New similarity measures between interval-valued fuzzy sets

CHAO-MING HWANG[a], MIIN-SHEN YANG[b,*]

[a]Department of Applied Mathematics, Chinese Culture University, Yangminshan, Taipei, Taiwan
[b]Department of Applied Mathematics, Chung Yung Christian University, Chung-Li 32023, Taiwan
[*]E-mail: msyang@math.cycu.edu.tw

Abstract: Interval-valued fuzzy set (IVFS) is an extension of fuzzy set in which it presents a fuzzy set with an interval-valued membership. A similarity measure is a useful tool for determining the similarity degree between two objects. It is also an important measure for fuzzy concepts. In this paper, we propose a new similarity measure between IVFSs. Some properties of this similarity measure are also made. We make numerical comparisons of the proposed similarity measure with some existing measures. These comparison results show the superiority of our proposed similarity measure.

Keywords: Fuzzy set; Interval-valued fuzzy set; Similarity measure

1 Introduction

Zadeh [13] first introduced the concept of fuzzy sets that give an approach for treating fuzziness which is a type of uncertainty different from randomness. In an ordinary set, an element of a universe either belongs to or does not belong to the set. That is, the membership value of an element in an ordinary set is 0 or 1. A fuzzy set is a generalization of an ordinary set in which it allows the membership value of an element in a fuzzy set is between 0 and 1. The idea of fuzzy sets could convey partial memberships of belongingness described by a membership function. Interval-valued fuzzy sets (IVFSs) or Φ-fuzzy sets proposed by Zadeh [14] and Sambuc [9] are one kind of extension of fuzzy sets (also see Turksen [10]). Since IVFSs present fuzzy sets with interval-valued membership functions, they could make more degree of freedom for uncertainties so that more wide uncertainty modeling can be provided by IVFSs. Up to date, IVFSs have been applied in various areas, such as medical diagnosis (Sambuc [9]), approximate reasoning (Bustince [1]), logic (Cornelis et al. [2]) and clustering (Carvalho and Tenório [3], D'Urso and Giordani [4], Guh et al. [6]), and so forth.

Similarity measures are generally used for determining the degree of similarity between two objects. Kaufman and Rousseeuw [7] presented some examples to illustrate traditional similarity measure applications in hierarchical cluster analysis. Pappis and Karacapilidis [8] proposed three similarity measures for fuzzy sets. Based on those three similarity measures, many researches and applications of similarity measures for fuzzy sets had been widely studied, such as Fan and Xie [5], Zwick et al. [17] and Yang et al. [11]. Zeng and Li [15] first gave a general definition of a similarity measure between IVFSs. However, similarity measures between IVFSs are not yet widespread studied.

In this paper, we first give two new similarity measures between IVFSs. We then consider a weighted average of the two proposed similarity measures between IVFSs. We give the properties of these similarity measures. The rest of this paper is organized as follows. In Section 2, the definitions of IVFSs are first reviewed. Axiomatic definitions for similarity measures between IVFSs are then introduced. In Section 3, new similarity measures between IVFSs are proposed. The properties of the proposed measures are also considered. In Section 4, some comparisons of the proposed similarity measures with existing measures are presented. Finally, conclusions are stated in Section 5.

2 Preliminaries

Zadeh [14] and Sambuc [9] proposed interval-valued fuzzy sets (IVFSs) that are extensions of fuzzy sets with interval-valued membership functions. Let $X=\{x_1,x_2,...,x_n\}$ be the discourse set and the notion *ivfs* denote the set of all IVFSs. Let L be the interval $[0,1]$ and $[L]$ be the set of all closed subintervals of the interval $[0,1]$. Let $\tilde{A}$ be an IVFS in X with $\tilde{A}: X \to [L]$. For any IVFS $\tilde{A}$ and $x \in X$, $\tilde{A}(x)=[\tilde{A}^-(x),\tilde{A}^+(x)]$ is used as the degree of membership of the element x to the IVFS $\tilde{A}$. Thus, $\tilde{A}^-: X \to [0,1]$ and $\tilde{A}^+: X \to [0,1]$ are called low and upper fuzzy sets of the IVFS $\tilde{A}$, respectively. For simplicity, we denote $\tilde{A}=[\tilde{A}^-,\tilde{A}^+]$. Let $\mathfrak{I}(X)$ and $P(X)$ denote the set of all fuzzy sets and the set of all crisp sets in X **,** respectively**.** For any two IVFS $\tilde{A}$ and $\tilde{B}$**,** the operations are defined (see also Zeng and Li [15]) as follows:

