMA(m, 2n; 2s, t)$, as noticed in [9], and the array $\begin{bmatrix} A & \\ & -A \end{bmatrix}$ is a signed magic array $SMG(2m, 2n; s, t)$.

The following two theorems can be found in [9].

Theorem 4. *An $SMR(m,n)$ exists precisely when $m = n = 1$, or when $m = 2$ and $n \equiv 0,3 \pmod 4$, or when $n = 2$ and $m \equiv 0,3 \pmod 4$, or when $m,n > 2$.*

In [9] the notation $SMS(n;k)$ is used for a signed magic square with k filled cells in each row and k filled cells in each column.

Theorem 5. *There exists an $SMS(n;k)$ precisely when $n = k = 1$ or $3 \leq k \leq n$.*

The existence of an $SMR(m,n;r,s)$ is an open problem. In [7] the authors study the signed magic rectangles with precisely two filled cells in each column. Below is the main theorem in [7].

Theorem 6. *There exists an $SMR(m,n;r,2)$ if and only if either $m = 2$ and $n = r \equiv 0,3 \pmod 4$ or $m,r \geq 3$ and $mr = 2n$.*

In this paper, we prove that a signed magic rectangle $SMR(m,n;\ k,3)$ exists if and only if $3 \leq m,k \leq n$ and $mk = 3n$.

Consider an $SMR(m,n;k,3)$. By definition, we have $mk = 3n$. So if n is odd, then m and k must be odd. In addition, the condition $mk = 3n$ implies that either $3|k$ or $3|m$. In Section 2 we study the existence of an $SMR(m,n;k,3)$ when n is odd and $3|k$. In Section 3 we study the existence of an $SMR(m,n;k,3)$ when n is odd and $3|m$. In Section 4 we study the existence of an $SMR(m,n;k,3)$ when n,k are even. In Section 5 we study the existence of an $SMR(m,n;k,3)$ when n is even and k is odd.

2 The existence of an $SMR(m,n;k,3)$ when n is odd and $3|k$

The following result shows a relationship between an $MR(m,n;r,s)$ and an $SMR(m,n;r,s)$ when mr is odd.

Lemma 7. *If there exists an $MR(m,n;r,s)$ with mr odd, then there exists an $SMR(m,n;r,s)$.*

Proof. Let m, r be odd and greater than 3. By assumption, there exists an $MR(m, n; r, s)$, say A. Since $mr = ns$, it follows that n and s are also odd. We will construct an $m \times n$ array B as follows. The cell (r, c) of B is filled with $a - (mr - 1)/2$ if and only if the cell (r, c) of A contains a. As the entries in A are precisely the integers 0 through $mr - 1$, it follows that the entries in B are precisely the integers $-\frac{mr-1}{2}$ through $\frac{mr-1}{2}$, the required set of integers for an $SMR(m, n; r, s)$. It remains to be shown that B has rows and columns summing to zero.

We know that the column sum of A is $(mr - 1)mr/(2n)$. So the column sum of B is

$$\frac{(mr-1)mr}{2n} - \frac{(mr-1)s}{2} = \frac{(mr-1)}{2}(\frac{mr}{n} - s) = 0.$$

Similarly, the row sum of B is also zero.

□

Theorem 8. *[8] Let k, m, s be positive integers. Then there exists a magic rectangle $MR(m, km; ks, s)$ if and only if $m = s = k = 1$ or $2 \leq s \leq m$ and either s is even or km is odd.*

By Lemma 7 and Theorem 8 we obtain the following:

Proposition 9. *Let k, m, s be positive odd integers. Then there exists an $SMR(m, km; ks, s)$ if $m = s = k = 1$ or $3 \leq s \leq m$. In particular, there exists an $SMR(m, km; 3k, 3)$.*

3 The existence of an $SMR(m, n; k, 3)$ when n is odd and $3|m$

Theorem 10. *[8] Let a, b, c be positive integers with $2 \leq a \leq b$. Let a, b, c be all odd, or let a and b both be even, c arbitrary, and $(a, b) \neq (2, 2)$. Then there exists an $MR(ac, bc; b, a)$.*

Proposition 11. *If n is odd and $3|m$, then there exists an $SMR(m, n; k, 3)$.*

Proof. If an $SMR(m, n; k, 3)$ exists, then $mk = 3n$. So by assumption m and k are odd. Let $m = 3c$. Then $mk = 3n$ implies that $n = kc$. Now apply Theorem 10, with $a = 3$, $b = k$ and c to obtain an $MR(m, n; k, 3)$. Now by Lemma 7 there exists an $SMR(m, n; k, 3)$. □

4 The existence of an $SMR(m, n; k, 3)$ when n, k are even

In this section and the next section we make use of the following result. Since the structure of the $SMR(3, 2)$ given below is crucial in our constructions of an $SMR(m, n; k, 3)$ we include the proof of this lemma here which can also be found in [9].

Lemma 12. *An $SMR(3, n)$ exists if n is even.*

Proof. An $SMR(3, 2)$ and an $SMR(3, 4)$ are given in Figure 1.

1	−1
2	−2
−3	3

1	−1	2	−2
5	4	−5	−4
−6	−3	3	6

Figure 1: An $SMR(3, 2)$ and an $SMR(3, 4)$

Now let $n = 2k \geq 6$ and $p_j = \lceil \frac{j}{2} \rceil$ for $1 \leq j \leq 2k$. Define a $3 \times n$ array $A = [a_{i,j}]$ as follows: For $1 \leq j \leq 2k$,

$$a_{1,j} = \begin{cases} -\left(\frac{3p_j-2}{2}\right) & j \equiv 0 \pmod 4 \\ \frac{3p_j-1}{2} & j \equiv 1 \pmod 4 \\ -\left(\frac{3p_j-1}{2}\right) & j \equiv 2 \pmod 4 \\ \frac{3p_j-2}{2} & j \equiv 3 \pmod 4. \end{cases}$$

For the third row we define $a_{3,1} = -3k$, $a_{3,2k} = 3k$ and when $2 \leq j \leq 2k-1$

$$a_{3,j} = \begin{cases} -3(k-p_j) & j \equiv 0 \pmod 4 \\ 3(k-p_j+1) & j \equiv 1 \pmod 4 \\ -3(k-p_j) & j \equiv 2 \pmod 4 \\ 3(k-p_j+1) & j \equiv 3 \pmod 4. \end{cases}$$

Finally, $a_{2,j} = -(a_{1,j} + a_{3,j})$ for $1 \leq j \leq 2k$ (see Figure 2). It is straightforward to see that array A is an $SMR(3, n)$. $\square$

1	−1	2	−2	4	−4	5	−5	7	−7
14	13	−14	11	−13	10	−11	8	−10	−8
−15	−12	12	−9	9	−6	6	−3	3	15

Figure 2: An $SMR(3,10)$ using the method given in Lemma 12

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_s\}$ be two partitions of a set S. We say the partitions $\mathcal{A}$ and $\mathcal{B}$ are *near orthogonal* if $|A_i \cap B_j| \leq 1$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$.

