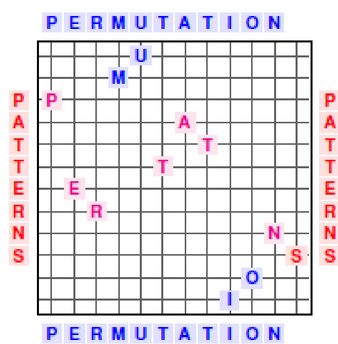
Permutation Patterns 2023

Université de Bourgogne

July 3-7, 2023, Dijon, France



Contents

Welcome

Welcome to University of Burgundy in Dijon for the 21st edition of Permutation Patterns! We hope that you enjoy the conference, Dijon and the captivating region of Burgundy.

We've included some suggestions of things to see this week in a subsequent section of this booklet.

If you need anything during your stay, please get in touch with one of the local members of the organizing committee: Jean-Luc Baril, Sergey Kirgizon, Rémi Maréchal, Vincent Vajnovszki, who will be happy to help you.

Many people have participated to the efforts to make the conference a success and a very enjoyable experience, I am grateful to all of them.

Thank you for joining us, we're glad you're here. We hope you will have an exciting journey.

Vincent Vajnovszki
(on behalf of the organizing committees)

Sponsors

Dijon Métropole, Région Bourgogne-Franche-Comté, Université de Bourgogne, Département Côte d'Or, Università degli Studi di Firenze, Université Franco-Italienne, Laboratoire d'Informatique de Bourgogne, ANR, NSF.

Local information

Important numbers

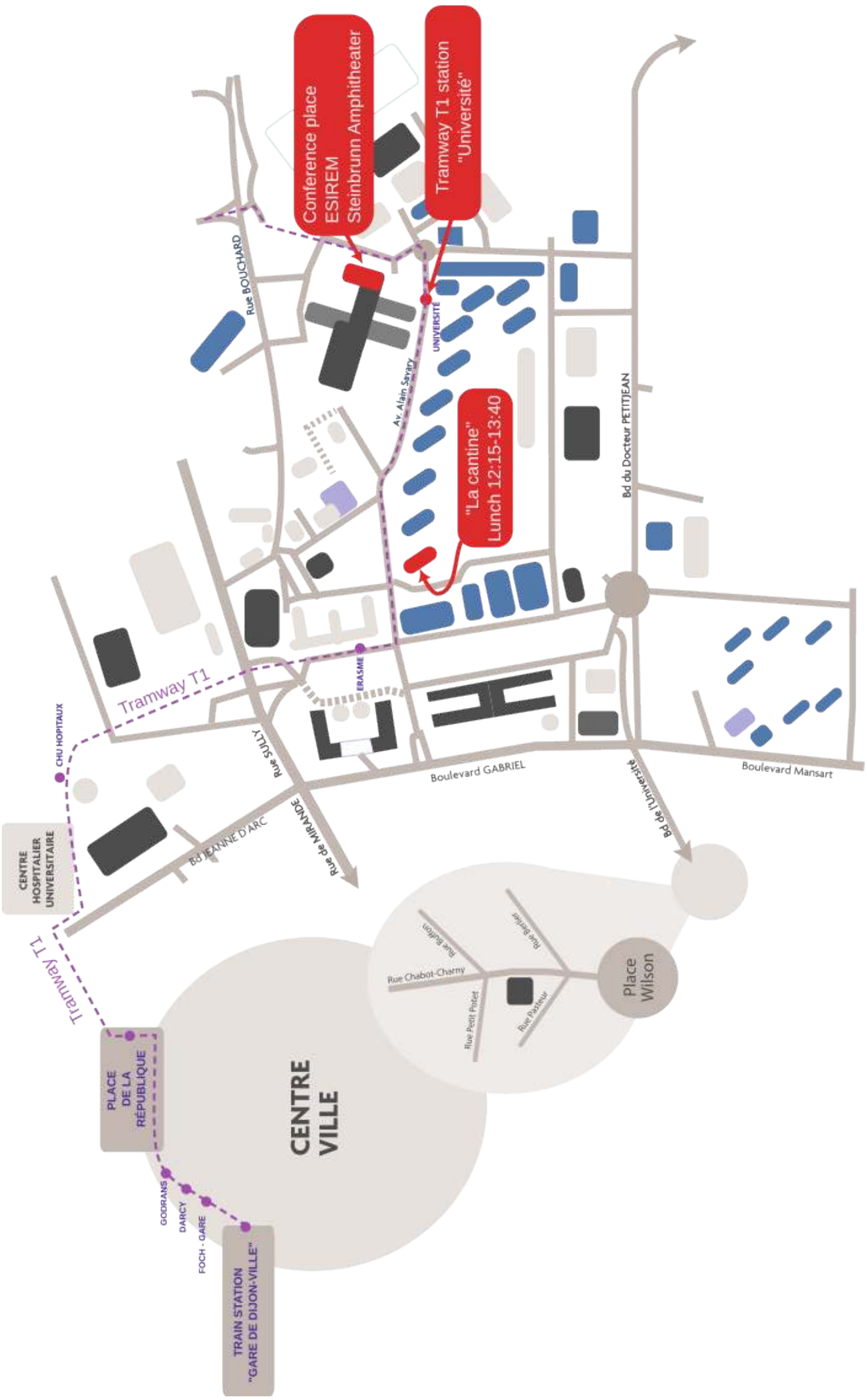
European emergency number is 112. It should be used for medical emergencies, in case of fire, threat to your security, or in all emergency situations.

Conference lunch

Participants of the conference will receive four lunch coupons, which are meant to be used from Monday to Thursday, in *La Cantine* cafeteria. They include three courses, water and coffee. Extra drink can be purchased; in this case you need to pay with your credit/debit card.

Lunch time is 12:15–13:40, except on Tuesday when it is 12:30–13:40 (just after the Conference photo session).

Campus map



Conference place
ESIREM
Steinbrunn Amphitheater
Tramway T1 station
"Université"
"La cantine"
Lunch 12:15-13:40
UNIVERSITÉ
ERASME
CHU HOPITAUX
Tramway T1
Rue BOUCHARD
Av. Alain Savary
Bd du Docteur PETITJEAN
Rue SULLY
Boulevard GABRIEL
Boulevard Mansart
Bd de l'Université
Bd JEANNE D'ARC
Rue de MIRANDE
CENTRE HOSPITALIER UNIVERSITAIRE
PLACE DE LA RÉPUBLIQUE
CENTRE VILLE
Place Wilson
Rue Chabot-Charny
GODRANS
DARCY
FOCH - GARE
TRAIN STATION
"GARE DE DIJON-VILLE"

Restaurants in Dijon

In the center of Dijon, you'll find numerous dining options to enjoy a meal. Here are some specific suggestions for locations to eat in the center of Dijon. We begin our listing by several squares (*places*) in Dijon center.

Place de la Libération and Place Darcy. These opposite ends of *Rue de la Liberté* are vibrant and charming hubs offering a variety of restaurants, bistros and cafes with outdoor seating. You can enjoy a meal and explore different dining options from traditional French cuisine to international flavors.

Place du Théâtre. Another magnificent place near the Place de la Libération with several good restaurants and brasseries.

Place Emile Zola. Located in the heart of Dijon, this square is surrounded by cafes and restaurants. It offers a pleasant atmosphere to sit down and have a meal while enjoying the surroundings.

Place des Halles is a lively square which is home to the *Les Halles* shopping center, housed in a beautiful one and half century old building. The square offers a vibrant atmosphere with outdoor seating, cafes, and restaurants around the building. It's an ideal place to wander and explore different dining options.

Rue Bossuet and rue Monge. Located near the Palace of the Dukes of Burgundy, they are home to several restaurants and eateries where you can find a range of cuisines.

The Cité of Gastronomy and Wine (*Cité de la Gastronomie et du Vin*): 10-minute away by walk from the city center, it offers a variety of meal opportunities for visitors. Inside the cultural center, you can find restaurants where you can savor gourmet meals inspired by the culinary traditions of the Burgundy region. Additionally, there are cafes and food stalls where you can indulge in regional specialties, snacks, and beverages.

These are some of the notable areas in the center or nearby of Dijon where you can find a variety of dining options. Remember to check the opening hours and make reservations if necessary, especially during peak times or for popular restaurants. Exploring these areas will give you a chance to discover the culinary delights that Dijon has to offer.

Things to do

Visiting Dijon, there are several notable attractions and landmarks that you should consider exploring. Here are some of the top recommendations.

Owl's Trail (*Parcours de la Chouette*). This is maybe the first approach to discover Dijon, it takes around une hour. Follow the bronze owl-shaped trail markers scattered throughout the city pavement (the owl is the fetish of Dijon). It leads you to the major attractions and historic sites of Dijon. This self-guided walking tour allows you to explore the city at your own pace while discovering its hidden gems. Alternatively, you can book a guided visit at the tourist office, located at *11 rue des Forges*.

Palace of the Dukes of Burgundy (*Palais des Ducs et des États de Bourgogne*). This impressive palace is a symbol of Dijon's rich history and was once the residence of the powerful dukes of Burgundy. Inside the palace, visitors can explore the Museum of Fine Arts (*Musée des Beaux-Arts de Dijon*), which houses a remarkable collection of art spanning various periods. Nearby, you may have a pleasant time around a glass or an ice cream, and you'll find a variety of options for dining to suit different tastes and budgets.

Rue de la Liberté, where a side of the Palace is located. If you're looking for shopping and intense street life, head to *Rue de la Liberté*. This bustling pedestrian street is lined with shops, boutiques, cafes, and restaurants, making it the perfect place for shopping, dining, and people-watching.

Rue des Forges. Take a stroll along this charming pedestrian street lined with picturesque half-timbered houses. Explore the unique boutiques, art galleries, and craft shops, and enjoy the vibrant atmosphere of one of Dijon's most picturesque streets.

Saint-Bénigne Cathedral known for its distinctive Gothic architecture, the Cathedral is a must-visit. Admire the intricate sculptures, stained glass windows, and the famous crypt of Saint-Bénigne, which dates back to the 6th century.

Jardin Darcy. Enjoy a moment of tranquility in this beautiful park, located near the city center. Admire the elegant fountain, lush greenery, and lovely flower beds. The park also offers a panoramic viewpoint over the city.

The Cité of Gastronomy and Wine (*Cité de la Gastronomie et du Vin*) offers a range of experiences related to gastronomy and wine. The Cité aims to provide visitors with a comprehensive experience that celebrates the gastronomy and wine culture of Burgundy. Whether you're a food lover, a wine enthusiast, or simply curious about the region's culinary traditions, this cultural center offers an engaging and educational visit.

These are just a few highlights of what Dijon has to offer. The city also boasts numerous churches, charming neighborhoods, and a vibrant food scene with traditional Burgundian cuisine. Don't forget to try local specialties like *boeuf bourguignon*, *escargots*, and the famous gingerbread *pain d'épices*. Enjoy your visit to Dijon!

More about Dijon the main city of Burgundy

Dijon is the capital city of the Burgundy historical region in France. The area has a long history dating back to ancient times when it was inhabited by the Gauls. In Roman times, Dijon was known as Divio and served as an important trade center.

During the Middle Ages, the region of Burgundy gained prominence under the rule of the powerful Dukes of Burgundy. Dijon became the capital of their duchy and experienced a period of economic growth and cultural flourishing. The dukes commissioned grand architectural projects, such as the Palace of the Dukes of Burgundy, which still stands as a testament to their influence.

In the 15th century, the duchy reached its peak under Duke Philip the Good and his successor Charles the Bold. Burgundy was a major political and economic force in Europe, known for its wealth and strategic alliances. However, the death of Charles the Bold and subsequent conflicts led to the decline of Burgundy's power.

Over the centuries, Burgundy became integrated into the Kingdom of France. Dijon continued to develop as a regional center, particularly in the fields of wine production and trade. The region's wines, such as those from the Côte d'Or vineyards, gained international renown and contributed to Burgundy's reputation as a gastronomic destination.

Today, Dijon and Burgundy continue to preserve their rich heritage. Dijon is a vibrant city that blends its historical charm with a modern urban landscape. The region of Burgundy remains renowned for its exquisite wines, picturesque countryside, and cultural traditions, attracting visitors from around the world.

Schedule

	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
	3 July	4 July	5 July	6 July	7 July
9:00-9:30	Regist. & 9:20 Welcome	Ramírez		Corsini	Weiner
9:30-10:00	Bevan	Williams	Adenbaum	Kirgizov	Slivken
10:00-10:30	Asinowski	Smith	Bean	Elvey Price	Troyka
10:30-11:00	Coffee break	Coffee break	Coffee break	Coffee break	Coffee break
11:00-11:30	Testart	Pantone	Problem session	Mütze	Jones
11:30-12:00	Yıldırım				Opler
		Conference Photo			Next PP and closing
	Lunch break	Lunch break	Lunch break	Lunch break	
14:00-14:30	Henriet	Dukes	Cultural visit	Deb	
14:30-15:00	Vatter	Cerbai		Sagan	
15:00-15:30	Coffee break	Coffee break		Coffee break	
15:30-16:00	Jarvis	Naima		Hämäläinen	
16:00-16:30	Brignall	Úlfarsson		Bouvel	
16:30-17:00	First poster session	Blanco		Second poster session	
17:00-17:30					
19:00-		Banquet			

Updated version: https://2023.permutationpatterns.com/program.html

Abstracts

Efficient algorithms for generating pattern-avoiding combinatorial objects

Torsten Mütze University of Warwick & Charles University Prague

This talk is based on joint work with Petr Gregor, Elizabeth Hartung, Hung P. Hoang, Arturo Merino, Namrata, Aaron Williams

In mathematics and computer science, we frequently encounter different classes of combinatorial objects characterized by pattern avoidance. In this talk I focus on algorithms for efficiently generating these objects, i.e., we seek an algorithm that visits each of the objects from the class exactly once. I present our recent framework for solving this problem for a large variety of different classes of objects by encoding the objects via permutations. I will focus in particular on generating different classes of pattern-avoiding permutations (described by classical patterns, vincular patterns, mesh patterns, monotone and geometric grid classes etc.), pattern-avoiding rectangulations, and pattern-avoiding binary trees.

Computational and experimental methods in permutation patterns

Jay Pantone Marquette University

For most of its existence, a hallmark of permutation patterns research has been the use of computers. Our research is regularly made possible by the ability to write a simple script to generate permutations with some certain property, helping us to discover an interesting theorem; or to open up Sage, or Maple, or Mathematica and perform some large generating function calculation; or to use one of the several existing large permutation patterns software libraries to test some intriguing conjectures.

The quest to understand permutation classes has led to the import of computational methods from others areas into the field of permutation patterns, as well as the development of a number of new techniques. Some of these methods produce rigorous results, assuming the correctness of the software implementation. Others are experimental in the sense that their output should be considered conjectural. The popularity of permutation patterns has even led to some of these computational techniques making the jump to other areas of combinatorics. This talk will survey a collection of these methods, including some developed by myself and my collaborators.

Amrita Acharyya University of Toledo

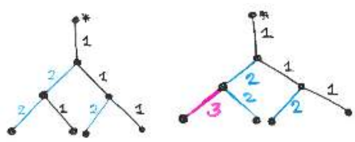
It is known from section 6 in [1], when we call a poset P, a $\mathcal{P}-chain-permutational$ given a subset of permutations $\mathcal{P}$ of S_n. In this work, I have been using the same idea to study subset of words with a visual presentation (by the hass-diagram of the poset) that are not necessarily permutations for example especially when they are certain classes of restricted growth functions. Let us define Chain-Word Poset

Definition 1. A poset P will be called $\mathcal{P}-chain-word$, where $\mathcal{P}$ is a subset of the set of all words of length n, if it is possible to label the covering relations of P (i.e. the edges of the Hasse diagram of P) with numbers from $\{1,2,...,n\}$ in such a way that along different maximal chains of P (whose length is necessarily n) the labels form different words from $\mathcal{P}$, and every word $w \in \mathcal{P}$ arises in this manner. For example, it is well known that the Boolean lattice B_n is a chain word poset where the set of words is S_n.

We denote the set of all restricted growth functions of length n as R_n and the set of all restricted growth functions of length n avoiding certain pattern v as $R_n(v)$.

Lemma 2. *The hass diagrams of the chain word posets of $R_n(121), R_n(122), R_n(123)$ are always rooted binary tree for any $n \geq 3$, and the respective vertex degrees except the leaves and the corresponding roots are always two. For $n \geq 3$, $R_n(112)$ is an n ary tree and only one vertex has n-many children.*

Following are the Hass diagrams of the posets $R_3(123)$ (the left one) and $R_3(121)$. The vertices are not labeled. The root is labeled is by *. Since restricted growth functions



always have the first digit "1" these posets from $R_n(v)$ are always rooted tree with the top edge leveled by 1. Since these posets are always rooted tree, in order to get a lattice structure from there with each leaf at the bottom level an additional edge ϵ can be attached. And we call the resulting lattice as $R_n(L)(v)$.

Theorem 3. *For $n \geq 3$ the Hass diagrams of $R_n(L)(121), R_n(L)(122), R_n(L)(123)$ give distributive lattice, whereas that of $R_n(L)(111)$ and $R_n(L)(112)$ does not in general.*

We discuss the Rank Generating Functions of those posets along with cardinality results of the corresponding graphs' vertex and edge sets.

We also consider these restricted growth functions of length n specifying the number of blocks k of the corresponding set partitions inducing the restricted growth functions. We denote the resulting lattice (ommiting ϵs' they can be considered as posets or trees again without the lattice structure) as $R_{nk}(L)(v)$.

Theorem 4. *For $n \geq 3$, $2 \leq k \leq n$, the Hass diagrams of $R_{nk}(L)(121), R_{nk}(L)(122)$, give distributive lattice, whereas that of and $R_{nk}(L)(112)$ are so iff $k \leq 2, n$ and $R_{nk}(L)(123)$(For the pattern 123 this lattice is defined only when $k = 1$ or $k = 2$) is distribitive if and only if $k = 1, 2$. $R_{nk}(L)(111)$ is defined iff $\lceil \frac{n}{2} \rceil \leq k \leq n$. And this lattice is distributive iff $k = \lceil \frac{n}{2} \rceil, n-1, n$.*

We construct the rank generating function, vertex and edge cardinality of some $R_{nk}(v)$ for some specifically given k. Attaching the ϵs', the same can be found for the corresponding $R_{nk}(L)(v)$.

Theorem 5. *i The rank generating function of $R_{n2}(121)$ is* $1 + x + 2x^2 + 3x^3 + \cdots + (n-2)x^{(n-2)} + (n-1)x^{(n-1)} + (n-1)x^n$.

ii The vertex cardinality of $R_{n2}(121)$ is sum of the coefficients in the above rank generating functions which is $\frac{n(n+1)}{2}$. And the edge cardinality is 1 less than that.

iii The rank generating function of $R_{n3}(122)$ is $1 + x + 2x^2 + 4x^3 + \sum_{r=4}^{n} \alpha_r x^r$, where $\forall r, 4 \leq r \leq n-2, \alpha_r = \frac{r^2-r+2}{2}$ and $\alpha_n = \alpha_{n-1} = \frac{n^2-3n+2}{2}$.

iv For $k \geq 2$ in the rank generating function of $R_{nk}(121), R_{nk}(122)$, the coefficient of x^n is same with that of x^{n-1}.

v The lattices $R_{nk}(L)(121), R_{nk}(L)(122), R_{nk}(L)(112)$ are embedded in a natural way inside $R_{(n+1)k}(L)(121), R_{(n+1)k}(L)(122), R_{(n+1)k}(L)(112)$ respectively. The same is true if we consider their corresponding tree structures ommiting the ϵ.

vi In the rank generating function of $R_{n3}(121), R_{n3}(122)$, the coefficients stabilizes as n grows.

vii The Hass diagram of $R_{nk}(L)(121)$ (and of $R_{nk}(121)$) are symmetric on both sides of the vertical line drawn through the top edge leveled by "1" iff $n = 2k-1$.

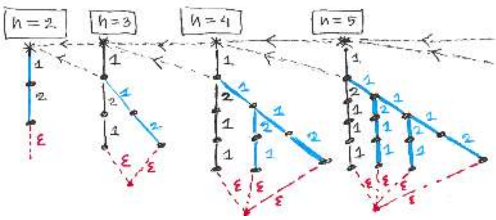
viii The rank generating function of $R_{n(n-1)}(122)$ is the same as that of $R_{n2}(121)$ as in i.

ix The rank generating function of $R_{n(n-2)}(112)$ is $\sum_{i=0}^{n-2} x^i + (n-2)x^{n-1} + \frac{n^2-3n+2}{2}x^n$.

Next we observe that for any pattern v, we get a collection of these Lattices ordered by set inclusion as $R_n(L)(v) \subseteq R_{n+1}(L)(v) \subseteq R_{n+2}(L)(v) \cdots$ (as long as they are defined) and the same follows if we specify the number of blocks k and also if we consider these Lattices induced by multipattern avoidence classes. Additionally, if the ϵs' are ommitted the same follows in the corresponding tree structures as well. And due to that fact from any pattern avoidence class (and for multi pattern avoidence as well) we can construct a countable family of projective system of Lattices (and the corresponding trees) giving a projective limit. Each map of the countable projective system being surjective this construction resembles in many ways to Projective Frasse limit of graphs and trees as in [3]. Although here to get a projective system of maps of graphs we considered non rigid map of graphs as in [4], where an edge can be mapped onto a vertex.

Theorem 6. *Let the pattern $v \in \{112, 121, 122, 123\}$. Then for any k (k=1 or 2 for the pattern 123) we can define a projective system of maps of graphs (possibly non rigid) $\{\phi_{ij} : R_{jk}(L)(v) \to R_{ik}(L)(v)\}_{i,j \in \mathcal{N}, j \geq i}$ in an algorithmic way.($\mathcal{N}$ is the set of all possitive integers). The corresponding projective limit is a connected (as each quotient is finite discrete path connected graph) second countable profinite graph Γ having a subgraph T, where T is a projective limit of the underlying trees inside each $R_{ik}(L)(v)$ ommitting all ϵs but one. And vertex set of Γ is same as the vertex set of T.*

Following is a picture of part of the projective system for the pattern 122 and $k = 2$.



Examples of other chainword posets where the word may not be restricted growth function: For any possitive integer n, D_n, the set of all possitive integer divisors of n form a poset with respect to $'' \leq ''$ where $a \leq b$, iff a divides b. Using the unique prime factorization of a possitive divisor of n we get a maximal chain whose edge level form a word representing that divisor.

- **Question i.** Is this possible to find rank generating functions, of the chainword posets of any $R_{nk}(L)(v)$ for any pattern v, in terms of n and k?
- **Question ii.** Is this possible to find algorithm to define the maps of graphs of those projective systems for any pattern v in terms of n only, when the number of blocks is not specified, and in terms of n and k as well?

References

[1] Rodica Simion, Frank W. Schmidts, *RESTRICTED PERMUTATIONS*, European J. Combin. 6 (1985), no. 4, 383-406.

[2] Lindsey R. Campbell, Samantha Dahlberg, Robert Dorward, Jonathan Gerhard,Thomas Grubb, Carlin Purcell, Bruce E Sagan (2021) published by Elsevier, *RESTRICTED GROWTH FUNCTION PATTERNS AND STATISTICS*

[3] Wlodzimierz J. Charatonik AND Robert P. Roe *PROJECTIVE FRA ̈ISSE LIMITS OF TREES*

[4] Amrita Acharyya, Jon M. Corson, AND Bikash C. Das (2019) published by Communications in Algebr, Volume 47-Issue 10 *VARIETIES OF PROFINITE GRAPHS*

[5] L. Ribes Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 66, Springer International Publishing, 2017 *PROFINITE GRAPHS and GROUPS*.

Rowmotion on 321-avoiding permutations

Ben Adenbaum Dartmouth College

This talk is based on joint work with Sergi Elizalde

We give a natural definition of rowmotion for 321-avoiding permutations, by translating, through bijections involving Dyck paths and the Lalanne–Kreweras involution, the analogous notion for antichains of the positive root poset of type A. We prove that some permutation statistics, such as the number of fixed points, are homomesic under rowmotion, meaning that they have a constant average over its orbits. Finally, we show that the Armstrong–Stump–Thomas equivariant bijection between antichains in types A and B and non-crossing matchings can be described more naturally in terms of the Robinson–Schensted–Knuth correspondence on permutations. Let $\mathcal{S}_n(321)$ denote the set of 321-avoding permutations in $\mathcal{S}_n$. We can represent $\pi \in \mathcal{S}_n(321)$ as an $n \times n$ array with crosses in squares $(i, \pi(i))$ for $1 \leq i \leq n$; we call this the *array* of π. Rows and columns are indexed using cartesian coordinates, so that (i, j) denotes the cell in the ith column from the left and jth row from the bottom. We say that $(i, \pi(i))$ is a *fixed point* (respectively *excedance*, *weak excedance*, *deficiency*, *weak deficiency*) if $\pi(i) = i$ (respectively $\pi(i) > i$, $\pi(i) \geq i$, $\pi(i) < i$, $\pi(i) \leq i$).

Let P be a finite poset, and let $\mathcal{A}(\mathsf{P})$ denote the set of antichains of P. *Antichain rowmotion* is the map $\rho_{\mathcal{A}} : \mathcal{A}(\mathsf{P}) \to \mathcal{A}(\mathsf{P})$ defined as follows: for $A \in \mathcal{A}(\mathsf{P})$, let $\rho_{\mathcal{A}}(A)$ be the minimal elements of the complement of the order ideal generated by A.

Restricting our attention to the poset of positive roots for the type A root system, using that antichains of this poset are in bijection with 321-avoiding permutations, we define a natural rowmotion operation on $\mathcal{S}_n(321)$.

To define rowmotion on 321-avoiding permutations, first consider the following bijection to antichains of $\mathbf{A}^{n-1}$.

Definition 1. Let Exc : $\mathcal{S}_n(321) \to \mathcal{A}(\mathbf{A}^{n-1})$ be the bijection where, for $\pi \in \mathcal{S}_n(321)$, we define

$$\mathrm{Exc}(\pi) = \{[i, \pi(i) - 1] : (i, \pi(i)) \text{ is an excedance of } \pi\}.$$

We then define Rowmotion on 321-avoiding permutations as follows.

Definition 2. Rowmotion on 321-avoiding permutations is the map $\rho_{\mathcal{S}} : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$ defined by $\rho_{\mathcal{S}} = \mathrm{Exc}^{-1} \circ \rho_{\mathcal{A}} \circ \mathrm{Exc}$.

When studying rowmotion, it is common to look for statistics that exhibit a property called *homomesy* [4]. Given a set S and a bijection $\tau : S \to S$ so that each orbit of the action of τ on S has finite order, we say that a statistic on S is *homomesic* under this action if its average on each orbit is constant. More specifically, the statistic is said to be *c-mesic* if its average over each orbit is c. We prove that the following statistics on 321-avoiding permutations are homomesic under rowmotion.

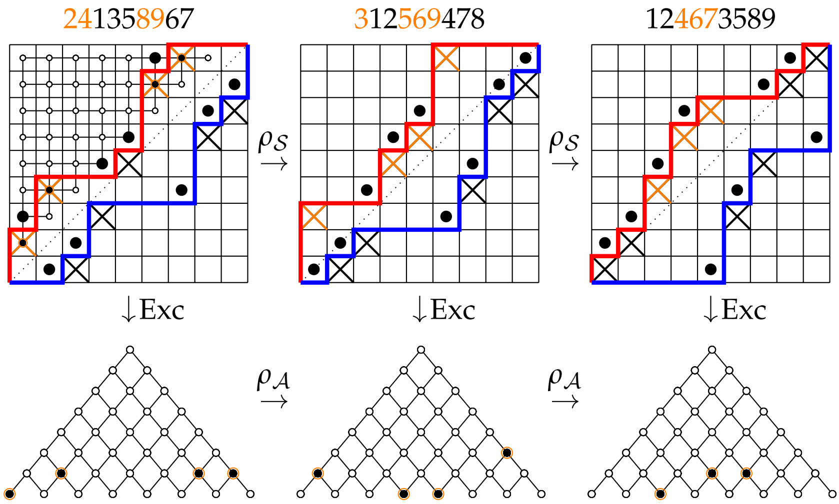


Figure 1: Two applications of rowmotion starting at $\pi = 241358967 \in \mathcal{S}_9(321)$ computed using Definition 2.

The first statistic that we consider is the number of fixed points of a permutation π, denoted by $\mathrm{fp}(\pi) = |\{i : \pi(i) = i\}|$. Let $\mathrm{exc}(\pi) = |\{i : \pi(i) > i\}|$ and $\mathrm{wexc}(\pi) = |\{i : \pi(i) \geq i\}|$ denote the number of excedances and weak excedances of π, respectively.

Theorem 3. *The statistic* fp *is 1-mesic under the action of* $\rho_\mathcal{S}$ *on* $\mathcal{S}_n(321)$.

Next we consider two families of statistics on 321-avoiding permutations, and show that they are also homomesic under rowmotion. The first family are the statistics h_i introduced by Hopkins and Joseph [3]. For $1 \leq i \leq n-1$, they define h_i on antichains $A \in \mathcal{A}(\mathbf{A}^{n-1})$ as

$$h_i(A) = \sum_{j=1}^{i} 1_{[j,i]}(A) + \sum_{j=i}^{n-1} 1_{[i,j]}(A),$$

where $1_{[i,j]}(A)$ is the indicator function that equals 1 if $[i,j] \in A$ and 0 otherwise. For $\pi \in \mathcal{S}_n(321)$, we now define $h_i(\pi)$ by $h_i(\mathrm{Exc}(\pi))$. In terms of the array of $\pi \in \mathcal{S}_n(321)$, $h_i(\pi)$ counts the number of crosses of the form $(j, i+1)$ with $1 \leq j \leq i$, or (i,j) with $i+1 \leq j \leq n$.

Hopkins and Joseph prove in [3, Thm. 4.3] that the statistics h_i on antichains are 1-mesic under $\rho_\mathcal{A}$. This result can be translated in terms of 321-avoiding permutations as follows.

Theorem 4 ([3]). *For* $1 \leq i \leq n-1$, *the statistic* h_i *is* 1*-mesic under the action of* $\rho_\mathcal{S}$ *on* $\mathcal{S}_n(321)$.

Next we define a new family of permutation statistics, that we denote by ℓ_i for $1 \leq i \leq n$. For $\pi \in \mathcal{S}_n(321)$, let $\ell_i(\pi)$ be the number of crosses in the array of π of the form (j,i) with $1 \leq j \leq i$, plus the number of crosses of the form (i,j) with $i < j \leq n$. It turns out that the statistics ℓ_i are homomesic as well.

Theorem 5. *For $1 \leq i \leq n$, the statistic ℓ_i is 1-mesic under the action of $\rho_{\mathcal{S}}$ on $\mathcal{S}_n(321)$.*

Additionally we describe how rowmotion interacts with the sign of a 321-avoiding permutation.

Theorem 6. *For all $\pi \in \mathcal{S}_n(321)$,*

$$\operatorname{sgn}(\rho_{\mathcal{S}}(\pi)) = \operatorname{sgn}(\mathrm{LK}_{\mathcal{S}}(\pi)) = \begin{cases} \operatorname{sgn}(\pi) & \text{if } n \text{ is odd,} \\ -\operatorname{sgn}(\pi) & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, we use the viewpoint of 321-avoiding permutations to shed new light into a celebrated bijection of Armstrong, Stump and Thomas [1] between antichains in root posets of finite Weyl groups (also known as nonnesting partitions) and non-crossing partitions. We will show that, in the case of types A and B, the Armstrong–Stump–Thomas (AST) bijection has a simple interpretation in terms of the Robinson–Schensted–Knuth (RSK) correspondence applied to 321-avoiding permutations. In particular, we use this interpretation together with properties of RSK to give a new proof of results of [1] in type B without relying upon rowmotion. Additionally, these methods are applied to answer a conjecture of Hopkins and Joseph concerning the number of fixed points of a natural involution in type B posed in [2].

