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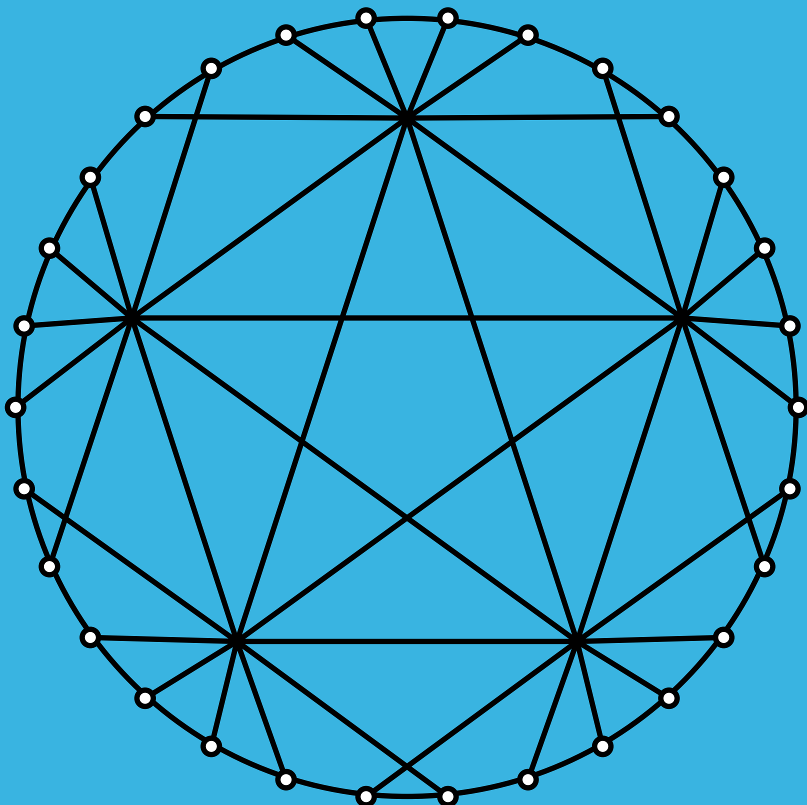

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Posets with series parallel orders and strict-double-bound graphs

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Abstract

For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* of P is the graph $\mathrm{sDB}(P)$ on $V(\mathrm{sDB}(P)) = X$ for which vertices u and v of $\mathrm{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. A poset P is a *series parallel order* if and only if it contains no induced subposet isomorphic to N-poset.

In this paper, we deal with strict-double-bound graphs of series parallel orders. In particular, we show that if P_3 is contained as an induced subgraph in a strict-double-bound graph of a series parallel order, it is contained in either of C_4, 3-pan, $K_{1,3}$ or $K_4 - e$. As a consequence of this result, we can claim that a strict-double graph of a series parallel order is P_4-free. Furthermore, we study sufficient conditions for a strict-double-bound graph of a series parallel order to be an interval graph, difference graph or Meyniel graph.

1 Introduction

In this paper we consider finite undirected simple graphs and finite posets. For a graph G and $S \subseteq V(G)$, $\langle S \rangle_G$ is the induced subgraph on S of G. For posets P and Q, Q is an induced subposet of P if $V(Q) \subseteq V(P)$ and $x, y \in V(Q)$, $x \leq_Q y$ if and only if $x \leq_P y$. For a poset P and $S \subseteq V(P)$, $\langle S \rangle_P$ is the induced subposet on S of P. For a poset P and elements u and v, $u \parallel v$ denotes that u is incomparable with v.

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For a poset $P = (X, \leq_P)$, the *strict-upper-bound graph* of P is the graph $\mathrm{sUB}(P)$ on $V(\mathrm{sUB}(P)) = X$ for which vertices u and v of $\mathrm{sUB}(P)$ are adjacent if and only if $u \neq v$ and there exists an element $x \in X$ distinct from u and v such that $u \leq_P x$ and $v \leq_P x$. We say that a graph G is a *strict-upper-bound graph* if there exists a poset whose strict-upper-bound graph is isomorphic to G.

For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* of P is the graph $\mathrm{sDB}(P)$ on $V(\mathrm{sDB}(P)) = X$ for which vertices u and v of $\mathrm{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. We say that a graph G is a *strict-double-bound graph* if there exists a poset whose strict-double-bound graph is isomorphic to G.

