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# Delay-dependent $\mathscr{H}^\infty$ filtering for time-delayed LPV systems

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**Abstract**

The present paper addresses the parameter-dependent $\mathscr{H}^\infty$ filter design problem for output estimation in linear parameter varying (LPV) plants that include constant delays in the state. We develop LMI-based delay-dependent conditions to guarantee stability and an induced $\mathscr{L}_2$ gain bound performance for the filtering error system. An explicit characterization of the filters' state–space representation is given in terms of the solutions to a convex optimization problem associated with the synthesis conditions. By taking the output estimation error into account as the $\mathscr{H}^\infty$ criterion, the developed filters are shown to be capable of tracking the desired outputs of the time-delayed parameter varying system in the presence of external disturbances. Two families of filters are examined: memoryless and state-delayed filters. The latter one which involves a delay term in its dynamics has the benefit of reducing the conservatism in the design and improving performance. Illustrative examples are provided to demonstrate the feasibility and advantages of the proposed methodologies for memoryless and state-delayed filter design and to validate the superiority of using the state-delayed configuration compared to the conventional memoryless filters.


*Keywords:* $\mathscr{H}^\infty$ filtering; Time-delayed systems; Linear parameter varying (LPV) systems; Linear matrix inequalities (LMIs)

## 1. Introduction

State estimation has been widely studied and has found many practical applications in the past four decades. When *a priori* statistical information on the external disturbance signals is not known, Kalman filtering cannot be employed. To address this issue, $\mathscr{H}^\infty$ filtering was introduced, in which the external disturbance signal is assumed to be only energy bounded, and the main objective of the design is to minimize the $\mathscr{H}^\infty$ norm of the filtering error system (see [5,6] and the references therein). Recently, a lot of research has been conducted on the design of $\mathscr{H}^\infty$ filters, including the treatment of system uncertainty and time delays [15,11,2] by means of Riccati-oriented approaches, as well as, linear matrix inequality (LMI)-based formulations.

In control design for systems that operate over a wide operating range, a common approach is to schedule various fixed operating point designs. Unfortunately, there are no known systematic techniques for scheduling such controllers that provide guarantees on the resulting performance or even stability of the combined closed-loop system. Moreover, unacceptable transients may occur while switching between the fixed-point designed controllers [13]. However, recent advances in optimal and robust control theory provide a design methodology that results in optimal parameter-dependent controllers that guarantee stability and performance over the full operating envelope [14]. The controllers are scheduled based upon the varying parameter values, which are not known *a priori* but can be measured in real time. The corresponding controllers are designed such that they achieve a level of performance against worst-case variations of the parameters. These controllers initially introduced in [14] are referred to as linear parameter varying (LPV) controllers.

Time delays generally occur in communication systems, transmission systems, chemical processing systems, power systems, and many other engineering processes. It is well known that if the presence of delays is not considered in the controller design, it can cause instability or serious deterioration in the performance of the resulting closed-loop system. The study of time-delay systems has received significant attention in the past decade (see [10] and numerous reference therein).

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Current research efforts in this area are divided into two main directions, namely, delay-independent stability and delay-dependent stability criteria. Although the delay-independent analysis conditions are easy to check, the absence of information on the delay causes conservativeness of the criterion, especially when the delay size is small. Delay-dependent stability conditions, which take the size of the delay into account, are typically less conservative than the delay-independent ones [10]. Recently, some appreciable work has been completed to analyze and synthesize time-delay LPV controlled systems (e.g., see [17,16]). The present work uses Lyapunov–Krasovskii functionals for the delay-dependent analysis and filter design. It generalizes the work in [9], that is concerned with the state estimation of LPV time-delay systems from an $\mathscr{H}^\infty$ perspective. Also, a more elaborate Lyapunov–Krasovskii functional that includes additional terms is used in the present paper resulting in improved performance. Comparisons with [9] are provided to illustrate the reduced conservatism of the results of the proposed designs. The work in [8] also formulates a general delay-independent/rate-dependent filter design methodology for continuous time-delayed LPV systems. Notice that the results of [8,11] are known to provide conservative results because of lack of information on the size of the state delay in the synthesis conditions. A short version of the current paper has also been recently appeared in [7].

This paper is concerned with delay-dependent analysis and design of $\mathscr{H}^\infty$ filters for continuous-time LPV systems that include a state delay in the state–space representation of the system. For this purpose, we use a parameter-dependent Lyapunov–Krasovskii functional and an $\mathscr{H}^\infty$ performance criterion, that depends on the LPV parameters. Both memoryless (proper) and state-delayed (nonrational) filters are examined, and the corresponding filter designs are formulated in the form of convex optimization problems, which can be effectively solved using the well-developed interior-point algorithms [3]. Finally, simulation results demonstrate the benefits and capabilities of employing our proposed methodologies for filter design.

The paper is organized as follows. Section 2 states the class of LPV delay systems for which the proposed LPV filters are designed. Section 3 presents sufficient analysis conditions for a time-delayed LPV system to be stable and to provide a prescribed level of induced $\mathscr{L}_2$ gain. Section 4 presents our design methodology and the synthesis conditions for the calculation of memoryless and state-delayed $\mathscr{H}^\infty$ filters based on the preliminary formulations given in Section 3. Section 5 illustrates the capability of our design method to estimate the desired outputs in selected numerical examples compared to past approaches, and Section 6 concludes the paper.

## 2. Plant formulation and a useful lemma

We consider a general class of the state-delayed LPV systems, that includes delay terms in the measurement and the plant output, as follows:

$$\begin{cases} \dot{x}(t) = A(\rho(t))x(t)+A_h(\rho(t))x(t-h)+B_1(\rho(t))w(t), \\ z(t) = C_1(\rho(t))x(t)+C_{1h}(\rho(t))x(t-h)+D_{11}(\rho(t))w(t), \\ y(t) = C_2(\rho(t))x(t)+C_{2h}(\rho(t))x(t-h)+D_{21}(\rho(t))w(t), \\ x(t) = \phi(t); \quad t \in [-h, 0]. \end{cases} \tag{1}$$

In this formulation $x(t) \in \mathbf{R}^n$ is the state vector, $w(t) \in \mathbf{R}^{n_w}$ is the vector of external disturbances, $y(t) \in \mathbf{R}^{n_y}$ is the measurements vector, $z(t) \in \mathbf{R}^{n_z}$ is the plant output vector, which is to be estimated, and $h$ denotes the constant time-delay. Also, we assume that $\Phi(t)$ is a given continuous function, and that all the state–space matrices are known functions of a time-varying parameter vector $\rho(t) \in \mathscr{F}^{v}_{\mathscr{P}}$ measured in real time. $\mathscr{F}^{v}_{\mathscr{P}}$ is the set of allowable parameter trajectories defined as

$$\mathscr{F}^{v}_{\mathscr{P}} \triangleq \{\rho \in C(\mathbf{R}, \mathbf{R}^s) : \rho(t) \in \mathscr{P}, |\dot{\rho}_i(t)| \leqslant v_i, \\ i = 1, 2, \ldots, s, \forall t \in \mathbf{R}_+\},$$

where $\mathscr{P}$ is a compact subset of $\mathbf{R}^s$, and $\{v_i\}_{i=1}^{s}$ are nonnegative numbers.

In the following lemma, we state an integral inequality which will play an important role in the proofs of our results.