Theorem 13. *Let A be an $SMR(m,n)$ with entries from the set X. Let $\mathcal{P}_1 = \{C_1, C_2, \ldots, C_n\}$, where C_i's are the columns of A. Let $k \geq m$ and $k|mn$. If there exists a partition $\mathcal{P}_2 = \{D_1, D_2, \ldots, D_\ell\}$ of X, where $\ell = mn/k$, such that $|D_i| = k$, the sum of members in each D_i is zero for $1 \leq i \leq \ell$, and $\mathcal{P}_1$ and $\mathcal{P}_2$ are near orthogonal, then there exists an $SMR(mn/k, n; k, m)$.*

Proof. Let B be an mn/k by n empty array. We use the members of D_i to fill ℓ cells of row i of B. Let $d \in D_i$. Then there is a unique column C_j of A which contains d. We place d in row i and column j of B. Note that the members used in B are precisely the members in X because $\mathcal{P}_2$ is a partition of X. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are near orthogonal, each cell of B has at most one member. By construction, row i of B and D_i have the same members and column j of B and C_j have the same members. So the sum of each row and each column of B is zero. Hence, B is an $SMR(mn/k, n; k, m)$. □

Proposition 14. *Let k and n be even integers, $k \geq 4$ and $k|3n$. Then there exists an $SMR(3n/k, n; k, 3)$.*

Proof. Let A be the $SMR(3,n)$ constructed in the proof of Lemma 12 with elements in $X = \{\pm 1, \pm 2, \ldots, \pm 3n/2\}$. Let $\mathcal{P}_1 = \{C_1, C_2, \ldots, C_n\}$, where C_i's are the columns of A. Obviously, $\mathcal{P}_1$ is a partition of X. By the proof of Lemma 12, we see that if x appears in row i of A, then $-x$ also appears in row i.

We construct a partition $\mathcal{P}_2 = \{D_1, D_2, \ldots, D_\ell\}$ of X, where $\ell = 3n/k$, such that $|D_i| = k$, the sum of members in each D_i is zero for $1 \leq i \leq \ell$, and $\mathcal{P}_1$ and $\mathcal{P}_2$ are near orthogonal. Then by Theorem 13, there exists an $SMR(3n/k, n; k, 3)$, as desired. Let $n = kq + r$ with $0 \leq r < k$. By assumption, it is easy to see that $r = 0$, $r = k/3$ or $r = 2k/3$. So we consider 3 cases.

Case 1: $r = 0$. Since k and n are even and for $x \in X$ both x and $-x$ appear in the same row of A, we can partition each row of A into n/k k-subsets such that if x is in a k-subset, then $-x$ is also in that k-subset. Hence the sum of members in each k-subset is zero. Let $\mathcal{P}_2 = \{D_1, D_2, \ldots, D_\ell\}$, $\ell = 3n/k$, be the collection of all these k-subsets. Since D_i is a subset of a row of A for $1 \leq i \leq \ell$, it follows that the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ are near orthogonal. So the result follows by Theorem 13.

Case 2: $r = k/3$. Then $\ell = 3n/k = 3q + 1$. We partition each row of A into q k-subsets and one $k/3$-subset. Note that $k/3$ is an even number because k is even. First we form a $k/3$-subset E_i of row i for $i = 1, 2, 3$, such that if x is in E_i, then $-x$ is also in E_i. In addition, no two members of $D_\ell = E_1 \cup E_2 \cup E_3$ are in the same column of A. Now consider the set F_i which consists of the elements in row i that are not in E_i for $i = 1, 2, 3$. The size of F_i is even and the members of F_i can be paired as $x, -x$ for some $x \in F_i$. Hence we can partition each F_i into k-subsets $D^i_1, D^i_2, \ldots, D^i_q$ such that if $a \in D^i_j$, then $-a \in D^i_j$, where $1 \leq i \leq 3$ and $1 \leq j \leq q$. Consider the partition $\mathcal{P}_2 = \{D^i_1, D^i_2, \ldots, D^i_q, D_\ell \mid 1 \leq i \leq 3\}$. By construction, no two members of a k-subset $Y \in \mathcal{P}_2$ belong to the same column of A, so the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ are near orthogonal. In addition, the sum of the members of Y is zero. Now the result follows by Theorem 13.

Case 3: $r = 2k/3$. Then $\ell = 3n/k = 3q + 2$. We partition each row of A into q k-subsets and two $k/3$-subsets. Note that $k/3$ is an even number because k is even. First we form two $k/3$-subsets E^1_i, E^2_i of row i for $i = 1, 2, 3$, such that if x is in E^1_i or E^2_i, then $-x$ is also in E^1_i or E^2_i, respectively. In addition, no two members of $D_{\ell-1} = E^1_1 \cup E^1_2 \cup E^1_3$ or of $D_\ell = E^2_1 \cup E^2_2 \cup E^2_3$ are in the same column of A. Now consider the set F_i which consists of the elements in row i that are not in $E^1_i \cup E^2_i$ for $i = 1, 2, 3$. The size of F_i is even and the members of F_i can be paired as $x, -x$ for some x. Hence, we can partition each F_i into k-subsets $D^i_1, D^i_2, \ldots, D^i_q$ such that if $a \in D^i_j$, then $-a \in D^i_j$, where $1 \leq i \leq 3$ and $1 \leq j \leq q$. Consider the partition $\mathcal{P}_2 = \{D^i_1, D^i_2, \ldots, D^i_q, D_{\ell-1}, D_\ell \mid 1 \leq i \leq 3\}$. By construction, no two members of a k-subset $Y \in \mathcal{P}_2$ belong to the same column of A, so the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ are near orthogonal. In addition, the sum of the members of Y is zero. Now the result follows by Theorem 13. $\square$