(a) $\tilde{A} \subseteq \tilde{B}\ iff\ \forall x \in X,\ \tilde{A}^-(x) \le \tilde{B}^-(x)$

$$and\ \tilde{A}^+(x) \le \tilde{B}^+(x)$$

(b) $\tilde{A} = \tilde{B}\ iff\ \forall x \in X,\ \tilde{A}^-(x) = \tilde{B}^-(x)$

$$and\ \tilde{A}^+(x) = \tilde{B}^+(x)$$

(c) $(\tilde{A})^c(x) = [(\tilde{A}^+(x))^c, (\tilde{A}^-(x))^c],\ \ x \in X$

(d) $(\tilde{A} \cap \tilde{B})(x) = [\tilde{A}^-(x) \wedge \tilde{B}^-(x),\ \tilde{A}^+(x) \wedge \tilde{B}^+(x)],\ \ x \in X$

(e) $(\tilde{A} \cup \tilde{B})(x) = [\tilde{A}^-(x) \vee \tilde{B}^-(x),\ \tilde{A}^+(x) \vee \tilde{B}^+(x)],\ \ x \in X$

The axiomatic definition of a similarity measure between IVFSs was introduced by Zeng and Li [15] and Zeng and Guo [16] as follows.

Definition 1 (Zeng and Li [15]**)** A real function $S: ivfs \times ivfs \to [0,1]$ is called a similarity measure of IVFSs, if, for any $\tilde{A}, \tilde{B}, \tilde{C} \in ivfs$**,** it satisfies the following properties:

$(i)\ S(\tilde{A},\tilde{B}) = S(\tilde{B},\tilde{A})$

$(ii)\ S(\tilde{A},\tilde{A}^c) = 0\ \ if\ \tilde{A}\ is\ a\ crisp\ set$

$(iii)\ S(\tilde{A},\tilde{B}) = 1\ \ iff\ \tilde{A} = \tilde{B}$

$(iv)\ If\ \ \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C},\ then\ S(\tilde{A},\tilde{C}) \le S(\tilde{A},\tilde{B})$

$$and\ S(\tilde{A},\tilde{C}) \le S(\tilde{B},\tilde{C}).$$

3 New similarity measures between interval-valued fuzzy sets

For any two fuzzy sets *A* and *B* on the finite set $X = \{x_1, x_2, \cdots, x_n\}$, Pappis and Karacapilidis [8] proposed the similarity measure $S_{pk}(A,B)$ between the two fuzzy sets *A* and *B* as follows:

$$S_{pk}(A,B) = \frac{\sum_{x \in X} \min\{\mu_A(x), \mu_B(x)\}}{\sum_{x \in X} \max\{\mu_A(x), \mu_B(x)\}}$$

Based on the similar concept, we first propose a new measure between any two IVFSs $\tilde{A}$ and $\tilde{B}$ as follows:

$$S_1(\tilde{A},\tilde{B}) = \min\left(\frac{\sum_{x\in X}\min(\tilde{A}^-(x),\tilde{B}^-(x))}{\sum_{x\in X}\max(\tilde{A}^-(x),\tilde{B}^-(x))}, \frac{\sum_{x\in X}\min((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)}{\sum_{x\in X}\max((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)}\right)$$

We find that the above proposed measure $S_1(\tilde{A},\tilde{B})$ is a similarity measure between IVFSs $\tilde{A}$ and $\tilde{B}$ as shown in the following property.