References

[1] D. Armstrong, C. Stump, and H. Thomas. A uniform bijection between nonnesting and non-crossing partitions. Transactions of the American Mathematical Society, 365(8):4121–4151, 2013.

[2] S. Hopkins and M. Joseph, The birational Lalanne–Kreweras involution, preprint available as 2012.15795v1 on arXiv, 2020.

[3] S. Hopkins and M. Joseph, The birational Lalanne–Kreweras involution, Algebraic Combinatorics, Volume 5(2), 2022.

[4] J. Propp and T. Roby, Homomesy in products of two chains, Electron. J. Combin. 22 (2015), no. 3, Paper 3.4, 29 pp.

Guillotine rectangulations and mesh patterns

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This talk is based on joint work with Jean Cardinal (Université libre de Bruxelles), Stefan Felsner (Technische Universität Berlin), and Éric Fusy (Université Gustave Eiffel).

Abstract. We present mesh patterns p_1, p_2 such that (p_1, p_2)-avoiding permutations correspond to guillotine rectangulations in standard mappings between permutations and rectangulations. In particular, this yields a size-preserving bijection between guillotine generic rectangulations and permutations that avoid certain mesh patterns.

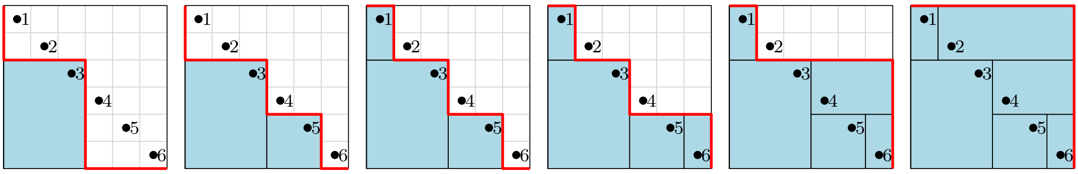
Introduction. A *rectangulation* is a partition of a rectangle into finitely many rectangles. It is assumed that segments meet in $\top$-shape joints, but never cross.

There are two ways to formalize "combinatorially identical" rectangulations. The *weak equivalence* preserves rectangle–segment contacts. The *strong equivalence* additionally preserves contacts between rectangles. Equivalence classes of the weak equivalence are called *mosaic rectangulations*; those of the strong equivalence, *generic rectangulations*. The *size* of a rectangulation is the number of rectangles.

A rectangulation is *guillotine* if it either has size 1, or can be recursively obtained by "gluing" two smaller guillotine rectangulations along a horizontal or a vertical cut. It is well known that guillotine rectangulations are characterized by avoiding *windmills* — quadruples of segments that form a ⊤⊥ or a ⊣⊢ shape.

The following figure demonstrates these concepts. Rectangulations A, B, and C are all weakly equivalent, but only A and B are strongly equivalent. Only D is guillotine.

Mappings between permutations and rectangulations. Here we sketch (mainly after [3, 6]) mappings γ_w and γ_s from S_n to mosaic and, respectively, to generic rectangulations of size n. One writes the numbers $1, 2, \ldots, n$ along the diagonal of a square. Then, given $\pi \in S_n$, one inserts rectangles labeled by these numbers, in the order prescribed by π, so that the North-East boundary of the union of already inserted rectangles is always a *staircase* — a horizontally and vertically monotone polygonal line (shown in red below). The obtained mosaic rectangulation is $\gamma_w(\pi)$. To obtain the generic rectangulation $\gamma_s(\pi)$, *wall shuffles* (shifts of segments) might further be needed.



$\pi = 3\ 5\ 1\ 6\ 4\ 2$ $\qquad$ $R = \gamma_w(\pi)$

The preimage of any rectangulation R consists of linear extensions of a poset on $[n]$ — the *weak poset* $P_w(R)$ for γ_w, and the *strong poset* $P_s(R)$ for γ_s. These posets can be

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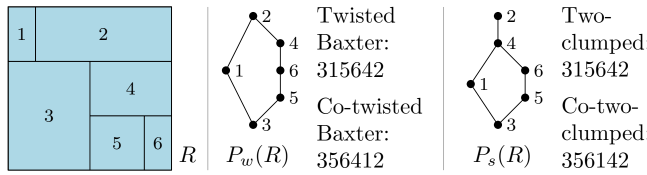
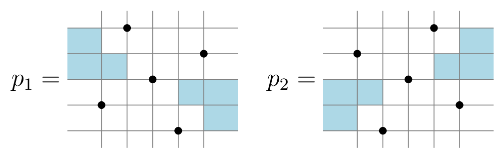
described by certain geometric relations between the rectangles. In $P_w(R)$, the minimum (w.r.t. the weak Bruhat order) linear extension is a unique *twisted Baxter* permutation[2]; the maximum is a unique *co-twisted Baxter* permutation; and it also contains a unique *Baxter* permutation. In $P_s(R)$, the minimum linear extension is a unique *two-clumped permutation*; and the maximum is a unique *co-two-clumped* permutation. Hence, γ_w restricts to size-preserving bijections between mosaic rectangulations and three families of permutations: Baxter, twisted Baxter, and co-twisted Baxter; and γ_s to size-preserving bijections between generic rectangulations and two families of permutations: two-clumped, and co-two-clumped. See [2, 3, 4, 6] for more details.



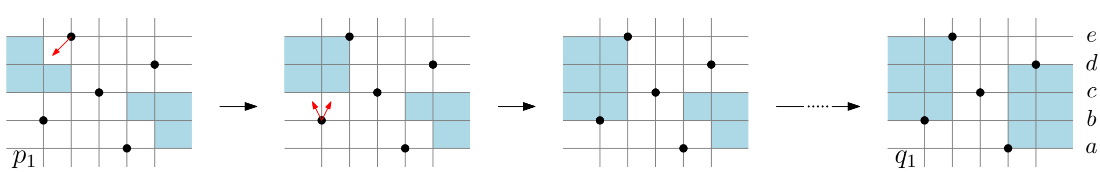
Guillotine rectangulations. The bijection between mosaic rectangulations and Baxter permutations specializes to a bijection between *guillotine mosaic rectangulations* and separable permutations [1]. In contrast, we are not aware of any results concerning *guillotine generic rectangulations*. Our main result is a joint treatment of mosaic and generic rectangulations, which allows to uniformly restrict all the bijections mentioned above to the guillotine case. In Theorem 1, we provide mesh patterns[3] that characterize permutations corresponding to guillotine rectangulations, under both γ_w and γ_s.

Theorem 1. *Let $\pi \in S_n$. The rectangulations $\gamma_w(\pi)$ and $\gamma_s(\pi)$ are guillotine if and only if π avoids the following mesh patterns (refer to the figure at the right):*



$p_1 = (25314, \{(0,3),(0,4),(1,3),(4,2),(5,1),(5,2)\})$ and
$p_2 = (41352, \{(0,1),(0,2),(1,2),(4,3),(5,3),(5,4)\})$.

Proof (sketch). Part 1: Equivalent patterns. We introduce two new mesh patterns: $q_1 = (25314,\ \{0,1\} \times \{2,3,4\}\ \cup\ \{4,5\} \times \{1,2,3\})$ and $q_2 = (41352,\ \{0,1\} \times \{1,2,3\} \cup \cup\ \{4,5\} \times \{2,3,4\})$. Using standard techniques, we show that a permutation contains p_1 (resp. p_2) if and only if it contains q_1 (resp. q_2). The following figure illustrates the proof that an occurrence of p_1 implies an occurrence of q_1 (the converse is trivial):

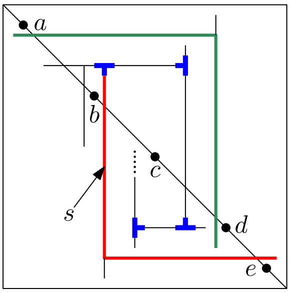


Part 2: q_1 implies $\dashv\!\perp$. Let π be a permutation that contains q_1, and consider $\gamma_w(\pi)$. Let a, b, c, d, e be the letters of π that correspond to $1, 2, 3, 4, 5$ in the pattern. The shaded areas of q_1 imply that no rectangle labelled x with $b < x < e$ is inserted earlier than e, and that no rectangle labelled x with $a < x < d$ is inserted later than a. It follows that just after inserting e / just before inserting a, the staircase contains edges as shown by

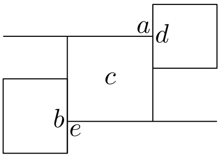
[2] Separable permutations = Av(2413, 3142); Baxter permutations = Av$(2\underline{41}3, 3\underline{14}2)$; twisted Baxter permutations = Av$(2\underline{41}3, 3\underline{41}2)$; co-twisted Baxter permutations = Av$(2\underline{14}3, 3\underline{14}2)$; two-clumped permutations = Av$(24\underline{51}3, 42\underline{51}3, 3\underline{51}24, 3\underline{51}42)$; co-two-clumped permutations = Av$(24\underline{15}3, 42\underline{15}3, 3\underline{15}24, 3\underline{15}42)$.

[3] These mesh patterns were proposed by Merino and Mütze [5], who in fact conjectured Corollary 2.

red / green in the figure. Then one can show that if we consider segment s that contains the right side of rectangle b, traverse it from below to above, turn to the right at its upper endpoint, and then similarly continue clockwise (right→down→left→up→...), then we never reach the boundary of the base rectangle. Since the process is finite, a windmill $\dashv\!\perp$ will be eventually obtained. The proof is valid for generic rectangulations as well, since being guillotine is invariant to wall shuffles. Similarly, q_2 implies $\vdash\!\top$.



Part 3: $\dashv\!\perp$ implies p_1. Let R be a generic rectangulation with $\dashv\!\perp$. Label by a, b, d, e the rectangles as shown at the right, and by c any rectangle inside the windmill. Then we have $a < b < c < d < e$. In *any* corresponding permutation π (any linear extension of $P_w(R)$ or of $P_s(R)$) these rectangles occur in the order b, e, c, a, d, which gives the pattern 25314. It remains to show that there are no points in the shaded areas from the plot of p_1. Suppose we have a point in the area $(0, 3) \cup (1, 3)$. Then there is a rectangle x such that $c < x < d$, which is inserted earlier than e. However, any rectangle that satisfies the first condition lays above e and hence cannot satisfy the second condition. Similar arguments and symmetry considerations apply for other shaded areas in p_1, as well as for the proof that $\vdash\!\top$ implies p_2. $\square$



Implications. Since $P_w(R)$ corresponding to any mosaic rectangulation R contains a unique Baxter / twisted Baxter / co-twisted Baxter permutation, adding p_1 and p_2 to respective patterns yields three families of permutations in bijection with guillotine mosaic rectangulations. In particular, separable permutations — Av(2413, 3142) — can be described as Av($2\underline{41}3$, $3\underline{14}2$, p_1, p_2), which can also be proven directly. Similar considerations applied to generic rectangulations and two-clumped permutations yield the following result, which is — to the best of our knowledge — the first characterization of guillotine generic rectangulations by means of permutation patterns:

Corollary 2 (conjectured by Merino and Mütze [5])**.** *There is a size-preserving bijection between guillotine generic rectangulations and two-clumped permutations that avoid p_1 and p_2.*

References

[1] E. Ackerman, G. Barequet, and R. Pinter. A bijection between permutations and floorplans, and its applications. Discret. Appl. Math., 154:12 (2006), 1674–1684.

[2] J. Cardinal, V. Sacristán, and R. Silveira. A note on flips in diagonal rectangulations. Discret. Math. Theor. Comput. Sci., 20:2 (2018), paper 14.

[3] S. Law and N. Reading. The Hopf algebra of diagonal rectangulations. J. Comb. Theory, Ser. A, 119:3 (2012), 788–824.

[4] A. Merino and T. Mütze. Combinatorial generation via permutation languages. III. Rectangulations. Discret. Comput. Geom., published online 4 November 2022.

[5] A. Merino and T. Mütze. Personal communication, 2021.

[6] N. Reading. Generic rectangulations. Eur. J. Comb., 33:4 (2012), 610–623.

Permutations avoiding bipartite partially ordered patterns have a regular insertion encoding

Christian Bean Keele University

This talk is based on joint work with Émile Nadeau, Jay Pantone, and Henning Ulfarsson

A *partially ordered pattern (POP)* of size k is a poset on k elements labeled with the symbols $\{1, \ldots, k\}$. We say that a permutation π of size n contains a POP P of size k if π contains a (not necessarily consecutive) subword $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ with $i_1 < i_2 < \cdots < i_k$ such that $\pi(i_\ell) < \pi(i_m)$ if $\ell < m$ in P [4]. Compare this to the definition of pattern avoidance; for two elements ℓ and m that are incomparable in P, there is no restriction on the relative order of the values $\pi(i_\ell)$ and $\pi(i_m)$ in an occurrence of the pattern P in π. The set of permutations avoiding a POP is a permutation class. We show that the generating function of a POP class whose poset is bipartite is rational and can be computed using techniques from the field of permutation patterns.

Theorem 1. *A POP class has a regular insertion encoding if and only if it is bipartite.*

The Tilescope software package is an implementation of Combinatorial Exploration for the field of permutation patterns [1, 2]. Although Tilescope is guaranteed to find a specification for every permutation class with a regular insertion encoding, there is no such guarantee in general. Tilescope was used to enumerate every permutation class whose basis contained only size 4 patterns except for Av(1324), which has no known enumeration. As a result, all other size 4 POP classes have been enumerated. All of these enumerative results are catalogued on the Permutation Pattern Avoidance Library (PermPAL), which can be found at https://permpal.com.

We applied TileScope to size 5 POP classes. In total there are 4231 size 5 POP classes, but only 1068 after considering symmetric equivalence. We computed the rational generating function for the 223 which have a regular insertion encoding. For the remaining 845, TileScope found a specification for 590 of them. We computed the generating function for 223 of these using automatic methods, and confirm several conjectures from Gao and Kitaev [3]. Our results have been added to PermPAL.

References

[1] M. H. Albert, C. Bean, A. Claesson, É. Nadeau, J. Pantone, and H. Ulfarsson (2022). *Combinatorial Exploration: An algorithmic framework for enumeration*. https://arxiv.org/abs/2202.07715

[2] C. Bean, J. S. Eliasson, T. K. Magnusson, É. Nadeau, J Pantone, and H. Ulfarsson (2021). *Tilings, Version 3.0.0*. https://github.com/PermutaTriangle/tilings

[3] A. Gao and S. Kitaev (2019). *On Partially Ordered Patterns of Length 4 and 5 in Permutations*. The Electronic Journal of Combinatorics, Volume 26, Issue 3

[4] S. Kitaev (2005). *Partially ordered generalized patterns*, Discrete Mathematics, Volume 298, Issues 1-3, Pages 212-229.

Mesh patterns in random permutations

David Bevan University of Strathclyde, Glasgow, Scotland

This talk is based on joint work with Jason Smith (Nottingham Trent University).

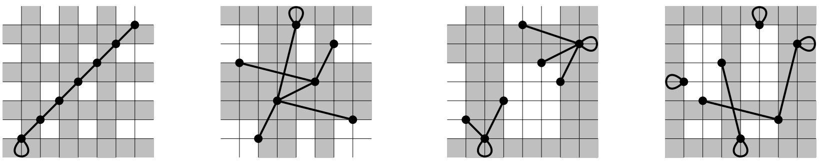
If p is a mesh pattern, we say that its *likelihood*, denoted $\kappa(p)$, is the asymptotic probability that a random permutation contains an occurrence of p. Thus,

$$\kappa(p) \;=\; 1 - \lim_{n\to\infty} |\mathsf{Av}_n(p)|/n!,$$

assuming the limit exists. (We have no reason to believe that there is a pattern for which this limit doesn't exist, but no proof yet that every mesh pattern has a likelihood.)

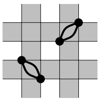
In this talk we investigate the likelihoods of various patterns. Our main results primarily concern bivincular patterns (although they extend to some other patterns), and include the following:

- By associating a graph with each pattern, the *Small Anchors Theorem* establishes a trichotomy on bivincular patterns, distinguishing between those for which $\kappa(p) = 0$, those for which $\kappa(p) = 1$, and those for which $0 < \kappa(p) < 1$.



FORESTS OF ANCHORED TREES

- We determine the likelihood of any bivincular pattern that is formed of *anchored trees*, including all vincular and covincular patterns and all *frames* and *ladders*. All these likelihoods are rational.



$\kappa(p) = 1 - e^{-1}I_0(2)$, where I_0 is a modified Bessel function of the first kind.

- Other bivincular patterns have irrational likelihoods. The likelihood of a *small ascent* or a *small descent* (the intervals of size two) is $1 - e^{-1}$, a result which has been known since the 1940s. By generalising a proof of this using the Chen–Stein method, we establish the likelihoods of a wide variety of bivincular patterns, often yielding values expressed using special functions.
- We also give a general result concerning the *nesting* of patterns.

Restricted permutations and bounded Dyck paths

Antonio Bernini Università degli Studi di Firenze

This talk is based on joint work with Elena Barcucci, Stefano Bilotta, Renzo Pinzani

In this talk, we will present a combinatorial interpretation of 312-avoiding permutations having particular constraints on *left-to-right maxima* in terms of bounded Dyck paths.

Preliminaries

We denote the set of Dyck paths having length $2n$ (or equivalently semilength n) by $\mathcal{D}_n$. A Dyck path can be codified by a string over the alphabet $\{U, D\}$, where U and D replace the up and down steps, respectively. The empty Dyck path is denoted by ε.

The height of a Dyck path P is the maximum ordinate reached by one ore more of its steps. A *valley* of P is an occurrence of the substring DU while a *peak* is an occurrence of the substring UD. The height of a valley (peak) is the ordinate reached by D (U).

The set $\mathcal{D}_n$ of Dyck paths having semilength $n \geq 0$ is enumerated by the n-Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

We are going to briefly recall a well-known bijection φ, useful in the rest of the paper, between the classes $\mathcal{D}_n$ and $\mathcal{S}_n(312)$ of 312-avoiding permutations (see for example [1, 2]). Fix a Dyck path P and label its up steps by enumerating them from left to right (so that the ℓ-th up step is labelled ℓ). Next assign to each down step the same label of the up step it corresponds to. Now consider the permutation whose entries are constituted by the labels of the down steps read from left to right. Such a permutation $\pi = \varphi(P)$ is easily seen to be 312-avoiding. As far as the inverse map φ^{-1} is concerned, once fixed a 312-avoiding permutation $\pi = \pi_1\pi_2 \dots \pi_n$ we can consider its factorization in terms of descending sub-sequences whose first elements coincide with the *left-to-right maxima* of π. Denoting $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_\ell}$ such left-to-right maxima, the corresponding Dyck path $P = \varphi^{-1}(\pi)$ is obtained as follows:

- write as many U's as $\pi_{i_1} (= \pi_1)$ followed by as many D's as the cardinality of the first descending sub-sequence headed by π_{i_1};
- for each $j = 2, \dots, \ell$, add as many U's as $\pi_{i_j} - \pi_{i_{j-1}}$ followed by as many D's as the cardinality of the sub-sequence headed by π_{i_j}.

A restricted class of $\mathcal{S}_n(312)$ and its enumeration

The set of Dyck paths having semilength n with height at most h is denoted by $\mathcal{D}_n^{(h)}$. It is easy to see that the restriction of φ to $\mathcal{D}_n^{(h)}$ gives the permutation of $\mathcal{S}_n(312)$ such that $\pi_{i_j} - i_j \leq h-1$, for each left-to-right maxima π_{i_j}, denoted by $\mathcal{S}_n^{(h)}(312)$. Indeed, $\pi_{i_j} - i_j$ is the height reached by the down step corresponding to π_{i_j} according to φ.

We denote by $\mathcal{D}_n^{(h,k)}$ the set of Dyck paths having semilength n and height at most h, and avoiding $k-1$ consecutive valleys at height $h-1$. In particular, when $k=2$, the set $\mathcal{D}_n^{(h,2)}$ represents the set of Dyck paths of $\mathcal{D}_n^{(h)}$ without valleys at height $h-1$.

As far as the permutations corresponding to the paths of $\mathcal{D}_n^{(h,2)}$ are concerned, denoted by $\mathcal{S}_n^{(h,2)}(312)$, we have the following:

Proposition 1. *$P \in \mathcal{D}_n^{(h,2)}$ if and only if in the corresponding permutation $\pi = \varphi(P)$ there is no left-to-right maximum π_{i_j} such that*

1. $\pi_{i_j} - i_j = h-1$ *and*
2. $\pi_{i_{j+1}} = \pi_{i_j} + 1$.

The cardinalities of $\mathcal{S}_n^{(h,2)}(312)$ and $\mathcal{D}_n^{(h,2)}$ are indicated by $S_n^{(h,2)}(312)$ and $D_n^{(h,2)}$, respectively.

The exhaustive generation of the objects of the above classes can be performed by means of an ECO operator [4] and described by the following succession rule:

$$\Omega_h : \begin{cases} (1) & \\ (1) & \rightsquigarrow \quad (2) \\ (k) & \rightsquigarrow \quad (2)(3)\cdots(k)(k+1), \; 2<k<h \\ (h) & \rightsquigarrow \quad (2)(3)\cdots(h-1)^2(h)\;. \end{cases}$$

According to the theory developed in [3], the production matrix P_h associated to Ω_h is $P_h = \begin{pmatrix} 0 & u^t \\ 0 & P_{h-1} + eu^t \end{pmatrix}$ where u^t is the row vector $(1\;0\;0\ldots)$ and e is the column vector $(1\;1\;1\;\ldots)^t$, of appropriate size.

Denoting by $f_h(x) = \sum_{n\leq 0} D_n^{(h,2)} x^n$ the generating function for $\mathcal{D}_n^{(h,2)}$, we show that, for $h \geq 2$:

$$f_h(x) = \frac{1}{1 - x f_{h-1}(x)}\;. \tag{1}$$

Since $f_1(x) = 1 + x$ is rational, describing the path UD, thanks to (1) it is possible to deduce that also $f_h(x)$ is rational, too. Therefore, we can consider the following general form $f_h(x) = \frac{p_h(x)}{q_h(x)}$, where

$$q_h(x) = a_{h,0} - a_{h,1}x - a_{h,2}x^2 - \ldots - a_{h,j}x^j \quad \text{with} \quad j = \left\lceil \frac{h+1}{2} \right\rceil.$$

We have an explicit formula for the coefficients $a_{h,j}$ thanks to the following proposition.

Proposition 2. *For $h \geq 2$ and for $j = 1, 2, \ldots, \left\lceil \frac{h+1}{2} \right\rceil$ we have:*

$$a_{h,j} = \frac{3j - h - 2}{j} \binom{h - j + 1}{j - 1} (-1)^j .$$

Finally, it is possible to deduce the following recurrence relation for $D_n^{(h,2)}$,

$$D_n^{(h,2)} = \begin{cases} 1 & \text{for } n = 0 \\ \sum_{j=1}^{\lceil \frac{h+1}{2} \rceil} D_{n-j}^{(h,2)} \frac{3j - h - 2}{j} \binom{h - j + 1}{j - 1} (-1)^j - \frac{3n - h - 1}{n} \binom{h - n}{n - 1} (-1)^n & \text{for } n \geq 1 . \end{cases}$$

A very interesting note arises when, once h is fixed, we ask for the number $D_n^{(h,2)}$ of Dyck paths having semilength $n \leq h$. Clearly, in this case, it is $D_n^{(h,2)} = C_n$ since all the Dyck paths of a certain semilegth $n \leq h$ have height at most equal to n. Thanks to the above argument it is possible to derive interesting relations involving Catalan numbers. Indeed, for the above remark, posing $h = n + \alpha$, we can write $D_n^{(n+\alpha,2)} = C_n$, where $\alpha \geq 0$ is integer. Then, it is possible to deduce the following combinatorial identity involving Catalan numbers:

$$C_n = \sum_{j=1}^{n} C_{n-j} \frac{3j - n - \alpha - 2}{j} \binom{n + \alpha - j + 1}{j - 1} (-1)^j - \frac{2n - \alpha - 1}{n} \binom{\alpha}{n - 1} (-1)^n .$$

References

[1] E. Deutsch. Dyck path enumeration. *Discrete Mathematics*, vol. 204 (1999), p. 167–202.

[2] C. Krattenthaler. Permutations with Restricted Patterns and Dyck Paths. *Advances in Applied Mathematics*, vol. 27 (2001), p. 510—530.

[3] E. Deutsch, L. Ferrari, and S. Rinaldi. Production matrices. *Advances in Applied Mathematics*, vol. 34 (2005), p.101–122.

[4] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. ECO: a methodology for the Enumeration of Combinatorial Objects. *Journal of Difference Equations and Applications*, vol. 5 (1999), p.435–490.

Enumerating grid signed permutation classes

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This talk is based on joint work with Daniel E. Skora

In this extended abstract, we present an algorithm that enumerates a certain class of signed permutations, referred to as grid signed permutation classes. In the case of permutations, the corresponding grid classes are of interest because they are equivalent to the permutation classes that can be enumerated by polynomials. Furthermore, we apply our results to the burnt pancake problem and genome rearrangements. We establish that the number of signed permutations with fixed prefix reversal and reversal distance is given by polynomials that can be computed by our algorithm.

Introduction

The study of the cardinality of permutation classes, namely either bounding or providing a closed expression for $|\mathcal{C} \cap S_n|$, where $\mathcal{C}$ denotes a permutation class, is a well-known combinatorial problem (see, for example [2, 3] and references within.) One of the most celebrated results is the proof by Markus and Tardos of the Stanley-Wilf Conjecture in [4] stating that unless $\mathcal{C}$ is the set of all permutations, $|\mathcal{C} \cap S_n|$ has at most exponential growth. One question that has gathered interest is which permutation classes have polynomial growth or those that can be enumerated by polynomials. In answering this question, another celebrated result in the area is the so-called *Fibonacci dichotomy*, first proved by Kaiser and Klazer [3], stating that for every permutation class $\mathcal{C}$, then either $|\mathcal{C} \cap S_n| \geq F_n$ for all n, where F_n denotes the nth Fibonacci number, or $|\mathcal{C} \cap S_n|$ is given by a polynomial for sufficiently large values of n. Furthermore, the permutation classes enumerated by polynomials are *exactly* the so-called *grid classes* (see [2, Theorem 1.3]) and Homberger and Vatter [2] provided an algorithm that computes the polynomial enumerating $|\mathcal{C} \cap S_n|$ if $\mathcal{C}$ is a grid class.

In this project, we are interested in *signed permutations classes*. A signed permutation can be written as a string of n characters taken from the set $\{1, 2, \ldots, n\}$ where each of the characters has a sign, either positive or negative. Similarly to S_n, the set of all signed permutations, denoted by B_n, is endowed with a group structure under composition. In this context, B_n is known as the *hyperoctahedral group*. A *signed permutation class* is a subset of signed permutations that is closed under containment. We define a particular type of signed permutation class, which we call *grid sign permutation class* in a similar manner to grid classes as defined in [2].

Our contribution

Given a signed permutation $\pi = \pi_1 \cdots \pi_n$, a *prefix reversal* r_i, with $1 \leq i \leq n$, applied to π reverses the first i characters of π, swaps their sign (from plus to minus and vice versa), and leaves any other character and sign unchanged. Similarly, a *block reversal*

$r_{i,j}$, with $1 \leq i \leq j \leq n$, reverses all characters from π_i to π_j in π, inclusively, and swaps their sign. The study of signed permutations and prefix and block reversals is of interest in genome rearrangements, parallel computing, and in the study of the classic *pancake problem*.

In this project, our main contributions are (i) extending the notion of grid class to signed permutations. Then, we (ii) present and implement an algorithm, Algorithm 1, that enumerates these grid signed permutation classes. Examples of these grid classes include classes with connections to the burnt pancake problem and genome rearrangements. In particular, we (iv) show that the classes of signed permutations with a fixed prefix-reversal and reversal distance are enumerated by polynomials and compute some of these polynomials.

We describe the pseudocode in Algorithm 1 and implement it here https://github.com/skora7/SignedPermutationClasses

Algorithm 1: Enumerating Grid(Π)

Input: Set Π of signed permutations
Output: Polynomial $P(n)$ which enumerates $|\text{Grid}(\Pi) \cap B_n|$ with $n > 1$

```
/* Completion Step:  Add all permutations ≤ π ∈ S in the
   containment order.                                        */
Set S = ∅;
for π ∈ Π do
  | Add to S all permutations π' ≤ π
end for
/* Compacting Step:  Remove all permutations π ∈ S with 12, 2̲1̲ ≤ π.
   This is equivalent to checking if π(i+1) − π(i) equals 1 for
   some index i                                              */
for π ∈ S do
  | for i from 1 to len(π) do
  |   | if π(i+1) − π(i) = 1 then
  |   |   | Remove π from S
  | end for
end for
/* Enumeration Step:  Obtain the enumerating polynomial       */
gen_fcn = 0;
for π ∈ S do
  | gen_fcn ← gen_fcn + (x/(1−x))^len(π)
end for
/* poly returns the polynomial that gives the coefficient of x^n in
   gen_fcn                                                   */
P(n) ← poly(gen_fcn);
return P(n)
```

Example 1. Let us illustrate the running of Algorithm 1 with input $\Pi = \{\underline{2}13\}$.

Completion The first step is to add to S all the signed permutations $\pi \leq \underline{2}13$. At the end of the completion step, $S = \{\varepsilon, 1, \underline{1}, 12, \underline{1}2, \underline{2}1, \underline{2}13\}$, where we use ε to denote the empty permutation, which is contained in any permutation.

Compacting We remove the signed permutations in S that are not compact. It turns out that this is equivalent to removing those permutations that contain intervals that are order isomorphic to 12 or $\underline{2}\,\underline{1}$. In this case, we remove 12 resulting in $S = \{\varepsilon, 1, \underline{1}, \underline{1}2, \underline{2}1, \underline{2}13\}$.

Enumeration The generating functions are as follows

- 1 for the empty permutation ε
- $\frac{x}{1-x} = \sum_{n\geq 0} x^{n+1}$ for 1 and $\underline{1}$,
- $\frac{x^2}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^{n+2}$ for $\underline{1}2$ and $\underline{2}1$, and
- $\frac{x^3}{(1-x)^3} = \sum_{n\geq 0}\binom{n+2}{2}x^{n+3}$ for $\underline{2}13$.

Therefore, by isolating the coefficient of x^n with $n \geq 1$ in the expansion of the sum

$$1 + \frac{2x}{1-x} + \frac{2x^2}{(x-1)^2} + \frac{x^3}{(1-x)^3},$$

we obtain the polynomial

$$P(n) = 2 + 2(n-1) + \binom{n-1}{2} = \frac{n^2}{2} + \frac{n}{2} + 1.$$

A more interesting example is counting the number of signed permutations that require exactly four prefix reversals to be sorted. Our algorithm outputs $\frac{1}{2}(n-1)^2(2n-3)$. This result was first proven using a counting argument in [1, Theorem 4.3].

In a subsequent project, we aim to characterize the signed permutation classes that are enumerated by polynomials. Is there an equivalent version of the Fibonacci dichotomy for signed permutation classes?

References

[1] Saúl A. Blanco, Charles Buehrle, and Akshay Patidar. On the number of pancake stacks requiring four flips to be sorted. *Discrete Mathematics & Theoretical Computer Science*, Vol. 21 no. 2, Permutation Patters 2018, November 2019.

[2] Cheyne Homberger and Vincent Vatter. On the effective and automatic enumeration of polynomial permutation classes. *Journal of Symbolic Computation*, 76:84–96, 2016.

[3] Tomá Kaiser and Martin Klazar. On growth rates of closed permutation classes. *Electron. J. Combin.*, 9(2):R10, 2002-3. Permutation Patterns (Otago, 2003).

[4] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the Stanley–Wilf conjecture. *Journal of Combinatorial Theory, Series A*, 107(1):153–160, 2004.

Baxter tree-like tableaux

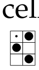
Mathilde Bouvel Université de Lorraine, CNRS, INRIA, LORIA, F-54000 Nancy, France

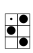
This talk is based on joint work with Jean-Christophe Aval, Adrien Boussicault, Olivier Guibert, Matteo Silimbani, see [1].