McMorris and Zaslavsky [10] introduced concepts of strict-upper-bound graphs and strict-double-bound graphs, and obtained some properties on strict-upper-bound graphs and strict-double-bound graphs. Note that maximal elements of a poset P are isolated vertices of $\mathrm{sUB}(P)$. So, a connected graph with $p \geq 2$ vertices is not a strict-upper-bound graph. McMorris and Zaslavsky [10] showed as follows: any graph that is the disjoint union of a non-trivial component and enough number of isolated vertices is a strict-upper-bound graph.

In [2], [7] and [13] we see some kinds of posets as follows: A poset is a series parallel order if and only if it contains no induced subposet isomorphic to N-poset, show in Figure 1. A poset is an interval order if and only if it contains no induced subposet isomorphic to $2+2$-poset, show in Figure 1. A poset is a semiorder if and only if it contains no induced subposet isomorphic to $2+2$-poset and $1+3$-poset, show in Figure 1.

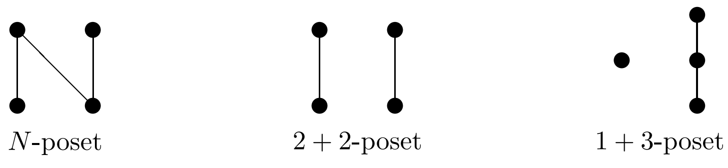

Figure 1: Forbidden subposets

In [6] Kim and Roberts gave a characterization of strict-upper-bound graphs of semiorders and interval orders.

Theorem 1.1 (Kim and Roberts [6]). *Let G be a graph. Then the following statements are equivalent.*
(1) G is the strict-upper-bound graph of a semiorder,
(2) G is the strict-upper-bound graph of an interval order,
(3) $G = K_r \cup \overline{K_q}$, where if $r \geq 2$, then $q \geq 1$.

In [7] Langley et al. gave some properties on strict-upper-bound graphs of series parallel orders.

Proposition 1.2 (Langley et al. [7]). *The strict-upper-bound graph of a series parallel order contains no induced subgraph isomorphic P_3.*

Theorem 1.3 (Langley et al. [7]). *The strict-upper-bound graph of a series parallel order is an interval graph.*

For a total order P with p elements, $\mathrm{sUB}(P) = K_{p-1} \cup K_1$. So, we obtain the following result from Proposition 1.2.

Proposition 1.4. *For a graph G, G is a strict-upper-bound graph of a series parallel order if and only if $G = K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_m} \cup \overline{K_q}$, where $r_i \geq 2$ ($i = 1, 2, \ldots, m$) and $q \geq m$.*

Proof. Let P be a union of total orders. Then P is a series parallel order and $\mathrm{sUB}(P)$ is $K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_m} \cup \overline{K_q}$. Thus $G = K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_m} \cup \overline{K_q}$ is a strict-upper-bound graph of a series parallel order.

We assume that there exist vertices x, y of $V(G)$ such that $xy \notin E(G)$ and x, y in same component C. Then there exists a $x - y$ path in C. For a minimal $x - y$ path $W : x, v_1, v_2, \cdots, v_k, y$, $\langle \{x, v_1, v_2\} \rangle_G \cong P_3$, which is a contradiction. Thus, each component of G is a complete graph and $G = K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_m} \cup \overline{K_q}$. □

Sano [11] gave a characterization of strict-double-bound graphs of semiorders and interval orders.

Theorem 1.5 (Sano [11]). *Let G be a graph. Then the following statements are equivalent.*
(1) G is the strict-double-bound graph of a semiorder,
(2) G is the strict-double-bound graph of an interval order,
(3) $G = K_r \cup \overline{K_q}$, where if $r \geq 2$, then $q \geq 2$.

In this paper we consider strict-double-bound graphs of series parallel orders. We know that there exist strict-double-bound graphs of series parallel orders without the union of complete graphs, for example, $(K_4 - e) \cup \overline{K_3}$, $K_{1,3} \cup \overline{K_4}$ and so on. And we also know that there exist some types of strict-double-bound graphs of series parallel orders contains P_3 as an induced subgraph. So, we consider several properties of induced subgraphs of strict-double-bound graphs on series parallel orders and interval strict-double-bound graphs on series parallel orders, as is the case with Proposition 1.2 and Theorem 1.3.

In particular, the following is our main theorem.