5 The existence of an $SMR(m, n; k, 3)$ when n is even and k is odd

Let n be a positive integer. Let A be the $SMR(3, n)$ constructed in the proof of Lemma 12 with elements in $X = \{\pm 1, \pm 2, \ldots, \pm 3n/2\}$. For $i = 1, 2, 3$, let R_i consist of the entries in row i of A. By the proof of Lemma 12, if $n \equiv 0 \pmod 4$, then

$$\begin{aligned} R_1 &= \{\pm(3i+1), \pm(3i+2) \mid 0 \leq i \leq (n-4)/4\}, \\ R_2 &= \{\pm(3i+1), \pm(3i+2) \mid n/4 \leq i \leq (n-2)/2\}. \end{aligned} \tag{1}$$

If $n \equiv 2 \pmod 4$, then

$$\begin{array}{rcl} R_1 & = & \{\pm(3i+1) \mid 0 \leq i \leq (n-2)/4\} \\ & \cup & \{\pm(3i+2) \mid 0 \leq i \leq (n-6)/4\}, \\ R_2 & = & \{\pm(3i+1) \mid (n+2)/4 \leq i \leq (n-2)/2\} \\ & \cup & \{\pm(3i+2) \mid (n-2)/4 \leq i \leq (n-2)/2\}. \end{array} \tag{2}$$

And

$$R_3 = \{\pm 3i \mid 1 \leq i \leq n/2\}. \tag{3}$$

For a set of numbers L we define $L' = \{-a \mid a \in L\}$.

Remark 15. *Let $n > k$, k odd and n even. If $k|3n$ and $n = kq + r$, where $0 \leq r < k$, then $r = 0, k/3$ or $2k/3$.*

Lemma 16. *Let $n > k \geq 5$ with k odd and n even such that $k|3n$. Let $n = kq + r$, where $0 \leq r < k$. Then there exist two sets of 3-subsets of $X = \{\pm 1, \pm 2, \ldots, \pm 3n/2\}$, say S_1 and S_2, such that*

1. *the member sum of each 3-subset is zero;*
2. *each 3-subset of S_1 has one member in R_1 and two members in R_2 and each 3-subset of S_2 has one member in R_2 and two members in R_1;*
3. *if $\{a, b, c\}$ is a member of S_1 or of S_2, then $\{-a, -b, -c\}$ is also a member of S_1 or S_2, respectively;*
4. *$|S_1| = |S_2| \geq q, q+1, q+2$ if $r = 0, k/3, 2k/3$, respectively;*
5. *the 3-subsets of $S_1 \cup S_2$ are all disjoint.*

Table 1: Small cases for Lemma 16

(n,k)	3-subsets of $R_1 \cup R_2$
$(10,5)$	$M_1=\{1,13,-14\}$, $M_1'=\{-1,-13,14\}$ $N_1=\{4,7,-11\}$, $N_1'=\{-4,-7,11\}$
$(12,9)$	$M_1=\{1,16,-17\}$, $M_1'=\{-1,-16,17\}$ $N_1=\{4,7,-11\}$, $N_1'=\{-4,-7,11\}$
$(14,7)$	$M_1=\{1,19,-20\}$, $M_1'=\{-1,-19,20\}$ $N_1=\{4,10,-14\}$, $N_1'=\{-4,-10,14\}$
$(18,9)$	$M_1=\{1,25,-26\}$, $M_1'=\{-1,-25,26\}$ $N_1=\{4,13,-17\}$, $N_1'=\{-4,-13,17\}$
$(20,5)$	$M_1=\{1,28,-29\}$, $M_1'=\{-1,-28,29\}$ $M_2=\{2,23,-25\}$, $M_2'=\{-2,-23,25\}$ $N_1=\{4,13,-17\}$, $N_1'=\{-4,-13,17\}$ $N_2=\{5,11,-16\}$, $N_2'=\{-5,-11,16\}$
$(20,15)$	$M_1=\{1,28,-29\}$, $M_1'=\{-1,-28,29\}$ $N_1=\{4,13,-17\}$, $N_1'=\{-4,-13,17\}$
$(22,11)$	$M_1=\{1,31,-32\}$, $M_1'=\{-1,-31,32\}$ $N_1=\{4,16,-20\}$, $N_1'=\{-4,-16,20\}$
$(24,9)$	$M_1=\{1,34,-35\}$, $M_1'=\{-1,-34,35\}$ $M_2=\{2,29,-31\}$, $M_2'=\{-2,-29,31\}$ $N_1=\{4,16,-20\}$, $N_1'=\{-4,-16,20\}$ $N_2=\{5,14,-19\}$, $N_2'=\{-5,-14,19\}$
$(26,13)$	$M_1=\{1,37,-38\}$, $M_1'=\{-1,-37,38\}$ $N_1=\{4,19,-23\}$, $N_1'=\{-4,-19,23\}$
$(28,7)$	$M_1=\{1,40,-41\}$, $M_1'=\{-1,-40,41\}$ $M_2=\{2,35,-37\}$, $M_2'=\{-2,-35,37\}$ $N_1=\{4,19,-23\}$, $N_1'=\{-4,-19,23\}$ $N_2=\{5,17,-22\}$, $N_2'=\{-5,-17,22\}$
$(28,21)$	$M_1=\{1,40,-41\}$, $M_1'=\{-1,-40,41\}$ $N_1=\{4,19,-23\}$, $N_1'=\{-4,-19,23\}$
$(30,5)$	$M_1=\{1,43,-44\}$, $M_1'=\{-1,-43,44\}$ $M_2=\{2,38,-40\}$, $M_2'=\{-2,-38,40\}$ $M_3=\{4,31,-35\}$, $M_3'=\{-4,-31,35\}$ $N_1=\{7,22,-29\}$, $N_1'=\{-7,-22,29\}$ $N_2=\{10,16,-26\}$, $N_2'=\{-10,-16,26\}$ $N_3=\{8,20,-28\}$, $N_3'=\{-8,-20,28\}$
$(30,9)$	$M_1=\{1,43,-44\}$, $M_1'=\{-1,-43,44\}$ $M_2=\{2,38,-40\}$, $M_2'=\{-2,-38,40\}$ $N_1=\{4,22,-26\}$, $N_1'=\{-4,-22,26\}$ $N_2=\{5,20,-25\}$, $N_2'=\{-5,-20,25\}$
$(30,15)$	$M_1=\{1,43,-44\}$, $M_1'=\{-1,-43,44\}$ $N_1=\{4,22,-26\}$, $N_1'=\{-4,-22,26\}$