We present a family of tree-like tableaux, defined by the avoidance of a pattern. We show that these tree-like tableaux are enumerated by Baxter numbers. We put these Baxter tree-like tableaux in bijections with several families of discrete objects already known to be enumerated by Baxter numbers: twisted Baxter permutations, mosaic floorplans, and triples of non-intersecting lattice paths. The goal of our work is to provide a unifying approach to bijections between Baxter objects, where Baxter tree-like tableaux play the central role. In this sense, it is similar to [3], where the central objects are however twin binary trees (and the family of pattern-avoiding permutations in bijective correspondence is the family of Baxter permutations).

Tree-like tableaux and their Baxter subfamily

Tree-like tableaux were introduced in [2], as a variant of alternative or permutation tableaux. A *tree-like tableau* (TLT) is a Ferrers diagram (drawn in the English notation) where each cell is either empty or pointed (i.e., occupied by a point), with the following conditions:
1. the top leftmost cell of the diagram is occupied by a point, called the *root point*;
2. for every non-root pointed cell c, there exists a pointed cell p either above c in the same column, or to its left in the same row, but not both; p is called the *parent* of c in the TLT;
3. every column and every row contains at least one pointed cell.
The *size* of a TLT is the number of pointed cells it contains. These objects were named tree-like tableaux because of the underlying tree structure they contain, the parent-child relation described above inducing indeed a tree structure.

In this work, we are interested in a family of TLTs restricted by pattern avoidance constraints. A TLT T is said to contain the pattern  if there exist two rows and three columns in T such that the restriction of T to the $2 \times 3 = 6$ cells at their intersection is equal to  or . We define in the same way the pattern  .

A *Baxter tree-like tableau* is a TLT which avoids (i.e., does not contain any of) the patterns  and  . Figure 1 shows an example of a Baxter TLT on the left, and an example of a TLT which is not Baxter in the middle.

As explained in [2], there is a natural way of labeling the points of a TLT of any size n with the integers from 1 to n (see Figure 1, right). It is not too complicated to define, but requires to introduce the additional notion of *ribbon* of a TLT, and we will explain it in the talk but not in this short abstract. This numbering is essential in defining the first two bijections below.

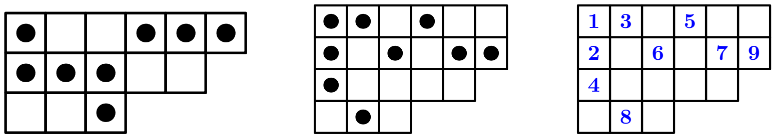


Figure 1: Two TLTs, the left one only being a Baxter TLT. We also show on the right the labeling of the points of the TLT in the middle.

Bijection between Baxter TLTs and twisted Baxter permutations

Twisted Baxter permutations are characterized by the avoidance of the two vincular patterns $2-41-3$ and $3-41-2$. We show that inverses of twisted Baxter permutations are in bijection with Baxter TLTs. The bijection is actually a restriction of a bijection between TLTs and permutations defined in [2]. It can be described as follows. Starting with a TLT, we label its points as in [2] and propagate this integer labeling to all cells of the TLT according to some local rules (omitted here). Then, the image permutation is obtained by reading labels on the southeast border of the TLT, from left to right and bottom to top. It holds that the produced permutation is the inverse of a twisted Baxter permutation if and only if the TLT we started from is a Baxter TLT. An example is shown in Figure 2.

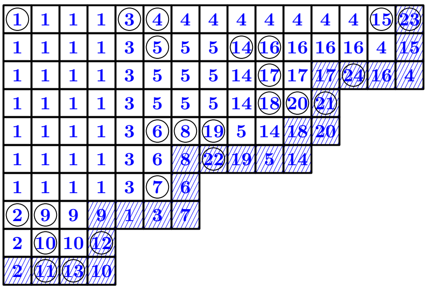


Figure 2: A Baxter TLT T (circled entries represent pointed cells), and its labeling by integers propagated to all cells. It corresponds to the permutation read on the southeast border, namely 2 11 13 10 12 9 1 3 7 6 8 22 19 5 14 18 20 21 17 24 16 4 15 23.

Bijection between Baxter TLTs and mosaic floorplans

Mosaic floorplans can be defined as follows. A *mosaic floorplan* of size n is a partition of a rectangle of semi-perimeter n into $n-1$ rectangular tiles whose sides have integer lengths, and such that the pattern $\lrcorner\ulcorner$ is avoided, meaning that: for every pair of tiles (t_1, t_2), denoting (x_1, y_1) the coordinates of the bottom rightmost corner of t_1 and (x_2, y_2) those of the top leftmost corner of t_2, it is not possible to have both $x_1 \leq x_2$ and $y_1 \geq y_2$.

Our bijection from TLTs to mosaic floorplans is easily described. Starting from a TLT of size n, whose points are labeled by integers from 1 to n as in [2], we obtain a mosaic floorplan of size n by considering the bounding rectangle of the TLT, and for each i from n to 1, by inserting a tile which is largest possible inside this bounding rectangle, whose northwest corner is the point of the TLT with label i. See Figure 3 for an example.

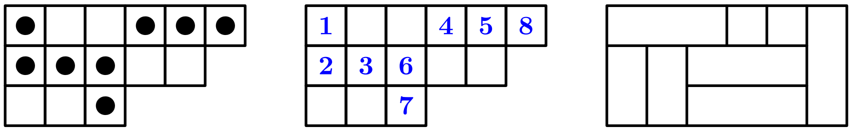


Figure 3: A Baxter TLT, the integer labeling of its points, and the corresponding mosaic floorplan.

Bijection between Baxter TLTs and triples of non-intersecting lattice paths

A *triple of non-intersecting lattice paths of size n* is a set of three lattice paths, with unitary $N = (0,1)$ and $E = (1,0)$ steps, that never meet, which respectively start at $(1,0)$, $(0,1)$ and $(-1,2)$ and end at $(n-i,i)$, $(n-i-1,i+1)$ and $(n-i-2,i+2)$ for some $i \in [0..(n-1)]$ (thus each of the three paths has $n-1$ steps).

Unlike the previous ones, our third bijection between Baxter TLT and triples of non-intersecting lattice paths does not rely on the integer labeling of the points of a TLT. Instead, it relies on the tree structure of the points of the TLT. More precisely, it extends a bijection between binary trees and pairs of non-crossing lattice paths (corresponding to the blue and green paths, with the blue path corresponding to the internal edges and the green path to the leaves). The third (red) path is determined by the southeast boundary of the TLT. An example is shown in Figure 4.

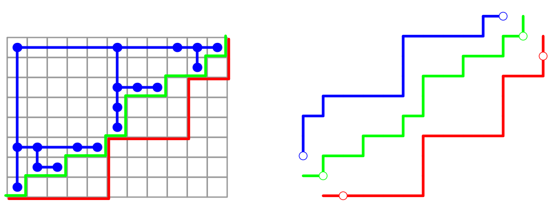


Figure 4: A TLT (with its underlying tree represented with blue edges, and its southeast boundary in red) with its corresponding triple of lattice paths.

References

[1] J.-C. Aval, A. Boussicault, M. Bouvel, O. Guibert, M. Silimbani, Baxter Tree-like Tableaux, to appear in the Australasian Journal of Combinatorics, https://arxiv.org/abs/2108.06212, (2023).

[2] J.-C. Aval, A. Boussicault, P. Nadeau, Tree-like tableaux, *FPSAC 2011, Reykjavík, Iceland, DMTCS proc.* (2011), 63–74.

[3] S. Felsner, E. Fusy, M. Noy, D. Orden, Bijections for Baxter families and related objects, *J. Combin. Theory Ser. A* **118**(3) (2011), 993–1020.

Uncountably many well-quasi-ordered permutation classes

Robert Brignall The Open University

This talk is based on joint work with Vincent Vatter

Notions of 'nice' structure tend to imply 'nice' enumeration. For example, geometrically griddable permutation classes have rational generating functions, while classes with finitely many simple permutations have a recursive structure that admit algebraic generating functions.

One 'nice' property that a class can possess is that of being *well-quasi-ordered*: that is, the class does not contain an infinite set of pairwise incomparable permutations (commonly called an *infinite antichain*). Must a well-quasi-ordered permutation class have a nice enumeration?

We say that a class is *strongly algebraic* if it and all of its subclasses have algebraic generating functions. Since there are only countably many different generating functions, it follows by a simple counting argument that strongly algebraic classes must be well-quasi-ordered.

We show that the converse is false, thus disproving a conjecture made in [3]. We do this by constructing an uncountably large collection of permutation classes with the following properties:

- every class is well-quasi-ordered,
- no two classes have the same enumeration sequence.

Since this collection gives rise to uncountably many different generating functions, it follows that there exist classes in the collection that do not have algebraic generating functions. (Indeed, there exist classes in the collection that do not have D-finite generating functions.)

The construction uses two ingredients: from the world of permutation classes, we need the notion of *pin sequences*, and from the world of infinite binary words, we need a construction from Maurice Pouzet's PhD thesis [2], which can be thought of as a generalisation of the Prouhet–Thue–Morse sequence,

$$1001\,0110\,0110\,1001\,0110\,1001\,1001\,0110\,\ldots.$$

From each of the uncountably many infinite binary words obtained from this construction, we construct an infinite pin sequence, and each infinite pin sequence naturally gives rise to a permutation class (namely the set of all finite permutations contained in the pin sequence).

The classes we construct, though well-quasi-ordered, are not *labelled-well-quasi-ordered* (a stronger notion involving labelling the points in the permutation – see [1]). Indeed, there can only be countably many labelled-well-quasi-ordered permutation classes, and we ask the following.

Question Does every labelled-well-quasi-ordered permutation class have an algebraic generating function?

References

[1] R. Brignall and V. Vatter, Labeled well-quasi-order for permutation classes. *Combinatorial Theory* **2(3)** (2022), #14 (55pp)

[2] M. Pouzet, *Sur la théorie des relations*. PhD thesis, Université Claude-Bernard (Lyon 1), 1978.

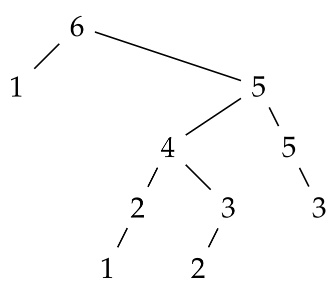
[3] V. Vatter, Permutation classes. In *Handbook of Enumerative Combinatorics*, M. Bóna, Ed. CRC Press, Boca Raton, Florida, 2015, pp. 754–833.

Fishburn trees

Giulio Cerbai University of Iceland

This talk is based on joint work with Anders Claesson

Permutations avoiding the bivincular pattern $\mathfrak{f} = (231, \{1\}, \{1\})$ are called Fishburn permutations. The 2010 paper [1] by Bousquet-Mélou, Claesson, Dukes and Kitaev, showed bijections between certain, apparently unrelated, objects: Fishburn permutations; unlabeled $(\mathbf{2}+\mathbf{2})$-free posets; and ascent sequences. Modified ascent sequences ($\hat{\mathcal{A}}$) were also introduced to make the relation with the level distribution of the corresponding poset more transparent. The current authors [2] showed that the relation between Fishburn permutations and modified ascent sequences can be described in terms of Burge transposition of Burge words. This approach lead to the development of a theory of transport of patterns between the two structures. Finally, Dukes and Parviainen [4] found a bijection between ascent sequences and upper triangular matrices with nonnegative integer entries whose every row and column contains at least one positive entry, the so-called Fishburn matrices. Here we define a new structure of this kind, namely *Fishburn trees*. We show surprisingly straightforward bijections relating them to modified ascent sequences, Fishburn matrices and unlabeled $(\mathbf{2}+\mathbf{2})$-free posets. By composing these new maps we obtain simplified versions of those previously known in the literature. In this sense, Fishburn trees provide a transparent encoding of other Fishburn structures, and we may regard them as central objects from which the others are derived. As an application, in the full paper [3] we provide a more direct answer to the *flip problem* [4]: Duality acts as an involution on unlabeled $(\mathbf{2}+\mathbf{2})$-free posets. On Fishburn matrices, this is equivalent to reflecting a matrix in its antidiagonal. What is the corresponding operation on ascent sequences?



$$\begin{bmatrix} 1 & & & & & \\ 1 & 0 & & & & \\ 0 & 1 & 0 & & & \\ 0 & 1 & 0 & 0 & & \\ 0 & 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 2 & 1 \end{bmatrix}$$

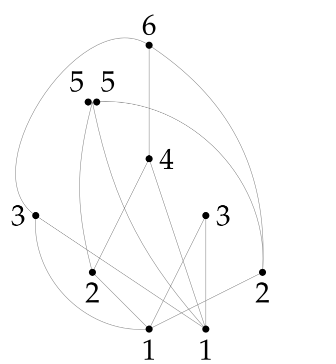


Figure 1: Fishburn tree, Fishburn matrix and $(\mathbf{2}+\mathbf{2})$-free poset corresponding to the modified ascent sequence $x = 1612423553$.

Fishburn trees and modified ascent sequences

A *binary tree* is either the empty tree or a triple $T = (L, r, R)$, where r is a node called the *root* of T and L and R are binary trees called the *left subtree* and the *right subtree* of

T, respectively. We denote by $\mathrm{V}(T)$ the set of nodes of T and by $\mathrm{r}(T)$ its root. The *size* of T is the number of its nodes. Now, suppose that T is equipped with a *vertex labeling* $\mathfrak{l} : \mathrm{V}(T) \to \{1, 2, \dots\}$ and let $\max(T)$ denote the largest value among the labels of T. Then T is *decreasing* if on any path from the root to a leaf we encounter the labels in weakly decreasing order. It is strictly decreasing to the left if on any such path when we take a left turn we encounter a smaller label. The *in-order sequence* of T is defined recursively as $\alpha(T) = \alpha(L)\mathfrak{l}(r)\alpha(R)$. Alternatively, we let $\alpha(T) = x_1 \cdots x_n$, where v_i is the i-th visited node in the *in-order traversal* of T and $x_i = \ell(v_i)$. In particular, $x_{i-1} < x_i$ if and only if v_i has a left child. Such an x_i is called an ascent top; by convention and convenience we will also include x_1 among the ascent tops. This justifies us defining

$$\mathrm{asctops}(T) = \{v_1\} \cup \{v_i : v_i \text{ has a left child}\}.$$

We also define

$$\mathrm{nub}(T) = \{v_j : \mathfrak{l}(v_i) \neq \mathfrak{l}(v_j) \text{ for each } i < j\}$$

as the set of nodes v_j whose label $\ell = \mathfrak{l}(v_j)$ is the first occurrence of ℓ in the in-order sequence of T. We are now ready to give the definition of Fishburn tree. From now on, let $[n] = \{1, \dots, n\}$.

Definition 1. A decreasing binary tree T of size n is said to be an *endotree* if it is strictly decreasing to the left and $\mathfrak{l}(v) \in [n]$ for each $v \in \mathrm{V}(T)$.

It is easy to see that if T is an endotree, then $\alpha(T) = x_1 \cdots x_n$ encodes the endofunction $x : [n] \to [n]$, where $x(i) = x_i$ for each $i \in [n]$.

Definition 2. A *Fishburn tree* is an endotree T in which $\mathrm{Im}(\mathfrak{l}) = [k]$, for some $k \leq n$, and $\mathrm{nub}(T) = \mathrm{asctops}(T)$. We denote by $\mathcal{T}$ the set of Fishburn trees.

Lemma 3. *Let $T \in \mathcal{T}$ and $k = \max(T)$. Then $|\mathrm{asctops}(T)| = k$ and $\mathfrak{l}\big(\mathrm{asctops}(T)\big) = [k]$.*

Proposition 4. *Let T be an endotree and let $x = \alpha(T)$. Then T is a Fishburn tree if and only if x is a modified ascent sequence. Therefore, the in-order sequence is a size-preserving bijection between $\mathcal{T}$ and $\hat{\mathcal{A}}$.*

The inverse map $\bar{\alpha}$ of α is defined as follows. The *max-decomposition* of an endofunction $x = x_1 \dots x_n$ is $x = \mathrm{pref}(x)\, x_m\, \mathrm{suff}(x)$, where $\mathrm{pref}(x) = x_1 \dots x_{m-1}$, $\mathrm{suff}(x) = x_{m+1} \dots x_n$ and $m = \min\big(x^{-1}(\max(x))\big)$ is the index of the leftmost occurrence of $\max(x) = \max\{x_i : i \in [n]\}$ in x. The tree $\bar{\alpha}(x)$ is then defined using recursion: If x is the empty word, then $\bar{\alpha}(x)$ is the empty tree. Otherwise, x is nonempty and using the max-decomposition we can write $x = \mathrm{pref}(x) x_m \mathrm{suff}(x)$. Finally, we let $\bar{\alpha}(x) = \big(L, r, R\big)$ be the tree with root r labeled $\mathfrak{l}(r) = x_m$, left subtree $L = \bar{\alpha}\big(\mathrm{pref}(x)\big)$ and right subtree $R = \bar{\alpha}\big(\mathrm{suff}(x)\big)$. The Fishburn tree corresponding to the modified ascent sequence $x = 1612423553$ is depicted in Figure 1.

Fishburn trees, Fishburn matrices and unlabeled $(2+2)$-free posets

Let T be a Fishburn tree. A *maximal right path* of T is a nonempty sequence of nodes $W = (w_1, w_2, \dots, w_k)$ such that w_{i+1} is the right child of w_i, for each $i = 1, \dots, k-1$;

and W is maximal in the sense that the first node w_1 is not the right child of any node and the last node w_k has no right child. Given $v \in \mathrm{V}(T)$, we denote by $\mathrm{rpath}(v)$ the only maximal right path that contains v. *Maximal left path* and $\mathrm{lpath}(v)$ are defined analogously. The *diagonal* of T is $\mathrm{rpath}\big(r(T)\big)$. Note that $\mathrm{diag}(T) \subseteq \mathrm{asctops}(T)$, and let $\overline{\mathrm{diag}}(T) = \mathrm{asctops}(T) \setminus \mathrm{diag}(T)$. Furthermore, $\mathrm{V}(T)$ can be partitioned as

$$\mathrm{V}(T) = \bigcup_{v \in \mathrm{diag}(T)} \mathrm{rpath}(v) \;\cup \bigcup_{v \in \overline{\mathrm{diag}}(T)} \mathrm{rpath}\big(\mathrm{lchild}(v)\big)$$

where all the unions are disjoint and $\mathrm{lchild}(v)$ denotes the left child of v. In particular, each maximal right path is associated with a unique node $v \in \mathrm{asctops}(T)$. Since $|\mathrm{asctops}(T)| = k$ and $\mathfrak{l}\big(\mathrm{asctops}(T)\big) = [k]$, where $k = \max(T)$, there are exactly k maximal right paths in T, each of which is associated uniquely with the integer $\mathfrak{l}(v) \in [k]$. Let W_i be the maximal right path assigned to $i \in [k]$ in this manner. Then for each node $u \in \mathrm{V}(T)$ we let $\mathfrak{b}(u)$ be the index of the maximal right path that contains u; i.e. $u \in W_{\mathfrak{b}(u)}$. For instance, the (labels of the) maximal right paths of the Fishburn tree of Figure 1 are

$$\begin{array}{lll} W_1 = (1) & W_2 = (1) & W_3 = (2) \\ W_4 = (2) & W_5 = (4,3) & W_6 = (6,5,5,3). \end{array}$$

Now, to a Fishburn tree T with $\max(T) = k$ we associate a $k \times k$ matrix $A = (a_{ij})$ by letting, for $i, j \in [k]$,

$$a_{ij} = \big|\{v \in \mathrm{V}(T) : \mathfrak{l}(v) = i \text{ and } \mathfrak{b}(v) = j\}\big|.$$

Finally, we define an unlabeled $(\mathbf{2}+\mathbf{2})$-free poset as follows. It is well known that any such poset is completely determined by its levels, which are strictly ordered by inclusion, and strict down-sets. We define them, respectively and for $i \in [k]$, as

$$L_i = \{v \in \mathrm{V}(T) : \mathfrak{l}(v) = i\} \quad \text{and} \quad D_i = \{v \in \mathrm{V}(T) : \mathfrak{b}(v) < i\}.$$

The maps defined above are bijections between Fishburn trees and Fishburn matrices and Fishburn trees and $(\mathbf{2}+\mathbf{2})$-free posets (see also Figure 1). For a proof of this fact, as well as for an explicit construction of the inverse maps, we refer once again to our full paper [3].

References

[1] M. Bousquet-Mélou, A. Claesson, M. Dukes and S. Kitaev, *(2+2)-free posets, ascent sequences and pattern avoiding permutations*, Journal of Combinatorial Theory, Series A, Vol. 117, pp. 884–909, 2010.

[2] G. Cerbai and A. Claesson, *Transport of patterns by Burge transpose*, European Journal of Combinatorics, Vol. 108, 2023.

[3] G. Cerbai and A. Claesson, *A solution to the duality problem on modified ascent sequences*, arXiv: 2211.07200, 2022.

[4] M. Dukes and R. Parviainen, *Ascent sequences and upper triangular matrices containing non-negative integers*, Electronic Journal of Combinatorics, Vol. 17, 2010.

The operad structures of poset matrices associated to natural partial ordering

Gi-Sang Cheon Sungkyunkwan University

This talk is based on joint work with Hong Joon Choi and Arnauld M. Mwafise

Operads are mathematical devices which describe algebraic structures of many varieties in various categories. In most cases, it involves the possible ways of gluing together two or more disjoint structures of the same type to form a new structure of that same type, which can lead to significantly new algebraic or combinatorial interpretations. More formally, an operad is a collection $\mathcal{O} := \bigsqcup_{n\geq 1} \mathcal{O}(n)$ together with partial composition operations $\circ_i$ such that

$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1), \qquad n,m \geq 1, i \in [n], \tag{1}$$

and $\circ_i$ is associative and has an identity element $id \in \mathcal{O}(1)$ called the unit of $\mathcal{O}$. Recently, operad theory develops many interesting connections with combinatorics, for example see [3]. Indeed, by endowing a set of combinatorial objects such as *posets* with an operad structure, the operad helps to establish the combinatorial properties of objects and leads to new ways of understanding them. The theory of linking operads derived from finite posets represented by Hasse diagrams was recently studied in [2].

This work is devoted to enrich a connection between operads and (possibly infinite) posets by establishing a new link between them by means of poset matrices. We say that a partial order $\preceq$ on a finite set $X = [n]$ or infinite set $X = \mathbb{N}$ is *natural* if $x \preceq y$ implies $x \leq y$. These natural labelled posets on X are in bijection with $|X| \times |X|$ binary (possibly infinite) lower triangular matrices with ones on the main diagonal that contain no $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ submatrix whose upper right entry is on the main diagonal, which will be called *poset matrices* [1]. This motivated to reexamine the poset operads in terms of poset matrices. We describe several new operad structures of poset matrices in constructions using the partial composition operations. These results are expected to open up new directions for working with poset matrices from combinatorial and algebraic perspectives that were previously unattainable.

References

[1] G.-S Cheon, B. Curtis, G. Kwon and A.M. Mwafise, Riordan posets and associated incidence matrices. *Linear Algebra and its Applications*, vol.632 (2022), n.1, 308-331.

[2] F. Fauvet, L. Foissy and D. Manchon, Operads of Finite Posets. *The Electronic Journal of Combinatorics*, vol.25 (2018), n.1, P1.44.

[3] S. Giraudo, Operads in Algebraic Combinatorics. (2017), Available at: https://arxiv.org/abs/1712.03782.

Descent statistics with the M-binomial property

William Clark Davidson College

This talk is based on joint work with Yan Zhuang

In this talk, we will present the M-binomial property of the peak set (Pk) and peak number (pk), which are two descent statistics.

Background

Let $\pi = \pi_1 \cdots \pi_n$ be an n-permutation. The peak set of π is defined by

$$\mathrm{Pk}(\pi) = \{i \in \{2, \cdots, n-1\} \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Meanwhile, $\mathrm{pk}(\pi) = |\mathrm{Pk}(\pi)|$. Since the peak set and peak number are descent statistics, they are defined on compositions. Let $J = (j_1, \cdots, j_m)$ be a composition of n. One formula for the peak set of a composition is

$$\mathrm{Pk}(J) = \left\{ \sum_{i=1}^{k} j_i \mid k \in [m-1] \text{ and } j_k \geq 2 \right\}.$$

Given a descent statistic st, we say that two compositions J and K are st-equivalent, denoted $J \sim_{\mathrm{st}} K$, if $\mathrm{st}(J) = \mathrm{st}(K)$ and $|J| = |K|$.

Let $\{x_1, x_2, \cdots\}$ be a countably infinite set of commuting indeterminates. A formal power series $f \in \mathbb{Q}[[x_1, x_2, \cdots]]$ of bounded degree is a quasisymmetric function if for every list $a_1, \cdots, a_k$ of positive integers,

$$[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}]f$$

whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. The quasisymmetric functions form a graded $\mathbb{Q}$-algebra (graded by the degree of a monomial) called QSym. Two important bases of QSym are indexed by compositions. The first is the basis of fundamental quasisymmetric functions: $\{F_J \mid J \text{ is a composition}\}$. The second is the basis of monomial quasisymmetric functions: $\{M_J \mid J \text{ is a composition}\}$.

The Kernel and M-binomial Property

For a descent statistic st, Grinberg [2] defines the kernel of st, denoted $\mathcal{K}_{\mathrm{st}}$, as the $\mathbb{Q}$-vector subspace of QSym spanned by all elements of the form $F_J - F_K$ where J and K are st-equivalent compositions. Proposition 101 [2] says that $\mathcal{K}_{\mathrm{st}}$ is an ideal of QSym if and only if st is shuffle-compatible (a property of permutation statistics introduced by Gessel and Zhuang [1]).

Grinberg [2] coins the term M-binomial. We say that a descent statistic st is M-binomial if $\mathcal{K}_{\mathrm{st}}$ is spanned by binomials of the form $\lambda M_J + \alpha M_K$, where $\lambda, \alpha \in \mathbb{Q}$ and J, K

are compositions. Grinberg [2] shows that the exterior peak set is M-binomial and conjectures that Pk, pk, and several other descent statistics are M-binomial.

The Peak Set is M-binomial

First we introduce the $\triangleright_i$ relation on compositions.

Definition 1. Suppose that $J = (j_1, \cdots, j_m)$ and K are compositions.

- Suppose there exists $l \in [m]$ such that $j_l > 2$. If
$$K = (j_1, \cdots, j_{l-1}, 2, j_l - 2, j_{l+1}, \cdots, j_m),$$
then we write $J \triangleright_1 K$. For example, $(3,4,3) \triangleright_1 (3,2,2,3)$.
- Suppose that $j_m = 2$ and there exists $l \in [m-1]$ such that $j_l = 2$. If
$$K = (j_1, \cdots, j_{l-1}, 1, 1, j_{l+1}, \cdots, j_{m-1}, 2),$$
then we write $J \triangleright_2 K$. For example, $(3,4,2,3,2) \triangleright_2 (3,4,1,1,3,2)$.
- If $J = (1^{m-1}, 2)$ and $K = (1^{m+1})$, then we write $J \triangleright_3 K$. For example, $(1,1,1,1,2) \triangleright_3 (1,1,1,1,1,1)$.

Let $\mathcal{C}$ be the set of all compositions. The following theorem confirms Grinberg's conjecture about Pk being M-binomial.

Theorem 2. *The following set spans $\mathcal{K}_{\mathrm{Pk}}$:*

$$\{M_J + M_K \mid J, K \in \mathcal{C}, J \triangleright_1 K \text{ or } J \triangleright_2 K\} \cup \{M_J \mid J \in \mathcal{C} \text{ and there exists } K \in \mathcal{C} \text{ such that } J \triangleright_3 K\}.$$

The Peak Number is M-binomial

Conveniently, we can reuse the relations $\triangleright_1$, $\triangleright_2$, and $\triangleright_3$ to show that the peak number is M-binomial. However, we still have to introduce $\triangleright_4$.

Definition 3. Suppose that $J = (j_1, \cdots, j_m)$ and K are compositions.

- Suppose $j_m = 1$, $j_i \leq 2$ for all $i \in [m-1]$, and there exists $l \in [m-2]$ such that $(j_l, j_{l+1}) = (1,2)$. If
$$K = (j_1, \cdots, j_{l-1}, j_{l+1}, j_l, j_{l+2}, \cdots, j_m),$$
then we write $J \triangleright_4 K$. For example, $(2,1,2,2,1,2) \triangleright_4 (2,2,1,2,1,2)$.

The following theorem confirms Grinberg's conjecture about pk being M-binomial.

Theorem 4. *The following set spans $\mathcal{K}_{\mathrm{pk}}$:*

$$\begin{aligned}&\{M_J + M_K \mid J, K \in \mathcal{C}, J \triangleright_1 \text{ or } J \triangleright_2 K\}\\ &\cup \{M_J \mid J \in \mathcal{C} \text{ and there exists } K \in \mathcal{C} \text{ such that } J \triangleright_3 K\}\\ &\cup \{M_J - M_K \mid J, K \in \mathcal{C}, J \triangleright_4 K\}.\end{aligned}$$

References

[1] Ira M. Gessel, Yan Zhuang. Shuffle-compatible permutation statistics. *Advances in Mathematics*, Volume 332, 9 July 2018, pp. 85–141. Also available at arXiv:1706.00750v3.

[2] Grinberg, Darij. Shuffle-compatible permutation statistics II: the exterior peak set. arXiv preprint arXiv:1806.04114 (2018).

Mallows processes and the expanded hypercube

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In this talk, we present a recently introduced process on the set of permutations, called *continuous-time Mallows process*, corresponding to a continuous-time random process with Mallows-distributed marginals. We then define its graph of possible transitions, which appears to have a lot of interesting properties: its transitions only correspond to transpositions; it is a supergraph of the permutohedron (graph of adjacent transpositions) and a subgraph of the graph of all transpositions, both inclusions being strict whenever $n \geq 3$; it strongly relates to sorting networks; and it has a hypercube-like structure. For this last reason, we call this graph the *expanded hypercube* (see Figure 1), Finally, we state a challenging open problem regarding the existence of a Mallows process which is a Markov process and whose jumping process is also a Markov chain. This problem relates to combinatorial properties of the expanded hypercube, in particular the existence of edge weights with equal in-going and out-going sums over all nodes with the same number of inversions (see Figure 2).

Continuous-time Mallows processes

For any $n \in \mathbb{N}$ and $q \in [0, \infty)$, the *Mallows distribution* [1] $\pi_{n,q}$ on the symmetric set $\mathcal{S}_n$ of size n is defined by

$$\pi_{n,q}(\sigma) = \frac{q^{\mathrm{inv}(\sigma)}}{Z_{n,q}},$$

where $\mathrm{inv}(\sigma) = |\{i < j : \sigma(i) > \sigma(j)\}|$ is the number of inversion of σ and $Z_{n,q} = \sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = \prod_{1 \leq i \leq n} (\sum_{0 \leq j < i} q^j)$ is a normalizing constant. This family of random permutations has a lot of very interesting properties, but in particular one can easily check that the individual inversion numbers, defined by $\mathrm{inv}_j(\sigma) = \{i < j : \sigma(i) > \sigma(j)\}$ for $1 \leq j \leq n$, are independent random variables and that they are all stochastically increasing with q.

Say that $(\mathcal{M}_t)_{t \in [0,\infty)}$ is a *continuous-time Mallows processes* if it is a càdlàg continuous-time processes on $\mathcal{S}_n$ such that, for all $t \in [0, \infty)$, $\mathcal{M}_t$ is $\pi_{n,t}$-distributed. Moreover, following properties of Mallows permutations, we say that a Mallows process $(\mathcal{M}_t)_{t \in [0,\infty)}$ has *independent inversions* if the processes $(\mathrm{inv}_j(\mathcal{M}_t))_{t \in [0,\infty)}$ for $j \in [n]$ are all independent of each other and we say that it is *smooth* if the previous processes are increasing with increments of size 1. These two definitions might seem restrictive but actually provide us with a good set of properties for Mallows processes as stated in the following theorem.