Property 1 The proposed $S_1(\tilde{A},\tilde{B})$ is a similarity measure between IVFSs $\tilde{A}$ and $\tilde{B}$.

Proof:

$(i)\ S_1(\tilde{A},\tilde{B})$

$$= \min\left(\frac{\sum_{x\in X}\min(\tilde{A}^-(x),\tilde{B}^-(x))}{\sum_{x\in X}\max(\tilde{A}^-(x),\tilde{B}^-(x))}, \frac{\sum_{x\in X}\min((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)}{\sum_{x\in X}\max((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)}\right)$$

$$= \min\left(\frac{\sum_{x\in X}\min(\tilde{B}^-(x),\tilde{A}^-(x))}{\sum_{x\in X}\max(\tilde{B}^-(x),\tilde{A}^-(x))}, \frac{\sum_{x\in X}\min((\tilde{B}^+(x))^c,(\tilde{A}^+(x))^c)}{\sum_{x\in X}\max((\tilde{B}^+(x))^c,(\tilde{A}^+(x))^c)}\right)$$

$$= S_1(\tilde{B},\tilde{A})$$

$(ii)\ If\ \ \tilde{A}\ is\ a\ crisp\ set,$

$i.e.\ \ \tilde{A}^-(x) = \tilde{A}^+(x) = 1\ or\ \tilde{A}^-(x) = \tilde{A}^+(x) = 0,$ then

$S_1(\tilde{A},\tilde{A}^c)$

$$= \min\left(\frac{\sum_{x\in X}\min(\tilde{A}^-(x),(\tilde{A}^+(x))^c)}{\sum_{x\in X}\max(\tilde{A}^-(x),(\tilde{A}^+(x))^c)}, \frac{\sum_{x\in X}\min((\tilde{A}^+(x))^c,\tilde{A}^-(x))}{\sum_{x\in X}\max((\tilde{A}^+(x))^c,\tilde{A}^-(x))}\right)$$

$$= 0$$

$(iii)\ S_1(\tilde{A},\tilde{B})$

$$= \min\left(\frac{\sum_{x\in X}\min(\tilde{A}^-(x),\tilde{B}^-(x))}{\sum_{x\in X}\max(\tilde{A}^-(x),\tilde{B}^-(x))}, \frac{\sum_{x\in X}\min((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)}{\sum_{x\in X}\max((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)}\right)$$

$$= 1$$

$iff\ \min(\tilde{A}^-(x),\tilde{B}^-(x)) = \max(\tilde{A}^-(x),\tilde{B}^-(x)),$

$\min((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c) = \max((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)$

$iff\ \tilde{A}^-(x) = \tilde{B}^-(x)\ \ and\ \tilde{A}^+(x) = \tilde{B}^+(x)\ \ \forall x \in X$ (3)

$iff\ \ \tilde{A} = \tilde{B}$

$(iv)\ if\ \ \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$**,** then

$S_1(\tilde{B},\tilde{A})$

$$= \min\left(\frac{\sum_{x\in X}\min(\tilde{A}^-(x),\tilde{B}^-(x))}{\sum_{x\in X}\max(\tilde{A}^-(x),\tilde{B}^-(x))}, \frac{\sum_{x\in X}\min((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)}{\sum_{x\in X}\max((\tilde{A}^+(x))^c\tilde{B}^+(x))^c}\right)$$

$$= \min\left(\frac{\sum_{x\in X}\tilde{A}^-(x)}{\sum_{x\in X}\tilde{B}^-(x)}, \frac{\sum_{x\in X}(\tilde{B}^+(x))^c}{\sum_{x\in X}(\tilde{A}^+(x))^c}\right)$$

$$\ge \min\left(\frac{\sum_{x\in X}\tilde{A}^-(x)}{\sum_{x\in X}\tilde{C}^-(x)}, \frac{\sum_{x\in X}(\tilde{C}^+(x))^c}{\sum_{x\in X}(\tilde{A}^+(x))^c}\right)$$