Proof. Table 1 displays the sets $S_1 = \{M_i, M_i'\}$ and $S_2 = \{N_i, N_i'\}$ for $(n,k) \in \{(10,5),(12,9),(14,7),(18,9),(20,5),$
$(20,15),(22,11),(24,9),(26,13),(28,7),(28,14),(30,5),(30,9),$
$(30,15)\}$. It is easy to see that the 3-sets in S_1 and S_2 satisfy items 1-5. So, we may assume from now on that $n \geq 34$.

Let $p = \lceil q/4 \rceil$ and note that $q = \lfloor n/k \rfloor \leq n/k$, hence

$$p = \lceil q/4 \rceil \leq \lceil n/(4k) \rceil < n/(4k) + 1. \tag{4}$$

For $0 \leq i \leq p-1$ we define

$$\begin{aligned} M_{i_1} &= \{3i+1, 3n/2-2-12i, -(3n/2-1-9i)\} \text{ and} \\ M_{i_2} &= \{3i+2, 3n/2-7-9i, -(3n/2-5-6i)\}. \end{aligned} \tag{5}$$

By construction, the member sums of M_{i_1} and of M_{i_2} are zero.

We also note that if $n \equiv 0 \pmod 4$, by (4),

$$3i+1, 3i+2 \leq 3(n-4)/4+2$$

and

$$3n/2-2-12i \geq 3n/2-2-12(p-1) > 3n/2-3n/k-2.$$

So $3n/2-2-12i > 3(n-4)/4+2$. Similarly,

$$3n/2-1-9i, 3n/2-7-9i, 3n/2-5-6i > 3(n-4)/4+2,$$

for $0 \leq i \leq p-1$.

If $n \equiv 2 \pmod 4$, by (4),

$$3i+1, 3i+2 \leq 3(n-2)/4+1$$

and

$$3n/2-2-12i > 3n/2-3n/k-2 > 3(n-2)/4+1.$$

Similarly,

$$3n/2-1-9i, 3n/2-7-9i, 3n/2-5-6i > 3(n-2)/4+1,$$

for $0 \leq i \leq p-1$.

Hence, M_{i_1} and M_{i_2} each have one member in R_1 and two members in R_2.

If $3n/2-2-12i = 3n/2-5-6j$, then $-4i+2j=-1$ which is impossible.

If $3n/2-7-9i = 3n/2-1-9j$, then $3(j-i)=2$ which is impossible.

Therefore the $2p$ 3-subsets M_{i_1} and M_{i_2} are disjoint and if a appears in this collection of 3-subsets, $-a$ does not appear in this collection.

We now define $2p$ 3-subsets each consisting of two members in R_1 and one member in R_2. First let $n \equiv 0 \pmod 4$. For $0 \le i \le p-1$ define

$$\begin{aligned} N_{i_1} &= \{3i+3p+1, (3n/4)-2-6i, -[(3n/4)-1+3(p-i)]\} \\ N_{i_2} &= \{3i+3p+2, (3n/4)-4-6i, -[(3n/4)-2+3(p-i)]\}. \end{aligned} \tag{6}$$

By construction, the member sums of N_{i_1} and of N_{i_2} are zero. We also note that $3p+1+3i, 3n/4-2-6i, 3p+2+3i, 3n/4-4-6i \le 3(n-4)/4+2$ and $3n/4-1+3(p-i), 3n/4-2+3(p-i) > 3(n-4)/4+2$, where $0 \le i \le p-1$.

Hence, N_{i_1} and N_{i_2} each have two members in R_1 and one member in R_2.

If $3p+1+3i = 3n/4-2-6j$, then $n/4 = p+i+2j+1 \le 4p-2 < n/k+2$, by (4). If $k \ge 7$, then $n/4 < n/7+2$ which is false because $n \ge 34$. If $k=5$, then $n/4 < n/5+2$ is false for $n \ge 40$.

If $3p+2+3i = 3n/4-4-6j$, then $n/4 = p+i+2j+2 \le 4p-1 < n/k+3$, by (4). If $k \ge 7$, then $3p+2+3i = 3n/4-4-6i$, then $n/4 \le 4p-1 < n/4+3$. If $k \ge 7$, then $n/4 < n/7+3$ is false because $n \ge 34$. If $k=5$, then $n/4 < n/5+3$ is false for $n \ge 60$. The remaining case is $(n,k)=(40,5)$ in which case $p=2$, giving the contradiction $10 < 4p-1 = 7$.

Therefore the $2p$ 3-subsets N_{i_1} and N_{i_2} are disjoint, and if a appears in this collection of 3-subsets, $-a$ does not appear in this collection.

Second, let $n \equiv 2 \pmod 4$. For $0 \le i \le p-1$, define

$$\begin{aligned} N_{i_1} &= \{3i+3p+1, (3n-2)/4-6i, -[(3n-2)/4+1+3(p-i)]\} \\ N_{i_2} &= \{3i+3p+2, (3n-2)/4-2-6i, -[(3n-2)/4+3(p-i)]\}. \end{aligned} \tag{7}$$

By construction, the member sums of N_{i_1} and of N_{i_2} are zero.

We also note that $3p+1+3i, (3n-2)/4-6i, 3p+2+3i, (3n-2)/4-2-6i \le 3(n-2)/4+1$ and $(3n-2)/4+1+3(p-i), (3n-2)/4+3(p-i) > 3(n-2)/4+1$, where $0 \le i \le p-1$.

Hence, N_{i_1} and N_{i_2} each have two members in R_1 and one member in R_2.