Theorem 1 ([2])**.** *There exists a unique Markov process on $\mathcal{S}_n$ which is also a smooth Mallows process with independent inversions.*

Graph of Mallows processes and expanded hypercube

Given a Mallows process $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$, write $\mathcal{G}_\mathcal{M} = (\mathcal{S}_n, E_\mathcal{M})$ for its transition graph defined by

$$E_\mathcal{M} = \left\{(\sigma,\sigma') : \mathbb{P}\Big(\exists t \in (0,\infty) : \mathcal{M}_{t-} = \sigma, \mathcal{M}_t = \sigma\Big)\right\},$$

where $\mathcal{M}_{t-}$ is the left limit of the process at time t. In other words, $(\sigma,\sigma') \in E_\mathcal{M}$ if and only if it is possible for $\mathcal{M}$ to directly transition from σ to σ'. Further write $\mathcal{H}_n = (\mathcal{S}_n, E)$ for the *expanded hypercube* defined by

$$E = \left\{(\sigma,\sigma') : \sum_{j=1}^{n} \left|\mathrm{inv}_j(\sigma) - \mathrm{inv}_j(\sigma')\right| = 1\right\}.$$

We refer to Figure 1 for a representation of the first few expanded hypercubes. The following theorem relates the transition graph of Mallows processes to the expanded hypercube.

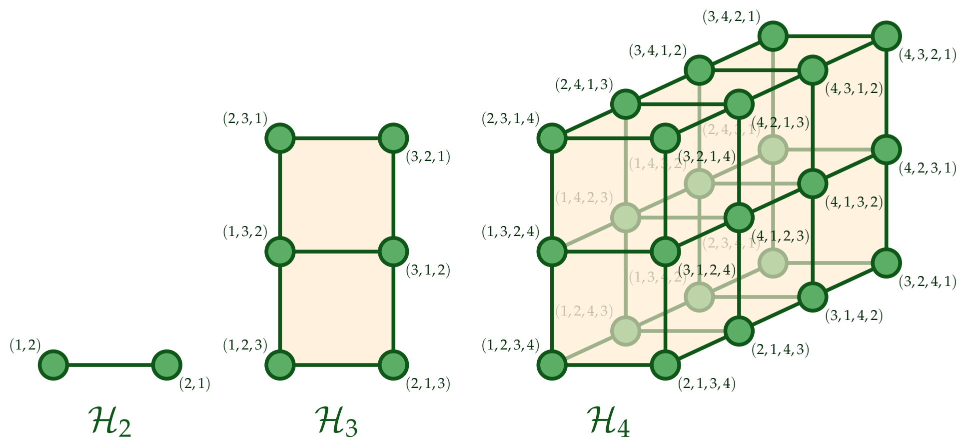

Figure 1: A representation of the first expanded hypercubes $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$. This figure explains where the name of this graph comes from, since it is structured as a hypercube expanded k-times in dimension k.

Theorem 2 ([2]). *Let $\mathcal{M}$ be a smooth Mallows process with independent inversions. Then*

$$\mathcal{G}_\mathcal{M} = \mathcal{H}_n\,.$$

Open problem on the expanded hypercube

Given a smooth Mallows process $\mathcal{M}$, write $\hat{\mathcal{M}} = (\hat{\mathcal{M}}_k)_{0\le k\le\binom{n}{2}}$ for its jumping process, that it is the sequence of distinct permutations taken by $\mathcal{M}$. Note that since the number of inversions of the process only increases by exactly 1, it is indeed parameterized by k taking values from 0 to $\binom{n}{2}$. We now state our conjecture regarding Markov properties of Mallows processes.

Conjecture 3. *The only markovian smooth Mallows processes with independent inversions $\mathcal{M}$ (given by Theorem 1) does not have a corresponding jumping process $\hat{\mathcal{M}}$ which is also markovian. However, there exists a smooth Mallows process $\mathcal{M}$ such that $\mathcal{M}$ is a Markov process and $\hat{\mathcal{M}}$ is a Markov chain (but it does not have independent inversions).*

The first part of this conjecture relies on proving that complex integrals are not equal to one another and is only stated here to show that a Markov process does not trivially correspond to a jumping Markov chain. The second part of this conjecture might look complicated but boils down to organizing the hypercube in *levels* according to the number of inversions of each permutation and defining weights on each edge such that the exiting and entering weights between each inversion level is the same over all the permutations. We provide an example of a possible choice for such weights in the case of $\mathcal{H}_4$ in Figure 2. Proving that such weights exists, and possibly characterizing them, is at the heart of Conjecture 3.

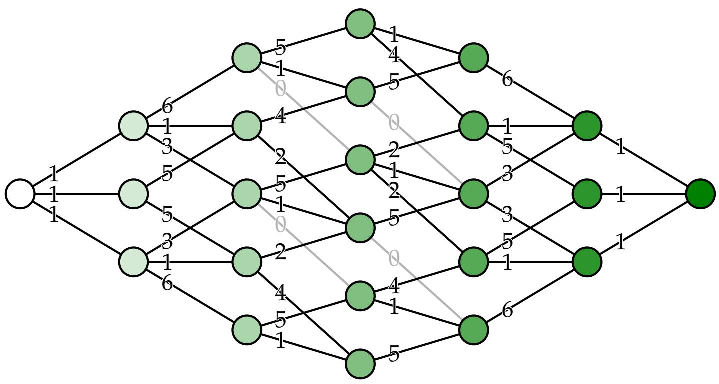

Figure 2: Possible weights for the expanded hypercube $\mathcal{H}_4$ such that, for each given number of inversions (ie column of nodes), the weight going to the right or coming from the left is the same for all nodes.

References

[1] C.L. Mallows. Non-null ranking models. I. *Biometrika, 44 (1/2)*, 114-130 (1957).

[2] B. Corsini. Continuous-time Mallows processes. *arXiv:2205.04967* (2022).

Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants

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A continued fraction of Jacobi-type (J-fraction) is of the form

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{1}{1-\gamma_0 t - \cfrac{\beta_1 t^2}{1-\gamma_1 t - \cfrac{\beta_2 t^2}{1-\cdots}}}\,, \tag{1}$$

where a_n are its coefficients when expanded as a formal power series. Euler [4, section 21] discovered a Stieltjes-type continued fraction for $a_n = n!$ which can be contracted (see [13, p. V-31] for the contraction formula) to obtain a J-fraction for $a_n = n!$ with coefficients $\gamma_n = 2n+1$ and $\beta_n = n^2$. One can introduce new variables in this J-fraction by replacing

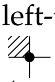
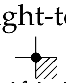
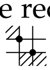
- $\gamma_n = 2n+1$ with $\gamma_0 = z$, $\quad \gamma_n = ([x_2 + (n-1)u_2] + [y_2 + (n-1)v_2] + w$ for $n \geq 1$;
- and $\beta_n = n^2$ with $\beta_n = [x_1 + (n-1)u_1][x_2 + (n-1)v_1]$;

and then ask what permutation statistics are enumerated by the 10 variables x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z. Sokal and Zeng systematically answered this question in [11]. In fact, they provide two interpretations for this J-fraction. However, their second interpretation was left as a conjecture [11, Conjecture 2.3] and they could only prove it with a specialisation. We have proved this conjecture in [2].

Statement of result

Given a permutation $\sigma \in \mathfrak{S}_n$, an index i can be classified as per the ***cycle classification*** into the following five disjoint categories: cycle peak if $\sigma^{-1}(i) < i > \sigma(i)$; cycle valley if $\sigma^{-1}(i) > i < \sigma(i)$; cycle double rise if $\sigma^{-1}(i) < i < \sigma(i)$; cycle double fall if $\sigma^{-1}(i) > i > \sigma(i)$; and fixed point if $\sigma^{-1}(i) = i = \sigma(i)$.

Additionally, an index i can also be classified using the ***record classification***. Following [8, p. 4] we also reformulate these statistics in terms of mesh patterns.

- record (or left-to-right maximum) if $\sigma(j) < \sigma(i)$ for all $j < i$; i.e., an occurrence of pattern  ;
- antirecord (or right-to-left minimum) if $\sigma(j) > \sigma(i)$ for all $j > i$; i.e., an occurrence of pattern  ;
- exclusive record if it is a record and not also an antirecord; i.e., an occurrence of pattern  ;
- exclusive antirecord if it is an antirecord and not also a record; i.e., an occurrence of pattern  ;
- record-antirecord if it is both a record and an antirecord; i.e., an occurrence of pattern  ;
- neither-record-antirecord if it is neither a record nor an antirecord ; i.e., an occurrence of pattern  , which is the pattern 321.

Every index i thus belongs to exactly one of the latter four types.

Furthermore, one can apply the record and cycle classifications simultaneously, to obtain 10 disjoint categories of the ***record-and-cycle classification***: exclusive records that are either cycle valleys (ereccval) or cycle double rises (ereccdrise); exclusive antirecords that are either cycle peaks (eareccpeak) or cycle double falls (eareccdfall); record-antirecords (these are always fixed points) (rar); neither-record-antirecords that are either cycle peaks (nrcpeak) or are cycle valleys (nrcval) or cycle double rises (nrcdrise) or cycle double falls (nrcdfall) or fixed points (nrfix).

Using the record-and-cycle classification and the count of cycles the following 11-variable polynomial $\widehat{Q}_n$ [11, Equation (2.29)] can be defined

$$\begin{aligned}\widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,z,w,\lambda) \quad = \quad & \sum_{\sigma\in\mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times \\ & u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}\end{aligned} \tag{2}$$

The polynomials $\widehat{Q}_n$ have a nice J-fraction:

Theorem 1 ([11, Conjecture 2.3], [2, Theorem 3.1])**.** *The ordinary generating function of the polynomials $\widehat{Q}_n$ specialised to $v_1 = y_1$ has a J-type continued fraction with coefficients*

- $\gamma_0 = \lambda z$ *and* $\gamma_n = [x_2+(n-1)u_2] + [y_2+(n-1)v_2] + \lambda w$ *for* $n \geq 1$,
- *and* $\beta_n = (\lambda + n - 1)[x_1+(n-1)u_1]y_1$

Overview of proof

We first provide an overview of the Foata–Zeilberger bijection [7], and then briefly mention how we reinterpet it to obtain the count of cycles in a permutation.

Let $\sigma \in \mathfrak{S}_n$ be a permutation on n letters. This permutation σ partitions the set $[n]$ into excedance indices ($F = \{i \in [n] : \sigma(i) > i\}$), anti-excedance indices ($G = \{i \in [n] : \sigma(i) < i\}$), and fixed points ($H$). Similarly, σ also partitions $[n]$ into excedance values ($F' = \{i \in [n] : i > \sigma^{-1}(i)\}$), anti-excedance values ($G' = \{i \in [n] : i < \sigma^{-1}(i)\}$), and fixed points. Clearly, $\sigma \upharpoonright F\colon F \to F'$, $\sigma \upharpoonright G\colon G \to G'$, and $\sigma \upharpoonright H\colon H \to H$ are bijections, and the permutation σ can be obtained from the following data:

- Two partitions of the set $[n] = F \cup G \cup H = F' \cup G' \cup H$.
- The two subwords of σ: $\sigma(x_1)\sigma(x_2)\ldots\sigma(x_m)$ and $\sigma(y_1)\sigma(y_2)\ldots\sigma(y_l)$, where $G = \{x_1 < x_2 < \ldots < x_m\}$ and $F = \{y_1 < y_2 < \ldots < y_l\}$.

In their construction, Foata and Zeilberger [7] use this data to describe a bijection between $\mathfrak{S}_n$ to a set of labelled Motzkin paths of length n. One then uses Flajolet's theorem [5] to obtain continued fractions from this bijection while keeping track of a multitude of simultaneous permutation statistics.

The Foata–Zeilberger bijection consists of the following steps (following [11, Section 6.1]):

- Step 1: A Motzkin path ω is described from σ. The description of ω completely depends on the sets F, F', G, G', H.
- Step 2: The labels ξ associated to ω are obtained from σ. It turns out that the description of the labels depend on $\sigma \upharpoonright F\colon F \to F'$, $\sigma \upharpoonright G\colon G \to G'$, and the set H, separately.

- Step 3: This step describes the construction of the inverse map $(\omega, \xi) \mapsto \sigma$ and can be further broken down as follows:
  - Step 3(a): The sets F, F', G, G', H are read off from the path ω.
  - Step 3(b): This description is the crucial part of the construction (at least for our purposes). We use the notion of *inversion tables* to construct the words σ: $\sigma(x_1)\sigma(x_2)\ldots\sigma(x_m)$ and $\sigma(y_1)\sigma(y_2)\ldots\sigma(y_l)$, the former is constructed using "right-to-left" inversion table and the latter is constructed using "left-to-right" inversion table.

It is, a priori, unclear how one might be able to track the number of cycles of σ in this construction. We resolve this issue by reinterpreting Step 3(b). We describe a "history" of this construction using *Laguerre digraphs* [6, 10].

A Laguerre digraph of size n is a directed graph where each vertex has a distinct label from the label set $[n]$ and has indegree 0 or 1 and outdegree 0 or 1. Clearly, any subgraph of a Laguerre digraph is also a Laguerre digraph. A permutation σ in cycle notation is equivalent to a Laguerre digraph L ([12, pp. 22-23]). The directed edges of L are precisely $u \to \sigma(u)$.

For a subset $S \subseteq [n]$, we let $L|_S$ denote the subgraph of L containing the same set of vertices $[n]$, but only the edges $u \to \sigma(u)$, with $u \in S$ (we are allowed to have $\sigma(u) \notin S$). Let $u_1, \ldots, u_n$ be a rewriting of $[n]$. We consider the "history" $L|_\emptyset \subset L|_{\{u_1\}} \subset L|_{\{u_1,u_2\}} \subset \ldots \subset L|_{\{u_1,\ldots,u_n\}} = L$ as a process of building up the permutation σ by successively considering the status of vertices $u_1, u_2, \ldots, u_n$. Thus, at each step we insert a new edge into the digraph, and at the end of this process, the resulting digraph obtained is the digraph of σ.

The crucial part of our construction is that the rewriting $u_1, \ldots, u_n$ is obtained as follows: we first go through H in increasing order (we call this stage (a)), we then go through G in increasing order (stage (b)), finally we go through F but in decreasing order (stage (c)). This total order is suggested by the inversion tables. On building up the permutation σ using this history, we will see that the cycles can only be formed during stage (c) and we can now count the number of cycles. Our total order on $[n]$ only depends on the sets F, G, H, and hence, only on the path ω and not on the labels ξ which is important for our proof to work.

Twist in the story and final remarks.

The continued fractions for permutations in [11] were classified as "second" or "first" depending on whether or not they involved the count of cycles. The proofs of the first and second continued fractions involved two different bijections: the first continued fractions used a variant of the Foata–Zeilberger bijections, whereas the second continued fractions used the Biane bijection [1]. However, our proof for the conjectured "second" continued fraction proceeds by employing the "first" bijection but then reinterpreting it differently. This was a surprise to us.

We can adapt our proof technique to also resolve [9, Conjecture 12] from 1996, and [3, Conjecture 4.1]; both of these are continued fractions generalising the Genocchi and median Genocchi numbers, respectively. More details can be found in [2].

References

[1] P. Biane, Permutations suivant le type d'excédance et le nombre d'inversions et interprétation combinatoire d'une fraction continue de Heine, European J. Combin. **14**, 277–284 (1993).

[2] B. Deb, Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants, preprint (April 2023), arxiv: 2304.14487 [math.CO] at arXiv.org.

[3] B. Deb and A.D. Sokal, Classical continued fractions for some multivariate polynomials generalizing the Genocchi and median Genocchi numbers, preprint (December 2022), arXiv:2212.07232 [math.CO] at arXiv.org.

[4] L. Euler, De seriebus divergentibus, Novi Commentarii Academiae Scientiarum Petropolitanae **5**, 205–237 (1760); reprinted in *Opera Omnia*, ser. 1, vol. 14, pp. 585–617. [Latin original and English and German translations available at http://eulerarchive.maa.org/pages/E247.html]

[5] P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. **32**, 125–161 (1980).

[6] D. Foata and V. Strehl, Combinatorics of Laguerre polynomials, in *Enumeration and Design*, edited by D.M. Jackson and S.A. Vanstone (Academic Press, Toronto, 1984), pp. 123–140.

[7] D. Foata and D. Zeilberger, Denert's permutation statistic is indeed Euler-Mahonian, Stud. Appl. Math. **83**, 31–59 (1990).

[8] B. Han and J. Zeng, Equidistributions of mesh patterns of length two and Kitaev and Zhang's conjectures, Adv. Appl. Math. **127**, 102149 (June 2021).

[9] A. Randrianarivony and J. Zeng, Some equidistributed statistics on Genocchi permutations, Electron. J. Combin. **3**, no. 2, Research Paper #22 (1996), 11 pp.

[10] A.D. Sokal, Multiple Laguerre polynomials: Combinatorial model and Stieltjes moment representation, Amer. Math. Soc. **150**, 1997–2005 (2022).

[11] A.D. Sokal and J. Zeng, Some multivariate master polynomials for permutations, set partitions, and perfect matchings, and their continued fractions, Adv. Appl. Math. **138**, 102341 (2022).

[12] R.P. Stanley, *Enumerative Combinatorics*, vol. 1, Second Edition, Cambridge University Press, 2012.

[13] G. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux, Notes de conférences données à l'Université du Québec à Montréal, septembre-octobre 1983. Available on-line at http://www.xavierviennot.org/xavier/polynomes_orthogonaux.html

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This talk is based on [1] joint work with Mahir Bilen Can.

In this work, we will present an important contribution to Algebraic Combinatorics.

We study *partition Schubert* varieties that are closely related to *toric varieties*. Specifically, we provide a characterization of *spherical* partition Schubert varieties using *Dyck paths*. We introduce the concept of a *nearly toric* variety. We identify both the nearly toric partition Schubert varieties and all *singular* nearly toric Schubert varieties. Furthermore, we determine the cardinalities of these sets of Schubert varieties.

Notation and Preliminaries

In order to maintain consistency and clarity, we will adhere to the following notation throughout these notes. The following algebraic groups and representations are defined over $\mathbb{C}$.

$\mathbf{G}$: connected reductive group	S : Coxeter generators of $(\mathbf{G}, \mathbf{B}, \mathbf{T})$
$\mathbf{B}$: Borel subgroup of $\mathbf{G}$	$\mathbf{P_I}$: parabolic subgroup generated by $\mathbf{I} \subseteq \mathrm{S}$
$\mathbf{T}$: maximal torus in $\mathbf{B}$	$w_0(\mathbf{I})$: longest element of $\mathbf{P_I}$
$\mathbf{W}$: Weyl group of $(\mathbf{G}, \mathbf{T})$	$\mathbf{L_I}$: Levi subgroup of $\mathbf{P_I}$ containing $\mathbf{T}$

Definition 1. An irreducible normal $\mathbf{G}$-variety $\mathbf{Y}$ is **spherical** if a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ has an open orbit in $\mathbf{Y}$.

Definition 2. Let $\mathbf{Y}$ be a spherical variety. The $\mathbf{T}$-complexity of $\mathbf{Y}$, denoted by $c_{\mathbf{T}}(\mathbf{Y})$, is the codimension of the maximal torus $\mathbf{T}$ in $\mathbf{Y}$. If the $\mathbf{T}$-complexity of $\mathbf{T}$ is 1, we call $\mathbf{Y}$ a **nearly toric variety**.

Remark 3. It is well-known result that if $c_{\mathbf{T}}(\mathbf{Y}) = 0$, then $\mathbf{Y}$ is a *toric variety*. This observation motivates the definition of nearly toric varieties. Additionally, Vinberg [2] established that if $c_{\mathbf{B}}(\mathbf{Y}) = 0$ for some Borel subgroup $\mathbf{B}$ in $\mathbf{G}$, then $\mathbf{Y}$ is spherical.

Example 4 (Flag variety)**.** If $\mathbf{G}$ is the **general linear group** GL_n, the Borel subgroup and maximal torus are the upper triangular matrices and the diagonal matrices respectively. Using the Bruhat-Chevalley decomposition, we obtain the **full flag variety** as the disjoint union of double cosets $\mathbf{B}\, w\, \mathbf{B} \,/\, \mathbf{B}$, i.e.,

$$\mathrm{GL}_n \,/\, \mathbf{B} = \bigsqcup_{w \in \mathfrak{S}_n} \mathbf{B}\, w\, \mathbf{B} \,/\, \mathbf{B}$$

where the Weyl group is the symmetric group $\mathfrak{S}_n$. In particular, the $\mathbf{B}$-orbit $\mathbf{B}\, w_0\, \mathbf{B} \,/\, \mathbf{B}$ is open in $\mathrm{GL}_n \,/\, \mathbf{B}$. Hence, $\mathrm{GL}_n \,/\, \mathbf{B}$ is a spherical variety.

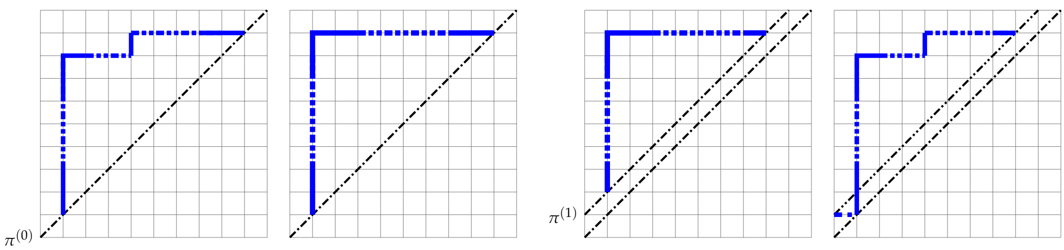
Definition 5. Let w be an element in $\mathfrak{S}_n$. The **Schubert variety associated** with w is the $\mathbf{B}$-orbit (Zariski) closure $X_{w\mathbf{B}} := \overline{\mathbf{B}\, w\, \mathbf{B} \,/\, \mathbf{B}}$ in the flag variety $\mathrm{GL}_n \,/\, \mathbf{B}$. Moreover, $X_{w\mathbf{B}}$ is said to be a **partition** Schubert variety if w is a 312-avoiding permutation. Let $\mathfrak{S}_n^{312}$ denote the set of all 312-avoiding permutations.

Definition 6. A **Dyck path** of size n is a lattice path in $\mathbb{Z}^2$ consisting of north steps $\mathsf{N}(1,0)$ and east steps $\mathsf{E}(0,1)$, starting at $(0,0)$ and ending at (n,n) , and never crossing below the zero diagonal $y = x$. A Dyck path π is an **elbow** if its *Dyck word* has the form NN...NEE...E, where the number of N's and E's are equal. A Dyck path π is an **ledge** if its Dyck word has the form NN...NE...ENE....EE starting with $(n-1)$-N steps followed by n-E steps, a unique N step, and ends with at least two E steps.

Let $\pi = a_1a_2...a_r$ be a Dyck word. We say that a Dyck path π' is a E_+ *extension* of π if $\pi' = \mathsf{E}\,\pi$. A portion τ of $\pi^{(r)}$ is said to be a **connected component** if τ starts and ends at the r-th diagonal, and it intersects the r-th diagonal exactly twice, for $0 \leq r \leq n-1$.

Definition 7. Let **L** denote the standard Levi factor of the parabolic subgroup of $X_{w\,\mathbf{B}}$ in GL_n. Let $\mathbf{B_L}$ be Borel subgroup of **L** containing **T**. The Schubert variety $X_{w\,\mathbf{B}}$ is spherical if $\mathbf{B_L}$ has only finitely many orbits in $X_{w\,\mathbf{B}}$.

Definition 8. A Dyck path π is said to be **spherical** if every connected component on the first diagonal $\pi^{(0)}$ is either an elbow or a ledge as depicted in Figure 1(a), or every connected component of $\pi^{(1)}$ is an elbow, or a ledge whose E_+ extension is the final step of a connected component of $\pi^{(0)}$ as depicted in Figure 1(b).



(a) Ledge and elbow of $\pi^{(0)}$ (b) Elbow and ledge of $\pi^{(1)}$

Figure 1: Spherical Dyck paths

Remark 9. Recently, Gao, Hodge, and Yong explored this question in [3] and established that a Schubert variety $X_{w\,\mathbf{B}}$ is spherical if and only if $w_0(\mathrm{J}(w))w$ is a *Coxeter element*, where $\mathrm{J}(w)$ denotes the *left descents* of w. Remarkably, Gaetz provided a new characterization of spherical Schubert varieties in [4] in terms of *pattern avoidances*. Specifically, he showed a Schubert variety $X_{w\,\mathbf{B}}$ is spherical if and only if w avoids the following 21 patterns

$$\mathscr{P} := \left\{\begin{matrix} 24531, & 25314, & 25341, & 34512, & 34521, & 35412, & 35421, \\ 42531, & 45123, & 45213, & 45231, & 45312, & 52314, & 52341, \\ 53124, & 53142, & 53412, & 53421, & 54123, & 54213, & 54231 \end{matrix}\right\}.$$

In short, this provides a different perspective on the study of spherical Schubert varieties and further deepens our understanding of these objects.

Main contribution

Theorem 10 (Can-D.). *Let w be in $\mathfrak{S}_n^{312}$. Let π denote the Dyck path of size n corresponding to w. Then $X_{w\,\mathbf{B}}$ is a spherical Schubert variety if and only if π is spherical.*

Proof. Building on the work presented in papers [5], [3], and [4] the proof is developed. □

Corollary 11 (Can-D.). *If $X_{w\,\mathbf{B}}$ is a partition Schubert variety of $\mathbf{T}$-complexity 1, then $X_{w\,\mathbf{B}}$ is nearly toric variety. In particular, the cardinality of this family is $2^{n-3}(n-2)$ for $n \geq 4$.*

Theorem 12 (Can-D.). *Let $X_{w\,\mathbf{B}}$ be a singular Schubert variety of $\mathbf{T}$-complexity 1. Then $X_{w\,\mathbf{B}}$ is nearly toric variety. Furthermore, let b_n be the cardinality of this family. Then the generating series of b_n is given by A001871 in the OEIS.*

Remark 13. Theorem 12 sheds light on an intriguing connection between $\mathbf{T}$-complexity and patterns of a permutation. Specifically, Lee, Masuda, and Park established in [6] that a singular Schubert variety has $\mathbf{T}$-complexity one if and only if its associated permutation w contains exactly one occurrence of the pattern 3412 and avoids the pattern 321.

Shortcoming and future work

It is worth noting there exist smooth Schubert varieties of $\mathbf{T}$-complexity one that are not nearly toric varieties. For instance, consider the Schubert variety $X_{w\,\mathbf{B}}$, where $w = 25314$. Despite being smooth and having $c_{\mathbf{T}}(X_{w\,\mathbf{B}}) = 1$, this variety is not nearly toric since $c_{\mathbf{B_L}}(X_{w\,\mathbf{B}}) \neq 0$ by using Gaetz's criterion.

Conjecture 14. *Let c_n denote the cardinality of smooth nearly toric Schubert variety in* $\mathrm{GL}_n\,/\,\mathbf{B}$. *Then the generating series of c_n is given by A317408.*

Question 15. *Find the generating series for t_n, where t_n denotes the number of spherical Dyck paths of size n.*

References

[1] M. B. Can and N. D. Morera, "Nearly toric Schubert varieties and Dyck paths," 2022. https://arxiv.org/abs/2212.01234.

[2] E. B. Vinberg, "Complexity of actions of reductive groups," *Funktsional. Anal. i Prilozhen.*, vol. 20, no. 1, pp. 1–13, 96, 1986.

[3] Y. Gao, R. Hodges, and A. Yong, "Classifying Levi-spherical Schubert varieties," *Sém. Lothar. Combin.*, vol. 86B, pp. Art. 29, 12, 2022.

[4] C. Gaetz, "Spherical Schubert varieties and pattern avoidance," *Selecta Math. (N.S.)*, vol. 28, no. 2, pp. Paper No. 44, 9, 2022.

[5] J. Bandlow and K. Killpatrick, "An area-to-inv bijection between Dyck paths and 312-avoiding permutations," *Electron. J. Combin.*, vol. 8, no. 1, pp. Research Paper 40, 16, 2001.

[6] E. Lee, M. Masuda, and S. Park, "On Schubert varieties of complexity one," *Pacific J. Math.*, vol. 315, no. 2, pp. 419–447, 2021.

Weak ascent sequences, permutations, matrices, and posets

Mark Dukes University College Dublin

This talk is based on joint work with Beáta Bényi and Anders Claesson

In this talk we introduce *weak ascent sequences*, a class of number sequences that properly contains ascent sequences. These new sequences uniquely encode each of the following objects:

- permutations avoiding a particular length-4 bivincular pattern;
- upper-triangular binary matrices that satisfy a column-adjacency rule;
- factorial posets that are weakly (3+1)-free.

Weak ascent sequences are related to a class of pattern avoiding inversion sequences that has been a topic of recent research by Auli and Elizalde [1]. We enumerate these new sequences and give a closed form expression for the number of weak ascent sequences having a prescribed length and number of weak ascents. Full details can be found in our paper [2].

Ascent sequences [3] are rich number sequences in that they uniquely encode four different combinatorial objects and thereby induce bijections between these objects. These objects are (2+2)-free posets; Fishburn permutations; upper-triangular matrices of non-negative integers having neither columns nor rows of only zeros; and Stoimenow matchings. Statistics on those objects have been shown to be related to natural considerations on the ascent sequences to which they correspond.

Our work constructs and presents results similar in spirit to those mentioned above. Given a sequence of integers $x = (x_1, \ldots, x_n)$, we say there is a *weak ascent* at position i if $x_i \leq x_{i+1}$. We denote by $\mathrm{wasc}(x)$ the number of weak ascents in the sequence x. We will use the notation $[a, b]$ for $\{a, a+1, a+2, \ldots, b\}$.

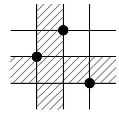
We call a sequence of integers $a = (a_1, \ldots, a_n)$ a *weak ascent sequence* if $a_1 = 0$ and $a_{i+1} \in [0, 1 + \mathrm{wasc}(a_1, \ldots, a_i)]$ for all $i \in [0, n-1]$. Let WAsc_n be the set of weak ascent sequences of length n.

The set WAsc_4 consists of the following:

$(0,0,0,0)$, $(0,0,0,1)$, $(0,0,0,2)$, $(0,0,0,3)$, $(0,0,1,0)$, $(0,0,1,1)$, $(0,0,1,2)$, $(0,0,1,3)$, $(0,0,2,0)$, $(0,0,2,1)$, $(0,0,2,2)$, $(0,0,2,3)$, $(0,1,0,0)$, $(0,1,0,1)$, $(0,1,0,2)$, $(0,1,1,0)$, $(0,1,1,1)$, $(0,1,1,2)$, $(0,1,1,3)$, $(0,1,2,0)$, $(0,1,2,1)$, $(0,1,2,2)$, $(0,1,2,3)$.

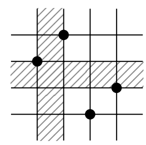
Weak Fishburn permutations

Let S_n be the set of permutations of the set $\{1, \ldots, n\}$. Given a pattern P, in the pattern-avoidance literature the convention is to denote by $S_n(P)$ the set of permutations in S_n that do not contain the pattern P. The set of Fishburn permutations [3], $\mathcal{F}_n = S_n(F)$, are those that avoid the bivincular pattern

$$F = \big(231,\ [0,3]\times\{1\} \cup \{1\}\times[0,3]\,\big) =$$  ,

here defined and depicted as a mesh pattern [4]. Shaded rows and columns indicates that in an occurrence of such a pattern in a permutation, there should be no other permutation dots in the shaded zones when this pattern is placed over a permutation. Bousquet-Mélou et al. [3] gave a bijection between ascent sequences and Fishburn permutations.

We define the bivincular pattern

$$W = \big(3412,\ [0,4]\times\{2\} \cup \{1\}\times[0,4]\,\big) =$$ 

and call $\mathcal{W}_n = S_n(W)$ the set of *weak Fishburn permutations*.

If $\pi_i\pi_j\pi_k\pi_\ell$ is an occurrence of W then $\pi_i\pi_j\pi_\ell$ is an occurrence of F. In other words, every Fishburn permutation is a weak Fishburn permutation and we have $\mathcal{F}_n \subseteq \mathcal{W}_n$.