**Lemma 1** ([1 2]). *As*[illegible] $a(\theta) \in \mathbf{R}^p$ *and* $b(\theta) \in \mathbf{R}^p$ *a*[illegible] [illegible]. *Th*[illegible], *for any* [illegible] $R$ *and* [illegible] $W$ [illegible], [illegible]

$$-2\int_{t-h}^{t} a^{\mathrm{T}}(\theta)b(\theta)\,\mathrm{d}\theta \leqslant \int_{t-h}^{t} [\,b^{\mathrm{T}}(\theta) \quad a^{\mathrm{T}}(\theta)\,] \begin{bmatrix} R & RW \\ W^{\mathrm{T}}R & T \end{bmatrix} \begin{bmatrix} b(\theta) \\ a(\theta) \end{bmatrix} \mathrm{d}\theta$$

[illegible], *where* $T = (W^{\mathrm{T}}R + I)R^{-1}(RW + I)$.

## 3. $\mathscr{L}_2$-gain analysis of LPV time-delayed systems

In this section, we obtain sufficient conditions to guarantee asymptotic stability and a prescribed $\mathscr{L}_2$-gain performance for the time-delayed parameter varying plant (1). In (1), the transfer matrix from the vector of disturbances $w(t)$ to the output vector $z(t)$, assuming frozen LPV parameters, is given by

$$H(\mathrm{j}\omega) = (C_1 + C_{1h}\mathrm{e}^{-\mathrm{j}\omega h})(\mathrm{j}\omega I - A - A_h\mathrm{e}^{-\mathrm{j}\omega h})^{-1}B_1 + D_{11}. \tag{2}$$

We use the relation $\sigma_{\max}(H(\mathrm{j}\omega)) = \sigma_{\max}(H^{\mathrm{T}}(-\mathrm{j}\omega))$ in our analysis. The following equation is straightforward to achieve:

$$G(\mathrm{j}\omega) = H^{\mathrm{T}}(-\mathrm{j}\omega) \\ = B_1^{\mathrm{T}}(\mathrm{j}\omega I + A^{\mathrm{T}} + A_h^{\mathrm{T}}\mathrm{e}^{\mathrm{j}\omega h})^{-1}(-C_1^{\mathrm{T}} - C_{1h}^{\mathrm{T}}e^{\mathrm{j}\omega h}) + D_{11}^{\mathrm{T}}.$$

Hence, $G(\mathrm{j}\omega)$ has the following state–space representation:

$$\begin{cases} \dot{\tilde{x}}(\tau) = -A^{\mathrm{T}}(\rho(\tau))\tilde{x}(\tau) - A_h^{\mathrm{T}}(\rho(\tau))\tilde{x}(\tau+h) - C_1^{\mathrm{T}}(\rho(\tau))\tilde{z}(\tau) \\ \qquad\quad - C_{1h}^{\mathrm{T}}(\rho(\tau))\tilde{z}(\tau+h), \\ \bar{w}(\tau) = B_1^{\mathrm{T}}(\rho(\tau))\tilde{x}(\tau) + D_{11}^{\mathrm{T}}(\rho(\tau))\tilde{z}(\tau), \\ \tilde{x}(\tau) = \phi(\tau); \quad \tau \in [0, h]. \end{cases}$$

This is the backward adjoint equivalent of system (1). Change of variable in the form of $\bar{x}(\tau)=\tilde{x}(h-\tau)$ gives the forward adjoint model as follows:

$$\begin{cases} \dot{\bar{x}}(\bar{t}) = A^{\mathrm{T}}(\rho(\bar{t}))\bar{x}(\bar{t}) + A_h^{\mathrm{T}}(\rho(\bar{t}))\bar{x}(\bar{t}-h) + C_1^{\mathrm{T}}(\rho(\bar{t}))\bar{z}(\bar{t}) \\ \qquad + C_{1h}^{\mathrm{T}}(\rho(\bar{t}))\bar{z}(\bar{t}-h), \\ \bar{w}(\bar{t}) = B_1^{\mathrm{T}}(\rho(\bar{t}))\bar{x}(\bar{t}) + D_{11}^{\mathrm{T}}(\rho(\bar{t}))\bar{z}(\bar{t}), \\ \bar{x}(\bar{t}) = \Psi(\bar{t}); \quad \bar{t}\in[-h,0]. \end{cases} \tag{3}$$

Note that the above state–space model has the same $\mathscr{L}_2$-gain as the original system (1). Now, assuming independence of noise signals $\bar{z}(t)$ and $\bar{z}(t-h)$, which is a reasonable assumption, we define the augmented vector $\tilde{z}(t)=[\bar{z}^{\mathrm{T}}(t),\bar{z}^{\mathrm{T}}(t-h)]^{\mathrm{T}}$ and hence the new system representation reads

$$\begin{cases} \dot{\bar{x}}(t) = A^{\mathrm{T}}(\rho(t))\bar{x}(t) + A_h^{\mathrm{T}}(\rho(t))\bar{x}(t-h) \\ \qquad + [C_1^{\mathrm{T}}(\rho(t)) \;\; C_{1h}^{\mathrm{T}}(\rho(t))]\tilde{z}(t), \\ \bar{w}(t) = B_1^{\mathrm{T}}(\rho(t))\bar{x}(t) + [D_{11}^{\mathrm{T}}(\rho(t)) \;\; 0]\tilde{z}(t), \\ \bar{x}(t) = \Psi(t); \quad t\in[-h,0]. \end{cases} \tag{4}$$

The obtained form of state–space representation of the system does not include a state delay in the output equation. We now determine sufficient conditions that guarantee asymptotic stability and $\mathscr{H}^\infty$ performance for plants in the form of (4).

Let us consider a time-delayed LPV plant given by

$$\begin{cases} \dot{x}(t) = A_0(\rho(t))x(t) + A_1(\rho(t))x(t-h) + B(\rho(t))w(t), \\ z(t) = L(\rho(t))x(t) + D(\rho(t))w(t), \\ x(t) = \phi(t); \quad t\in[-h,0]. \end{cases} \tag{5}$$

Given system (5) and a prescribed scalar $\gamma>0$, we define the performance index

$$J(w) = \int_0^\infty (z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t))\,\mathrm{d}t. \tag{6}$$

Now, we use the forward adjoint system, as pointed out before, which is equivalent to (5)

$$\begin{cases} \dot{\tilde{x}}(t) = A_0^{\mathrm{T}}(\rho(t))\tilde{x}(t) + A_1^{\mathrm{T}}(\rho(t))\tilde{x}(t-h) + L^{\mathrm{T}}(\rho(t))\tilde{z}(t), \\ \tilde{w}(t) = B^{\mathrm{T}}(\rho(t))\tilde{x}(t) + D^{\mathrm{T}}(\rho(t))\tilde{z}(t). \end{cases} \tag{7}$$

The adjoint system (7) and its forward and backward adjoint systems are equivalent with respect to the $\mathscr{H}^\infty$ norm from the disturbance vector to the output vector. Following [2], we represent (7) in an equivalent descriptor model form as

$$\begin{aligned} &\dot{\tilde{x}}(t) = \eta(t), \\ &0 = -\eta(t) + (A_0+A_1)^{\mathrm{T}}\tilde{x}(t) - A_1^{\mathrm{T}}\int_{t-h}^{t}\eta(s)\,\mathrm{d}s + L^{\mathrm{T}}\tilde{z}(t), \end{aligned} \tag{8}$$

where we have used the Leibniz–Newton formula instead of the time-delayed state term. This form enables us to formulate the synthesis conditions in terms of linear matrix inequalities as we will see later.