Theorem 1.6. *Let P be a series parallel order and L be a connected component of* $\mathrm{sDB}(P)$ *with at least four vertices. If P_3 is an induced subgraph of L, then P_3 is contained in either of C_4, 3-pan, $K_{1,3}$ or $K_4 - e$, which are shown in Figure 2, as an induced subgraph.*

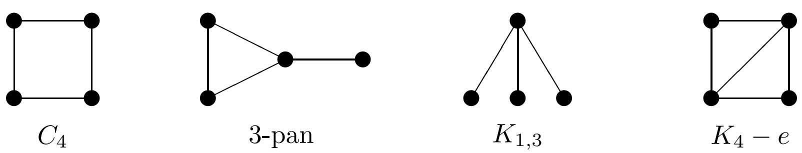

Figure 2: Graphs on Proposition 1.6

2 Strict-double-bound graphs of series parallel orders.

We give a proof of Theorem 1.6 below.

Proof of Theorem 1.6. Let P be a series parallel order, G be the strict-double-bound graph $\mathrm{sDB}(P)$ and L be a connected component of G with at least four vertices. Let a, b, c be vertices of L such that $ab, bc \in E(G)$ and $ac \notin E(G)$, that is, $\langle\{a, b, c\}\rangle_G$ is P_3. Then a is not comparable with c.

Case 1: $a \parallel b$ and $b \parallel c$.
Since $ac \notin E(G)$, $ab \in E(G)$ and $bc \in E(G)$, there exist elements α_1 and α_2 such that $\alpha_1 \neq \alpha_2$, $\alpha_1 \parallel \alpha_2$, $a, b \leq_P \alpha_1$ and $b, c \leq_P \alpha_2$ or there exists α_3 such that $a, b, c \leq_P \alpha_3$. If $a \leq_P \alpha_2$, that is, $a, b, c \leq_P \alpha_3 = \alpha_2$, then there exist elements β_1 and β_2 such that $\beta_1 \neq \beta_2$, $\beta_1 \parallel \beta_2$, $\beta_1 \leq_P a, b$ and $\beta_2 \leq_P b, c$, because $ac \notin E(G)$. Since $a \parallel \alpha_2$ or $a \parallel \beta_2$, $\langle\{a, \alpha_1, b, \alpha_2\}\rangle_P$ is an N-poset or $\langle\{a, \beta_1, b, \beta_2\}\rangle_P$ is an N-poset, which is a contradiction.

Case 2: $a \parallel b$ and b is comparable with c.
Since $ab \in E(G)$, there exist common upper bounds and common lower bounds of a and b.

Subcase 2-1: $c \leq_P b$.
Then there exists an element β such that $\beta \leq_P a, b$ and $\beta \parallel c$. Thus $\langle\{a, \beta, b, c\}\rangle_P$ is an N-poset, which is a contradiction.

Subcase 2-2: $b \leq_P c$.
Then there exists an element α such that $a, b \leq_P \alpha$ and $\alpha \parallel c$. Thus $\langle\{a, \alpha, b, c\}\rangle_P$ is an N-poset, which is a contradiction.

Case 3: b is comparable with a and $b \parallel c$.
By similar way of Case 2, we have an N-poset as an induced subposet, which is a contradiction.

Case 4: b is comparable with a and c.
Since a and c are incomparable, $a, c \leq_P b$ or $b \leq_P a, c$. Since L has at least four vertices and L is connected, we can choose $d \in V(L) - \{a, b, c\}$ such that d is adjacent to one of a, b, c.

Subcase 4-1: $a, c \leq_P b$ and d is comparable with c or a.
We consider the case d is comparable with c without loss of generality.

Subsubcase 4-1-1: $c \leq_P d$.
If $b \parallel d$, then $a \leq_P d$, because P does not contain N-posets as an induced subposets. Then $\langle\{a, b, c, d\}\rangle_G \cong K_4 - e$ if there exists an element α such that $b, d \leq_P \alpha$, or $\langle\{a, b, c, d\}\rangle_G \cong C_4$ if there exist no common upper bounds of b and d. If $b \leq_P d$, then $\langle\{a, b, c, d\}\rangle_G \cong K_4 - e$. If $d \leq_P b$ and $a \leq_P d$, then $\langle\{a, b, c, d\}\rangle_G \cong K_4 - e$. If $d \leq_P b$ and $a \parallel d$, then there exists an element β such that $\beta \leq_P a$. Then $\beta \parallel c$, because $ac \notin E(G)$. If $\beta \parallel d$, then $\langle\{a, b, c, d\}\rangle_G \cong$ 3-pan. If $\beta \leq_P d$, then $\langle\{a, \beta, d, c\}\rangle_G$ is an N-poset, which is a contradiction. If $d \leq_P \beta$, then $c \leq_P a$, which is a contradiction. If $d \leq_P b$ and $d \leq_P a$, then $c \leq_P a$, which is a contradiction.