A class of upper-triangular binary matrices

Dukes and Parviainen [6] showed how the set of upper triangular integer matrices whose entries sum to n and which contain no zero rows or columns are in one-to-one correspondence with ascent sequences. We do something similar for weak ascent sequences, however this correspondence is different to [6] in that the matrix entries are binary and rows of zeros will be allowed.

Let WMat_n be the set of upper triangular square 0/1-matrices A that satisfy the following three properties: (a) there are n 1s in A; (b) there is at least one 1 in every column of A; (c) for every pair of adjacent columns, the topmost 1 in the left column is weakly above the bottommost 1 in the right column.

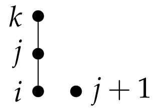
A class of factorial posets

A poset P on the elements $\{1,\dots,n\}$ is *naturally labeled* if $i <_P j$ implies $i < j$. A naturally labeled poset P on $[1,n]$ such that, for all $i,j,k \in [1,n]$, we have

$$i < j \text{ and } j <_P k \implies i <_P k$$

is called a *factorial poset* (see [5]). Factorial posets are known to be (2+2)-free, and further properties of factorial posets can be found in [5]. Let P be a factorial poset on $[1,n]$. We say that P contains a *special 3+1* if there exist four distinct elements

$i < j < j+1 < k$ such that the poset P restricted to $\{i, j, j+1, k\}$ induces the 3+1 poset with $i <_P j <_P k$:



If P does not contain a special 3+1 we say that P is *weakly* $(3+1)$*-free*. Let WPoset_n be the set of weakly $(3+1)$-free factorial posets on $[1, n]$.

Highlighted Results

With much to say, but no space to say it, the story of weak ascent sequences is equally as rich as that of ascent sequences.

Theorem 1.

(a) *There exist bijections between the set* WAsc_n *and the set of weak Fishburn permutations* $\mathcal{W}_n$*, the set of upper triangluar matrices* WMat_n*, and the set of weakly (3+1)-free factorial posets* WPoset_n*. Moreover, these bijections preserve several statistics on each of the structures.*

(b) *The number of weak ascent sequences of length n having k weak ascents,* $a_{n,k+1}$*, satisfies:*

$$a_{n,k} = \sum_{i=0}^{n} \sum_{j=0}^{k-1} (-1)^j \binom{k-j}{i} \binom{i}{j} a_{n-i,k-j-1},$$

where $a_{0,0} = 1$, $a_{n,0} = a_{0,k} = 0$.

References

[1] J.S. Auli and S. Elizalde. Wilf equivalences between vincular patterns in inversion sequences. *App. Math. Comput.* **388** (2021), 125514.

[2] B. Bényi, A. Claesson, and M. Dukes. Weak ascent sequences and related combinatorial structures. *Europ. J. Comb.* vol. 108 (2023), 103633.

[3] M. Bousquet-Mélou, A. Claesson, M. Dukes and S. Kitaev. (2+2)-free posets, ascent sequences and pattern avoiding permutations. *J. Combin. Theory Ser. A* **117** (2010), no. 7, 884–909.

[4] P. Brändén and A. Claesson. Mesh patterns and the expansion of permutation statistics as sums of permutation patterns. *Elec. J. Combin.* **18** (2011), no. 2, #P5.

[5] A. Claesson and S. Linusson. $n!$ matchings, $n!$ posets. *Proc. Amer. Math. Soc.* **139** (2011), 435–449.

[6] M. Dukes and R. Parviainen. Ascent sequences and upper triangular matrices containing non-negative integers. *Elec. J. Combin.* **17** (2010), no. 1, #R53.

On flattened parking functions

Jennifer Elder Missouri Western State University

This talk is based on joint work with Pamela E. Harris, Zoe Markman, Izah Tahir, Amanda Verga

In this presentation, we will give new results related to a subset of parking functions called *flattened parking functions*.

Let $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and let $\mathfrak{S}_n$ denote the set of permutations of $[n] := \{1, 2, \ldots, n\}$. Recall that a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in \mathfrak{S}_n$ has a *descent* (an *ascent*) at index i if $\sigma_i > \sigma_{i+1}$ (if $\sigma_i < \sigma_{i+1}$). A *run of length* p in $\sigma \in \mathfrak{S}_n$ is a subword $\sigma_i\sigma_{i+1} \ldots \sigma_{i+p-1}$ where $i, i+1, \ldots, i+p-2$ are consecutive ascents, and $\sigma_{i+p-1} > \sigma_{i+p}$, or $i + p - 1 = n$. Ascents, descents, and the number of runs of a permutation are often referred to as *permutation statistics*. The study of permutation statistics is a robust area of research in combinatorics and has gathered much interest as its connections to other areas of math are plentiful.

A permutation σ is said to be a *flattened partition* if it consists of runs arranged from left to right such that their leading values are in increasing order.

Example 1. The permutation $\sigma = 1423$ is flattened, as its runs are 14 and 23, and $1 \leq 2$. However, the permutation $\tau = 4321$ is not flattened, as each element is in its own run, and these elements form a decreasing sequence.

The research teams of Nabawanda, Rakotondrajao, Bamunoba and Beyene, Mantaci studied the distribution of runs for flattened partitions [1, 2]. We will focus specifically on the work done in the paper by Nabawanda, Rakotondrajao, and Bamunoba [2]. In the paper, they give several recursive formulas for the number of flattened partitions which have k runs, denoted $f_{n,k}$. One such formula appears below, for $n \geq 3$ and $k \geq 1$:

$$f_{n,k} = kf_{n-1,k} + (n-2)f_{n-2,k-1}.$$

They also establish a bijection between flattened partitions of length n with k runs and set partitions of the set $[n-1]$ with exactly $k-1$ subsets containing at least two elements.

In this work, we extend the definition of flattened partitions to parking functions by allowing ascents to be *weak* ascents, i.e., a word $a_1a_2 \ldots a_n$ has a *weak ascent* at index i if $a_i \leq a_{i+1}$.

We begin by recalling that parking functions are tuples in $[n]^n$ describing the parking preferences of n cars (denoted $c_1, c_2, \ldots, c_n$) entering a one-way street (in sequential order) with a preference for a parking spot among n spots on the street. If $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$ is such that a_i denotes the parking preference of car c_i, then c_i drives to a_i and parks there if the spot is not previously occupied by an earlier car c_j, with $1 \leq j < i$. If such a car has already parked in spot a_i, then c_i drives down the street parking in the first available spot. Given the parking preferences in α, if the

n cars are able to park in the n spots on the street, then we say that α is a parking function.

There is much known about parking functions, including their enumeration, their characterization in terms of inequalities, and statistics such as ascents and descents, not to mention their numerous generalizations. However, there is little to nothing known about the set of flattened parking functions of length n, which we denote as $\text{flat}(\text{PF}_n)$.

Example 2. When $n = 3$, there are eight flattened parking functions: 111, 112, 121, 113, 131, 122, 123, and 132.

For $1 \leq n \leq 8$, the numbers $|\text{flat}(\text{PF}_n)|$ are 1, 2, 8, 46, 336, 2937, 29629, 336732. This sequence does not appear as a known sequence in the Online Encyclopedia of Integer Sequences (OEIS). In fact, it remains an open problem to provide a recursive formula for this sequence. Given this challenge, we restrict our study by considering a specific subset of parking functions that are created from permutations and (through an insertion process) made into parking functions.

Let $\mathcal{S}$ be a multiset whose elements are in $[n+1]$ and let $|\mathcal{S}|$ denote the cardinality of $\mathcal{S}$ with multiplicity. Given a permutation $\pi \in \mathfrak{S}_n$, let $\mathcal{I}(\mathcal{S}, \pi)$ be the set of words of length $n + |S|$ constructed by inserting the elements of $\mathcal{S}$ into the permutation π in all possible ways.

Example 3. If $\mathcal{S} = \{1, 1\}$ and $\pi = 12$, then $\mathcal{I}(\mathcal{S}, \pi) = \{1112, 1121, 1211\}$.

We define the set of *$\mathcal{S}$-insertion parking functions* as the set

$$\mathcal{PF}_n(\mathcal{S}) := \bigcup_{\pi \in \mathfrak{S}_n} \mathcal{I}(\mathcal{S}, \pi).$$

A parking function $\alpha \in \mathcal{PF}_n(\mathcal{S})$ is said to be a *$\mathcal{S}$-insertion flattened parking function* if it satisfies the properties of a flattened parking function; we denote the set of these parking functions by

$$\text{flat}(\mathcal{PF}_n(\mathcal{S})) := \{\alpha \in \mathcal{PF}_n(\mathcal{S}) : \alpha \text{ is flattened}\},$$

and when we require there to be exactly k runs we denote the set by

$$\text{flat}_k(\mathcal{PF}_n(\mathcal{S})) := \{\alpha \in \mathcal{PF}_n(\mathcal{S}) : \alpha \text{ is flattened with } k \text{ runs}\}.$$

For ease of notation, henceforth we let

$$|\text{flat}(\mathcal{PF}_n(\mathcal{S}))| = f(\mathcal{S}; n) \text{ and } |\text{flat}_k(\mathcal{PF}_n(\mathcal{S}))| = f(\mathcal{S}; n, k).$$

We considered a special subset of parking functions arising from inserting r ones into a permutation of length n. This family of parking functions gives a direct generalization of the work of Nabawanda, Rakotondrajao, and Bamunoba.

Example 4. For $n = 3$, and $\mathcal{S} = \{1\}$, there are five flattened parking functions in $\text{flat}(\mathcal{PF}_3(\{1\}))$: 1123, 1213, 1231, 1132, and 1312.

First, we show that the elements in $\text{flat}(\mathcal{PF}_n(\mathcal{S}))$ are in fact parking functions. Next, we consider the multiset $\mathbf{1}_r = \{1, \dots, 1\}$ containing $r \geq 1$ ones, and the set of flattened parking functions, $\text{flat}(\mathcal{PF}_n(\mathbf{1}_r))$. We give several recursive formulas for the number of these parking functions having k runs, $f(\mathbf{1}_r; n+1, k)$. One such formula is given below:

$$f(\mathbf{1}_r; n+1, k) = k \cdot f(\mathbf{1}_r; n, k) + (n-1) \cdot f(\mathbf{1}_r; n-1, k-1) + r \cdot f(\mathbf{1}_{r-1}; n, k-1).$$

The similarities between our recursions and those in [2] is not a coincidence. In fact, in the special case of $r = 1$,

$$f(\{1\}; n, k) = f_{n+1,k}.$$

Our main result has to do with set partitions of $[n+r]$, where

1. the first r elements all appear in distinct subsets of $[n+r]$
2. there are exactly k subsets containing at least two elements.

We define the (r,k)-Bell numbers, denoted $B_k(n,r)$, as the cardinality of the set partitions described above, which generalize the sequence found in A124234 for $r > 1$.

Theorem 5. *The set* $\text{flat}_{k+1}(\mathcal{PF}_{n+1}(\mathbf{1}_r))$ *is enumerated by* $B_k(n,r)$.

We remark that in our work we establish that the statistic k, which accounts for $k+1$ runs in the $\mathbf{1}_r$-insertion flattened parking functions and the k blocks of size at least 2 in the set partitions is preserved by our bijections. While it was noted by Nabawanda, Rakotondrajao, Bamunoba that the bijection was statistic preserving, they did not note that $f_{n,k}$ is enumerated by A124234 for all k.

Our final set of results concern the $\mathbf{1}_r$-insertion flattened parking functions whose first s integers belong to different runs. This is a generalization of a section in [2]. In particular, we provide a generalization which involves compositions of r delineating the position of ones among the k runs.

To conclude, we provide a variety of directions for future work, including work that is currently being edited for journal submission.

References

[1] F. Beyene and R. Mantaci, Merging-Free Partitions and Run-Sorted Permutations, preprint, 2021, https://arxiv.org/abs/2101.07081.

[2] O. Nabawanda, F. Rakotondrajao, A. S. Bamunoba, Run distribution over flattened partitions, *Journal of Integer Sequences* **23** (2020).

Combinatorial Stieltjes moment sequences

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It is now fairly widely believed that the counting sequence of any principal permutation class is a Stieltjes moment sequence [1, 2, 3, 4, 5]. In other words, for any permutation π there is a measure Ω_π supported on $\mathbb{R}_{\geq 0}$ such that the number $|\mathrm{Av}_n(\pi)|$ of permutations of length n avoiding π is equal to the nth moment of Ω_π.

By analysing the known terms of all 303199 sequences in the OEIS with at least 15 terms, we show that at most 6686 are Stieltjes moment sequences. Amongst these 6686 potential Stieltjes moment sequences are the 21 different sequences $|\mathrm{Av}_n(\pi)|_{n\geq 0}$ for π a pattern of length at most 5, as expected.

For classes avoiding two patterns, Stieltjes moment sequences appear significantly less frequently: Amongst the nine Wilf classes of permutations avoiding one pattern of length 3 and one of length 4, only one (A001519) is counted by a Stieltjes moment sequence. Amongst the 38 Wilf classes of permutations avoiding two patterns of length 4, eight appear to be counted by Stieltjes moment sequences: A006318, A032351, A047849, A109033, A164651, A165538, A165543, A165546, while the remaining 30 are not.

We will describe the algorithm used as well as some consequences of our analysis.

References

[1] N Blitvić and E Steingrímsson. *Permutations, moments, measures*, Transactions of the American Mathematical Society, 374(08) (2021), 5473-5508.

[2] N Blitvić, M S Kammoun and E Steingrimsson. *Consecutive permutation patterns, point processes, and moment sequences*. Formal Power Series and Algebraic Combinatorics 2023.

[3] A Bostan, A Elvey Price, A J Guttmann and J-M Maillard. *Stieltjes moment sequences for pattern-avoiding permutations*. The Electronic Journal of Combinatorics, 27(4) (2020), 1-59.

[4] N Clisby, A R Conway, A J Guttmann and Y Inoue. *Classical Length-5 Pattern-Avoiding Permutations*. The Electronic Journal of Combinatorics, 29(3) (2022).

[5] A Elvey Price, *Selected problems in enumerative combinatorics: permutation classes, random walks and planar maps*, PhD Thesis, School of Mathematics and Statistics, University of Melbourne (2018).

A q,r-analogue of Stirling numbers of type B of the first kind

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This talk is based on joint work with Eli Bagno

Summary

The *(unsigned) Stirling numbers of the first kind* $s(n,k)$ are defined by the following identity: $t(t+1)\cdots(t+n-1) = \sum_{k=1}^{n} s(n,k)t^k$. A known combinatorial interpretation for these numbers is given by considering them as the number of permutations of the set $[n] = \{1,2,\ldots,n\}$ having k cycles.

Bala [1] presented a generalization of the Stirling numbers of the first kind to the framework of Coxeter groups of type B, a.k.a. the group of signed permutations. We denote these numbers by $s^B(n,k)$. The definition of these numbers is as follows:

$$(t+1)(t+3)\cdots(t+(2n-1)) = \sum_{k=0}^{n} s^B(n,k)\cdot t^k. \quad (1)$$

A generalization of Stirling numbers was given by Broder in [2], which is called the *r-Stirling number*. Also, some q-analogues were given by several authors, see e.g. [7].

In this work, we suggest a q,r-analogue for the Stirling numbers of the first kind for the Coxeter groups of type B, together with a combinatorial interpretation and some identities.

Background on signed permutations

The definition of the group of signed permutations is as follows:

Definition 1. Denote $[\pm n] := \{\pm 1,\ldots,\pm n\}$. A *signed permutation* is a bijective function: $\pi : [\pm n] \to [\pm n]$, satisfying: $\pi(-i) = -\pi(i)$ for all $1 \le i \le n$. *The group of signed permutations of the set* $[\pm n]$ (with respect to the composition of functions), also known as the *hyperoctahedral group* or the *Coxeter group of type B*, is denoted by B_n.

Considering B_n as a subgroup of the symmetric group S_{2n} in a natural way, we can also write every signed permutation as a product of disjoint cycles. We consider the pairs of cycles C and $-C = \{-x \mid x \in C\}$ (if $-C \neq C$) as one unit, i.e. although C and $-C$ are two disjoint cycles, we consider them as two parts of the same cycle (since they act on the same set of absolute values). For the uniqueness of the presentation, we use the convention that the minimal positive element of the pair of cycles C and

$-C$ appears in C. We distinguish between two types of cycles: a cycle C will be called a *non-split cycle* (or an *odd cycle*) if the following condition holds: $i \in C$ if and only if $-i \in C$, and will be called a *split cycle* (or an *even cycle*) otherwise. A signed permutation, written as a sequence of disjoint cycles, is presented in *standard form* if its cycles are ordered in such a way that the sequence composed by the smallest absolute value of the elements of each cycle increases.

Restricted growth words and q,r-Stirling numbers of type B of the first kind

The notion of *restricted growth words* was introduced for ordinary set partitions, see Hutchinson [4] and Milne [5]; see also [3, Sec. 1.7].

Restricted growth words of type B of the first kind

For introducing a q-analogue for the Stirling number of the first kind of type B, we define *restricted growth words* for this context (based on a version of *restricted growth words* for type A using their cycle structure, introduced to us by Bruce Sagan [6]):

Definition 2. Let $\Sigma_B = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \le |i|,|j| \le n\}$. A *restricted growth word (RG word) of type B of the first kind* is a word $\omega = \omega_1 \cdots \omega_n = (i_1,j_1)\cdots(i_n,j_n)$ in the alphabet Σ_B, which satisfies the following conditions: we have either $(i_1,j_1) = (1,1)$ or $(i_1,j_1) = (-1,1)$; for each $2 \le t \le n$, the following inequality holds:

$$|i_t| \le \max\{|i_1|,\ldots,|i_{t-1}|\} + 1.$$

If $|i_t| = \max\{|i_1|,\ldots,|i_{t-1}|\} + 1$, we have: $j_t = 1$. If $|i_t| \le \max\{|i_1|,\ldots,|i_{t-1}|\}$, one of the pairs $(i_t, j_t - 1)$ or $(i_t, -(j_t - 1))$ exists in ω.

We denote by $R_B(n,k)$ the set of all RG-words of type B of the first kind satisfying $\#\{i_t \mid 1 \le t \le n,\ i_t < 0\} = k$.

We define a mapping Φ_B between the signed permutations in B_n and restricted growth words of type B of the first kind as follows: Let $\pi = c_1 \cdots c_k$ where for each $1 \le \ell \le k$, c_ℓ is a signed cycle, and the cycles are ordered in standard form. Let $\Phi_B(\pi) = \omega_1 \cdots \omega_n$ be defined according to the following rule, for each $1 \le i \le n$: $\omega_i = ((-1)^{\mathrm{par}(C_t)} \cdot t\ ,\ \mathrm{sign}(i,C_t) \cdot \mathrm{loc}(i,C_t))$, where: i (or $-i$) appears in the cycle C_t; $\mathrm{par}(C_t)$ is 0 if C_t is a split cycle, and 1 otherwise; $\mathrm{sign}(i,C_t)$ is the sign of the first appearance of i or $-i$ in C_t; and $\mathrm{loc}(i,C_t)$ is the location of the first appearance of i or $-i$ in the cycle C_t.

The other way around, i.e. how to obtain a signed permutation from a restricted growth word of type B of the first kind, is straightforward.

Next, we define the *r-variant of restricted growth words*:

Definition 3. An *r-restricted growth (RG-)word of type B of the first kind* is an RG-word of type B of the first kind $\omega = \omega_1 \cdots \omega_n = (i_1, j_1) \cdots (i_n, j_n)$ in the alphabet Σ_B, satisfying the following additional condition: the first r pairs of ω are: $(-1,1)(-2,1)\cdots(-r,1)$. We denote by $R^r_B(n,k)$ the set of all r-RG-words of type B of the first kind satisfying $\#\{i_t \mid 1 \le t \le n,\ i_t < 0\} = k$.

In terms of cycle decomposition of signed permutations, the additional condition is translated to the requirement that the elements $1, 2, \ldots, r$ are placed in different non-split cycles.

The definition of q, r-Stirling numbers and their properties

We now define a q-version for the Stirling numbers of type B of the first kind. We start by defining a linear order $\preceq_{\text{abslex}}$ on Σ_B as follows:

$$(x, y) \preceq_{\text{abslex}} (x', y') \qquad \Longleftrightarrow \qquad \big(|x| < |x'|\big) \ \text{ or } \ \big(|x| = |x'| \ \text{ and } \ |y| \le |y'|\big),$$

with the convention that $\omega_i \prec_{\text{abslex}} \omega_j$ means that $\omega_i \preceq_{\text{abslex}} \omega_j$ and $\omega_i \neq \omega_j$.

Definition 4. Let $\omega = \omega_1 \cdots \omega_n \in R_B(n,k)$. Define the *inversion of* ω by

$$\mathrm{inv}_B(\omega) = \#\left\{(\omega_i, \omega_j) \mid i < j,\ \omega_j \prec_{\text{abslex}} \omega_i\right\}.$$

Definition 5. We define the *q-Stirling number $s^B_q(n,k)$ of type B of the first kind* as follows:

$$s^B_q(n,k) := \sum_{\omega \in R_B(n,k)} q^{2\mathrm{inv}_B(\omega) + \mathrm{neg}(\omega)},$$

where $\mathrm{neg}(\omega) = \#\{\omega_t = (x_t, y_t) \mid y_t < 0\}$.

By the bijection between the set $R_B(n,k)$ and the group of signed permutations B_n, we have: $s^B_q(n,k)|_{q=1} = s^B(n,k)$.

Definition 6. We define the *q, r-Stirling number $s^B_{q,r}(n,k)$ of type B of the first kind* as follows:

$$s^B_{q,r}(n,k) := \sum_{\omega \in R^r_B(n,k)} q^{2\mathrm{inv}_B(\omega) + \mathrm{neg}(\omega)}.$$

It is known that the Stirling number of the first kind $s^B(n,k)$ satisfies the following recursion:

$$s^B(n,k) = s^B(n-1,k-1) + (2n-1)s^B(n-1,k).$$

We now present a Stirling-type recursion for $s^B_q(n,k)$:

Proposition 7. *For each $1 \le k \le n$,*

$$s^B_q(n,k) = s^B_q(n-1,k-1) + \big(1 + [2n-2]_q\big) \cdot s^B_q(n-1,k), \tag{2}$$

with the boundary conditions: $s^B_q(n,0) = \sum\limits_{\ell=1}^{n} s^A_{q^2}(n,\ell) \cdot (1+q)^{n-\ell}$ for $n \ge 1$ (where $s^A_q(n,\ell)$ is the ordinary q-Stirling number of type A of the first kind), and $s^B_q(0,k) = \delta_{0k}$.

Remark 8. Note that recursion (2) is different from the recursion of the first kind q-Stirling number of type B defined by Sagan and Swanson [7] (in their recursion, the coefficient $1+[2n-2]_q$ is replaced by $[2n-1]_q$). Our coefficient reflects two different cases for adding the last element ω_n.

Next, we present a recursion for the r-variant:

Proposition 9. *For each* $1 \le k \le n$*,*

$$s^B_{q,r}(n,k) = s^B_{q,r}(n-1,k-1) + \left(1+[2n-2]_q\right) \cdot s^B_{q,r}(n-1,k),$$

with the boundary conditions: $s^B_{q,r}(n,r) = \sum\limits_{\ell=0}^{n-2r} s^A_{q^2}(n-r,\ell+r)\cdot(1+q)^{n-r-\ell}$*,* $s^B_{q,r}(n,k)=0$ *for* $0 \le k < r$*, and* $s^B_{q,r}(0,k) = \delta_{0k}$*.*

Now we present a q,r-analogue of Equation (1):

Proposition 10.

$$\left(t+1+[2r]_q\right)\left(t+1+[2r+2]_q\right)\cdots\left(t+1+[2n-2]_q\right) = \sum_{k=0}^{n-r} s^B_{q,r}(n,r+k)\, t^k$$

Finally, some connections between Stirling numbers of the first and the second kinds of type B are presented, which are analogous to known connections for type A, and their proofs use symmetric polynomials:

Proposition 11. *For all* $m,n \ge 1$*, we have:*

1. $\sum\limits_{j=0}^{m} (-1)^j s^B(n,n-j) S^B(n-1+m-j,n-1) = 0.$
2. $\sum\limits_{j=1}^{m} (-1)^{j-1} j\, s^B(n,n-j) S^B(n-1+m-j,n-1) = 1^m + 3^m + \cdots + (2n-1)^m.$

References

[1] P. Bala, *A 3-parameter family of generalized Stirling numbers* (2015). Electronic version: https://oeis.org/A143395/a143395.pdf

[2] A. Z. Broder, *The r-Stirling numbers*, Discrete Math. **49** (1984), 241–259.

[3] Ö. Eğecioğlu and A.M. Garsia, *Lessons in Enumerative Combinatorics*, Graduate Texts in Math. Vol. **290**, Springer, 2021.

[4] G. Hutchinson, *Partitioning algorithms for finite sets*, Commun. ACM **6** (1963), 613–614.

[5] S. Milne, *Restricted growth functions and incidence relations of the lattice of partitions of an n-set*, Adv. Math. **26** (1977), 290–305.

[6] B. Sagan, Private communication, 30.1.2022.

[7] B. Sagan and J. Swanson, *q-Stirling numbers in type B*, submitted [arXiv:2205.14078].

Characterizing Rothe Diagrams

Ben Gillen and Jonathan Michala University of Southern California

In this talk, we will present full characterizations of Rothe diagrams, which track the inversions of a permutation and are used in several areas of combinatorics.

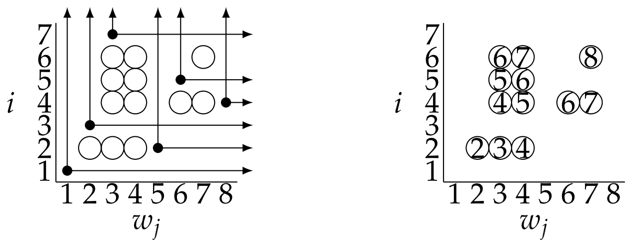

Figure 1: The Rothe diagram $\mathbb{D}(152869347)$ and a numbering of the diagram

Rothe Diagrams

One way to visualize the inversions of a permutation w is with its ***Rothe diagram***, defined to be the following subset of cells in the first quadrant:

$$\mathbb{D}(w) = \{(i, w_j) \colon i < j, w_i > w_j\} \subset \mathbb{Z}^+ \times \mathbb{Z}^+. \tag{1}$$

In Figure 1, we represent cells in $\mathbb{D}(w)$ by bubbles and use the French convention that the i coordinates are written on the vertical axis and the w_j coordinates are written on the horizontal axis.

Characterizations

Rothe Diagrams satisfy many nice properties defined for arbitrary bubble diagrams, one of which can be seen in the second part of Figure 1:

Definition 1. Consider two procedures for labeling the bubbles in a bubble diagram with numbers. First, we give the bubbles a ***horizontal numbering*** where in the ith row, we label the bubbles from left to right $i, i+1, i+2$, and so on. Second, we give the bubbles a ***vertical numbering*** where in the jth column, we label the bubbles from bottom to top $j, j+1, j+2$, and so on. We say a bubble diagram satisfies the ***numbering condition*** and is an ***enumerated diagram*** if the horizontal numbering and vertical numbering yield the same labels for each bubble.

Rothe diagrams satisfy the numbering condition, and the motivation for this property comes from the fact that, on $\mathbb{D}(w)$, when you read the labeling out like a book, you get a reduced word for w.

There exist some diagrams of bubbles which are not Rothe diagrams of permutations but still satisfy the numbering condition. We asked if there is a collection of nice properties like the numbering condition that can tell us if an arbitrary bubble diagram is the Rothe diagram of some permutation, and we developed the following characterizations:

Theorem 2. *Given a diagram D, the following are equivalent:*

(i) D is a Rothe diagram.

(ii) D satisfies the vertical popping and emtpy cell gap rules.

(iii) D satisfies the numbering and dot rules.

(iv) D satisfies the dot and southwest rules.

(v) D satisfies the numbering rule and is step-out avoiding.

Some of these properties extend to new situations as well. In a more general setting, instead of being given a diagram, we are only given a collection of columns of bubbles ordered from left to right. We ask if we can construct a Rothe diagram with these columns in this order; see Figure 2. Based on our characterization of Rothe diagrams, we get the following answer with appropriate reformulations of the numbering and step-out avoiding rules.

Corollary 3. *An ordered collection of free columns may be placed uniquely into a Rothe diagram if and only if the collection satisfies the numbering rule and is step-out avoiding.*

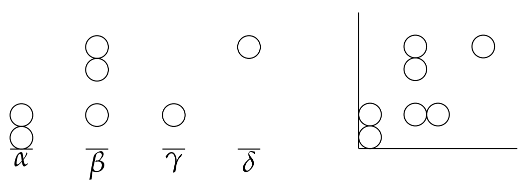

Figure 2: Free columns α, β, γ, and δ, and the Rothe diagram $\mathbb{D}(251463)$.

On the Erdős-Szekeres problem for convex permutations and orthogonally convex point sets

Rimma Hämäläinen

This talk is based on joint work with Heather S. Blake, Stefan Felsner and Marcin Witkowski

In this talk, we determine the largest integer $t(n)$ such that every permutation of length n contains a convex sub-permutation of size $t(n)$. It turns out that $t(n)$ is equal to the largest integer such that every generic set of n points contains an orthogonally convex subset of that size.

Background

In 1935, Erdős and Szekeres [3] proved that for each $n \geq 3$ there exists a smallest positive integer $N(n)$ such that each finite set of at least $N(n)$ points in general position contains a subset of n points in convex position. Albert et al [1] looked at an analogous problem to the Erdős-Szekeres conjecture in terms of permutations by considering the largest integer $t(n)$ such that each permutation σ of length $|\sigma| = n$ contains a subpermutation of length $t(n)$ which is a *convex* permutation, i.e. is order isomorphic to some finite generic set of points in convex position. Going forward, we consider generic sets of $n = \binom{m(n)}{2}$ points and for convenience, we write $m = m(n)$.

Theorem 1. *(Albert et al. [1]) For every positive integer m and $n = \binom{m}{2}$ it holds that $2\sqrt{n} - 1 \leq t(n) \leq 2m - 2 \leq 2\sqrt{2(n+1)} - 2$.*

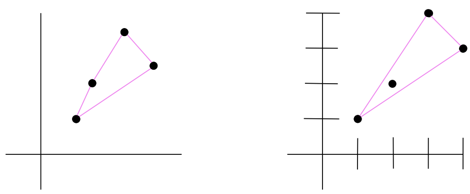

Figure 1: The permutation 1243 is convex as can be seen by the drawing on the left.

A set X in the plane is *orthogonally convex* if and only if $X \cap L$ is connected when L is a horizontal or vertical straight line. Moreover, any generic set of points is in orthogonally convex position if and only if it forms a convex permutation. Our main result is improving the bound in Theorem 1 by the following:

Theorem 2. *For every positive integer m and $n = \binom{m}{2}$ it holds that $t(n) = 2m - 3$.*

Sketch of the proof of Theorem 2

The proof of Theorem 2 is based on considering a Young diagram of shape $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ with k rows and λ_i boxes on the i^{th} row and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. The following two results by Greene and Kleitman [6] and Frank [5] are a key to our proof:

Theorem 3 (Greene and Kleitman [6]). *For all posets $\mathcal{P}$ on n elements, there exists a shape λ of size n such that: the number of cells in the k largest columns of λ is the maximum size of a k-chain in $\mathcal{P}$. Similarly, the number of cells in the ℓ largest rows of λ is the maximum size of an ℓ-antichain in $\mathcal{P}$.*

Theorem 4 (Frank [5]). *If λ contains a grid point (k, ℓ), then poset $\mathcal{P}$ contains a k-chain $\mathcal{C} = \{C_1, \ldots, C_k\}$ and an ℓ-antichain $\mathcal{A} = \{A_1, \ldots, A_\ell\}$ which are orthogonal, i.e. $|A \cap C| = 1$ for all $C \in \mathcal{C}$ and $A \in \mathcal{A}$.*

Lemma 5 (Felsner [4]).