If $3p+1+3i = (3n-2)/4-6j$, then $(n+2)/4 = p+1+i+2j \leq 4p-2 < n/k+2$, by (4), which is impossible because $n \geq 34$.

If $3p + 2 + 3i = (3n - 2)/4 - 2 - 6j$, then $(n + 2)/4 = p + 2 + i + 2j \leq 4p - 1 < n/k + 3$, by (4). If $k \geq 7$, then $(n+2)/4 < n/7+3$ is false because $n \geq 34$, If $k = 5$, then $(n + 2)/4 < n/5 + 3$ is false for $n \geq 50$. (There are no exceptional cases, since $n \geq 34$, $n \equiv 2 \pmod 4$ and $5|n$ imply $n \geq 50$.)

Therefore the $2p$ 3-subsets N_{i_1} and N_{i_2} are disjoint and if a appears in this collection of 3-subsets, $-a$ does not appear in this collection.

We now prove that no member of X appears more than once in the $4p$ 3-subsets constructed above. First we prove this claim when $n \equiv 0 \pmod 4$. Consider the 3-subsets given in (5) and (6).

Obviously, $3i + 1 \neq 3p + 1 + 3j$ and $3i + 2 \neq 3p + 2 + 3j$ because $p \geq 1$.

If $3i + 1 = (3n/4) - 2 - 6j$, then $n/4 = i + 2j + 1 \leq 3p - 2 < 3n/4k + 1$, by (4), which is impossible because $n \geq 34$.

If $3i + 2 = 3n/4 - 4 - 6j$, then $n/4 = i + 2j + 2 \leq 3p - 1 < 3n/4k + 2$, by (4), which is also impossible because $n \geq 34$.
If $3n/2 - 2 - 12i = 3n/4 - 2 + 3p - 3j$, then $n/4 = p + 4i - j \leq 5p - 4 < 5n/4k + 1$, which is impossible because $n \geq 34$.

If $3n/2 - 7 - 9i = 3n/4 - 1 + 3p - 3j$, then $n/4 = p + 3i - j + 2 \leq 4p - 1 < n/k + 3$, which is impossible because $n \geq 34$.

If $3n/2-1-9i = 3n/4-1+3p-3j$, then $n/4 = p+3i-j \leq 4p-3 < n/k+1$, by (4), which is impossible because $n \geq 34$.

If $3n/2 - 5 - 6i = 3n/4 - 2 + 3p - 3j$, then $n/4 = 2i - j + p + 1 \leq 3p - 1 < 3n/4k + 2$, by (4), which is not possible if $n \geq 34$.

Second, we prove the claim above when $n \equiv 2 \pmod 4$.

Consider the 3-subsets given in (5) and (7).

If $3i+1 = (3n-2)/4-6j$, then $(n+2)/4 = i+2j+1 \leq 3p-2 < 3n/4k+1$, which is impossible because $n \geq 34$.

If $3i+2 = (3n-2)/4-2-6j$, then $(n+2)/4 = i+2j+2 \leq 3p-1 < 3n/4k+2$, which is impossible because $n \geq 34$.

If $3n/2-2-12i=(3n-2)/4+3p-3j$, then $(n+2)/4=p-j+4i+1\leq 5p-3<5n/4k+2$, which is impossible for $n\neq 10(2a+1)$, where a is a nonnegative integer. If $n=10(2a+1)$, then the inequality $(n+2)/4\leq 5p-3$ does not hold.

If $3n/2-1-9i=(3n-2)/4+1+3(p-j)$, then $(n+2)/4=3i+p-j+1\leq 4p-2<n/k+2$, which is impossible because $n\geq 34$.

If $3n/2-7-9i=(3n-2)/4+1+3p-3j$, then $(n+2)/4=3i+p-j+3\leq 4p<n/k+4$, which is impossible because $n\geq 34$.

If $3n/2-5-6i=(3n-2)/4+3(p-j)$, then $(n+2)/4=p-j+2i+2<3p<3n/4k+3$, which is impossible because $n\geq 34$.

We are now ready to construct the sets S_1 and S_2. Recall that for a set of numbers L we defined $L'=\{-a\mid a\in L\}$. When $n=kq$, $n=kq+k/3$ or $n=kq+2k/3$ and $q\equiv 2\pmod 4$ the desired S_1 and S_2 are given by:

$$\begin{aligned} S_1&=\{M_{i_1},M_{i_2},M'_{i_1},M'_{i_2}\mid 0\leq i\leq p-1\}\\ S_2&=\{N_{i_1},N_{i_2},N'_{i_1},N'_{i_2}\mid 0\leq i\leq p-1\}.\end{aligned}\tag{8}$$

Then $|S_1|+|S_2|=4p$. Now if $n=kq$, then $4p\geq q$, if $n=kq+k/3$, then $4p\geq q+1$ because q is odd, if $n=kq+2k/3$ and $q\equiv 2\pmod 4$ then $4p=q+2$ and if $n=kq+2k/3$ and $q\equiv 0\pmod 4$ then $4p=q$. We now add two 3-subsets to S_1 and two 3-subsets to S_2 to obtain $|S_1|=|S_2|=q+2$. Define

$$\bar{M}_{i_1}=M_{i_1}\cup\{\{6p+1,\frac{3n-2}{4}+3p+3,-(\frac{3n-2}{4}+9p+4)\}\},$$

$$\bar{N}_{i_1}=N_{i_1}\cup\{\{9p+1,\frac{3n-2}{4}-6p+3,-(\frac{3n-2}{4}+3p+4)\}\}$$

and

$$\begin{aligned} S_1&=\{\bar{M}_{i_1},M_{i_2},\bar{M}'_{i_1},M'_{i_2}\mid 0\leq i\leq p-1\}\\ S_2&=\{\bar{N}_{i_1},N_{i_1},\bar{N}'_{i_1},N'_{i_2}\mid 0\leq i\leq p-1\}.\end{aligned}\tag{9}$$

It is straightforward to see that S_1 and S_2 satisfy the required conditions. $\square$