Subsubcase 4-1-2: $d \leq_P c$.
Since $ac \notin E(G)$, $a \parallel d$. Then $\langle\{a, b, c, d\}\rangle_G$ is a 3-pan.

Subcase 4-2: $a, c \leq_P b$, $c \parallel d$ and $a \parallel d$.

Subsubcase 4-2-1: d is comparable with b.

Then $d \leq_P b$ and there exist elements β_1, β_2 and β_3 such that $\beta_1 \leq_P a$, $\beta_2 \leq_P c$ and $\beta_3 \leq_P d$. Since $ac \notin E(G)$, $\beta_1 \neq \beta_2$. If $\beta_1 \neq \beta_3$ and $\beta_2 \neq \beta_3$, then $\langle\{a,b,c,d\}\rangle_G \cong K_{1,3}$. If $\beta_1 = \beta_3$ and $\beta_2 \neq \beta_3$ (or $\beta_1 \neq \beta_3$ and $\beta_2 = \beta_3$), then $\langle\{a,b,c,d\}\rangle_G \cong$ 3-pan. If there exist β_3 and β_3' such that $\beta_3, \beta_3' \leq_P d$, $\beta_3 = \beta_1$ and $\beta_3' = \beta_2$, then $\beta_3 \neq \beta_3'$, $\beta_3 \parallel \beta_3'$, $a \parallel \beta_3'$ and $c \parallel \beta_3$, because $ac \notin E(G)$. Then $\langle\{a, \beta_3, d, \beta_3'\}\rangle_p$ is an N-poset, which is a contradiction.

Subsubcase 4-2-2: $b \parallel d$.

In the case $cd \in E(L)$ (or $ad \in E(L)$), there exists an element β such that $\beta \leq_P c, d$ (or $\beta \leq_P a, d$). Then $\beta \parallel a$ (or $\beta \parallel c$), because $ac \notin E(G)$. Thus $\langle\{a,b,\beta,d\}\rangle_P$ (or $\langle\{c,b,\beta,d\}\rangle_P$) is an N-poset, which is a contradiction. In the case $cd \notin E(L)$ and $ad \notin E(L)$, then $bd \in E(L)$ and there exist elements α and β such that $\beta \leq_P b, d \leq_P \alpha$. Then $c \parallel \beta$ and $c \parallel d$, because $cd \notin E(L)$. Thus $\langle\{c,b,\beta,d\}\rangle_P$ is an N-poset, which is a contradiction.

Subcase 4-3: $b \leq_P a, c$ and d is comparable with c or a.

By similar way of Subcase 4-1, we obtain C_4, $K_4 - e$ or 3-pan, which contains $P_3 \cong \langle\{a,b,c\}\rangle_G$, or we have an N-poset as an induced subposet, which is a contradiction.

Subcase 4-4: $b \leq_P a, c$, $c \parallel d$ and $a \parallel d$.

By similar way of Subcase 4-2, we obtain $K_{1,3}$ or 3-pan, which contains $P_3 \cong \langle\{a,b,c\}\rangle_G$, or we have an N-poset as an induced subposet, which is a contradiction. □

The proof above also shows that if L contains four vertices a, b, c, d such that $ab, bc, cd \in E(G)$, the induced subgraph $\langle a,b,c,d\rangle_G$ contains either C_4 or 3-pan. Hence, we have the following result.

Corollary 2.1. *For a series parallel order P,* sDB(P) *is a P_4-free graph.*

3 Interval graphs, Split graphs and other graphs

In this section we consider interval graphs, split graphs, threshold graphs, difference graphs and Meyniel graphs.

First, we deal with interval graphs. Lekkerkerker and Boland [8] gave a characterization of interval graphs as follows:

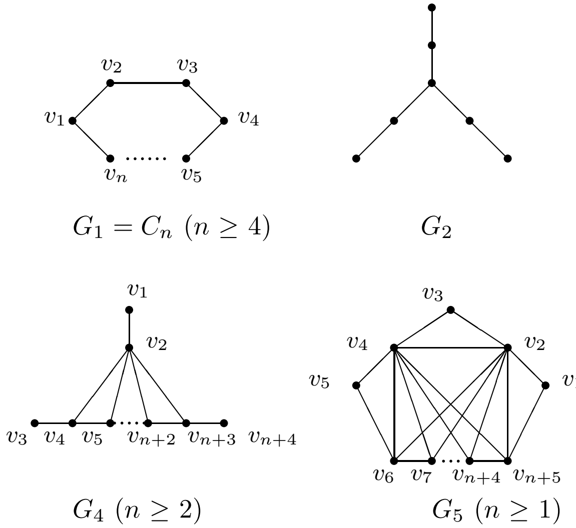

Figure 3: Forbidden subgraphs of interval graphs.