1. *Let $\mathcal{A}$ and $\mathcal{C}$ be an orthogonal pair of $\mathcal{P}$ and let $\mathcal{P}_{\mathcal{A}}$ be the order induced by $\mathcal{P}$ on the set $X_{\mathcal{A}} = \bigcup\{A \ : \ A \in \mathcal{A}\}$. If $\mathcal{A}'$ is the canonical antichain partition of $\mathcal{P}_{\mathcal{A}}$, then $\mathcal{A}'$ and $\mathcal{C}$ are again an orthogonal pair of $\mathcal{P}$.*
2. *Let $\mathcal{A}$ and $\mathcal{C}$ be an orthogonal pair of $\mathcal{P}$ and let $\mathcal{P}_{\mathcal{C}}$ be the order induced by $\mathcal{P}$ on the set $X_{\mathcal{C}} = \bigcup\{C \ : \ C \in \mathcal{C}\}$. If $\mathcal{C}'$ is the canonical chain partition of $\mathcal{P}_{\mathcal{C}}$, then $\mathcal{C}'$ and $\mathcal{A}$ are again an orthogonal pair of $\mathcal{P}$.*

For an increasing (decreasing) sequence I (respectively D), let $I(j)$ (respectively $D(j)$)) denote the *j-th* elements of the sequence. We also say that families of disjoint increasing and decreasing sequences $\mathcal{I}$ and $\mathcal{D}$ are orthogonal if $|I \cap D| = 1$ for each $I \in \mathcal{I}$ and $D \in \mathcal{D}$.

Definition 6. A core of the permutation is the set of $k \cdot \ell$ points that can be partitioned into k disjoint increasing subsequences $\mathcal{I} = \{I_1, .., I_k\}$, each of length ℓ and into ℓ disjoint decreasing subsequences $\mathcal{D} = \{D_1, .., D_\ell\}$, each of length k, such that $\mathcal{I}$ and $\mathcal{D}$ are orthogonal and elements of those sequences are ordered.

Lemma 7. *Let $\mathcal{I}$ and $\mathcal{D}$ be an orthogonal partition of a core of permutation π and let $\mathcal{D}' = \{D_1', D_2', ..., D_\ell'\}$ be the partition with property $D_1'(i) \leq D_2'(i) \leq ... \leq D_k'(i)$ and $\mathcal{I}' = \{I_1', I_2', ..., I_k'\}$ be the partition with property $I_1'(i) \leq I_2'(i) \leq ... \leq I_k'(i)$, for every i, then $\mathcal{D}'$ and $\mathcal{I}'$ are again an orthogonal pair.*

Corollary 8. *An orthogonal partition of a core of the permutation into increasing and decreasing sequences forms a grid $G_{k\ell}$. The boundary of $G_{k\ell}$ consists of points in o-convex position, creating an o-convex set of $2(k + \ell) - 4$ points for $k, \ell \geq 2$.*

The grid point (k, ℓ) in the Young Tableaux of the permutation implies the existence of the core of this permutation of the size $k \cdot \ell$ [2]. Note that the grid point with maximum sum of x and y coordinates in the Young tableaux is defined by the maximum over c of the intersection points of the Young tableaux with the line $y = -x + c$ in $\mathbb{R}^2$ when the tableaux is placed at the origin $(0, 0)$. It is not difficult to observe that this value is uniquely minimized by the triangular shaped Young diagram, as seen in Figure 2. The following two propositions establish the lower and upper bounds (respectively) for Theorem 2:

Proposition 9. *Each permutation of length n contains a convex subpermutation of length $2m - 3$.*

Proof. Consider an arbitrary tableau of size n which is not the triangular shape. Then the tableau contains a grid point (k, ℓ) where $k + \ell \geq m + 1$. By Corollary 8, we have an orthogonally convex set of size $\lceil 2m - 2 \rceil$.

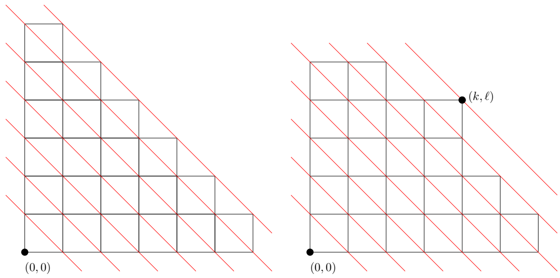

Figure 2: (left) A tableau of size $\binom{m}{2}$ in a triangular shape, (right) a tableau of size $\binom{m}{2}$ not in triangular shape, there exists a grid point (k, ℓ) where $k + \ell \geq m + 1$.

On the other hand, if we have a tableaux with the triangular shape, then the sum of the lengths of the two longest columns or rows in the Young Tableaux is $(m-1)+(m-2) = 2m-3$. Further, was already noticed by Albert in the proof of Theorem 1 that the two longest increasing or two longest decreasing permutations give the size of an orthogonally convex set. $\square$

Proposition 10. *There exists a permutation of $\binom{m}{2}$ points whose largest convex subpermutation is $2m-3$.*

Proof. The construction is similar to the one given by Albert et. al [1]. In our example instead of the decreasing sequences of length $2+4+\cdots+2m+2m+\cdots+4+2$ we consider the sequences of length $1+3+5+\cdots+(m-2)+(m-1)+\cdots+4+2$. If a convex subpermutation of this permutation contains more than two points from any layer, then it cannot contain points both in layers above and below that one. It follows that the longest convex subpermutation has length $2m-3$. $\square$

References

[1] M. H. Albert, S. Linton, N. Ruskuc, V. Vatter, and S. Waton. On convex permutations. *Discrete Math.*, 311(8-9):715–722, 2011.

[2] H. S. Blake, S. Felsner, R. Hämäläinen, and M. Witkowski. On the Erdős-Szekeres problem for convex permutations and orthogonally convex point sets. *In preparation.*

[3] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compos. Math.*, 2:463–470, 1935.

[4] Stefan Felsner. 4-connected triangulations on few lines. *J. Comput. Geom.*, 11(1):476–492, 2020.

[5] A. Frank. On chain and antichain families of a partially ordered set. *J. Combin. Theory Ser. B*, 29(2):176–184, 1980.

[6] C. Greene and D. J. Kleitman. The structure of Sperner k-families. *J. Combin. Theory Ser. A*, 20(1):41–68, 1976.

Flattened Stirling permutations and type B set partitions

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This talk is based on joint work with Azia Barner, Adam Buck, Jennifer Elder, Pamela E. Harris, and Anthony Simpson

In this talk, we let $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, and $[n] := \{1, 2, \cdots, n\}$, and $[n]\dot{\cup}[n] := \{1, 1, 2, 2, 3, 3, \ldots, n, n\}$ be the multiset with each element in $[n]$ appearing twice. We recall that a Stirling permutation (of order n) is a permutation on the multiset $[n]\dot{\cup}[n]$ such that any numbers appearing between repeated values of i must be greater than i; we refer to such values as being *nested* between i. Stirling permutations were first defined by Gessel and Stanley in [3] where they study the coefficients of the polynomial $(1-x)^{2k+1} \sum_{n=0}^{\infty} f_k(n) x^n$ analogous to the combinatorial interpretation of the Eulerian numbers, described in OEIS A008292, in terms of descents of permutations.

Throughout, we let ST_n denote the set of Stirling permutations of order n. For example, the 15 Stirling permutations in ST_3 are: 112233, 122133, 221133, 112332, 122331, 221331, 113322, 123321, 223311, 133122, 133221, 233211, 331122, 331221, and 332211.

In our work, we aim to enumerate Stirling permutations which are flattened. That is, given $\sigma \in \mathsf{ST}_n$ we begin by defining the *runs* of σ to be the maximal contiguous weakly increasing subwords of σ, and write $\sigma = \sigma_1 \sigma_2 \cdots \sigma_r$ where σ_i is a run. We let $\sigma_{i,1}$ be the initial value of each run in σ and call these values the *leading terms of the runs* of σ. If the sequence of leading terms are in weakly increasing order, namely they satisfy $\sigma_{1,1} \leq \sigma_{2,1} \leq \cdots \leq \sigma_{r,1}$, then we say the Stirling permutation σ is *flattened*. We denote the set of flattened Stirling permutations of order n by $\mathrm{flat}(\mathsf{ST}_n)$. For example, $123321 \notin \mathrm{flat}(\mathsf{ST}_3)$, as the leading values of the runs are $1 \leq 2 \not\leq 1$, while $122133 \in \mathrm{flat}(\mathsf{ST}_3)$ as the leading terms of the runs satisfy $1 \leq 1$.

Our definitions above are motivated by the work of Nabawanda, Rakotondrajao, and Bamunoba who established a recursive formula for the number of flattened partitions of $[n]$ with exactly r runs [4]. We recall that flattened partitions of $[n]$ are permutations of $[n]$, which satisfy the leading terms condition described above. In their work, they established that for all integers n and r satisfying $2 \leq r \leq n$, the numbers $f_{n,r}$ of flattened partitions over $[n]$ with r runs satisfy the recurrence relation

$$f_{n,r} = \sum_{m=1}^{n-2} \left(\binom{n-1}{m} - 1 \right) f_{m,r-1} \, . \tag{1}$$

Observe $f_{n,1} = 1$ for all $n \geq 1$. Moreover, they establish a combinatorial bijection between flattened partitions over $[n+1]$ and set partitions of $[n]$. We remark that much of the work in [4] was recently generalized by Elder, Harris, Markman, Tahir, and Verga in [2], where they gave recursive formulas for $\mathbf{1}_r$-insertion flattened parking functions, parking functions of length n with r additional ones. Furthermore, they established a bijection between $\mathbf{1}_r$-insertion flattened parking functions and set partitions of $[n+r]$ with the first r integers in different subsets.

In our setting, note that for $1 \leq n \leq 6$, the cardinalities of $\mathrm{flat}(\mathsf{ST}_n)$ are 1, 2, 6, 24, 116, and 648. This sequence begins identically to the Dowling numbers, described in OEIS A007405, which enumerate type B set partitions defined by Adler in [1, page 13] as follows: A set partition π of $\{-n, -n+1, \ldots, n-1, n\}$ is called a *type B set partition* if

1. $\pi = -\pi$. That is for any block β of π, $-\beta$ is also a block of π.
2. There is exactly one zero-block, which is defined to be a block β such that $\beta = -\beta$. Note that 0 must be an element of the zero-block.

We let Π_n^B denote the collection of all type B set partitions of $\{-n, -n+1, \ldots, n-1, n\}$, and $\Pi_{n,m}^B$ denote the collection of all type B set partitions with m block pairs. For example, $\pi = \{\{0, 1, -1, 2, -2\}, \{3, -4\}, \{-3, 4\}\} \in \Pi_{4,1}^B$.

We can now state our main result:

Theorem 1. *For $n \geq 1$, the set* $\mathrm{flat}(\mathsf{ST}_n)$ *is in bijection with the set* Π_{n-1}^B.

Given the technicality of the bijection, we instead present the following example illustrating the bijection and its inverse. We begin by setting needed notation.

We encode $\pi \in \Pi_{n,m}^B$ as a sequence of subwords $\pi_0|\pi_1|\cdots|\pi_m$ from the alphabet $\{0, 1, \ldots, n\}$. This is similar to the notation of Adler [1], but adapted for our purposes. Negative numbers $a < 0$ are written as $\overline{a}$, and we refer to these as *barred* elements. The word π_0 is formed by listing the non-barred elements of the zero-block in increasing order. To form π_1, we find the block with minimal non-barred element from the remaining subsets. We first list the barred elements of this block in increasing order, then we list the non-barred elements in increasing order. We let a_1 be number of barred elements and b_1 be the size of the block. Thus $b_1 - a_1$ is the number of non-barred elements in this block.

$$\pi_1 = \underbrace{\pi_{1,1} \cdots \pi_{1,a_1}}_{\text{barred elts}} \underbrace{\pi_{1,a_1+1} \cdots \pi_{1,b_1}}_{\text{non-barred elts}}$$

We form π_2, ..., π_m in the same way. First we find the next block of Π_{n-1}^B with minimal non-barred element, excluding previously used blocks and their block pair. Then we put the $\pi_0, \pi_1, \ldots, \pi_m$ into a set of blocks separated by dividers to get

$$\pi_{0,1} \cdots \pi_{0,b_0} | \pi_{1,1} \cdots \pi_{1,a_1} \cdots \pi_{1,b_1} | \cdots | \pi_{m,1} \cdots \pi_{m,a_m} \cdots \pi_{m,b_m} \ . \tag{2}$$

With this notation at hand, we now proceed to illustrate the bijection.

<u>**Step 1:**</u> **From $\pi \in \Pi_{10}^B$ to $\sigma \in \mathrm{flat}(\mathsf{ST}_{11})$.** Consider the following element of Π_{10}^B:

$\{0\}, \{1\}, \{-1\}, \{2, 7, -8\}, \{-2, -7, 8\}, \{3, 5, 6, -9, -10\}, \{-3, -5, -6, 9, 10\}, \{4\}, \{-4\}$

First, we designate negative integers with a bar:

$$\{0\}, \{1\}, \{\overline{1}\}, \{2, 7, \overline{8}\}, \{\overline{2}, \overline{7}, 8\}, \{3, 5, 6, \overline{9}, \overline{10}\}, \{\overline{3}, \overline{5}, \overline{6}, 9, 10\}, \{4\}, \{\overline{4}\}.$$

Then the blocks are organized as described in (2), which results in:

$$0|1|\overline{8}27|\overline{9}(\overline{10})356|4.$$

Increasing the magnitude of all values by one and duplicating each value gives:

$$11|22|\overline{9}3388|(\overline{10})(\overline{10})(\overline{11})(\overline{11})446677|55.$$

Next, for blocks with more than one distinct positive value, place all larger positive values in increasing order, nesting them between the smallest positive numbers in the block. Then eliminate bars over the barred elements. In our example, this gives:

$$11|22|993883|(10)(10)(11)(11)466774|55.$$

Finally, omitting the block dividers recovers an element of $\text{flat}(\mathsf{ST}_{11})$, as desired:

$$1122993883(10)(10)(11)(11)46677455.$$

Step 2: From $\sigma \in \text{flat}(\mathsf{ST}_{11})$ to $\pi \in \Pi_{10}^{B}$. Consider the following element of $\text{flat}(\mathsf{ST}_{11})$:

$$1122993883(10)(10)(11)(11)46677455.$$

Divide the permutation into blocks by nestings

$$11|22|99|3883|(10)(10)|(11)(11)|466774|55.$$

Next, eliminate the second instance of each value

$$1|2|9|38|(10)|(11)|467|5.$$

We note that the first number of the divided blocks are not in increasing order. Fix this by merging the larger values from a previous block with the next block to the right and place those larger values at the end of the block marking them with a bars. Iterate this process until all blocks have first entries in increasing order. This yields:

$$1|2|38\overline{9}|467(\overline{10})(\overline{11})|5$$

Finally, we decrease the magnitude of every value by one, and then denote the barred elements as negatives:

$$0|1|27(-8)|356(-9)(-10)|4.$$

The result of this process produces an element in Π_{10}^{B}, as desired:

$\{0\}, \{1\}, \{-1\}, \{2, 7, -8\}, \{-2, -7, 8\}, \{3, 5, 6, -9, -10\}, \{-3, -5, -6, 9, 10\}, \{4\}, \{-4\}.$

Steps 1 and 2 define the bijective map and its inverse, and together they establish the proof of Theorem 1. We conclude with the immediate enumerative result.

Corollary 2. *The set of flattened Stirling permutations of order n are enumerated by the Dowling numbers.*

References

[1] V. È. Adler. Set partitions and integrable hierarchies *Teoret. Mat. Fiz.*, 187(3):455-486, 2016

[2] Jennifer Elder, Pamela E. Harris, and Zoe Markman, Izah Tahir, and Amanda Verga. On Flattened Parking Functions. arXiv preprint

[3] Ira Gessel and Richard P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24-33, 1978.

[4] Olivia Nabawanda, Fanja Rakotondrajao, and Alex Bamunoba. Run Distribution Over Flattened Partitions. *Journal of Integer Sequences*, 23:Article20.9.6, 10 2020.

Corentin Henriet Università degli Studi di Firenze

This talk is based on joint work with Lapo Cioni and Luca Ferrari

In this talk, we will present a direct bijection between two-stack sortable permutations and fighting fish. It extends the rich bijective connections between the numerous combinatorial structures counted by the numbers $\frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ (sequence A000139 in the OEIS), we refer to the introduction of [1] and references therein for an overview of this bijective world. Along the way, we encounter a new class of labeled trees, that we call labeled sorting trees, which appears to be interesting in itself.

Preliminaries

The *stack-sorting operator* $\mathbf{S}$ on permutations is defined recursively by $\mathbf{S}(\varepsilon) = \varepsilon$ and $\mathbf{S}(\sigma_L n \sigma_R) = \mathbf{S}(\sigma_L)\mathbf{S}(\sigma_R)n$, where n is the largest letter of $\sigma = \sigma_L n \sigma_R$. It can alternatively be described as the result of a certain procedure passing the permutation into a non-increasing stack (see [2]). A *two-stack sortable permutation* is a permutation σ such that $\mathbf{S}^2(\sigma) = \sigma$.

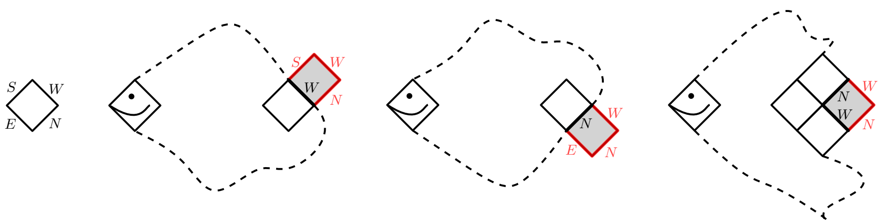

Figure 1: The fighting fish $ENWS$ and operations of upper, lower and double gluing.

While fighting fish have been introduced in [3] in terms of gluings of square cells, we present them here as words on the alphabet $\{E, N, W, S\}$ (see Figure 1):

Definition 1. A word $w \in \{E, N, W, S\}^*$ is a *fighting fish* if it can be obtained from the word $ENWS$ using a finite sequence of the following 3 operations:

- Upper gluing: replace a subword W by NWS.
- Lower gluing: replace a subword N by ENW.
- Double gluing: replace a subword WN by NW.

The size of a fighting fish is half of its length minus 1. We denote by $\mathcal{FF}_n$ the set of fighting fish of size n.

The bijection

From permutations to labeled sorting trees

Let $\sigma \in \mathfrak{S}_n$, we define $\hat{\sigma} \in \mathfrak{S}_{n+1}$ by setting $\hat{\sigma}(1) = n+1$ and $\hat{\sigma}(i) = \sigma(i-1)$ for $2 \leq i \leq n+1$. We represent $\hat{\sigma}$ as the set of points $\{(i, \hat{\sigma}(i))\}$ in $\mathbb{Z}^2$ (its *grid representation*) and we construct a rooted plane tree on this set of points. The *sorting tree* $\mathrm{ST}(\sigma)$ associated to σ is the rooted plane tree obtained by the following top-to-bottom process:

- Define the root to be $(1, n+1)$.
- At step $j \geq 1$, we insert the point $(k, \hat{\sigma}(k))$ in the tree, where k is such that $\hat{\sigma}(k) = n - j + 1$. To do so, let us consider $1 = i_1 < i_2 < ... < i_j$ the x-coordinates of all points already inserted in the tree. There is then a maximal index m such that $i_m < k$. We distinguish two cases:
  - If $m = j$ or $\hat{\sigma}(i_m) < \hat{\sigma}(i_{m+1})$, we define the parent of $(k, \hat{\sigma}(k))$ to be $(i_m, \hat{\sigma}(i_m))$.
  - If $m < j$ and $\hat{\sigma}(i_m) > \hat{\sigma}(i_{m+1})$, we consider the greatest r such that $m+1 \leq r \leq j$ and $\hat{\sigma}(i_m) > \hat{\sigma}(i_{m+1}) > ... > \hat{\sigma}(i_r)$, and we set the parent of $(k, \hat{\sigma}(k))$ to be $(i_r, \hat{\sigma}(i_r))$.
- The process ends when all points have been inserted, i.e. after the n^{th} step.

The permutation $\hat{\sigma}$ can be split into maximal descending runs in a unique way. We associate to every element of $\hat{\sigma}$ its *run label* in the following way: if it is not the last element of its descending run, we label it by 0, else we label it by the number of elements in its descending run. The *labeled sorting tree* $\mathrm{LST}(\sigma)$ associated to σ is the plane rooted tree obtained by labeling each node of $\mathrm{ST}(\sigma)$ with the run label of the element it corresponds in the permutation.

Proposition 2. *Let T be a rooted labeled plane tree with root r, and having n non-root vertices (we say that it has* size *n). For a given node $v \in T$, we denote by $\lambda(v)$ its nonnegative label,* $\deg(v)$ *its number of children,* $\mathrm{sub}(v)$ *the subtree of T rooted at v and* $\mathrm{anc}(v)$ *the nodes w such that v belongs to* $\mathrm{sub}(w)$ *(the ancestors of v).*
Then LST *induces a bijection that sends $2\mathcal{SS}_n$ to the set $\mathcal{LST}_n$ of labeled rooted plane trees of size n verifying the following 3 conditions:*
$$\begin{cases} \sum\limits_{v \in T} \lambda(v) = n+1 \\ \forall v \in T, \lambda(v) \leq \sum\limits_{w \in \mathrm{anc}(v)} \big(2 - \deg(w)\big) - 1 \\ \forall v \in T \setminus \{r\}, \sum\limits_{w \in \mathrm{sub}(v)} \big(\lambda(w) - 1\big) \geq 1\,. \end{cases}$$

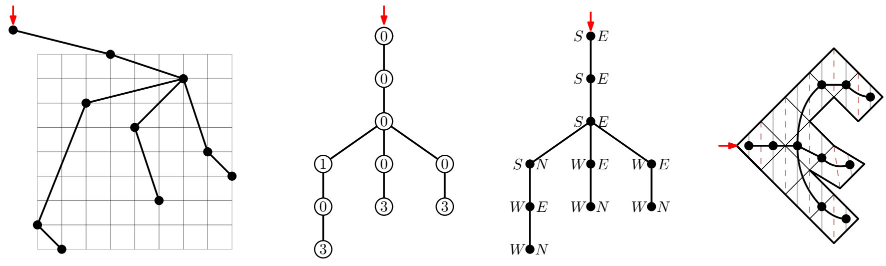

Figure 2: (i) A permutation in $2\mathcal{SS}_9$ and its sorting tree, (ii) its labeled sorting tree, (iii) its fish word displayed on the tree and (iv) the corresponding fighting fish

From labeled sorting trees to fighting fish

Let T be a tree in $\mathcal{LST}_n$. We define the *fish word* $\mathrm{FW}(T)$ of T to be the word w on the alphabet $\{E, N, W, S\}$ built by the following algorithm using a stack:

- Set w to be the empty word and the stack to be empty, and run a clockwise tour of the tree T, starting from the root.
- Every time we encounter a vertex v for the first time, we read its label $\lambda(v)$: if $\lambda(v) = 0$, then we put nothing in the stack and append E to w, else $\lambda(v) > 0$ and we insert (in this order) a letter S and $\lambda(v) - 1$ letters W in the stack and append N to w.
- Every time we encounter a vertex for the last time, we pop the top element of the stack and append it to w.
- The algorithm ends after we reach the root vertex for a second (and last) time.

Proposition 3. FW *is a bijection from* $\mathcal{LST}_n$ *to* $\mathcal{FF}_n$.

Combining the maps LST and FW, we then get a direct description of the recursive bijection of Fang (see [4]) between two-stack sortable permutations and fighting fish:

Theorem 4. $\mathrm{FW} \circ \mathrm{LST}$ *is a bijection from* $2\mathcal{SS}_n$ *to* $\mathcal{FF}_n$.

Comments and perspectives

- We want to highlight that fighting fish are very nice objects because they naturally encode two decompositions: the wasp-waist one isomorphic to the natural one on two-stack sortable permutations (see [4]) and the jaw one isomorphic to the natural decomposition on nonseparable planar maps (see [1]).
- For a general $\sigma \in \mathfrak{S}_n$, it can be shown that its labeled sorting tree $\mathrm{LST}(\sigma)$ is in $\mathcal{LST}_n$. It would be interesting to characterize the set of permutations σ such that $\mathrm{LST}(\sigma) = T$, for a given $T \in \mathcal{LST}_n$.
- The *shifted area* of a fighting fish is the number of square cells composing it, minus its size. It would be very interesting to have a direct interpretation of this statistic on two-stack sortable permutations.

References

[1] E. Duchi and C. Henriet. A bijection between rooted planar maps and generalized fighting fish. *arxiv e-prints*, arXiv:2210.16635, 2022.

[2] J. West. Sorting twice through a stack. *Theoret. Comp. Science*, 117(1):303–313, 1993.

[3] E. Duchi, V. Guerrini, S. Rinaldi, and G. Schaeffer. Fighting Fish. *Journal of Physics A*, 50(2):024002, 16, 2017.

[4] W. Fang, Fighting Fish and Two-Stack Sortable Permutations. *Séminaire Lotharingien de Combinatoire*, 80B.7, 2018.

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The powered Catalan numbers are an integer sequence counting $S_n(1\underline{23}4)$. This appears as OEIS A113227. In [2], Baxter and Shattuck conjecture that $S_n(\underline{23}14)$ is counted by the powered Catalan numbers [2, Conjecture 22]. Beaton et. al. refined this conjecture by the statistic lmin [3, Conjecture 23]. Lin and Fu further refined the conjecture by considering four statistics on permutations as well as four statistics on inversion sequences.

Conjecture 1. *[5, Conjecture 4.2] The quadruple $(rmin, lmin, rmax, asc)$ on $S_n(\underline{23}14)$ has the same distribution as $(zero, max, rmin, rep)$ on $I_n(110)$.*

Given that $I_n(110)$ has been proven to be counted by the powered Catalan numbers [4, Theorem 13], if one could find an explicit bijection that preserves these statistics, that would prove all the preceding conjectures true.

Definition 2. Let $S_{n,2}$ be the permutations on $[n]$ that have exactly two ascents. Let $I_{n,2}$ be the inversion sequences of length n that have exactly two repeats.

We can state our main contribution, which is a special case of Conjecture 1:

Theorem 3. *The quadruple $(rmin, lmin, rmax, asc)$ on $S_{n,2}(\underline{23}14)$ has the same distribution as $(zero, max, rmin, rep)$ on $I_{n,2}(110)$.*

We provide a bijective proof using the map φ defined in Definition 16. The map φ is a modification of the bcode of Baril and Vajnovszki.

Background and Properties

Definition 4. Let I_n be the set of inversion sequences. That is n-tuples $(e_1, \cdots, e_n)$ of nonnegative integers such that $e_i < i$ for all $1 \le i \le n$.

Definition 5. Let $b : S_n \to I_n$ be the bcode as defined in [1, Definition 2].

Let $\beta : S_n \to I_n$ be the map $b \circ \mathrm{inv}$, where $\mathrm{inv} : S_n \to S_n$ is the map that takes a permutation to its inverse.

Example 6.

$$\begin{array}{cccccccccc}
 & & & \bullet & & & & & & \\
\bullet & & & 3 & & & & & & 3 \\
6 & & \bullet & 3 & & & & & & 6 \\
5 & & 3 & 3 & & & \bullet & & & 3 \\
4 & & 3 & 3 & & & 2 & \bullet & & 2 \\
3 & & 2 & 2 & \bullet & & 0 & 0 & & 0 \\
2 & \bullet & 1 & 1 & 1 & & 0 & 0 & & 1 \\
1 & 1 & 1 & 1 & 1 & \bullet & 0 & 0 & & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0
\end{array}
\qquad \beta(72683154) = (0,1,1,0,2,3,6,3)$$

We have the following statistics defined on permutations.

Definition 7. Let the **A**scent **B**ottom **V**alues of a permutation x, be defined as $\mathrm{AscBV}(x) = \{x_i \mid x_i < x_{i+1}\}$. Let $\mathrm{asc}(x) = |\mathrm{AscBV}(x)|$. Let the **R**ight to Left **Max**imum **V**alues of a permutation x, be defined as $\mathrm{RmaxV}(x) = \{x_i \mid x_i > x_j \quad \text{for all} \quad j > i\}$. Let $\mathrm{rmax}(x) = |\mathrm{RmaxV}(x)|$.

Let DescTV, AscTV, RminV, rmin, LminV, and lmin be defined analogously. We refer the reader to [1] for some standard definitions of set-valued inversion sequence statistics such as Zero and Max. In addition, we use the following statistics.

Definition 8. Let $\overline{\mathrm{Row}(e)} = \{i \mid e_i \quad \text{does not occur in the suffix} \quad e_{i+1}e_{i+2}\cdots e_n\}$ Let the early **Rep**eated position set of an inversion sequence e, be defined as $\mathrm{Rep}(e) = \{i \mid e_i \quad \text{does occur in the suffix} \quad e_{i+1}e_{i+2}\cdots e_n\}$. Let $\mathrm{rep}(e) = |\mathrm{Rep}(e)|$. Let the **Emp**ty value set of an inversion sequence e, be defined as $\mathrm{Emp}(e) = \{i+1 \mid i \quad \text{does not occur in} \quad e\}$. Let $\mathrm{emp}(e) = |\mathrm{Emp}(e)| = \mathrm{rep}(e)$.

Example 9.

$$\begin{array}{rr}
\mathrm{RLminV}(72683154) = \{1,4\} & \mathrm{LRminV}(72683154) = \{7,2,1\} \\
\mathrm{RLmaxV}(72683154) = \{8,5,4\} & \mathrm{DesTV}(72683154) = \{7,8,3,5\} \\
\mathrm{AscBV}(72683154) = \{2,6,1\} & \mathrm{AscTV}(72683154) = \{6,8,5\}
\end{array}$$

Example 10.

$$\begin{array}{rr}
\mathrm{Zero}(01102363) = \{1,4\} & \mathrm{Max}(01102363) = \{1,2,7\} \\
\mathrm{RLmin}(01102363) = \{4,5,8\} & \mathrm{Row}(01102363) = \{3,5,7,8\} \\
\mathrm{Rep}(01102363) = \{1,2,6\} & \mathrm{Emp}(01102363) = \{5,6,8\}
\end{array}$$

The following can be proven using [1, Theorem 2] and the observation that inv swaps the role of positions and values.