Now, consider the following Lyapunov–Krasovskii functional

$$V(t) = [\tilde{x}^{\mathrm{T}}(t) \;\; \eta^{\mathrm{T}}(t)]FP(\rho)\begin{bmatrix}\tilde{x}(t)\\ \eta(t)\end{bmatrix} + \int_{-h}^{0}\int_{t+\theta}^{t}\eta^{\mathrm{T}}(s)R\eta(s)\,\mathrm{d}s\,\mathrm{d}\theta, \tag{9}$$

where

$$P(\rho)=\begin{bmatrix}P_1(\rho) & 0\\ P_2(\rho) & P_3(\rho)\end{bmatrix}, \quad F=\mathrm{diag}[I,0],$$
$$0<P_1(\rho)=P_1^{\mathrm{T}}(\rho), \quad R^{\mathrm{T}}=R>0.$$

The derivative of the first term of the above functional along the system trajectory results in

$$\begin{aligned} \dot{V}_1(t) &= \tilde{x}^{\mathrm{T}}(t)\dot{P}_1\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)P_1\dot{\tilde{x}}(t) = \tilde{x}^{\mathrm{T}}(t)\dot{P}_1\tilde{x}(t) \\ &\quad + 2[\tilde{x}^{\mathrm{T}}(t) \;\; \eta^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix}\dot{\tilde{x}}(t)\\ 0\end{bmatrix} \\ &= \tilde{x}^{\mathrm{T}}(t)\dot{P}_1\tilde{x}(t) + 2[\tilde{x}^{\mathrm{T}}(t) \;\; \eta^{\mathrm{T}}(t)]P^{\mathrm{T}} \\ &\quad \times\begin{bmatrix}\eta(t)\\ -\eta(t)+A_2^{\mathrm{T}}\tilde{x}(t) - A_1\int_{t-h}^{t}\eta(s)\,\mathrm{d}s + L^{\mathrm{T}}\tilde{z}(t)\end{bmatrix} \\ &= 2[\tilde{x}^{\mathrm{T}}(t) \;\; \eta^{\mathrm{T}}(t)]P^{\mathrm{T}}\left(\begin{bmatrix}0 & I\\ A_2^{\mathrm{T}} & -I\end{bmatrix}\begin{bmatrix}\tilde{x}(t)\\ \eta(t)\end{bmatrix} + \begin{bmatrix}0\\ L^{\mathrm{T}}\end{bmatrix}\tilde{z}(t)\right) \\ &\quad + \tilde{x}^{\mathrm{T}}(t)\dot{P}_1\tilde{x}(t) + \sigma(t), \end{aligned}$$

where $A_2=A_0+A_1$. Also, the time derivative of the second term of $V(t)$ is

$$\begin{aligned} \dot{V}_2(t) &= \int_{-h}^{0}(\eta^{\mathrm{T}}(t)R\eta(t) - \eta^{\mathrm{T}}(t+\theta)R\eta(t+\theta))\,\mathrm{d}\theta \\ &= h\eta^{\mathrm{T}}(t)R\eta(t) - \int_{t-h}^{t}\eta^{\mathrm{T}}(\theta)R\eta(\theta)\,\mathrm{d}\theta, \end{aligned}$$

where we have used the Leibniz integral rule. Now, making use of Lemma 1 provides us with the existence of a matrix $R>0$ such that

$$\begin{aligned} &-2\int_{t-h}^{t}[\tilde{x}^{\mathrm{T}}(t) \;\; \eta^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix}0\\ A_1^{\mathrm{T}}\end{bmatrix}\eta(s)\,\mathrm{d}s \\ &\quad\leqslant \int_{t-h}^{t}\eta^{\mathrm{T}}(s)R\eta(s)\,\mathrm{d}s \\ &\qquad + \int_{t-h}^{t}[\tilde{x}^{\mathrm{T}}(t) \;\; \eta^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix}0\\ A_1^{\mathrm{T}}\end{bmatrix}R^{-1}[0 \;\; A_1]P\begin{bmatrix}\tilde{x}(t)\\ \eta(t)\end{bmatrix}\mathrm{d}s \\ &\quad\leqslant \int_{t-h}^{t}\eta^{\mathrm{T}}(s)R\eta(s)\,\mathrm{d}s \\ &\qquad + h[\tilde{x}^{\mathrm{T}}(t) \;\; \eta^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix}0\\ A_1^{\mathrm{T}}\end{bmatrix}R^{-1}[0 \;\; A_1]P\begin{bmatrix}\tilde{x}(t)\\ \eta(t)\end{bmatrix}. \end{aligned}$$

Note that while using Lemma 1, we have assumed that the corresponding matrix $W=0$. A less conservative filter design can be obtained by making the matrix $W$ a free parameter. Then, a more complex but less conservative LMI formulation can be derived to design such filters. Using the obtained derivative terms provides us with the following result for $\dot{V}(t)$

$$\begin{aligned} \dot{V}(t) &= \psi^{\mathrm{T}}(t)\Sigma_0\psi(t) + h\eta^{\mathrm{T}}(t)R\eta(t) \\ &\quad + h[\tilde{x}^{\mathrm{T}}(t) \;\; \eta^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix}0\\ A_1^{\mathrm{T}}\end{bmatrix}R^{-1}[0 \;\; A_1]P\begin{bmatrix}\tilde{x}(t)\\ \eta(t)\end{bmatrix}, \end{aligned}$$

where

$$\psi(t)=\begin{bmatrix}\tilde{x}(t)\\ \eta(t)\\ \tilde{z}(t)\end{bmatrix},\quad \Sigma_0=\begin{bmatrix}\phi_1+\begin{bmatrix}\dot{P}_1 & 0\\ 0 & 0\end{bmatrix} & \star\\ [0\ \ L]P & 0\end{bmatrix},$$

$$\phi_1=P^{\mathrm{T}}\begin{bmatrix}0 & I\\ A_2^{\mathrm{T}} & -I\end{bmatrix}+\begin{bmatrix}0 & A_2\\ I & -I\end{bmatrix}P.$$

If our objective is to keep the $\mathscr{H}^\infty$ norm of the operator mapping external disturbances to the desired output less than a prescribed positive value $\gamma$, then we have to make the performance measure, as defined in (6), negative. We can rewrite the performance measure as

$$J'=\int_0^\infty(\tilde{w}^{\mathrm{T}}(t)\tilde{w}(t)-\gamma^2\tilde{z}^{\mathrm{T}}(t)\tilde{z}(t)+\dot{V}(t))\,\mathrm{d}t. \tag{10}$$