Lemma 17. *Let $n>k\geq 5$ with k odd and n even such that $k|3n$. Let $n=kq+r$, where $0\leq r<k$, and*

$$R_3=\{\pm 3i\mid 1\leq i\leq n/2\}.$$

Then there is a set S_3 of disjoint 3-subsets of R_3 such that

1. *the member sum of each* 3*-subset is zero;*
2. *if* $\{a, b, c\} \in S_3$*, then* $\{-a, -b, -c\} \in S_3$*;*
3. $|S_3| \geq q, q+1, q+2$ *if* $r = 0, k/3, 2k/3$*, respectively.*

Proof. If $n = 2k$ or $(n,k) \in \{(12,9),(20,15),(28,21)\}$ define

$$S_3 = \{\{3,6,-9\},\{-3,-6,9\}\}.$$

For $(n,k) \in \{(20,5),(24,9),(28,7),(30,9)\}$ define

$$S_3 = \{\{3,27,-30\},\{-3,-27,30\},\{6,12,-18\},\{-6,-12,18\}\}.$$

For $(n,k) = (30,5)$ define

$$S_3 = \left\{\begin{matrix}\{3,42,-45\},\ \{6,33,-39\},\ \{9,21,-30\}, \\ \{-3,-42,45\},\{-6,-33,39\},\{-9,-21,30\}\end{matrix}\right\}.$$

Hence, the statement is true for $n \leq 30$. For $n > 30$ we proceed as follows: Let $\alpha = \lfloor (n-8)/12 \rfloor$. Define

$$T_{i_1} = \{3+6i, 6\alpha+9+6i, -(6\alpha+12+12i)\},$$

for $0 \leq i \leq \alpha$. Since

$$12 + 6\alpha + 12i \leq 12 + 18\alpha \leq 12 + 18(n-8)/12 = 3n/2,$$

it follows that $T_{i_1}, T'_{i_1} \subseteq R_3$.

Now define

$$T_{i_2} = \{12\alpha+15+6i, 6\alpha-6-12i, -(18\alpha+9-6i)\},$$

for $0 \leq i \leq \lfloor (\alpha-2)/2 \rfloor$. See Figure 3 for illustration. Note that $6\alpha - 6 - 12\lfloor(\alpha-2)/2\rfloor \geq 6$ and $18\alpha + 9 \leq 3n/2$. Hence, $T_{i_2}, T'_{i_2} \subseteq R_3$. In addition, the absolute value of the odd numbers used in T_{i_1} are $\{3, 9, \ldots, 12\alpha+9\}$ and used in T_{i_2} are $\{12\alpha+15, 12\alpha+21, \ldots, 18\alpha+9\}$. The absolute value of the even numbers used in T_{i_1} are $\{6\alpha+12+12i \mid 0 \leq i \leq \alpha\}$ and used in T_{i_2} are $\{6\alpha-6-12i \mid 0 \leq i \leq \lfloor(\alpha-2)/2\rfloor\}$. Hence, the numbers used in T_{i_1} and T_{i_2} are all different. Define

$$S_3 = \{T_{i_1}, T'_{i_1} \mid 0 \leq i \leq \alpha\} \cup \{T_{i_2}, T'_{i_2} \mid 0 \leq i \leq \lfloor(\alpha-2)/2\rfloor\}.$$

It is straightforward to confirm that $|S_3| \geq q, q+1, q+2$ if $r = 0, k/3, 2k/3$, respectively. $\square$

$\{3, 45, -48\}$	$\{-3, -45, 48\}$
$\{9, 51, -60\}$	$\{-9, -51, 60\}$
$\{15, 57, -72\}$	$\{-15, -57, 72\}$
$\{21, 63, -84\}$	$\{-21, -63, 84\}$
$\{27, 69, -96\}$	$\{-27, -69, 96\}$
$\{33, 75, -108\}$	$\{-33, -75, 108\}$
$\{39, 81, -120\}$	$\{-39, -81, 120\}$

L_{i_1} and L'_{i_1}

$\{87, 30, -117\}$
$\{93, 18, -111\}$
$\{99, 6, -105\}$
$\{-87, -30, 117\}$
$\{-93, -18, 111\}$
$\{-99, -6, 105\}$

L_{i_2} and L'_{i_2}

Figure 3: 20 disjoint 3-subsets in R_3 when $(n, k) = (90, 5)$

Proposition 18. *Let k be odd and n be even integers, $k \geq 3$ and $k|3n$. Then there exists an $SMR(3n/k, n; k, 3)$.*

Proof. In [9] it is proved that there is an $SMR(n, n; 3, 3)$ for $n \geq 3$. Hence, the statement is true for $k = 3$. Now let $k \geq 5$. Let A be the $SMR(3, n)$ constructed in the proof of Lemma 12 with entries $X = \{\pm 1, \pm 2, \ldots, \pm 3n/2\}$. Let $\mathcal{P}_1 = \{C_1, C_2, \ldots, C_n\}$, where C_i's are the columns of A. Obviously, $\mathcal{P}_1$ is a partition of X. As in the proof of Proposition 14, we construct a partition $\mathcal{P}_2 = \{D_1, D_2, \ldots, D_\ell\}$ of X, where $\ell = 3n/k$, such that $|D_i| = k$, the sum of members in each D_i is zero for $1 \leq i \leq \ell$, and $\mathcal{P}_1$ and $\mathcal{P}_2$ are near orthogonal. Then by Theorem 13, the result follows.

Since $n = kq + r$, where $0 \leq r < k$, it follows that $r = 0$, $r = k/3$ or $r = 2k/3$ by Remark 15. We consider 3 cases.

Case 1: $r = 0$. So $n = kq$. Since k is odd and n is even, it follows that q is even. By Lemma 16, there are sets S_1 and S_2 each consisting of q 3-subsets of $R_1 \cup R_2$ having the properties given in this lemma. Consider the set $L_1 = \{x \in R_1 \mid x \text{ is not in any 3-subset of } S_1 \cup S_2\}$. Then $|L_1| = n - 3q = (k - 3)q$. Note that $k - 3$ is even and the members of L_1 are of the form $\pm x$ for some $x \in R_1$. Now we assign each 3-subset of S_2 to a $(k - 3)$-subset of L_1 as follows: Let $\{a_i, b_i, c_i\} \in S_2$ with $a_i \in R_2$ and $b_i, c_i \in R_1$, where $1 \leq i \leq q$. Let d_i be the member of R_1 which is in the same column as of a_i. We partition the set L_1 into q $(k - 3)$-subsets of L_1 such that if x is in a $(k - 3)$-subset, then $-x$ is also in this subset. In addition, the ith $(k - 3)$-subset misses d_i. It is obvious that a d_i can be in at most one of the q $(k - 3)$-subsets. Let D_i be the union of the ith $(k - 3)$-subset and $\{a_i, b_i, c_i\}$, where $1 \leq i \leq q$. Note that, by construction, the k-subsets D_i are disjoint, the member sums of D_i are zero and intersect each column of A in at most one member for $1 \leq i \leq q$.