Theorem 3.1 (Lekkerkerker and Boland [8]). *Let G be a graph. Then G is an interval graph if and only if G does not contain any of the graphs in Figure 3 as an induced subgraph.*

By Corollary 2.1 and Theorem 3.1, we obtain the following result.

Proposition 3.2. *Let G be a strict-double-bound graph of a series parallel order. If G is a C_4-free graph, then G is an interval graph.*

Proof. If G is not an interval graph, then G contains one of the graphs in Figure 3 as an induced subgraph. These graphs without C_4 contain P_4 as an induced subgraph. $\square$

Next we consider split graphs and threshold graphs.

A graph G is a *split graph* if its vertices can be partitioned into an independent set and the vertex set of a complete subgraph. Földes and Hammer [3] gave a characterization of split graphs.

Theorem 3.3 (Földes and Hammer [3]). *Let G be a graph. Then G is a split graph if and only if G does not contain $2K_2$, C_4 and C_5 as induced subgraphs.*

A graph G is *threshold graph* if there exist a labeling f of vertices by non-negative integers and an integer t such that for all $X \subseteq V(G)$, X is an independent set if and only if $\sum_{v \in X} f(v) \leq t$. Chvátal and Hammer [1] gave a characterization of threshold graphs.

Theorem 3.4 (Chvátal and Hammer [1]). *Let G be a graph. Then G is a threshold graph if and only if G does not contain $2K_2$, C_4 and P_4 as induced subgraphs.*

By Corollary 2.1, Theorem 3.3 and Theorem 3.4 we obtain the following result.

Proposition 3.5. *Let G be a strict-double-bound graph of a series parallel order. If G does not contain $2K_2$ and C_4 as induced subgraphs, then G is a split graph and a threshold graph.*

Next we consider difference graphs. A graph G is a *difference graph* if there exist a real number t and an assignment $a(v)$ of real numbers to vertices v of G such that (1) $|a(v)| < t$ for all vertices of G, (2) for $u, v \in V(G)$, $uv \in E(G)$ if and only if $|a(u) - a(v)| \geq t$. Mahadev and Peled [9] showed a characterization of difference graphs.

Theorem 3.6 (Hammer, Peled and Sun [4]). *Let G be a graph. Then G is a difference graph if and only if G does not contain $2K_2$, C_3 and C_5 as induced subgraphs.*

Using this result and Corollary 2.1, we also obtain the following result.

Proposition 3.7. *Let G be a strict-double-bound graph of a series parallel order. If G does not contain $2K_2$ and C_3 as induced subgraphs, then G is a difference graph.*

Finally, we consider Meyniel graphs. For a graph G, G is a *Meyniel graph* if each cycle of odd length at least 5 has at least two chords. We already knew the following result on Meyniel graphs.

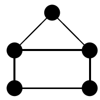

Figure 4: The house graph.

Theorem 3.8 (Hoàng [5]). *Let G be a graph. Then G and $\bar{G}$ are Meyniel graphs if and only if G does not contain C_5, P_5 and the house graph (see Figure 4) as induced subgraphs.*

Note that the house graph contains P_4 as an induced subgraph. Using Theorem 3.8 and Corollary 2.1, we also obtain the following result.

Proposition 3.9. *Let G be a strict-double-bound graph of a series parallel order. Then G and $\bar{G}$ are Meyniel graphs.*

4 Tree posets

In this section we consider strict-double-bound graphs of tree posets. A *tree poset* is a poset such that for $x, y, z \in V(P)$, if $x, y \leq_P z$, then x is comparable with y. Wolk [14], [15] dealt with comparability graphs of tree posets and Scott [12] dealt with upper bound graphs of tree posets. Rechecking the proof of Theorem 1.6, we know the following fact. For a series parallel order P, if the strict-double-bound graph sDB(P) contains $\langle\{a, b, c, d\}\rangle_G \cong C_4$, then the induced subposet on $\{a, b, c, d\}$ is a N_2-poset (see Figure 5). Then tree posets do not contains N-posets and N_2-posets as induced subposets. Thus, tree posets are series parallel orders and strict-double-bound graphs of tree posets are C_4-free. So, we obtain the following result by Proposition 3.2.

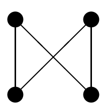

Figure 5: N_2-poset

Proposition 4.1. *The strict-double-bound graphs of tree posets are interval graphs.*

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