Substituting $\tilde{w}(t)$ from (7) results in the integrand in (10) being less than $\psi^{\mathrm{T}}(t)\Sigma_1\psi(t)$. Now, if $\Sigma_1<0$, then $J<0$ which implies that the $\mathscr{L}_2$-gain from the disturbance $\tilde{z}(t)$ to the system output $\tilde{w}(t)$ is less than $\gamma$. Using Schur complement on $\Sigma_1<0$ leads to

$$\Sigma_2=\begin{bmatrix}\phi_1+\mathrm{diag}(\dot{P}_1,hR)+h\phi_2 & \star & \star\\ [0\ \ L]P & -\gamma^2 I & \star\\ [B^{\mathrm{T}}\ \ 0] & D^{\mathrm{T}} & -I\end{bmatrix}<0, \tag{11}$$

with $\phi_2$ given by $\phi_2=P^{\mathrm{T}}\begin{bmatrix}0\\ A_1^{\mathrm{T}}\end{bmatrix}R^{-1}[0\ \ A_1]P$. But $\Sigma_2<0$ is a bilinear matrix inequality because of the presence of both $R$ and $R^{-1}$. However, using the Schur complement again we obtain the following LMI:

$$\Sigma_3=\begin{bmatrix}\phi_1+\mathrm{diag}(\dot{P}_1,hR) & \star & \star & \star\\ [0\ \ L]P & -\gamma^2 I & \star & \star\\ [B^{\mathrm{T}}\ \ 0] & D^{\mathrm{T}} & -I & \star\\ [0\ \ A_1]hP & 0 & 0 & -hR\end{bmatrix}<0. \tag{12}$$

An equivalent LMI can be obtained by utilizing the congruence transformation $\mathscr{T}_1=\mathrm{diag}(Q,I,I,I)$, where $Q=P^{-1}$, and applying it to LMI (12). First, note that

$$Q=\begin{bmatrix}Q_1 & 0\\ Q_2 & Q_3\end{bmatrix},\quad Q_1=P_1^{-1}>0,\quad \dot{Q}_1=-Q_1\dot{P}_1Q_1. \tag{13}$$

Based on the above transformation, we have

$$\Sigma_4=\begin{bmatrix}(1,1) & \star & \star & \star\\ [0\ \ L] & -\gamma^2 I & \star & \star\\ [B^{\mathrm{T}}\ \ 0]Q & D^{\mathrm{T}} & -I & \star\\ [0\ \ hA_1] & 0 & 0 & -hR\end{bmatrix}<0, \tag{14}$$

where $(1,1)=\begin{bmatrix}0 & I\\ A_2^{\mathrm{T}} & -I\end{bmatrix}Q+Q^{\mathrm{T}}\begin{bmatrix}0 & A_2\\ I & -I\end{bmatrix}+Q^{\mathrm{T}}\begin{bmatrix}0 & 0\\ 0 & hR\end{bmatrix}Q+\begin{bmatrix}-\dot{Q}_1 & 0\\ 0 & 0\end{bmatrix}$. Applying the Schur complement to (14) and noting that $\begin{bmatrix}0 & 0\\ 0 & hR\end{bmatrix}=\begin{bmatrix}0\\ I\end{bmatrix}hR[0\ \ I]$, the following inequality is obtained:

$$\Sigma_5=\begin{bmatrix}(1,1) & \star & \star & \star & \star\\ [0\ \ L] & -\gamma^2 I & \star & \star & \star\\ [B^{\mathrm{T}}\ \ 0]Q & D^{\mathrm{T}} & -I & \star & \star\\ [0\ \ hA_1] & 0 & 0 & -hR & \star\\ h[0\ \ I]Q & 0 & 0 & 0 & -hR^{-1}\end{bmatrix}<0, \tag{15}$$

where $(1,1)=\begin{bmatrix}0 & I\\ A_2^{\mathrm{T}} & -I\end{bmatrix}Q+Q^{\mathrm{T}}\begin{bmatrix}0 & A_2\\ I & -I\end{bmatrix}+\begin{bmatrix}-\dot{Q}_1 & 0\\ 0 & 0\end{bmatrix}$. The resultant inequality is nonlinear, so we use the congruence transformation $\mathscr{T}_2=\mathrm{diag}(I,I,I,S,I)$, where $S=S^{\mathrm{T}}=R^{-1}$. Application of the mentioned transformation to (15) leads to

$$\Sigma_6=\begin{bmatrix}(1,1) & \star & \star & \star & \star\\ [0\ \ L] & -\gamma^2 I & \star & \star & \star\\ [B^{\mathrm{T}}\ \ 0]Q & D^{\mathrm{T}} & -I & \star & \star\\ [0\ \ hSA_1] & 0 & 0 & -hS & \star\\ h[0\ \ I]Q & 0 & 0 & 0 & -hS\end{bmatrix}<0. \tag{16}$$

Now, we are in the position where we can state the following theorem which provides conditions to ensure the system stability and $\mathscr{H}^\infty$ performance.

**Theorem 2.** [illegible] (5) [illegible] $\phi=0$. *I* [illegible] $Q_1(\rho)$ [illegible] $S=S^{\mathrm{T}}>0$, $Q_2$ [illegible] $Q_3$ [illegible]

$$\Sigma=\begin{bmatrix}(1,1) & \star & \star & \star & \star\\ [0\ \ L(\rho)] & -\gamma^2 I & \star & \star & \star\\ [B^{\mathrm{T}}(\rho)Q_1(\rho)\ \ 0] & D^{\mathrm{T}}(\rho) & -I & \star & \star\\ [0\ \ hSA_1(\rho)] & 0 & 0 & -hS & \star\\ h[Q_2\ \ Q_3] & 0 & 0 & 0 & -hS\end{bmatrix}<0, \tag{17}$$

[illegible] $(1,1)=\begin{bmatrix}Q_2+Q_2^{\mathrm{T}}-\dot{Q}_1(\rho) & \star\\ A_2^{\mathrm{T}}(\rho)Q_1(\rho)-Q_2+Q_3^{\mathrm{T}} & -Q_3-Q_3^{\mathrm{T}}\end{bmatrix}$, [illegible] $\rho\in\mathscr{P}$, $|\dot{\rho}_i|\leqslant v_i$, [illegible] $\mathscr{L}_2$-[illegible] $\gamma$. [illegible], [illegible] LMI [illegible]

$$\begin{cases}\min & \gamma^2\\ \text{s.t.} & (17).\end{cases} \tag{18}$$

**Proof.** Using (16) and substituting $Q$ from (13) proves the theorem. □

Now, we turn our attention to the original problem, that is, the stability and performance analysis of plant (1) and the corresponding design of the $\mathscr{H}^\infty$ filters. The next lemma is a straightforward extension of Theorem 2 to analyze plant (1).

**Lemma 3.** [illegible] (1) [illegible] $\phi(t)=0$. *I* [illegible] $Q_1(\rho)$

[illegible] $S = S^{\mathrm{T}} > 0,\ Q_2,$ [illegible] $Q_3$ [illegible]

$$\begin{bmatrix} (1,1) & \star & \star & \star & \star \\ \begin{bmatrix} 0 & C_1(\rho) \\ 0 & C_{1h}(\rho) \end{bmatrix} & -\gamma^2 I & \star & \star & \star \\ [B_1^{\mathrm{T}} Q_1(\rho)\ \ 0] & [D_{11}^{\mathrm{T}}\ \ 0] & -I & \star & \star \\ [0\ \ hSA_h(\rho)] & 0 & 0 & -hS & \star \\ h[Q_2\ \ Q_3] & 0 & 0 & 0 & -hS \end{bmatrix} < 0, \tag{19}$$

[illegible] $(1,1)$ [illegible] 2, [illegible] $\rho \in \mathscr{P},\ |\dot{\rho}_i| \leqslant v_i,$ [illegible] $\mathscr{L}_2$-[illegible] $\gamma$.