Now consider $L_2 = \{x \in R_2 \mid x \text{ is not in any 3-subset of } S_1 \cup S_2\}$ and the 3-subsets in S_1. By a similar method described above, we find k-subsets $D_{q+1}, D_{q+2}, \ldots, D_{2q}$ which are disjoint, the member sums of D_i are zero and intersect each column of A in at most one member for $q+1 \leq i \leq 2q$. In addition, the set $\{D_1, D_2, \ldots, D_{2q}\}$ partitions $R_1 \cup R_2$.

Finally, consider the 3-subsets in S_3 given in Lemma 17 and the subset L_3 of R_3 which consists of members of R_3 which are not in any member of S_3. Then L_3 can be partitioned into q subsets of size $k-3$, say Q_i, where $1 \leq i \leq q$, such that if $x \in Q_i$, then $-x \in Q_i$. Pair q members of S_3 with Q_i's to obtain k-subset D_i for $2q+1 \leq i \leq 3q$. Then the sum of members of each D_i is zero. In addition, $\{D_i \mid 2q+1 \leq i \leq 3q\}$ is a partition of R_3. Hence, it has at most one member in common with each column of A.

Define $\mathcal{P}_2 = \{D_1, D_2, \ldots, D_\ell\}$, where $\ell = 3n/k = 3q$. Then $\mathcal{P}_2$ is a partition of X and is near orthogonal to the partition $\mathcal{P}_1 = \{C_1, C_2, \ldots, C_n\}$. So by Theorem 13, there exists an $SMR(3n/k, n; k, 3)$.

Case 2: $\boldsymbol{r = k/3}$. So $n = kq + k/3$. Since n is even and k is odd, it follows that q is odd. Figure 4 displays an $SMR(4, 12; 9, 3)$. In what follows, we assume $(n, k) \neq (12, 9)$. This case does not follow the construction given below. By Lemma 16, there are sets S_1 and S_2, each containing $q+1$ 3-subsets of $R_1 \cup R_2$ having the properties given in this lemma. In what follows, we partition $R_1 \cup R_2$ into $2q$ k-subsets and two $k/3$-subset such that the member sum of each subset is zero.

Consider the set

$$L_1 = \{x \in R_1 \mid x \text{ is not in any 3-subsets of } S_1 \cup S_2\}$$

and

$$L_2 = \{x \in R_2 \mid x \text{ is not in any 3-subset of } S_1 \cup S_2\}.$$

Also consider the 3-subsets in S_3 given in Lemma 17 and the subset L_3 of R_3 which consists of members of R_3 which are not in any member of S_3. Then

$$|L_1| = |L_2| = |L_3| = n - 3(q+1) = (k-3)q + (k/3 - 3).$$

It is easy to see that there exist 3-subsets E_i in S_i, $1 \leq i \leq 3$, such that the subset $E_1 \cup E_2 \cup E_3$ intersect each column of A in at most one member. Using the method described in Case 1 for construction of $(k-3)$-subsets of L_1 we construct $(k/3-3)$-subsets ${E'}_i$ of L_i for $i=1,2,3$ such that if $x \in {E'}_i$, then $-x \in {E'}_i$. In addition, no two members of

$$D_\ell = \bigcup_{i=1}^{3} ({E'}_i \cup E_i), \text{ where } \ell = 3q+1$$

are in the same column of A. By construction the member sum of D_ℓ is zero. Note that $(k/3-3) \geq 0$ because $k \nmid n$ and k is odd.

Consider the set $F_i = L_i \setminus {E'}_i$, $i=1,2,3$. Note that $|F_i| = (k-3)q$ which is even and the members of F_i can be paired as $x, -x$ for some $x \in F_i$. Now we pair the q $(k-3)$-subsets of F_1 with members of $S_2 \setminus \{E_2\}$ to obtain k-subsets, say D_i^1, $1 \leq i \leq q$ as described in Case 1. We also pair the q $(k-3)$-subsets of F_2 with members of $S_1 \setminus \{E_1\}$ to obtain k-subsets, say D_i^2, $1 \leq i \leq q$.

Finally, we pair the q $(k-3)$-subsets of L_3 with members of $S_3 \setminus \{E_3\}$, to obtain k-subsets, say D_i^3, $1 \leq i \leq q$. Note that the member sum of each of k-subset is zero. By construction,

$$\mathcal{P}_2 = \{D_1^i, D_2^i, \ldots, D_q^i, D_\ell \mid 1 \leq i \leq 3\}$$

is a partition of X and is near orthogonal to the partition

$$\mathcal{P}_1 = \{C_1, C_2, \ldots, C_n\}.$$

So by Theorem 13, there exists an $SMR(3n/k, n; k, 3)$. See Example 19 for illustration.

Case 3: $r = 2k/3$. So $n = kq + 2k/3$. Since n and $2k/3$ are even and k is odd, it follows that q is even. By Lemma 16, there are sets S_1 and S_2 each containing $q+2$ 3-subsets of $R_1 \cup R_2$ having the properties given in this lemma. We partition $R_1 \cup R_2$ into $2q$ k-subsets and four $k/3$-subset as follows: Let L_1, L_2 and L_3 be as defined in Case 2. Then

$$|L_1| = |L_2| = |L_3| = n - 3(q+2) = (k-3)q + 2(k/3-3).$$

It is easy to find two disjoint sets of 3-subsets E_i^1 and E_i^2 in S_i, $1 \leq i \leq 3$, such that the subset $E_1^1 \cup E_2^1 \cup E_3^1$ and $E_1^2 \cup E_2^2 \cup E_3^2$ intersect each column of A in at most one member.