Theorem 11.

$$(\mathrm{rmin}, \mathrm{lmin}, \mathrm{rmax}, \mathrm{asc})(x) = (\mathrm{zero}, \mathrm{max}, \mathrm{rmin}, \mathrm{rep})(\beta(x))$$

Results

Definition 12. Suppose that $x \notin S_{n,2}(\underline{23}14)$, $e = \beta(x) \in I_{n,2}(110)$. Let $\mathrm{Rep}(e) = \{u, w\}$ with $e_u = b$ and $e_w = a$. Let $i, k \in \overline{\mathrm{Row}(e)}$ such that $e_i = a$ and $e_k = b$. Let $\mathrm{Emp}(e) = \{j, z\}$ where $j < z$ with $e_j = c$ and $e_z = h$. If $j > k$, let $e_m = d$ be the unique value appearing in e, but not in the sub-inversion sequence $(e_1, \cdots, e_j)$. If $i < u$, let $e_v = a'$ be the unique value appearing in e, but not in the sub-inversion sequence $(e_1, \cdots, e_u)$. If we have an inversion sequence $e = (e_1, \cdots, e_n)$ and a permutation τ, let $\tau(e) = \tau((e_1, \cdots, e_n)) = (e_{\tau(1)}, \cdots, e_{\tau(n)})$. Let $\tau_1 : I_{n,2}(110) \to I_{n,2} \setminus I_{n,2}(110)$ be the following map that permutes the values of the inversion sequence e.

$$\tau_1(e) = \begin{cases} \begin{pmatrix} k & i & j \end{pmatrix}(e) & u < i, j < k \\ (k \ \ i)(m \ \ j)(e) & u < i, j > k \\ \begin{pmatrix} k & v & j \end{pmatrix}(e) & u > i, j < k \\ (k \ \ v)(m \ \ j)(e) & u > i, j > k \end{cases}$$

Definition 13. Suppose that $x \in S_{n,2}(\underline{23}14)$, $f = \beta(x) \notin I_{n,2}(110)$. Let $\mathrm{Rep}(f) = \{u, w\}$ with $f_u = b$ and $f_w = a$. Let $i, k \in \overline{\mathrm{Row}(f)}$ such that $f_i = a$ and $f_k = b$. Let $\mathrm{Emp}(f) = \{m, z\}$ where $m < z$ with $f_z = h$. Let $f_j = c$ be the unique value appearing in f, but not in the sub-inversion sequence $(f_1, \cdots, f_m)$. If $i < u$, let $f_v = a'$ be the unique value appearing in f, but not in the sub-inversion sequence $(f_1, \cdots, f_u)$. Let $\tau_2 : I_{n,2} \setminus I_{n,2}(110) \to I_{n,2}(110)$ be the following map (permuting values of the inversion sequence f).

$$\tau_2(f) = \begin{cases} (\,j \quad k \quad m\,)(f) & u < i,\ f_m < b\ (f_m = f_i = a) \\ (i\ \ k)(j\ \ m)(f) & u < i,\ f_m > b \\ (\,j \quad k \quad m\,)(f) & u > i,\ f_m < b\ (f_m = f_v = a') \\ (v\ \ k)(j\ \ m)(f) & u > i,\ f_m > b \end{cases}$$

Lemma 14. $(zero, max, rmin, rep)(e) = (zero, max, rmin, rep)(\tau_1(e))$

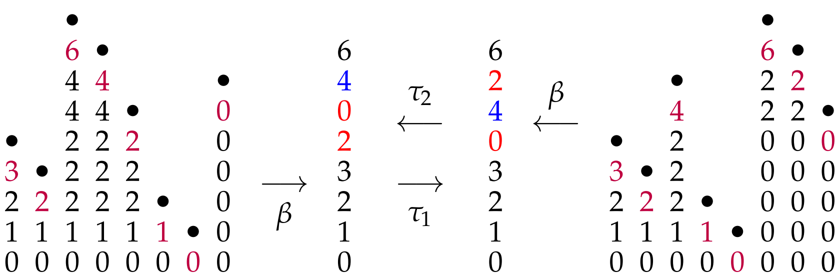
Lemma 15. $(zero, max, rmin, rep)(f) = (zero, max, rmin, rep)(\tau_2(f))$

Definition 16. Consider the following map $\varphi : S_{n,2} \to I_{n,2}$.

$$\varphi(x) = \begin{cases} \beta(x) & x \in S_n(\underline{23}14),\ \beta(x) \in I_n(110) \\ \beta(x) & x \notin S_n(\underline{23}14),\ \beta(x) \notin I_n(110) \\ (\tau_1 \circ \beta)(x) & x \in S_n(\underline{23}14),\ \beta(x) \notin I_n(110) \\ (\tau_2 \circ \beta)(x) & x \notin S_n(\underline{23}14),\ \beta(x) \in I_n(110) \end{cases}$$

Example 17.

$$\varphi(43875216) = 01230426 \quad \varphi(43621875) = 01232046$$



Lemma 18. *φ is a bijection.*

Lemma 19.

$$(rmin, lmin, rmax, asc)(x) = (zero, max, rmin, rep)(\varphi(x))$$

Proof. This follows from Property 11, Lemma 14, and Lemma 15. □

We now have everything to prove our main theorem.

Proof of 3. We restrict φ to $S_n(\underline{23}14)$. The result immediately follows from Lemma 18 and Lemma 19. □

References

[1] Jean-Luc Baril and Vincent Vajnovszki. A permutation code preserving a double Eulerian bistatistic. *Discrete Appl. Math.* 224:9-15, 2017

[2] Andrew M. Baxter and Mark Shattuck. Some Wilf-equivalences for vincular patterns. J. Comb., 6(1-2):19–45, 2015.

[3] Nicholas R. Beaton, Mathilde Bouvel, Veronica Guerrini, and Simone Rinaldi. Enumerating five families of pattern-avoiding inversion sequences; and introducing the powered Catalan numbers. Theoret. Comput. Sci., 777:69–92, 2019.

[4] Sylvie Corteel, Megan A. Martinez, Carla D. Savage, and Michael Weselcouch. Patterns in inversion sequences I. Discrete Math. Theor. Comput. Sci., 18(2):Paper No. 2, 21, 2016.

[5] Zhicong Lin and Shishuo Fu. On 120-avoiding inversion and ascent sequences. European J. Combin., 93:Paper No. 103282, 12, 2021.

On permutation classes defined by pin sequences

Ben Jarvis Open University

This talk is based on joint work with Robert Brignall

Pin sequences were introduced by Brignall, Huczynska and Vatter [1] as a means of studying simple permutations. Since then they have attracted interest as a method of constructing permutation classes with a large number of simples, and in connection with monotone griddability - see [3]. In this talk we consider so-called 'pin classes' - permutation classes consisting of all finite permutations contained in a given infinite pin sequence - with a focus on the smallest possible growth rates that these classes can have. We will conclude with a brief discussion of the application of this theory to the study of growth rates of permutation classes with bounded oscillations.

Classifying small pin classes

We begin with a definition, following Bassino, Bouvel, and Rossin [2]:

Definition 1. A **pin sequence** is an word (finite or infinite) over the language

$$\{1,2,3,4\}(\{\mathrm{l},\mathrm{r}\}\{\mathrm{u},\mathrm{d}\})^* \cup \{1,2,3,4\}(\{\mathrm{u},\mathrm{d}\}\{\mathrm{l},\mathrm{r}\})^*$$

A finite pin sequence can be converted into a 2-by-2 gridded permutation by the following procedure (see Fig. 1 for an illustration of this process):

1. Place an initial point in the quadrant specified by the initial number (counting anti-clockwise from the top-right);
2. At all subsequent steps, place a point either up, down, left or right (depending on the letter u, d, l, or r) of the bounding rectangle of all previous points (including a 'ghost point' at the origin) at the end of a 'pin' which separates the last point from all points before.

Note that this definition almost guarantees that a permutation produced from a pin sequence will be simple - and it is in fact this connection with simple permutations that has motivated much of the study of pin sequences. Given an (infinite) pin sequence we can define the corresponding **pin class** as the downward closure of the set of all permutations produced by a finite initial subsequence. In this talk we develop the theory of pin sequences and apply this to classify the growth rates of 'small' pin classes.

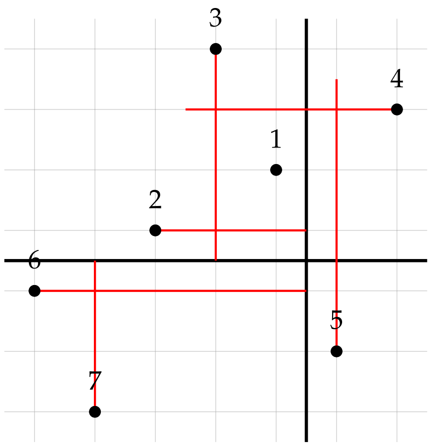


Figure 1: The permutation $3,1,4,7,5,2,6$ (in a 2-by-2 grid), constructed from the pin sequence 2lurdld. The numbers refer to the order in which the points were placed: the first point was placed in quadrant 2 (due to the 2 at the start of the pin sequence); then, the second point was placed to the left (due to the l) of the bounding rectangle of the first point and the origin, at the end of a pin separating point 1 from the origin; next, point 3 was placed above (or 'up', due ot the u) the bounding rectangle of the first two points and the origin, at the end of a pin separating point 2 from point 1 and the origin; and so on...

We begin with the class $\mathcal{O}$, the downwards closure of the increasing oscillations, which is also the pin class defined by the sequence $1(\mathtt{ur})^*$. The growth rate of this class is $\kappa \approx 2.20557$; it is known that this is the smallest possible growth rate of a pin class and that $\mathcal{O}$ is 'essentially' the only pin class that achieves it.

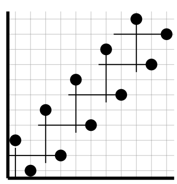


Figure 2: The first 11 points given by the pin sequence $1(\mathtt{ur})^*$, defining the permutation $3,1,5,2,7,4,9,6,11,8,10$. The downward closure of this pin sequence is the pin class $\mathcal{O}$, the class of **increasing oscillations**.

We shall show that, somewhat surprisingly, the next smallest pin class does not appear until the (significantly larger) growth rate $\nu \approx 3.069$, achieved by the class $\mathcal{V}$, defined by the pin sequence $1(\mathtt{ulur})^*$; see Figure 3. This is the first pin class to visit two quadrants infinitely often - though much more exotic behaviour is possible in two quadrants; for example, non-periodic pin sequences.

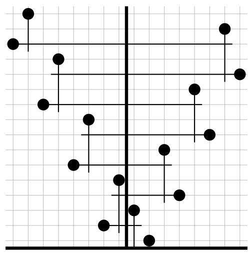


Figure 3: The first 16 points given by the pin sequence $1(\mathtt{ulur})^*$; the downward closure of this pin sequence is the pin class $\mathcal{V}$

In seeking to find the next possible growth rate of a pin class, we are naturally led to define the class $\mathcal{Y}$ from the pin sequence $1(\mathtt{uldlur})^*$; this has growth rate $\gamma \approx 3.366$ and can be shown to be the smallest pin class that visits three quadrants infinitely often. This leaves open the question of what happens between $\mathcal{V}$ and $\mathcal{Y}$; we will address this problem, as well as briefly looking beyond γ at the smaller pin classes in three and four quadrants, concluding with a classification of what we shall call the 'small' pin classes.

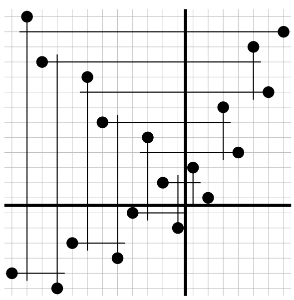


Figure 4: The first 17 points given by the pin sequence $1(\mathtt{uldlur})^*$; the downward closure of this pin sequence is the pin class $\mathcal{Y}$

REFERENCES

[1] Robert Brignall, Sophie Huczynska, and Vincent Vatter. Decomposing simple permutations, with enumerative consequences. Combinatorica , 28:385–400, 2008.

[2] Frédérique Bassino, Mathilde Bouvel, and Dominique Rossin. Enumeration of pin- permutations. Electron. J. Combin. , 18(1):Paper 57, 39, 2011.

[3] M.H. Albert, A. Atminas, and R. Brignall. Characterising inflations of monotone grid classes of permutations. J. Combin. Theory Ser. A, 154:444–463, 2018.

Evolutionary permutations

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Evolutionary algorithms can be used to explore optimization problems that are related to the topic of pattern containment. They mimic natural selection of the living world and find solutions that get closer to the optimum with each generation.

In order to describe an evolutionary algorithm, one must have a description of an individual organism, a measure for their fitness, and a means for repopulating. Individuals that are more fit than others have a better chance of surviving and passing along their genetic data leading to a population that becomes more fit as a whole, i.e., *survival of the fittest.*

An example of an optimization problem well-suited to using evolutionary algorithms is to find the permutation of length n with the most occurrences of a given pattern or set of patterns (or in other words, find the permutation of length n that is most densely packed with a given pattern or set of patterns).

Using the definition from [1], given a pattern q and an integer n, $M_{n,q}$ is the largest integer such that there exists an n-permutation with exactly $M_{n,q}$ occurrences of the pattern q.

This is an optimization problem that works well with evolutionary algorithms. There is no guarantee that the exact value of $M_{n,q}$ will be found for any n, q but what can be done is to find permutations with a lot of occurrences of q.

In order to describe an evolutionary algorithm (EA), one typically has to describe these important components (components taken from [2].):

- Representation (description of individual organisms)
  In general, the description of each individual is a member of the set that you are searching. In the case of finding the permutation with the most occurrences of a pattern we can describe each individual is a permutation of length n.

- Fitness function
  The fitness function is used to put an order on the individuals based on how close they are to a desired fitness. In the case of finding a permutation with many occurrences of a pattern, we can define the fitness function as $f_q(\pi)$ = the number of occurrences of the pattern q in the permutation π.

- Population size
  Having a larger population size means more variability which theoretically suggests that you are more likely to find the optimal individual or at least find individuals who are close to being optimal. But large population size also means that you need more resources (memory and time).

- Parent Selection
  Some evolutionary algorithms will select parents by choosing a m-element subset

from the population and then selecting the 2 most fit individuals from the subset. You can change the parameter m to achieve different results. If m is very small then it is less likely that the most fit individuals will have an opportunity to pass on their genetic data. If m is very large, then only the most fit individuals will pass on their genetic data and you will lose some variation.

- Variation operations
  - Recombination (cross-over)
    Cross-over is the process of creating offspring from parents. Usually this is a randomized process that should pass on a mix of genetic data from the parents to the offspring.
  - Mutation
    Mutation is a way to alter the genetic data of one individual. Typically it is not necessary to apply mutations to all individuals but instead apply mutations seldom enough that it introduces a bit of variability so that you don't get stuck in a local optimum.
- Survivor selection mechanism
  The survivor selection mechanism is used to find a way to keep the most fit individuals and get rid of least fit individuals. One way to do this is that whenever new individuals are "born", if they are more fit than the weakest individual of the population, they get swapped out.

Although evolutionary algorithms may not be able to achieve exact answers to optimization problems, they may offer a way to explore solutions that are close to the optimum and can be used to make conjectures.

Some optimization problems that are well-suited to study using evolutionary algorithms:

- Given a pattern q and a length n, find the permutation with the most occurrences of q.
- Given a set of patterns Q and a length n, find the permutation with the most occurrences of patterns in Q.
- Given two permutations q and r and a length n, find the permutation with the most occurrences of r while avoiding q.
- Given a set of patterns Q, find the shortest permutation that contains all occurrences of patterns in Q.

As an example: Given the patterns $q = (4, 2, 3, 5, 1)$ and $r = (1, 3, 2)$, here is the evolution of a permutation of length 75 with many occurrences of q and very few occurrences of $(1, 3, 2)$. The last permutation in the figure has 1115127 occurrences of q and 0 occurrences of $(1, 3, 2)$.

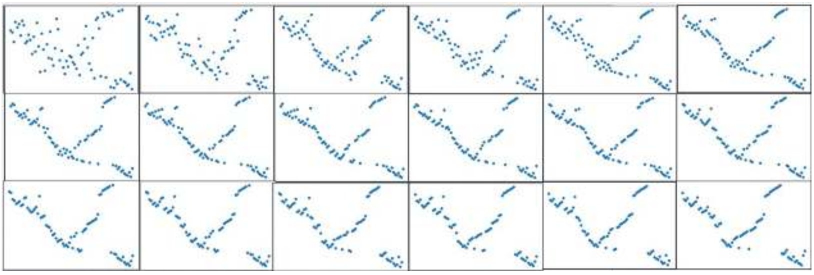


Figure 1: the evolution of permutations with many occurrences of $(4,2,3,5,1)$ and few occurrences of $(1,3,2)$.

References

[1] Miklós Bóna *Combinatorics of Permutations* (2004). Chapman and Hall.

[2] A. E. Eiben and J. E. Smith *Introduction to Evolutionary Computing* corrected second printing (2007). Springer.

Sturm meets Fibonacci in Minkowski's fractal bar

Sergey Kirgizov Université de Bourgogne

This talk is based on joint work with Sergey Dovgal

The set $\mathcal{W}_{q,n}$ of *q-decreasing words* of length n is the subset of all binary words of length n whose maximal factors of the form 0^a1^b satisfy $a = 0$ or $qa > b$. In the talk [1] presented at "Permutation Patterns 2021" we started to explore the case of natural q and asked what happens otherwise. Here, we deal with positive real q, construct words from $\mathcal{W}_{q,n}$ using prefixes of Sturmian words and explore the growth function $\Phi(q) = \lim_{n\to\infty} |\mathcal{W}_{q,n+1}|/|\mathcal{W}_{q,n}|$. The function $\Phi(q)$ is bounded, strictly increasing, discontinuous at every rational point, and exhibits a nice fractal structure related to the Stern–Brocot tree and Minkowski's question mark function.

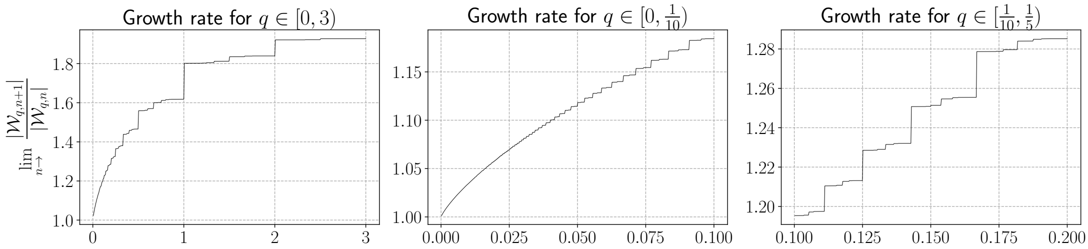


Figure 1: $\Phi(q)$ in three different intervals.

For natural q, $|\mathcal{W}_{q,n}|$ are enumerated by generalized $(q+1)$-step Fibonacci numbers: for $q = 1$ this corresponds to Fibonacci numbers, for $q = 2$ it is known as tribonacci sequence, etc. So, $\Phi(1)$ is the golden ratio and $\Phi(2)$ is tribonacci constant. For $q \in \mathbb{N}$, the set $\mathcal{W}_{q,n}$ is in bijection with the set of n-length binary words that avoid $q+1$ consecutive 1s [2]. As far as we know, the latter appears for the first time in Knuth's book [3].

Our work connects two different kinds of Fibonacci words: one from the Sturmian context (0100101001001010... see A3849 in OEIS) and other from Knuth's realm of pattern avoiders. The growth function $\Phi(q)$ generalizes the golden ratio by parametrizing it by any positive real number q.

References

[1] J.-L. Baril, S. Kirgizov, and V. Vajnovszki. *Qubonacci words*. Permutation Patterns 2021.

[2] J.-L. Baril, S. Kirgizov, and V. Vajnovszki. *Gray codes for Fibonacci q-decreasing words*. Theoretical Computer Science, vol.927 (2022), p.120-132.

[3] D.E. Knuth, *The Art of Computer Programming, Volume 3: Sorting and Searching* 2nd ed., Addison-Wesley, 1998.

Counting parking sequences and parking assortments through permutations

Jan Kretschmann University of Wisconsin–Milwaukee

This talk is based on joint work with Spencer J. Franks, Pamela E. Harris, Kim Harry, Megan Vance

In this talk, we present an enumerative formula to count different generalizations of parking functions called *parking sequences* and *parking assortments*.

Throughout, we let $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $[n] = \{1, 2, 3, \ldots, n\}$. *Parking sequences*, as introduced by Ehrenborg and Happ [4], are defined as follows. Suppose there are n cars $1, 2, \ldots, n$ of lengths $y_1, y_2, \ldots, y_n \in \mathbb{N}$, respectively. Let $m = \sum_{i=1}^{n} y_i$ be the number of parking spots on a one-way street. Sequentially label parking spots $1, 2, 3, \ldots, m$ increasingly along the direction of a one-way street. We let $x_i \in [m]$ denote the preferred spot of car i, for all $i \in [n]$, and we say $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the *preference list* for the cars with lengths $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. The cars enter the one-way street from the left in the order $1, 2, \ldots, n$; and car i seeks the first empty spot $j \geq x_i$. If all of the spots $j, j+1, \ldots, j + y_i - 1$ are empty, then car i parks there. If spot j is empty and at least one of the spots $j+1, j+2, \ldots, j + y_i - 1$ is occupied, then there is a collision; and car i fails to park. If all cars park successfully under the preference list $\mathbf{x}$, then $\mathbf{x}$ is a *parking sequence* for $\mathbf{y}$. We denote the set of parking sequences for $\mathbf{y}$ by $\mathsf{PS}_n(\mathbf{y})$. Figure 1 contains an example to illustrate the parking process.

Our first main result establishes a way to count the number of parking sequences for a given length vector $\mathbf{y}$[1]. To this effect, we let $\mathfrak{S}_n$ denote the set of permutations on $[n]$ and for a fixed $\mathbf{y}$ we define the outcome map $\mathcal{O}_{\mathsf{PS}_n(\mathbf{y})} : \mathsf{PS}_n(\mathbf{y}) \to \mathfrak{S}_n$ by $\mathcal{O}_{\mathsf{PS}_n(\mathbf{y})}(\mathbf{x}) = \sigma = \sigma_1\sigma_2 \cdots \sigma_n$ and, given $\sigma \in \mathfrak{S}_n$, we study the fibers of the outcome map:

$$\mathcal{O}^{-1}_{\mathsf{PS}_n(\mathbf{y})}(\sigma) = \{\mathbf{x} \in \mathsf{PS}_n(\mathbf{y}) : \mathcal{O}_{\mathsf{PS}_n(\mathbf{y})}(\mathbf{x}) = \sigma\}.$$

Theorem 1. *Fix $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$ and $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$. Then*

$$|\mathcal{O}^{-1}_{\mathsf{PS}_n(\mathbf{y})}(\sigma)| = \prod_{i=1}^{n} \left(1 + \sum_{k \in L(\mathbf{y}, \sigma_i)} y_k\right),$$

and

$$|\mathsf{PS}_n(\mathbf{y})| = \sum_{\sigma \in \mathfrak{S}_n} |\mathcal{O}^{-1}_{\mathsf{PS}_n(\mathbf{y})}(\sigma)|$$

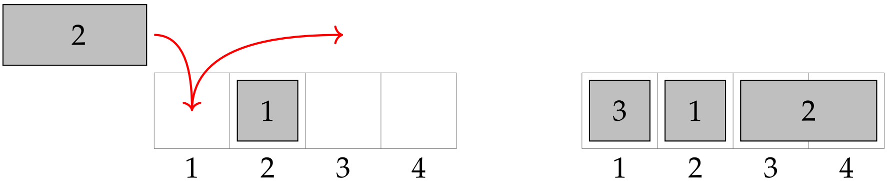
where $L(\mathbf{y}, \sigma_i) = \emptyset$ if $i = 1$ or if $\sigma_{i-1} > \sigma_i$, otherwise $L(\mathbf{y}, \sigma_i) = \{\sigma_t, \sigma_{t+1}, \ldots, \sigma_{i-1}\}$ with $\sigma_t\sigma_{t+1} \ldots \sigma_i$ being the longest subsequence of σ such that $\sigma_k < \sigma_i$ for all $t \leq k < i$.

Additionally, we introduce an enumerative formula for a generalization of parking sequences, called *parking assortments* (introduced in [1]).
Parking assortments are defined similarly to parking sequences, with cars of varying

[1]Proofs for all following results are available in the arXiv preprint, see [2]

lengths. The difference to parking sequences is the more intuitive rule of a car moving forward until it finds a gap large enough to fit in, instead of leaving the street if the first unoccupied spot is not followed by enough free spots to fit the entire car length.

We let $\mathsf{PA}_n(\mathbf{y})$ denote the set of all parking assortments for $\mathbf{y}$. Figure 1 illustrates an example.



$\mathbf{x} = (2,1,1) \notin \mathsf{PS}_3(\mathbf{y})$ $\quad\quad$ $\mathbf{x} = (2,1,1) \in \mathsf{PA}_3(\mathbf{y})$

Figure 1: Let $\mathbf{y} = (1,2,1)$. We show, that $\mathbf{x} = (2,1,1) \in \mathsf{PA}_3(\mathbf{y})$, but $\mathbf{x} \notin \mathsf{PS}_3(\mathbf{y})$. It is obvious, that for example $(2,2,1), (3,1,4) \in \mathsf{PS}_3(\mathbf{y})$.

We consider the analogous study of the set of parking assortments resulting in a particular parking order. To make this precise, we fix $\mathbf{y}$, and define the outcome map $\mathcal{O}_{\mathsf{PA}_n(\mathbf{y})} : \mathsf{PA}_n(\mathbf{y}) \to \mathfrak{S}_n$ by $\mathcal{O}_{\mathsf{PA}_n(\mathbf{y})}(\mathbf{x}) = \sigma = \sigma_1\sigma_2\cdots\sigma_n$, where $\sigma_j = i$ denotes that car i is the jth car parked on the street.

To handle the different cases that can occur with the increased complexity, we introduce the following definition to simplify our results.

Definition 2. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$. Fix $i \in [n]$ and partition the subword $T(\sigma_i) := \sigma_1\sigma_2\cdots\sigma_{i-2}\sigma_{i-1}$ as follows:

1. If $i = 1$, then $T(\sigma_1) = \emptyset$.

2. If $\sigma_{i-1} > \sigma_i$, then

$$T(\sigma_i) = \sigma_1\sigma_2\cdots\sigma_{i-2}\sigma_{i-1} = \begin{cases} \beta_\ell\alpha_\ell\cdots\beta_2\alpha_2\beta_1\alpha_1 & \text{if } \sigma_1 < \sigma_i \\ \alpha_{\ell+1}\beta_\ell\alpha_\ell\cdots\beta_2\alpha_2\beta_1\alpha_1 & \text{if } \sigma_1 > \sigma_i \end{cases} \tag{1}$$

   where α_1 is the longest contiguous subword consisting of $\sigma_j > \sigma_i$ and β_1 is the longest contiguous subword consisting of $\sigma_j < \sigma_i$. We iterate in this way until we consider σ_1 arriving at one of the two cases in (1).

3. If $\sigma_{i-1} < \sigma_i$, then

$$T(\sigma_i) = \sigma_1\sigma_2\cdots\sigma_{i-2}\sigma_{i-1} = \begin{cases} \alpha_\ell\beta_\ell\cdots\alpha_2\beta_2\alpha_1\beta_1 & \text{if } \sigma_1 > \sigma_i \\ \beta_{\ell+1}\alpha_\ell\beta_\ell\cdots\alpha_2\beta_2\alpha_1\beta_1 & \text{if } \sigma_1 < \sigma_i \end{cases} \tag{2}$$

   where β_1 is the longest contiguous subword consisting of $\sigma_j < \sigma_i$ and α_1 is the longest contiguous subword consisting of $\sigma_j > \sigma_i$. We iterate in this way until we consider σ_1 arriving at one of the two cases in (2).

We can now formally state the analogous result to Theorem 1 for parking assortments.

Theorem 3. *Fix* $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$ *and let* $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$*. Then*

$$|\mathcal{O}^{-1}_{\mathsf{PA}_n(\mathbf{y})}(\sigma)| = \prod_{i=1}^{n} |\mathsf{Pref}_{\mathsf{PA}_n}(\sigma_i)|,$$

where

$$|\mathsf{Pref}_{\mathsf{PA}_n}(\sigma_i)| = \begin{cases} 1 & \text{if } i = 1 \text{ or } \sigma_{i-1} > \sigma_i \\ 1 + \sum\limits_{\sigma_k \in \beta_1} y_{\sigma_k} & \text{if } T(\sigma_i) = \beta_1 \\ 1 + \sum\limits_{k=1}^{i-1} y_{\sigma_k} & \text{if } m(i) \text{ does not exist} \\ \sum\limits_{\sigma_k \in \beta_{m(i)}\alpha_{m(i)-1}\beta_{m(i)-1}\cdots\alpha_1\beta_1\sigma_i} y_{\sigma_k} & \text{if } m(i) \text{ exists} \end{cases} \tag{3}$$

with

$$m(i) = \min\left\{1 \le j \le \ell : \sum_{\sigma_k \in \alpha_j} y_{\sigma_k} \ge y_{\sigma_i}\right\}.$$

With the obtained enumerative formulas, we are able to examine the behavior of the number of parking sequences and assortments for a fixed $\mathbf{y}$. We show interesting results, which we obtained using well-known sequences for $\mathbf{y}$ and letting the number of cars increase. I.e., if $\mathbf{y} = (1, 1, 2, 3, 5, 8, \ldots)$ consists of the first n Fibonacci numbers, then the cardinality of $\mathcal{O}^{-1}_{\mathsf{PA}_n}(123\cdots n)$ as n increases is

$$1, 2, 6, 30, 240, 3120, 65520, 2227680, \ldots.$$

This sequence agrees with OEIS A003266: The product of the first $n+1$ nonzero Fibonacci numbers. We conclude by providing directions for future work.

References

[1] (2022). Chen, D. M., Harris, P. E., Martínez Mori, J. C., Pabón-Cancel, E. J., and Sargent, G. Permutation Invariant Parking Assortments. *arXiv preprint arXiv:2211.01063*.

[2] (2023) Franks S. J., Harris P. E., Harry K., Kretschmann J., Vance M. Counting Parking Sequences and Parking Assortments Through Permutations. *arXiv preprint arXiv:2301.10830*

[3] (2021). Colmenarejo, L., Harris, P. E., Jones, Z., Keller, C., Ramos Rodríguez, A., Sukarto, E., and Vindas-Meléndez, A. R. Counting k-Naples parking functions through permutations and the k-Naples area statistic. *Enumer. Comb. Appl.* 1(2): Paper No. S2R11, 16.

[4] (2016) Ehrenborg, R. and Happ, A. Parking Cars of Different Sizes. *Amer. Math. Monthly*. 123(10): 1045–1048.

[5] (1969). Riordan, J. Ballots and trees. *J. Combinatorial Theory*, 6:408–411.

Occurrences of subsequences in binary words

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This talk is based on joint work with Anurag Singh

The numbers we study in this paper are of the form $B_{n,p}(k)$, which is the number of binary words of length n that contain the word p (as a subsequence) exactly k times. Our motivation comes from the analogous study of pattern containment in permutations, especially [1]. In our first set of results, we obtain explicit expressions for $B_{n,p}(k)$ for small values of k. We then focus on words p with at most 3 runs and study the maximum number of occurrences of p a word of length n can have. We also study the internal zeros in the sequence $(B_{n,p}(k))_{k\geq 0}$ for fixed n and discuss the unimodality and log-concavity of such sequences.

Preliminaries

A *binary word* is a finite sequence $w = w_1w_2\cdots w_n$ where $w_i \in \{0,1\}$ for all $i \in [n]$. Here n is called the *length* of w. A *run* in a binary word is a maximal subsequence of consecutive terms that are equal. For instance, the word $11100001 = 1^30^41^1$ has three runs, which are of sizes 3, 4, and 1 respectively.

An *occurrence* of a binary word $p = p_1p_2\cdots p_l$ in a binary word $w = w_1w_2\cdots w_n$ is a subsequence of w that matches p, i.e., a choice of indices $1 \leq i_1 < i_2 < \cdots < i_l \leq n$ such that $w_{i_1}w_{i_2}\cdots w_{i_l} = p$. In this context, we usually call p a *pattern*. Analogous to the notation of [1], we denote the number of occurrences of the pattern p in w by $c_p(w)$. For example, $c_{10}(10010) = 4$.

For any binary word p and $n,k \geq 0$, we define $B_{n,p}(k)$ to be the number of binary words of length n that have exactly k occurrences of the pattern p. That is,

$$B_{n,p}(k) = \#\{w \in \{0,1\}^n \mid c_p(w) = k\}.$$

Just as for permutation patterns, we say that two patterns are *trivially equivalent* if one can be obtained from the other using reversal and complementation operations. One can check that if p and q are trivially equivalent, then $B_{n,p}(k) = B_{n,q}(k)$ for all $n,k \geq 0$.

For use in the sequel, we also recall the following definitions. A sequence of non-negative integers $(a_k)_{k=0}^m$ is said to have an *internal zero* if there exist $0 \leq k_1 < k_2 < k_3 \leq m$ such that $a_{k_1}, a_{k_3} \neq 0$ but $a_{k_2} = 0$. The sequence is said to be *unimodal* if there exists an $i \in [0,m]$ such that $a_0 \leq a_1 \leq \cdots \leq a_i \geq a_{i+1} \geq \cdots \geq a_m$. The sequence is said to be *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i \in [m-1]$.