$$= \min\left(\frac{\sum_{x\in X}\min(\tilde{A}^-(x),\tilde{C}^-(x))}{\sum_{x\in X}\max(\tilde{A}^-(x),\tilde{C}^-(x))}, \frac{\sum_{x\in X}\min((\tilde{A}^+(x))^c,(\tilde{C}^+(x))^c)}{\sum_{x\in X}\max((\tilde{A}^+(x))^c,(\tilde{C}^+(x))^c}\right)$$

$$= S_1(\tilde{C},\tilde{A})$$

Thus, we have $S_1(\tilde{C},\tilde{A}) \le S_1(\tilde{B},\tilde{A})$. Similarity, we have that $S_1(\tilde{A},\tilde{C}) \le S_1(\tilde{B},\tilde{C})$. By the conditions (i), (ii), (iii) and (iv), we prove the

ISBN: 978-1-61804-023-7

similarity property. □

We next propose another new measure between IVFSs $\tilde{A}$ and $\tilde{B}$ with

$$S_2(\tilde{A},\tilde{B})=\min(\frac{\sum_{x\in X}\min(\tilde{A}^+(x),\tilde{B}^+(x))}{\sum_{x\in X}\max(\tilde{A}^+(x),\tilde{B}^+(x))},\frac{\sum_{x\in X}\min((\tilde{A}^-(x))^c,(\tilde{B}^-(x))^c)}{\sum_{x\in X}\max((\tilde{A}^-(x))^c,(\tilde{B}^-(x))^c)})$$

We find that the above proposed measure $S_2(\tilde{A},\tilde{B})$ is also a similarity measure between IVFSs $\tilde{A}$ and $\tilde{B}$.

Property 2 The proposed $S_2(\tilde{A},\tilde{B})$ is a similarity measure between IVFSs $\tilde{A}$ and $\tilde{B}$.

Proof: The proof is similar as Property 1. □

If we compare the forms of $S_1(\tilde{A},\tilde{B})$ and $S_2(\tilde{A},\tilde{B})$, we find that they have a dual relation. In this sense, a compromise between $S_1(\tilde{A},\tilde{B})$ and $S_2(\tilde{A},\tilde{B})$ is meaningful so that we consider a new measure with a weighted average of $S_1(\tilde{A},\tilde{B})$ and $S_2(\tilde{A},\tilde{B})$ as follows:

$$S_3(\tilde{A},\tilde{B})=wS_1(\tilde{A},\tilde{B})+(1-w)S_2(\tilde{A},\tilde{B}),\ 0\le w\le 1$$

We find that this new measure $S_3(\tilde{A},\tilde{B})$ is also a similarity measure between IVFSs $\tilde{A}$ and $\tilde{B}$ as shown in the following property.

Property 3 The proposed $S_3(\tilde{A},\tilde{B})$ is a similarity measure between IVFSs $\tilde{A}$ and $\tilde{B}$.

Proof: For $w_1+w_2=1,\ w_1>0,\ w_2>0$,

$$(i)\ S_3(\tilde{A},\tilde{B})=w_1S_1(\tilde{A},\tilde{B})+w_2S_2(\tilde{A},\tilde{B})$$
$$=w_1S_1(\tilde{B},\tilde{A})+w_2S_2(\tilde{B},\tilde{A})$$
$$=S_3(\tilde{B},\tilde{A}).$$

(ii) *If* $\tilde{A}$ *is a crisp set,*

i.e. $\tilde{A}^-(x)=\tilde{A}^+(x)=1$ *or* $\tilde{A}^-(x)=\tilde{A}^+(x)=0$, then