Now we construct two disjoint sets of $(k/3-3)$-subset ${E'}_i^1$ and ${E'}_i^2$ of L_i for $i=1,2,3$ such that if $x \in {E'}_i^1$ or $x \in {E'}_i^2$, then $-x \in {E'}_i^1$ or $-x \in {E'}_i^2$, respectively. In addition, no two members of

$$D_{\ell-1} = \bigcup_{i=1}^{3} ({E'}_i^1 \cup E_i^1) \text{ and } D_\ell = \bigcup_{i=1}^{3} ({E'}_i^2 \cup E_i^2), \text{ where } \ell = 3q+2,$$

are in the same column of A. By construction, the member sums of $D_{\ell-1}$ and of D_ℓ are both zero.

Now we construct two $(k/3-3)$-subsets E_i^1 and E_i^2 of L_i for $i = 1, 2, 3$ such that if $x \in E_i^1$ or $x \in E_i^2$, then $-x \in E_i^1$ or $-x \in E_i^2$, respectively. In addition, no two members of $E_1^1 \cup E_2^1 \cup E_3^1$ or of $E_1^2 \cup E_2^2 \cup E_3^2$ are in the same column of A.

Consider the set $F_i = L_i \setminus ({E'}_i^1 \cup {E'}_i^2)$, $i = 1, 2, 3$. Note that $|F_i| = (k-3)q$, which is even and the members of F_i can be paired as $x, -x$ for some $x \in F_i$.

Now we pair the q $(k-3)$-subsets of F_1 with members of $S_2 \setminus \{E_2^1, E_2^2\}$ to obtain k-subsets, say D_i^1, $1 \le i \le q$ as described in Case 1. We also pair the q $(k-3)$-subsets of F_2 with members of $S_1 \setminus \{E_1^1, E_1^2\}$ to obtain k-subsets, say D_i^2, $1 \le i \le q$.

Finally, we pair the q $(k-3)$-subsets of L_3 with members of $S_3 \setminus \{E_3^1, E_3^2\}$, to obtain k-subsets, say D_i^3, $1 \le i \le q$. Note that the member sum of each of k-subset is zero.

By construction,

$$\mathcal{P}_2 = \{D_1^i, D_2^i, \ldots, D_q^i, D_{\ell-1}, D_\ell \mid 1 \le i \le 3\}$$

is a partition of X and is near orthogonal to the partition

$$\mathcal{P}_1 = \{C_1, C_2, \ldots, C_n\}.$$

So by Theorem 13, there exists an $SMR(3n/k, n; k, 3)$. □

Example 19. In order to illustrate the construction given in Proposition 18, Case 2 let consider the case $n = 30$ and $k = 9$. Then

$$\begin{aligned} S_1 &= \{\{1, 43, -44\}, \{-1, -43, 44\}, \{2, 38, -40\}, \{-2, -38, 40\}\}; \\ S_2 &= \{\{4, 22, -26\}, \{-4, -22, 26\}, \{5, 20, -25\}, \{-5, -20, 25\}\}; \\ S_3 &= \{\{3, 15, -18\}, \{-3, -15, 18\}, \{9, 21, -30\}, \{-9, -21, 30\}\}. \end{aligned}$$

So the unused members of R_1, R_2 and R_3, respectively, are

$$\begin{aligned} L_1 &= \{\pm 7, \pm 8, \pm 10, \pm 11, \pm 13, \pm 14, \pm 16, \pm 17, \pm 19\}; \\ L_2 &= \{\pm 23, \pm 28, \pm 29, \pm 31, \pm 32, \pm 34, \pm 35, \pm 37, \pm 41\}; \\ L_3 &= \{\pm 6, \pm 12, \pm 24, \pm 27, \pm 33, \pm 36, \pm 39, \pm 42, \pm 45\}. \end{aligned}$$

Note that since $k = 9$, then $k/3 - 3 = 0$. So $E_1 = E_2 = E_3 = \emptyset$. Define

$$\begin{aligned}
D_1^1 &= \{7, -7, 8, -8, 10, -10\} \cup \{-5, -20, 25\} \\
D_1^2 &= \{11, -11, 13, -13, 14, -14\} \cup \{-4, -22, 26\} \\
D_1^3 &= \{16, -16, 17, -17, 19, -19\} \cup \{5, 20, -25\}
\end{aligned}$$

$$\begin{aligned}
D_2^1 &= \{23, -23, 28, -28, 29, -29\} \cup \{-1, -43, 44\} \\
D_2^2 &= \{31, -31, 32, -32, 34, -34\} \cup \{-2, -38, 40\} \\
D_2^3 &= \{35, -35, 37, -37, 41, -41\} \cup \{2, 38, -40\} \\
D_1^3 &= \{6, -6, 12, -12, 24, -24\} \cup \{3, 15, -18\} \\
D_2^3 &= \{27, -27, 33, -33, 36, -36\} \cup \{9, 21, -30\} \\
D_3^3 &= \{39, -39, 42, -42, 45, -45\} \cup \{-9, -21, 30\} \\
D_{10} &= \{1, 43, -44, 4, 22, -26, -3, -15, 18\}.
\end{aligned}$$

It is straightforward to see that

$$\mathcal{P}_2 = \{D_1^i, D_2^i, D_3^i, D_{10} \mid 1 \leq i \leq 3\}$$

is a partition of $X = \{\pm 1. \pm 2, \dots, \pm 45\}$ and is near orthogonal to the partition $\mathcal{P}_1 = \{C_1, C_2, \dots, C_{30}\}$. So by Theorem 13, there exists an $SMR(10, 30; 9, 3)$.

1	16	−17	−12	12			−6	6	−3	3	
17	−1			−16	13	5	−5	−13		8	−8
		2	−2	4	−9	9		7	10	−11	−10
−18	−15	15	14		−4	−14	11		−7		18

Figure 4: An $SMR(4, 12; 9, 3)$

By Propositions 9, 11, 14 and 18 we achieve the main theorem of this paper.

Main Theorem 20. *Let m, n, k be positive integers and $3 \leq m, k \leq n$. Then there exists an $SMR(m, n; k, 3)$ if and only if $mk = 3n$.*

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