**Proof.** Making use of the forward adjoint of (4) and Theorem 2 proves the lemma. □

## 4. LPV $\mathscr{H}^\infty$ filter design

In this section, the problem of designing parameter-dependent filters for the time-delayed LPV system (1), which make the induced $\mathscr{L}_2$-gain bound of the filter error system minimum, is investigated. We propose two filters with similar structure, one of which has memory in its dynamics and another one that is memoryless. In the present work, we intend to design the filters for estimation of the plant output $z(t)$.

### 4.1. [illegible]

Consider a class of memoryless filters with the state–space equations given by

$$\begin{cases} \dot{x}_{\mathrm{F}}(t) = A_{\mathrm{F}}(\rho(t))x_{\mathrm{F}}(t) + B_{\mathrm{F}}(\rho(t))y(t), \\ z_{\mathrm{F}}(t) = C_{\mathrm{F}}(\rho(t))x_{\mathrm{F}}(t) + D_{\mathrm{F}}(\rho(t))y(t), \end{cases} \tag{20}$$

in which the parameter-dependent matrices $A_{\mathrm{F}}(\rho(t))$, $B_{\mathrm{F}}(\rho(t))$, $C_{\mathrm{F}}(\rho(t))$, and $D_{\mathrm{F}}(\rho(t))$ are the unknown filter parameters. In the above equations, we have assumed that $z_{\mathrm{F}}(t)$ is the estimation of the plant output $z(t)$. Let us define the estimation error as $e(t) = z(t) - z_{\mathrm{F}}(t)$.

Our goal is to develop an $\mathscr{H}^\infty$ filter of form (20) such that for all admissible parameter trajectories $\rho(t) \in \mathscr{F}_{\mathscr{P}}^{v}$:

- The filtering error system obtained from the interconnection of plant (1) and filter (20) is asymptotically stable.
- The above-mentioned filtering error system guarantees, under zero initial condition, that

$$\sup_{\rho \in \mathscr{F}_{\mathscr{P}}^{v}} \sup_{\|w\|_2 \neq 0} \frac{\|e(t)\|_2}{\|w(t)\|_2 + \|w(t-h)\|_2} \leqslant \gamma \tag{21}$$

for all energy-bounded disturbances and a prescribed positive value $\gamma$.

We use the analysis condition (19) to obtain the synthesis inequalities for the augmented system of the plant and the filter in terms of LMIs. The following theorem provides a sufficient condition to guarantee both asymptotic stability and induced $\mathscr{L}_2$-gain performance of the interconnection of plant (1) and the proposed memoryless filter (20).

**Theorem 4.** *[illegible]* $Q_a(\rho),\ Q_b(\rho),$ [illegible] $X$ [illegible] $Y$ [illegible] $S_1 = S_1^{\mathrm{T}} > 0,\ Q_2,\ Q_3,\ C_{\mathrm{F}},$ [illegible] $D_{\mathrm{F}}$ [illegible]

$$\begin{bmatrix} \sigma_{11} & \star & \star & \star & \star \\ [0\ \ \sigma_{21}] & -\gamma^2 I & \star & \star & \star \\ [\sigma_{31}\ \ 0] & [\sigma_{32}\ \ 0] & -I & \star & \star \\ [0\ \ h\sigma_{41}] & 0 & 0 & -hS & \star \\ h[Q_2\ \ Q_3] & 0 & 0 & 0 & -hS \end{bmatrix} < 0 \tag{22}$$

[illegible] (1) [illegible] (20) [illegible] $\mathscr{L}_2$-[illegible] $\gamma$. [illegible], [illegible] (20) [illegible]

$$A_{\mathrm{F}}(\rho) = (X(\rho)Q_b^{-1}(\rho))^{\mathrm{T}},$$
$$B_{\mathrm{F}}(\rho) = (Y(\rho)Q_b^{-1}(\rho))^{\mathrm{T}},$$
$$C_{\mathrm{F}}, D_{\mathrm{F}}: \text{[illegible] (22).} \tag{23}$$

The variables in (22) are defined as

$$\sigma_{11} = \begin{bmatrix} Q_2 + Q_2^{\mathrm{T}} - \begin{bmatrix} \dot{Q}_a & 0 \\ 0 & \dot{Q}_b \end{bmatrix} & \star \\ \begin{bmatrix} A_2^{\mathrm{T}} Q_a & A_2^{\mathrm{T}} Q_b - X - C_2^{\mathrm{T}} Y - C_{2h}^{\mathrm{T}} Y \\ 0 & X \end{bmatrix} - Q_2 + Q_3^{\mathrm{T}} & -Q_3 - Q_3^{\mathrm{T}} \end{bmatrix},$$

$$\sigma_{21} = \begin{bmatrix} C_1 - C_{\mathrm{F}} - D_{\mathrm{F}} C_2 & C_{\mathrm{F}} \\ C_{1h} - D_{\mathrm{F}} C_{2h} & 0 \end{bmatrix},$$
$$\sigma_{31} = [B_1^{\mathrm{T}} Q_a\ \ B_1^{\mathrm{T}} Q_b - D_{21}^{\mathrm{T}} Y],$$
$$\sigma_{41} = \begin{bmatrix} (S_1 + Q_b)A_h - Y^{\mathrm{T}} C_{2h} & 0 \\ 2Q_b A_h - Y^{\mathrm{T}} C_{2h} & 0 \end{bmatrix},$$
$$\sigma_{32} = D_{11}^{\mathrm{T}} - D_{21}^{\mathrm{T}} D_{\mathrm{F}}^{\mathrm{T}}, \quad S = \begin{bmatrix} S_1 & Q_b \\ Q_b & Q_b \end{bmatrix}.$$

**Proof.** The equations of the augmented system determined from the interconnection of (1) and (20) read

$$\begin{cases} \dot{x}_a(t) = A_a(\rho(t))x_a(t) + A_{ah}(\rho(t))x_a(t-h) \\ \qquad + B_a(\rho(t))w(t), \\ e(t) = C_a(\rho(t))x_a(t) + C_{ah}(\rho(t))x_a(t-h) \\ \qquad + D_a(\rho(t))w(t), \end{cases} \tag{24}$$

where $x_a(t) = [x^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t) - x_{\mathrm{F}}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and

$$A_a = \begin{bmatrix} A_0 & 0 \\ A - A_{\mathrm{F}} - B_{\mathrm{F}} C_2 & A_{\mathrm{F}} \end{bmatrix}, \quad A_{ah} = \begin{bmatrix} A_h & 0 \\ A_h - B_{\mathrm{F}} C_{2h} & 0 \end{bmatrix},$$

$$B_a = \begin{bmatrix} B_1 \\ B_1 - B_F D_{21} \end{bmatrix}, \quad C_a = [C_1 - C_F - D_F C_2 \;\; C_F],$$

$$C_{ah} = [C_{1h} - D_F C_{2h} \;\; 0], \quad D_a = D_{11} - D_F D_{21}.$$

Now, substituting the above matrices into (19), replacing $B_F^T Q_b = Y$, $A_F^T Q_b = X$, and using $Q_1 = \mathrm{diag}(Q_a, Q_b)$ and $S = \begin{bmatrix} S_1 & Q_b \\ Q_b & Q_b \end{bmatrix}$ (in order to eliminate the appeared nonlinear terms) the analysis condition to satisfy asymptotic stability and $\mathscr{H}^\infty$ performance results in (22), and the filter matrices are obtained from (23). □

### 4.2. [illegible]

Now, consider a class of delayed filters with the following state–space equations:

$$\begin{cases} \dot{x}_F(t) = A_F(\rho(t))x_F(t) + A_{hF}(\rho(t))x_F(t-h) \\ \qquad + B_F(\rho(t))y(t), \\ z_F(t) = C_F(\rho(t))x_F(t) + C_{hF}(\rho(t))x_F(t-h) \\ \qquad + D_F(\rho(t))y(t). \end{cases} \tag{25}$$

The filter representation (25) includes an additional state-delayed term. This term increases the computational complexity of the filter but results in reduced conservatism and improved performance. Writing the equations of the augmented system formed from plant (1) and filter (25), we obtain the following result that provides a sufficient condition in order to ensure stability and induced $\mathscr{L}_2$-gain performance.