$$S_3(\tilde{A},\tilde{A}^c)=w_1S_1(\tilde{A},\tilde{A}^c)+w_2S_2(\tilde{A},\tilde{A}^c)$$
$$=w_1(\min(\frac{\sum_{x\in X}\min(\tilde{A}^-(x),(\tilde{A}^+(x))^c)}{\sum_{x\in X}\max(\tilde{A}^-(x),(\tilde{A}^+(x))^c)},\frac{\sum_{x\in X}\min((\tilde{A}^+(x))^c,\tilde{A}^-(x))}{\sum_{x\in X}\max((\tilde{A}^+(x))^c,\tilde{A}^-(x)}))$$
$$+w_2(\min(\frac{\sum_{x\in X}\min(\tilde{A}^+(x),(\tilde{A}^-(x))^c)}{\sum_{x\in X}\max(\tilde{A}^+(x),(\tilde{A}^-(x))^c)},\frac{\sum_{x\in X}\min((\tilde{A}^-(x))^c,(\tilde{A}^+(x))}{\sum_{x\in X}\max((\tilde{A}^-(x))^c,(\tilde{A}^+(x))}))$$
$$=0$$

$$(iii)\ S_3(\tilde{A},\tilde{B})=w_1S_1(\tilde{A},\tilde{B})+w_2S_2(\tilde{A},\tilde{B})$$
$$=w_1(\min(\frac{\sum_{x\in X}\min(\tilde{A}^-(x),(\tilde{A}^+(x))^c)}{\sum_{x\in X}\max(\tilde{A}^-(x),(\tilde{A}^+(x))^c)},\frac{\sum_{x\in X}\min((\tilde{A}^+(x))^c,\tilde{A}^-(x))}{\sum_{x\in X}\max((\tilde{A}^+(x))^c,\tilde{A}^-(x)}))$$
$$+w_2(\min(\frac{\sum_{x\in X}\min(\tilde{A}^+(x),(\tilde{A}^-(x))^c)}{\sum_{x\in X}\max(\tilde{A}^+(x),(\tilde{A}^-(x))^c)},\frac{\sum_{x\in X}\min((\tilde{A}^-(x))^c,(\tilde{A}^+(x))}{\sum_{x\in X}\max((\tilde{A}^-(x))^c,(\tilde{A}^+(x))}))$$
$$=0$$

$$iff\ \ \min(\tilde{A}^-(x),\tilde{B}^-(x))=\max(\tilde{A}^-(x),\tilde{B}^-(x)),$$
$$\min((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)=\max((\tilde{A}^+(x))^c,(\tilde{B}^+(x))^c)$$
$$iff\ \tilde{A}^-(x)=\tilde{B}^-(x)\ and\ \tilde{A}^+(x)=\tilde{B}^+(x)\ \forall x\in X$$
$$iff\ \ \tilde{A}=\tilde{B}$$

(iv) *if* $\tilde{A}\subseteq\tilde{B}\subseteq\tilde{C}$, *then*

$$S_3(\tilde{A},\tilde{B})=w_1S_1(\tilde{A},\tilde{B})+w_2S_2(\tilde{A},\tilde{B})$$
$$=w_1S_1(\tilde{B},\tilde{A})+w_2S_2(\tilde{B},\tilde{A})$$
$$\ge w_1S_1(\tilde{C},\tilde{A})+w_2S_2(\tilde{C},\tilde{A})=S_3(\tilde{C},\tilde{A})$$

Thus, we have $S_3(\tilde{C},\tilde{A})\le S_3(\tilde{B},\tilde{A})$. Similarity, we have that $S_3(\tilde{A},\tilde{C})\le S_3(\tilde{B},\tilde{C})$. By the conditions (i), (ii), (iii) and (iv), we prove the similarity property. □

4 Comparisons

In this section, we compare the proposed similarity measures between IVFSs with Zeng and Guo [16] method. Assume that $X=\{x_1,x_2,...,x_n\}$. Let $\tilde{A}$ and $\tilde{B}$ be two IVFSs on X. Recall that Zeng and Guo [16] proposed several similarity measures between IVFSs $\tilde{A}$ and $\tilde{B}$ as follows:

$$N_1(\tilde{A},\tilde{B})=1-\frac{1}{2n}\sum_{i=1}^{n}(|\tilde{A}^-(x_i)-\tilde{B}^-(x_i)|+|\tilde{A}^+(x_i)-\tilde{B}^+(x_i)|)$$
$$N_2(\tilde{A},\tilde{B})=1-(\frac{1}{2n}\sum_{i=1}^{n}(|\tilde{A}^-(x_i)-\tilde{B}^-(x_i)|^2+|\tilde{A}^+(x_i)-\tilde{B}^+(x_i)|^2))^{\frac{1}{2}}$$
$$N_c(\tilde{A},\tilde{B})=(1-d(\tilde{A},\tilde{B}))\big/(1+d(\tilde{A},\tilde{B}))$$
$$N_d(\tilde{A},\tilde{B})=1-d^2(\tilde{A},\tilde{B})$$

and $N_e(\tilde{A},\tilde{B})=\dfrac{e^{-d(\tilde{A},\tilde{B})}-e^{-1}}{1-e^{-1}}$

where

$$d(\tilde{A},\tilde{B})=\frac{1}{2n}\sum_{i=1}^{n}(|\tilde{A}^-(x_i)-\tilde{B}^-(x_i)|+|\tilde{A}^+(x_i)-\tilde{B}^+(x_i)|).$$

Assume that there are three IVFS patterns on $X=\{x_1\}$ which are denoted as follows:

$$\tilde{A}_1=\{[\tilde{A}_1^-(x_1)=0.2,\tilde{A}_1^+(x_1)=0.6]\}$$
$$\tilde{A}_2=\{[\tilde{A}_2^-(x_1)=0.1,\tilde{A}_2^+(x_1)=0.3]\}$$
$$\tilde{B}=\{[\tilde{B}^-(x_1)=0.3,\tilde{B}^+(x_1)=0.4]\}$$

We calculate Zeng and Guo [16] similarity measures as follows:

$N_1(\tilde{A}_1, \tilde{B}) = N_1(\tilde{A}_2, \tilde{B}) = 0.85$,

$N_2(\tilde{A}_1, \tilde{B}) = N_2(\tilde{A}_2, \tilde{B}) = 0.842$

$N_c(\tilde{A}_1, \tilde{B}) = N_c(\tilde{A}_2, \tilde{B}) = 0.739$,

$N_d(\tilde{A}_1, \tilde{B}) = N_d(\tilde{A}_2, \tilde{B}) = 0.978$

$N_e(\tilde{A}_1, \tilde{B}) = N_e(\tilde{A}_2, \tilde{B}) = 0.780$.

According to these Zeng and Guo [16] similarity measures, they all consider the sample pattern $\tilde{B}$ has the same similarity to the patterns $\tilde{A}_1$ and $\tilde{A}_2$. Obviously, these results are not good.

We next calculate our proposed similarity measures as follows:

$S_1(\tilde{A}_1, \tilde{B}) = 0.667,\ S_1(\tilde{A}_2, \tilde{B}) = 0.333$

$S_2(\tilde{A}_1, \tilde{B}) = 0.667,\ S_2(\tilde{A}_2, \tilde{B}) = 0.750$

$S_3(\tilde{A}_1, \tilde{B}) = wS_1(\tilde{A}_1, \tilde{B}) + (1-w)S_2(\tilde{A}_1, \tilde{B}) = 0.667$

$S_3(\tilde{A}_2, \tilde{B}) = 0.333w + 0.75(1-w), 0 \le w \le 1$

For the similarity measure S_1, it give the result that the sample pattern $\tilde{B}$ is more similar to the pattern $\tilde{A}_1$ than the pattern $\tilde{A}_2$. This result is actually matched the true status of the given patterns $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{B}$. Since the similarity measure S_2 is considered as an alternative side of the similarity measure S_1, it actually pulls the similarity degree from $S_1(\tilde{A}_2, \tilde{B}) = 0.333$ up to $S_2(\tilde{A}_2, \tilde{B}) = 0.750$. If we consider a weighted similarity measure $S_3(\tilde{A}_2, \tilde{B})$ of $S_1(\tilde{A}_2, \tilde{B})$ and $S_2(\tilde{A}_2, \tilde{B})$, we find that the similarity S_3 can give a compromise degree with $S_3(\tilde{A}_2, \tilde{B}) = 0.333w + 0.75(1-w)$, $0 \le w \le 1$. Since $S_3(\tilde{A}_1, \tilde{B}) = 0.667$, we find that the similarity S_3 obviously have higher probability to give the result that the sample pattern $\tilde{B}$ is more similar to the pattern $\tilde{A}_1$ than the pattern $\tilde{A}_2$.