**Theorem 5.** *I f [illegible]* $Q_a(\rho)$, $Q_b(\rho)$, *[illegible]* $X$ , $Y$, *[illegible]* $Z$ , *[illegible]* $S_1^T = S_1 > 0$, $Q_2$, $Q_3$, $C_F$, $C_{hF}$, *[illegible]* $D_F$ *[illegible]*

$$\begin{bmatrix} \sigma_{11} & \star & \star & \star & \star \\ [0 \;\; \bar{\sigma}_{21}] & -\gamma^2 I & \star & \star & \star \\ [\sigma_{31} \;\; 0] & [\sigma_{32} \;\; 0] & -I & \star & \star \\ [0 \;\; h\bar{\sigma}_{41}] & 0 & 0 & -hS & \star \\ h[Q_2 \;\; Q_3] & 0 & 0 & 0 & -hS \end{bmatrix} < 0 \tag{26}$$

*[illegible]* (1) *[illegible]* (25) *[illegible]* $\mathscr{L}_2$-*[illegible]* $\gamma$. *[illegible]* , *[illegible]*

$$\begin{aligned} &A_F(\rho) = (X(\rho)Q_b^{-1}(\rho))^T - Q_b^{-1}(\rho)Z(\rho), \\ &A_{hF} = Q_b^{-1}(\rho)Z(\rho), \\ &B_F(\rho) = (Y(\rho)Q_b^{-1}(\rho))^T, \\ &C_F, C_{hF}, D_F: \text{[illegible] (26)}. \end{aligned} \tag{27}$$

*[illegible]* (26) *[illegible]* 4 *[illegible]* :

$$\bar{\sigma}_{21} = \begin{bmatrix} C_1 - C_F - D_F C_2 & C_F \\ C_{1h} - C_{hF} - D_F C_{2h} & C_{hF} \end{bmatrix},$$

$$\bar{\sigma}_{41} = \begin{bmatrix} (S_1 + Q_b)A_h - Z - Y^T C_{2h} & Z \\ 2Q_b A_h - Z - Y^T C_{2h} & Z \end{bmatrix}.$$

**Proof.** The equations of the augmented system determined from the interconnection of (1) and (25) are given by (24), in which

$$A_{ah} = \begin{bmatrix} A_h & 0 \\ A_h - A_{hF} - B_F C_{2h} & A_{hF} \end{bmatrix},$$
$$C_{ah} = [C_{1h} - C_{hF} - D_F C_{2h} \;\; C_{hF}]$$

and the rest of the matrices are computed as in Theorem 4. Now, using the above matrices, condition (19) and structured matrices $Q_1$ and $S$ similar to those in Theorem 4, and replacing $B_F^T Q_b = Y$, $(A_F + A_{hF})^T Q_b = X$, and $Q_b A_{hF} = Z$ to eliminate nonlinear terms result in the synthesis condition (26). □

**Remark 1.** Note that for convenience, in the above results, we have dropped the dependence of some matrices on the LPV parameter .

**Remark 2.** In LMIs (22) and (26) the derivative terms are computed by

$$\dot{Q}_a = \sum_{j=1}^{s} \left( \pm v_j \frac{\partial Q_a(\rho)}{\partial \rho_j} \right).$$

This notation means that every combination of + and − must be included in the inequalities, in which $\sum_{j=1}^{s} \pm$ is involved. Therefore, the LMIs actually represent $2^s$ different inequalities obtained from the corresponding $2^s$ different combinations in the summation.

**Remark 3.** The above-mentioned LMIs are infinite dimensional. An approach for solving an infinite-dimensional LMI problem due to its dependence on the parameters is to grid the parameter space [1]. To keep the computational effort feasible, the minimization for a defined grid should be performed, and the constraints for the optimal $\mathscr{H}^\infty$ attenuation level must be checked on a finer grid. If this check fails, the minimization needs to be repeated for a more dense grid. In spite of the relative efficiency of the available numerical algorithms for solving LMIs, the utility of this ad hoc approach is limited to systems with a small number of parameters [1].

**Remark 4.** The presented filter design methodology can be easily extended to address plants that include multiple delays by adding similar terms associated with multiple delays to the Lyapunov–Krasovskii functional (9).

**Remark 5.** In the cited theorems, the filter matrices $C_F$ and $D_F$ are not parameter dependent. Imposing a polynomial structure in terms of the LPV parameters on $C_F$ and $D_F$ and finding the corresponding polynomial matrix coefficients can provide improved results in terms of the $\mathscr{L}_2$-gain performance. Also, taking the matrix $R$ into account as a parameter-dependent Lyapunov matrix function can provide improved estimation results. Obviously, these extensions will result in an increased computational effort for the solution.

## 5. Simulation results

In this section, we demonstrate our filter design methodology using some illustrative numerical examples.

In the first example, we are concerned with the estimation of a noisy output using the filters proposed in the previous sections. We consider a plant given by

$$\dot{x}(t) = \begin{bmatrix} 0 & -1.5-0.3\cos(t/3) \\ 1 & -2+0.5\cos(t/3) \end{bmatrix} x(t) + \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0 \end{bmatrix} x(t-h) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t),$$

$$y(t) = [0 \;\; 1]x(t-h) + 0.1w(t),$$

$$z(t) = [0.5\cos(t/3) \;\; 0]x(t) + [0 \;\; 0.5]x(t-h).$$

We assume that the cosine term in the model corresponds to a system parameter whose functional representation is not known a priori, but can be measured in real time. Define $\rho(t) = \cos(t/3)$, and assume that the original system is reformulated as a state-delayed LPV system with parameter $\rho(t)$. Note that the parameter space is $[-1, 1]$. The synthesis problem is solved for both the memoryless filter (20) and the delayed filter (25), and the results are then compared. To solve the corresponding LMIs, we select an affine form for the matrix functions with basis functions

$$f_1(\rho) = 1, \quad f_2(\rho) = \rho(t). \tag{28}$$

Using the above basis functions, we obtain an LMI problem with respect to constant parameter matrices. For instance, $Q_a(\rho)$ in Theorems 4 and 5 is represented as $Q_a(\rho(t)) = Q_{a1} + \rho(t)Q_{a2}$ where $Q_{a1}$ and $Q_{a2}$ are symmetric matrices to be determined by solving LMIs. Gridding in order to solve the synthesis problem was uniform using seven points over the parameter space. Validation of the LMI constraints was performed on a finer grid of 15 points following [1].

Fig. 1 shows the worst case $\mathscr{L}_2$-gain performance $\gamma_{\text{lpv}}$ of the system for the two designed filters (the memoryless and the delayed one) vs. a constant value of the time delay. The plot illustrates that the delayed filter outperforms the memoryless filter from the $\mathscr{L}_2$-gain performance viewpoint. Note that $\gamma_{\text{lpv}}$ is obtained by solving an LMI problem similar to (18).

In the second example, we compare our proposed filter design method with a past method that addresses design of parameter-dependent $\mathscr{H}^\infty$ filters for a state-delay LPV system proposed in [9]. Note that the previous work [9] is only concerned with state estimation. Hence, for comparison, we consider a time varying plant represented by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1+0.2\cos(2t) \\ -2 & -3+0.3\cos(2t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2\cos(2t) & 0.1 \\ 0 & 0.1\cos(2t) \end{bmatrix} x(t-h) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t),$$

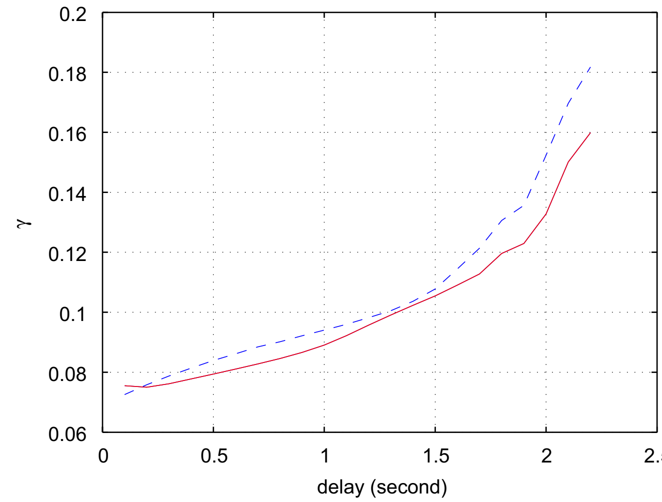


Fig. 1. Profile of the worst case performance of the system ($\gamma_{\text{lpv}}$) for delayed filter (solid line), and for memoryless filter (dashed line).

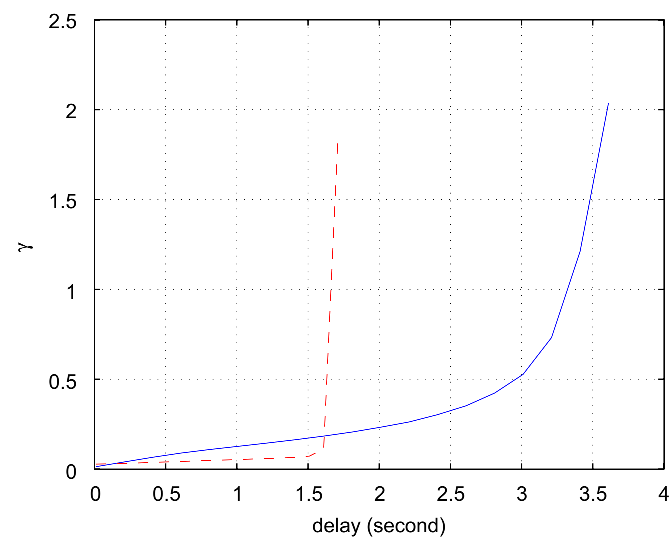


Fig. 2. Profile of the worst case performance of the system ($\gamma_{\text{lpv}}$) for our new delayed filter (solid line), and for delayed filter in [9] (dashed line).

$$y(t) = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x(t) + [0 \;\; 1]w(t),$$

$$z(t) = x(t).$$

Fig. 2 shows the induced $\mathscr{L}_2$-gain from the noise signal to the estimation error for the two filters, the one presented in [9] and the other one given in Section 4 of the present paper. With respect to the fact that the analysis and design approaches for the two methods are different, the profile of $\gamma$ vs. the delay is unpredictable in terms of its behavior. One point of comparison is the maximum allowable size of delay for which the $\mathscr{H}^\infty$ filter exists. These values and the corresponding induced norms are provided in Table 1. The table shows that our proposed filter design methodology in this paper handles delay sizes more than twice the size of those covered by the method suggested in [9]. The worst case performance of the method given in [9] is a bit

Table 1
Worst case performance of available techniques vs. maximum allowable state delay

| Method | Maximum allowable delay ($h_{max}$) | Worst case performance ($\gamma$) |
|---|---|---|
| Method in [9] | 1.712 | 2.1736 |
| Proposed method in this paper | 3.62 | 2.2153 |

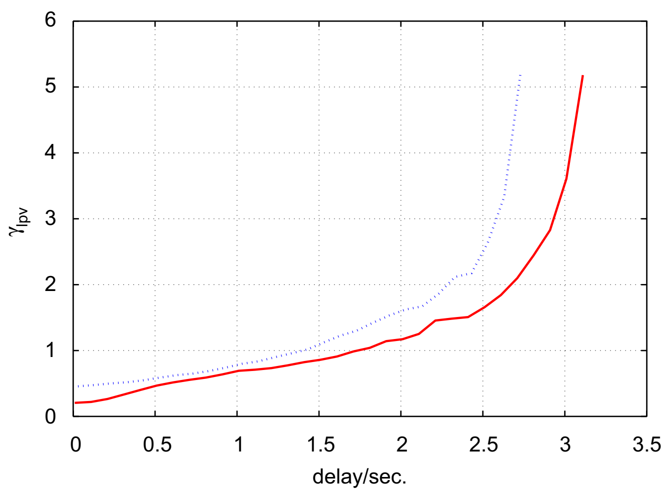


Fig. 3. Profile of the worst case robust performance of the system ($\gamma_{lpv}$) for delayed filter (solid line), and for memoryless filter (dotted line).

better for delay sizes $h < 1.62$ s, whereas the suggested filtering method in this paper considerably outperforms the previous one from the $\mathscr{L}_2$-gain performance viewpoint for $h > 1.62$ s.

As a third example, consider a time varying plant given by

$$A = \begin{bmatrix} 0 & 1 + 0.2\sin(t/2) \\ -2 & -3 + 0.1\sin(t/2) \end{bmatrix},$$

$$A_h = \begin{bmatrix} 0.2\sin(t/2) & 0.1 \\ -0.2 + 0.1\sin(t/2) & -0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.3 & 0.5\sin(t/2) \\ -0.4 - 0.5\sin(t/2) & 0.75 \end{bmatrix},$$

$$C_{1h} = \begin{bmatrix} 0 & -0.45 \\ 2.5 & -0.15 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix}, \quad C_{2h} = \begin{bmatrix} 0 & 1 \\ 0 & 0.7 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Similar to previous example, we define $\rho(t) = \sin(t/2)$ and reformulate the original system as a state-delayed LPV system with parameter $\rho(t)$. Note that the parameter space and the parameter derivative space are $[-1, 1]$ and $[-0.5, 0.5]$, respectively. We solve the synthesis problems for both the memoryless filter and the delayed filter using Theorems 4 and 5, and compare the results. To solve the LMIs, we select the same affine form for the matrix functions as in the previous example using the basis functions in (28).

Fig. 3 shows the worst case $\mathscr{L}_2$-gain performance $\gamma_{lpv}$ of the system for the two filters vs. a constant value of the time delay. The plot illustrates that the delayed filter outperforms the memoryless filter from the $\mathscr{L}_2$-gain performance viewpoint.