5 Conclusions

In this paper, we proposed new similarity measures between interval-valued fuzzy sets. We make comparisons of the proposed measures with Zeng and Guo [16] method. The comparison results show that the proposed measures present better results than those of Zeng and Guo [16] methods. For an application to cluster analysis, we suggest to adopt Yang and Shih [12] algorithm for clustering interval-valued fuzzy data based on our proposed similarity measures.

References:

[1] H. Bustince, Indicator of inclusion grade for interval-valued fuzzy sets: Application to approximate reasoning based on interval-valued fuzzy sets, International Journal of Approximate Reasoning, Vol. 23, 2000, pp. 137-209.

[2] C. Cornelis, G. Deschrijver, E.E. Kerre, Advances and challenges in interval-valued fuzzy logic, Fuzzy Sets and Systems, Vol. 157, 2006, pp. 622-627.

[3] F.D.A.T. De Carvalho, C.P. Tenório, Fuzzy k-means clustering algorithms for interval-valued data based on adaptive quadratic distances, Fuzzy Sets and Systems, Vol. 161, 2010, pp. 2978-2999.

[4] P. D'Urso, G. Giordani, A robust fuzzy k-means clustering model for interval valued data, Computational Statistics, Vol. 21, 2006, pp. 251-269.

[5] J. Fan, W. Xie, Some notes on similarity measure and proximity measure. Fuzzy Sets and Systems, Vol. 101, 1999, pp. 403-412.

[6] Y.Y. Guh, M.S. Yang, R.W. Po, E.S. Lee, Interval-valued fuzzy relation-based clustering with its application to performance evaluation, Computers and Mathematics with Applications, Vol. 57, 2009, pp. 841-849.

[7] L. Kaufman, P.J. Rousseeuw, *Finding Groups in Data*, Wiley, New York, 1990.

[8] C.P. Pappis, N.I. Karacapilidis, A comparative assessment of measures of similarity of fuzzy values, Fuzzy Sets and Systems, Vol. 56, 1993, pp. 171-174.

[9] R. Sambuc, Fonctions Φ-Floues, Application a l'aide au Diagnostic en Pathologie Thyroidienne, Ph.D. Thesis, Univ. Marseille, France, 1975.

[10] I.B. Turksen, Interval-valued fuzzy sets based on normal forms, Fuzzy Sets and Systems, Vol. 20, 1986, pp.191–210.

[11] M.S. Yang, W.L. Hung, S.J. Chang-Chien, On a similarity measure between LR-type fuzzy numbers and its application to database acquisition, International Journal of Intelligent Systems, Vol. 20, 2005, pp. 1001-1016.

[12] M.S. Yang, H.M. Shih, Cluster analysis based on fuzzy relations, Fuzzy Sets and Systems, Vol. 120, 2001, pp. 197-212.

[13] L.A. Zadeh, Fuzzy sets, Information and Control, Vol. 8, 1965, pp. 338-353.

[14] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision

processes, IEEE Trans. Syst. Man Cybernet., Vol. 3, 1973, pp. 28–44.
[15] W. Zeng, H. Li, Relationship between similarity measure and entropy of interval valued fuzzy sets, Fuzzy Sets and Systems, Vol. 157, 2006, pp. 1477–1484.
[16] W. Zeng, P. Guo, Normalized distance, similarity measure, inclusion measure and entropy of interval-valued fuzzy sets and their relationship, Information Science, Vol. 178, 2008, pp. 1334-1342.
[17] R. Zwick, E. Carlstein, D.V. Budescu, Measures of similarity among fuzzy concepts: a comparative analysis, International Journal of Approximate Reasoning, Vol. 1, 1987, pp. 221-242.