Fig. 4 illustrates the results of the time-domain simulation for a constant time delay $h = 1$ s in the presence of a uniformly distributed random signal varying in the interval $[-1, 1]$. The figure shows the estimation error $e(t)$ obtained using the two classes of filters. It is observed that the estimation error of the delayed filter is smaller compared to that of the memoryless filter. Nevertheless, both the delayed and the memoryless filters have satisfactory performance in tracking the plant outputs. Improved estimation performance using the delayed filter can be observed as expected.

Finally, we compare our LPV $\mathscr{H}^\infty$ filter design with corresponding robust $\mathscr{H}^\infty$ filters assuming that an uncertain time-delayed linear system with bounds on its uncertain parameter is given. There are some recent methods [4,15,11] in the literature to analyze such systems and to design robust $\mathscr{H}^\infty$ filters. In the above methods, the uncertain parameters are assumed to reside in a polytope. For comparison purposes, we treat the uncertain plant also as an LPV system, where the uncertain parameters are taken into consideration as the LPV parameters. The objective is to compare the performance of our filter design methodologies with the ones proposed in [4,15]. We consider the following linear uncertain system that includes a state delay. The example is borrowed from [15] and modified here.

$$\dot{x}(t) = \begin{bmatrix} 0 & 3 + \rho \\ -4 & -5 \end{bmatrix} x(t) + \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.2 + \rho \end{bmatrix} x(t - h) + \begin{bmatrix} -0.4545 \\ 0.9090 \end{bmatrix} w(t),$$

$$y(t) = [0 \;\; 100]x(t) + [0 \;\; 10]x(t - h) + w(t),$$

$$z(t) = [0 \;\; 100]x(t).$$

We assume that the uncertain real parameter $\rho$ varies in the region $|\rho| < 0.3$ that defines a two-vertex uncertainty domain. We compute the minimum noise attenuation level using the results of this paper and the previous work in [4,15] for different constant time delays. For different delay sizes, Fig. 5 illustrates the worst case $\mathscr{L}_2$-gain performance $\gamma$ that is achieved from the filtering error system using (i) the delay-independent conditions of [15], (ii) the robust filter proposed in [4], and (iii) the memoryless and delayed parameter-dependent filters proposed in the present paper. The disturbance attenuation levels illustrated in Fig. 5 show the improved performance and decreased conservatism of our design methodology compared to those of [4,15]. It should be pointed out, however, that the computational complexity for solving the optimization problems of this paper is higher than that of the optimization problems of [4,15] because of the larger number of LMI decision variables. The reason is that the descriptor form (8) adds extra states to the state vector of the original time-delay system.

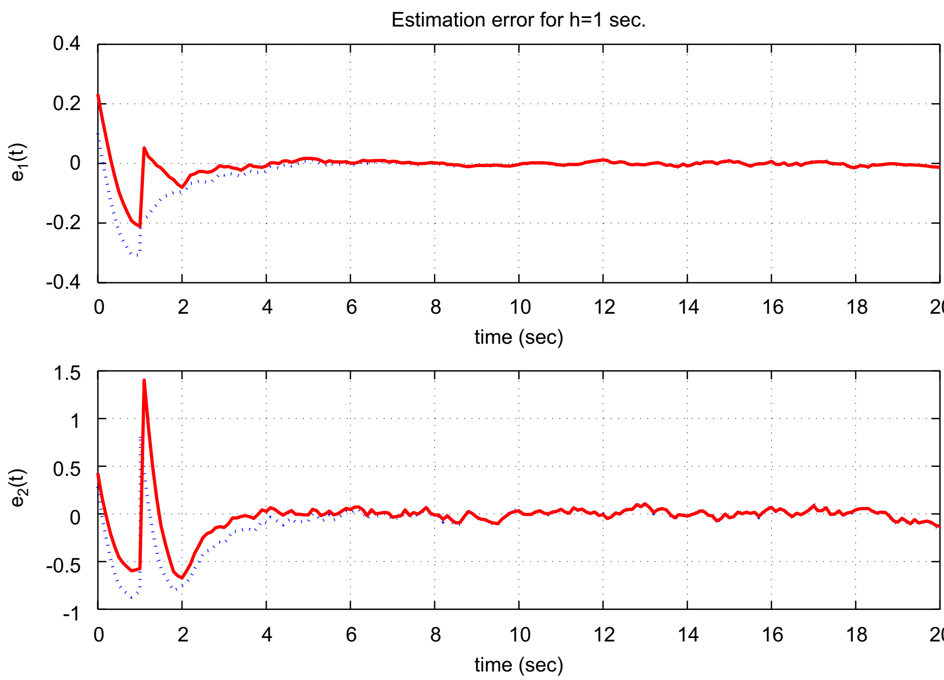


Fig. 4. Estimation results (for $h = 1$ s): estimation error of the memoryless filter (dotted line), and error of the delayed filter (solid line).

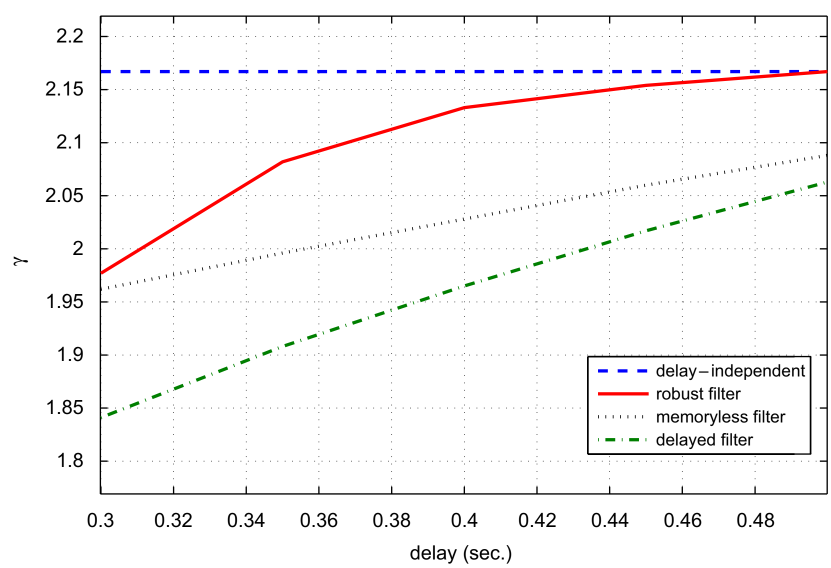


Fig. 5. Profile of worst case performance level of the filtering error system ($\gamma$) for results of [15] (dashed line), robust filter of [4] (solid line), our memoryless filter (dotted line), and our delayed filter (dash-dotted line).

## 6. Conclusion

We have presented a methodology to design two classes of parameter-dependent $\mathscr{H}^\infty$ filters for estimation of noisy outputs in state-delayed systems whose state–space information is parameter dependent. Based on our results, we have considered memoryless filters, as well as, filters whose dynamics is time delayed and infinite-dimensional. We have formulated the synthesis conditions to guarantee asymptotic stability and $\mathscr{H}^\infty$ performance in terms of LMIs which can be readily solved using available software packages. The various examples illustrate the superiority of the delayed filter because of providing more degrees of freedom and consequently reducing the conservatism in the design. We have also provided a comparison of our designs to other recently developed methods to show the qualifications of the proposed method.

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