Few occurrences of a pattern

Let p be a binary word of length l that has r runs, r_i of which are of size i for each $i \geq 1$. We have the following expressions for $B_{n,p}(k)$ for small values of k.

Proposition 1. *For any $n \geq 0$, we have the following.*

- $B_{n,p}(0) = \sum\limits_{j=0}^{l-1} \binom{n}{j}$.
- $B_{n,p}(1) = \binom{n-r+1}{l-r+1}$.
- $B_{n,p}(2) = r_1 \binom{n-r}{l-r+1}$ *if* $l \geq 2$, *and* $B_{n,p}(2) = \binom{n}{2}$ *if* $l = 1$.

We obtain the results for $k \geq 1$ by considering the spaces between the letters of the pattern p as slots and studying how inserting letters into these slots affects the number of occurrences of p. Similar ideas can be used to obtain expressions for $B_{n,p}(3)$ and $B_{n,p}(4)$ as well [2].

For example, if $p = \texttt{100011}$ then all words w such that $c_p(w) = 1$ can be obtained by adding appropriate letters in the slot diagram below.

$$\underset{0}{\sqcup}\ ^{1}\ \sqcup\ ^{0}\ \underset{1}{\sqcup}\ ^{0}\ \underset{1}{\sqcup}\ ^{0}\ \sqcup\ ^{1}\ \underset{0}{\sqcup}\ ^{1}\ \underset{0}{\sqcup}$$

Here the letters under the slots represent what types of letters can be inserted in them. For example, inserting appropriate letters, we get the word 00100101001 that contains exactly one occurrence of p (which has been highlighted).

We also note the following result which is easy to verify by studying how many occurrences of p there are in a word obtained by adding a letter to p.

Lemma 2. *For any $k \geq 2$, we have $B_{l+1,p}(k) = r_{k-1}$.*

Definition 3. We say that two patterns p, q are *strong Wilf-equivalent* if $B_{n,p}(k) = B_{n,q}(k)$ for all $n, k \geq 0$.

Clearly, strong Wilf-equivalent patterns must have the same length. A consequence of Lemma 2 is that two strong Wilf-equivalent patterns must have the same number of runs of each size. We have already noted that trivially equivalent patterns are strong Wilf-equivalent. Computations suggest that these are the only strong-Wilf equivalences.

Conjecture 4. *The patterns p and q are strong Wilf-equivalent if and only if they are trivially equivalent.*

We have verified the above conjecture for patterns of length up to 13 using Sage [3].

Maximum occurrences and internal zeroes

Given $n \geq 0$ and a pattern p, we set $M_{n,p}$ to be maximum possible number of occurrences of p in a binary word of length n. Hence,

$$M_{n,p} = \max\{c_p(w) \mid w \in \{0,1\}^n\} = \max\{k \mid B_{n,p}(k) \neq 0\}.$$

A binary word w of length n is said to be *p-optimal* if $c_p(w) = M_{n,p}$. We have the following result for patterns that have at most 3 runs. Note that any such pattern is trivially equivalent to a pattern of the form mentioned below.

Theorem 5. *Let $p = \mathtt{1}^i\mathtt{0}^j\mathtt{1}^k$ for some $i, k \geq 0$ and $j \geq 1$.*

- *For any $n \geq 0$, there exists a p-optimal word of length n that has the same number of runs as p. Hence, we have*

$$M_{n,p} = \max\left\{\binom{a}{i}\binom{b}{j}\binom{c}{k} \mid a+b+c = n\right\}.$$

- *The sequence $(M_{n,p})_{n\geq 0}$ is log-concave.*
- *If $\mathtt{1}^a\mathtt{0}^b\mathtt{1}^c$ is p-optimal, so is at least one word in $\{\mathtt{1}^{a+1}\mathtt{0}^b\mathtt{1}^c, \mathtt{1}^a\mathtt{0}^{b+1}\mathtt{1}^c, \mathtt{1}^a\mathtt{0}^b\mathtt{1}^{c+1}\}$.*

The key points behind the proof of the above theorem are that $\mathtt{0}$s play a special role when finding occurrences of p and that the binomial coefficients are log-concave.

Definition 6. A binary word p is said to have an *internal zero* at n if the sequence $(B_{n,p}(k))_{k\geq 0}$ has an internal zero.

For example, Lemma 2 shows that if p is of length l with maximum run size i, then p does not have an internal zero at $l+1$ if and only if p has a run of size j for all $j \in [i]$. We have the following result for patterns with at most 3 runs, where we say that a binary word is *alternating* if all its runs are of size 1.

Theorem 7. *Let p be a binary word of length l with at most 3 runs.*

- *If p is alternating, then p does not have an internal zero at any $n \geq 7$.*
- *If p is not alternating, then p has an internal zero at all $n \geq l+3$. In fact, we have $B_{n,p}(M_{n,p}-1) = 0$.*

Since sequences that have internal zeroes cannot be unimodal, when p has at most 3 runs, the sequence $(B_{n,p}(k))_{k\geq 0}$ can be unimodal only if p is alternating. If $p = \mathtt{0}$ or $\mathtt{1}$, this sequence is just $(\binom{n}{k})_{k\geq 0}$ which is not only unimodal, but log-concave. However, if p is an alternating pattern of length 2 or 3, using Proposition 1 we have $B_{n,p}(0) > B_{n,p}(1) < B_{n,p}(2)$ for $n \geq 4$ and hence the sequence $(B_{n,p}(k))_{k\geq 0}$ is not unimodal.

References

[1] M. Bóna, B. E. Sagan, and V. R. Vatter. Pattern frequency sequences and internal zeros. vol.28 (2002), p.395-420. Special issue in memory of Rodica Simion.

[2] K. Menon and A. Singh. Subsequence frequency in binary words. *in preparation*, 2023.

[3] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.5.0)*, 2022. `https://www.sagemath.org`

Increasing Schröder trees and restricted permutations

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In this resume we explore links between Increasing Schröder trees and permutations in term of nodes, cycles and depth.

Increasing Schröder trees

Increasing Schröder trees have been introduced as a model of phylogenetic trees to represent the evolutionary relationship between species. They have been studied in [2, 1] from different perspectives.

We define rooted trees as genealogical structures: the root is the unique common ancestor of all nodes of the tree, each node except the root has exactly one parent (the root has no parent), nodes that have no children are called *leaves*, nodes that have at least one child are called *internal nodes*. The *arity* of a node is it's number of children. We say that a tree is *plane* if siblings (nodes that have the same parent) are ordered.

Definition 1. A Schröder tree is a rooted plane tree whose internal nodes all have arity at least 2. The size of a Schröder tree is its number of leaves.

Schröder trees were introduced by Ernst Schröder in [3] and we are interested in an increasingly-labelled variation of Schröder trees.

Definition 2. An *increasing Schröder tree* has a Schröder tree structure and its internal nodes are labelled with the integers between 1 and ℓ (where ℓ is the number of internal nodes) in such a way that each label appears exactly once and each sequence of labels in the paths from the root to any leaf is strictly increasing.

We will denote by $\mathcal{T}$ the class of increasing Schröder trees. We now introduce an evolution process generating *increasing Schröder trees*:

- *Start with a single (unlabelled) leaf;*
- Iterate the following process: *at step ℓ (for $\ell \geq 1$), select one leaf and replace it by an internal node with label ℓ attached to an arbitrary sequence of at least 2 new leaves.*

Examples of trees from this process are depicted on Figure 1. It is not hard to find out a recurrence for the number of trees t_n of size n (having n leaves)

$$t_n = \begin{cases} 1, & n = 1, 2 \\ n\, t_{n-1}, & n \geq 3 \end{cases} \tag{1}$$

Then, the first values of t_n are

$$(t_n)_{n\geq 0} = 0, 1, 1, 3, 12, 60, 360, 2520, 20160, 181440, 1814400, \ldots$$

Theorem 3. *The number of Increasing Schröder trees of size n for all* $n \geq 2, t_n = \frac{n!}{2}$.

This last Theorem suggests a link between these trees and permutations. In the subsequent we will exhibit two bijections that show their links.

Bijection with cycles in permutations and relationship to internal nodes

The idea here is to present a bijection with permutations based on their number of cycles. Let us define the class $\mathcal{PR}$ (permutations with cycle restriction) to be the set of permutations in which the elements 1 and 2 belong to different cycles.

Lemma 4. *The number of permutations in* $\mathcal{PR}$ *of size n for* $n \geq 2, PR_n = \frac{n!}{2}$.

In this section, let σ be an n-permutation with k cycles. Then we denote

$$\sigma = c_1 \circ c_2 \circ \cdots \circ c_k,$$

as a product of cycles c_i. A cycle can be ordered canonically by putting the smallest integer in the beginning and by extension the product of cycles can ordered by the smallest element of each cycle. Let $c = (a_1, a_2, \ldots, a_i)$ be a cycle that contains i elements. A new element e can be added in i different places. We also denote by $c_{i,j}$ the element of position j in the cycle i.

Let us define also $\sigma^{-1}(k)$ the function that returns a pair of integers (i, j) where i is the cycle number that contains integer k and j the position of integer k in the cycle c_i. Finally let $|c_j|$ be the number of elements in the cycle c_j.

Finally we define $\sigma \backslash n$ to be the permutation σ from which the element n has been removed. We will define the mapping $\mathcal{N} : \mathcal{PR} \to \mathcal{T}$ and show that

$$\mathcal{PR} \overset{\mathcal{N}}{\cong} \mathcal{T}$$

Definition 5. We define the mapping $\mathcal{N}$ recursively as follow:

- If $\sigma = (1)(2)$ then $\mathcal{N}(\sigma)$ is the tree which is a binary root labelled 1.
- Else, Let $(i, j) = \sigma^{-1}(n)$ where n is the largest element in σ.
  - If $|c_i| = 1$ then, we set $\mathcal{N}(\sigma)$ to be the tree $\mathcal{N}(\sigma \backslash n)$ in which we add a new rightmost leaf to the last internal node of the tree (it is also the node with highest label).
  - Else, let $k = c_{i,j-1}$, we set $\mathcal{N}(\sigma)$ to be the tree $\mathcal{N}(\sigma \backslash n)$ in which a new binary node labelled ν with the smallest integer that does not appear as a label in $\mathcal{N}(\sigma \backslash n)$ and attach two new leaves to this internal node. Insert this binary node in $\mathcal{N}(\sigma \backslash n)$ by placing ν in the k-th leaf (we assume, for example, that the leaves are ordered in the depth-first order) of $\mathcal{N}(\sigma \backslash n)$.

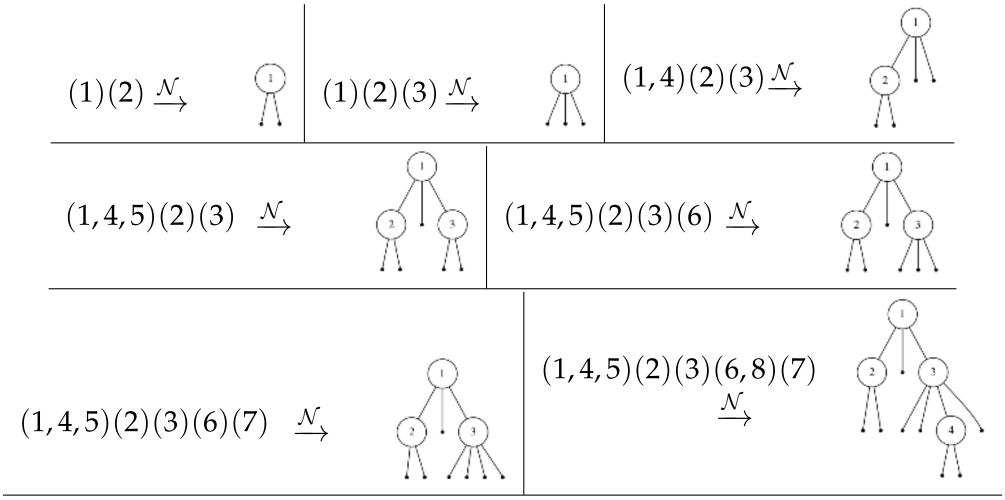

Figure 1: A size-8 example of the mapping $\mathcal{N}$

Figure 1 shows an example of this mapping.

Theorem 6. *The map $\mathcal{N}$ is a one-to-one correspondence between $\mathcal{PR}$ and $\mathcal{T}$.*

Theorem 7. *Let t be an increasing Schröder tree with n leaves and k internal nodes, and let $p = \mathcal{N}^{-1}(t)$ be its corresponding permutation. Let i be the number of cycles of p, then*

$$k = n + 1 - i.$$

We can find a similar connection between the cycles in a permutation and the depth of the leftmost leaf of the tree by defining a mapping $\mathcal{O} : \mathcal{PR} \to \mathcal{T}$ that is a one-to-one correspondence between $\mathcal{PR}$ and $\mathcal{T}$ based on the depth of the leftmost leaf rather than the number of internal nodes and we obtain the following.

Theorem 8. *Let t be an increasing Schröder tree with n leaves and k is the depth of the leftmost leaf, and let $p = \mathcal{O}^{-1}(t)$. Let i be the number of cycles of p, then*

$$k = i - 1.$$

References

[1] Olivier Bodini, Antoine Genitrini, Cécile Mailler, and Mehdi Naima. Strict monotonic trees arising from evolutionary processes: combinatorial and probabilistic study. *Advances in Applied Mathematics*, 133:102284, 2022.

[2] Olivier Bodini, Antoine Genitrini, and Mehdi Naima. Ranked Schröder Trees. In *2019 Proceedings of the Sixteenth Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*, pages 13–26. SIAM, 2019.

[3] Ernst Schröder. Vier combinatorische Probleme. *Zeitschrift für Mathematik und Physik*, 15:361–376, 1870.

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This talk is based on joint work with Vít Jelínek

Permutations can be viewed as pairs of linear orders, or more formally as models over a signature consisting of two binary relation symbols. This approach was adopted by Albert, Bouvel and Féray [1], who studied the expressibility of first-order logic in this setting. In this talk, we will focus our attention on monadic second-order logic.

Permutations as models

A *signature* is a set of relation and function symbols, each associated with an arity, which is a non-negative integer that represents the number of arguments the symbol can take. For the purposes of this talk, we are focused on the signature $\mathcal{S}_{\mathsf{TO}}$, which consists of two binary relation symbols $<_1$ and $<_2$. With this signature, any permutation can be represented as a finite structure, which is a set equipped with two linear orders: one describing the ordering of the indices, and the other describing the ordering of the values. In simpler terms, permutations are finite models of The *Theory of Two Orders (TOTO)*, a theory that requires both $<_1$ and $<_2$ to be linear orders.

A *first-order (FO) logic formula* is a logical expression built from atomic formulas, logical connectives (such as negation, conjunction, disjunction, implication, and equivalence), and quantifiers (both existential and universal). Atomic formulas over the signature $\mathcal{S}_{\mathsf{TO}}$ are either equality predicates ($x = y$), or predicates formed from the two relation symbols, i.e., $x <_1 y$ or $x <_2 y$. A *monadic second-order (MSO) formulas* then extend FO formulas by introducing set variables (denoted by capital letters), quantifications over them and set membership predicates ($x \in X$).

Example 1. We can define a predicate $\mathit{partition}(X, Y)$ enforcing that X and Y form a partition of the domain, and predicates $\mathit{increasing}(X)$ and $\mathit{decreasing}(X)$ enforcing that X is an increasing, respectively decreasing subsequence as follows

$$\begin{aligned}
\mathit{partition}(X,Y) &= \forall x \ [(x \in X \vee x \in Y) \wedge \neg(x \in X \wedge x \in Y)]\,,\\
\mathit{increasing}(X) &= \forall x,y\,[(x \in X \wedge y \in X) \to (x <_1 y \leftrightarrow x <_2 y)]\,,\\
\mathit{decreasing}(X) &= \forall x,y\,[(x \in X \wedge y \in X) \to (x <_1 y \leftrightarrow y <_2 x)]\,.
\end{aligned}$$

Using these predicates, we can define permutations obtained as a union of an increasing and a decreasing sequence (known as *skew-merged permutations*) by the MSO sentence $\exists X \exists Y\, (\mathit{partition}(X,Y) \wedge \mathit{increasing}(X) \wedge \mathit{decreasing}(Y))$.

Expressive power of MSO logic

We may easily generalize Example 1 to permutations that can be obtained as a union of k permutations, each coming from an arbitrary set defined by a fixed MSO sentence.

We say that a permutation π is a *merge* of permutations σ and τ if we can color the points of π with colors red and blue so that the red points are isomorphic to σ and the blue ones to τ. The *merge* of a class $\mathcal{C}$ and a class $\mathcal{D}$ is the class $\mathcal{C} \odot \mathcal{D}$ of permutations that can be obtained by merging an element of $\mathcal{C}$ with an element of $\mathcal{D}$. Thus, we show that a merge of a finitely many permutation classes, each definable by an MSO sentence, is itself MSO-definable.

Proposition 2. *For arbitrary MSO sentences $\varphi_1, \dots, \varphi_k$ in TOTO, we can construct an MSO sentence ρ such that π satisfies ρ if and only if π can be obtained as a merge of permutations $\pi_1, \dots, \pi_k$ such that π_i satisfies φ_i for every $i \in [k]$.*

On the other hand, we show that there are merges that are not definable by an FO sentence. In particular, there exists no FO sentence describing a merge of two copies of the same class avoiding a fixed simple pattern. To show this, we use a standard tool for proving inexpressibility in logic – the Ehrenfeucht–Fraïssé games.

Theorem 3. *Let α be a simple permutation of length at least 4. The class $\mathrm{Av}(\alpha) \odot \mathrm{Av}(\alpha)$ is not definable by an FO sentence in TOTO.*

Albert et al. [1] proved that the property of having a fixed point cannot be expressed by a FO sentence in TOTO. We strengthen their result by showing that this property is not even expressible by an MSO sentence in TOTO.

Proposition 4. *The property of having a fixed point is not MSO-definable in TOTO.*

MSO model checking

Additionally, we investigate the complexity of deciding MSO formulas of TOTO. On the positive side, we show that there exists an FPT algorithm for this problem parameterized by a parameter called grid-width and the length of the given formula. This is shown in two steps. First, we encode the permutation π as a labeled graph whose clique-width is at most four times the grid-width of π. Subsequently, we employ the classical tool for MSO model checking on graphs – Courcelle's theorem [2].

Theorem 5. *Given a permutation π of size n and an MSO sentence φ in TOTO, we can decide if π satisfies φ in time $f(|\varphi|, \mathrm{gw}(\pi)) \cdot n$ for some computable function f.*

On the other hand, we complement this algorithm with a negative result showing that checking MSO sentences on permutations is hard even when the permutations are restricted to most permutation classes. More precisely, we show that checking MSO sentences on permutations from any class with the so-called poly-time computable long path property is as hard as checking MSO sentences on unrestricted graphs. Let Theory of Graphs (TOG) be the theory capturing all undirected graphs over a signature with a single binary relation symbol E.

Theorem 6. *Let $\mathcal{C}$ be a permutation class with the poly-time computable long path property. There is a polynomial time algorithm that given a graph G on n vertices and an MSO sentence φ in TOG, computes a permutation $\pi \in \mathcal{C}$ of length $O(n^2)$ and an MSO sentence ψ in TOTO of length $O(|\varphi|)$ such that G satisfies φ if and only if π satisfies ψ.*

Note that we conjecture that a permutation class $\mathcal{C}$ has the long path property (dropping the computability requirements) if and only if $\mathcal{C}$ has unbounded grid-width. If the conjecture holds, we get a nice dichotomy of the complexity of MSO model checking inside a fixed permutation class.

References

[1] M. Albert, M. Bouvel, V. Féray. Two first-order logics of permutations. *J. Combin. Theory Ser. A* 171, 105158, 46 (2020).

[2] B. Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Inform. and Comput.* 85(1), 12–75 (1990).

On the Rajchgot index of permutations

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In this talk, we give some new combinatorial properties of the Rajchgot index, introducing two new classes of permutations related to dominant permutations and fireworks permutations, enumerating them, and beginning a generating function approach to understanding the distribution of the Rajchgot index over various classes of permutations. In doing so, we draw a connection to the well-studied q-Stirling numbers.

Introduction

Given a matrix Schubert variety X_w for $w \in \mathfrak{S}_n$, an important question at the intersection of algebraic combinatorics and algebraic geometry was to compute the Castelnuovo–Mumford regularity of X_w. This question was simplified by Rajchgot, Ren, Robichaux, St. Dizier, and Weigandt [9] to that of computing the degree of the corresponding Grothendieck polynomial $\mathfrak{G}_w(\mathbf{x})$. The question was then answered completely by Pechenik, Speyer, and Weigandt [8] who showed that the Castelnuovo–Mumford regularity of X_w is given by the quantity

$$\mathrm{raj}(w) - \mathrm{inv}(w) \tag{1}$$

where $\mathrm{inv}(w)$ is the number of inversions of w and $\mathrm{raj}(w)$ is the **Rajchgot index** of w, a novel permutation statistic introduced in the same paper. We recall this definition here.

Definition 1 ([8]). The **Rajchgot code** of a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ written in one-line notation is the tuple of nonnegative integers $(r_1, \ldots, r_n)$ defined as follows. For each i, find a longest increasing subsequence σ_i of w starting at (and including) w_i. Then r_i is the number of letters of w following w_i which do not appear in the subsequence σ_i:

$$r_i := \#\{\, w_j \,|\, j > i \text{ and } w_j \notin \sigma_i \,\}$$

We write $\mathrm{rajcode}(w) = (r_1, \ldots, r_n)$. The **Rajchgot index** of w is the sum of the entries in its Rajchgot code: $\mathrm{raj}(\omega) := \sum_{i=1}^{n} r_i$.

The goal of our study is to further explore this permutation statistic in a purely combinatorial way. In particular, we do this by pursuing two different directions: in the first, we give a few enumerative results on certain special classes of permutations; in the second, we investigate generating functions of the Rajchgot index and Rajchgot code to obtain various results. In what follows, we give an overview of these two directions and an outline for the rest of the paper.

Enumeration

A permutation is called a **fireworks permutation** if the maximum elements of the descending chains are in increasing order [8, Definition 3.5]. We denote the set of fireworks permutation in $\mathfrak{S}_n$ by $\mathcal{F}_n$. The cardinality of the set $\mathcal{F}_n$ is given by the nth **Bell number**, B_n, the number of set partitions of $[n]$ [8]. Further recall that the **major code** of a permutation w in $\mathfrak{S}_n$ is the tuple $(m_1, \ldots, m_n)$ where m_i is:

$$m_i := \#\{j \geq i \mid w_j > w_{j+1}\}.$$

The sum of all m_i in the major code is called the **major index** of w.

One of the main results connecting the raj and maj statistics is the following:

Proposition 2 ([8, Prop. 3.8])**.** *Given a permutation $w \in \mathfrak{S}_n$, we have $maj(w) = raj(w)$ if and only if w is a fireworks permutation. Furthermore, $r_i = m_i$ for all i.*

In a different direction, recall that the **inversion code** of a permutation $w \in \mathfrak{S}_n$ is the tuple $(\ell_1, \ldots, \ell_n)$ where

$$\ell_i := \#\{j > i \mid w_j < w_i\}.$$

The sum of all ℓ_i equals the number of inversions of w and is denoted $\mathrm{inv}(w)$. Moreover, we say that a permutation is a **dominant permutation** if it is 132-avoiding. The set of 132-avoiding permutations of length n is enumerated by the nth Catalan number, $Cat_n := \frac{1}{n+1}\binom{2n}{n}$ (c.f. [7]).

One of the main results relating the raj and inv statistics is the following:

Proposition 3 ([8, Prop. 3.2])**.** *Given a $w \in \mathfrak{S}_n$, $inv(w) = raj(w)$ if and only if w is dominant. Furthermore, $\ell_i = r_i$ for all i.*

We extend the results of Proposition 2 and Proposition 3 to the case that raj and maj (respectively raj and inv) differ by one instead of by zero.

Theorem 4. *Let $\widetilde{\mathcal{F}}_n$ be the set of permutations w in $\mathfrak{S}_n$ where $raj(w) - maj(w) = 1$. Then:*

$$|\widetilde{\mathcal{F}}_n| = nB_{n-1} - B_n. \tag{2}$$

This is the sequence [5, A278677].

Theorem 5. *Let $\widetilde{D}_n$ be the set of permutations in $\mathfrak{S}_n$ where $raj(w) - inv(w) = 1$. Then:*

$$|\widetilde{D}_n| = \binom{2n-2}{n+1} + \binom{2n-3}{n+1} = Cat_{n+1} \cdot \frac{n(1+3n)}{6+2n}. \tag{3}$$

This is the sequence [5, A265612].

We call elements of $\widetilde{\mathcal{F}}_n$ **almost-fireworks permutations** and elements of $\widetilde{D}_n$ **almost-dominant permutations**. In order to prove 5 we also give a characterization of almost-dominant permutations. We likewise give a classification of almost-fireworks permutation.

Generating functions

Our other direction of investigation pertains to generating functions that keep track of the Rajchgot index for classes of permutations. For this we set up the following notation:

Definition 6. Let S be a set of permutations. We define $R_S(q)$ to be the sum:

$$R_S(q) := \sum_{w \in S} q^{\mathrm{raj}(w)}.$$

In most of the paper we will take S to be a finite subset of $\mathfrak{S}_n$, so the sum is guaranteed to exist. The classes of permutations S that we give the most attention to are $\mathcal{F}_n$ and $\mathfrak{S}_n$ itself. Our first result in this direction motivates the utility of our generating function approach by showing that the specialization $S = \mathcal{F}_n$ recovers the well-studied q-Bell numbers:

Theorem 7.

$$R_{\mathcal{F}_n}(q) = B_n(q^{-1}) \cdot q^{\binom{n}{2}}$$

In other words, $R_{\mathcal{F}_n}(q)$ is the q-Bell number with reversed coefficients.

We show that the number of permutations in $\mathfrak{S}_\infty$ with a fixed Rajchgot index is finite, guaranteeing that the generating function $R_S(q)$ exists for any subset $S \subset \mathfrak{S}_\infty$. The classes of permutation we consider all descend nicely to subsets of $\mathfrak{S}_\infty$, so we compute the first few values of the generating functions in the case that $S \in \{\mathfrak{S}_\infty, \mathcal{F}_\infty, \widetilde{\mathcal{F}}_\infty, D_\infty, \widetilde{D}_\infty\}$.

References

[1] Miklós Bóna. Permutations with one or two 132-subsequences. *Discrete Mathematics*, 181:267 – 274, 1998.

[2] L. Carlitz. q-Bernoulli numbers and polynomials. *Duke Mathematical Journal*, 15(4):987 – 1000, 1948.

[3] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.7)*.

[4] A. M Garsia and J. B Remmel. Q-counting rook configurations and a formula of frobenius. *Journal of Combinatorial Theory, Series A*, 41(2):246–275, 1986.

[5] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. https://oeis.org.

[6] Pierre Leroux. Reduced matrices and q-log-concavity properties of q-stirling numbers. *Journal of Combinatorial Theory, Series A*, 54(1):64–84, 1990.

[7] L. Manivel. *Symmetric Functions, Schubert Polynomials and Degeneracy Loci*. Collection SMF. American Mathematical Society, 2001.

[8] Oliver Pechenik, David E. Speyer, and Anna Weigandt. Castelnuovo-mumford regularity of matrix schubert varieties. 2021.

[9] Jenna Rajchgot, Yi Ren, Colleen Robichaux, Avery St. Dizier, and Anna Weigandt. Degrees of symmetric grothendieck polynomials and castelnuovo-mumford regularity. 149:1405–1416, 2021.

[10] Bruce E. Sagan. A maj statistic for set partitions. *European Journal of Combinatorics*, 12(1):69–79, 1991.

[11] Bruce E. Sagan and Joshua P. Swanson. q-stirling numbers in type b.

[12] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics.

Descent distribution on Catalan words avoiding ordered pairs of relations

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This talk is based on joint work with Jean-Luc Baril, Université de Bourgogne, France

The word $w = w_1 w_2 \cdots w_n$ over the set of positive integers with $w_1 = 1$ and $1 \leq w_i \leq w_{i-1} + 1$, for $i = 2, \ldots, n$ is called a *Catalan word*. This type of words are enumerated by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. In this talk, we present several enumerative results for Catalan words avoiding a pair of consecutive patterns of length 3. More precisely, we provide a systematic bivariate generating function for the total number of Catalan words avoiding a given pair of relations with respect to the length and the number of descents. We also present several constructive bijections preserving the number of descents. As a byproduct, we deduce the generating function for the total number of descents on all Catalan words of a given length avoiding a pair of ordered relations.

References

[1] J.-L. Baril, S. Kirgizov, and V. Vajnovszki, Descent distribution on Catalan words avoiding a pattern of length at most three, Discrete Math. **341** (2018), 2608–2615.

[2] J.-L. Baril, C. Khalil, and V. Vajnovszki, Catalan words avoiding pairs of length three patterns, Discret. Math. Theor. Comput. Sci. **22**(2) (2021), # 5.

[3] J.-L. Baril, J.L. Ramírez, Descent distribution on Catalan words avoiding ordered pairs of relations, Adv. in Appl. Math. **149**, (2023), 102551.

[4] D. Callan, T. Mansour, and J. L. Ramírez, Statistics on bargraphs of Catalan words, J. Autom. Lang. Comb. **26** (2021), 177–196.

[5] S. Corteel, M.A. Martinez, C. Savage, and M. Weselcouch, Patterns in inversion sequences I, Discrete Math. Theor. Comput. Sci. **18**(2) (2016).

Block-wise simple permutations

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This talk is based on joint work with Eli Bagno, Estrella Eisenberg and Moriah Sigron

A permutation is called *block-wise simple* if it contains no interval of the form $p_1 \oplus p_2$ or $p_1 \ominus p_2$. We present this new set of permutations and explore some of its combinatorial properties. We present a generating function for this set, as well as a recursive formula for counting block-wise simple permutations. Following Tenner, who founded the notion of interval posets, we characterize and count the interval posets corresponding to block-wise simple permutations. We also present a bijection between these interval posets and certain tiling's of the n-gon.

Introduction

This study focuses on a new set of permutations called *block-wise simple permutations*, which we define in two different but equivalent ways. The first definition is recursive and is stated here. The other, as well as the proof of the equivalence of the two definitions, is presented in Section .

Definition 1. Let $\mathcal{S}_n$ be the group of permutations of the set $\{1, \ldots, n\}$.

1. The identity permutation $\pi = 1$ is *block-wise simple*.
2. A permutation $\pi \in \mathcal{S}_n$ is *block-wise simple* if there is $\sigma \in \mathcal{S}_k$ $(k \geq 4)$ which is a simple permutation, and there are $\alpha_1, \ldots, \alpha_k$ which are block-wise simple permutations, such that $\pi = \sigma[\alpha_1, \ldots, \alpha_k]$, the inflation of σ by $\alpha_1, \ldots, \alpha_k$.

There are no block-wise simple permutations of orders 2 and 3. For $n \in \{4, 5, 6\}$, a permutation is block-wise simple, if and only if it is simple. One of the first nontrivial examples of block-wise simple permutations is $2413[3142, 1, 1, 1] = 4253716$.

Enumeration of block-wise simple permutations

An alternate definition of block-wise simple permutations

Recall Definition 1. We begin this section with another definition of block-wise simple permutations.

Definition 2. A permutation $\pi \in \mathcal{S}_n$ is block-wise simple if and only if it has no interval of the form $p_1 \oplus p_2$ or $p_1 \ominus p_2$.

Theorem 3. *The two definitions of block-wise simple permutations are equivalent.*

Observation 4. *Let $w_n = |W_n|$ and $s_n = |Simp_n|$. Then, for $n \geq 4$ we have*

$$w_n = \sum_{l=4}^{n} s_l \sum_{\lambda=(\lambda_1,\ldots,\lambda_l)\in Comp(n,l)} w_{\lambda_1} \cdots w_{\lambda_l} \tag{1}$$

where $Comp(n,l)$ is the set of compositions of n in l parts.

Counting the number of interval posets

Following Tenner [4], we define an *interval poset* for each permutation as follows:

Definition 5. The interval poset of a permutation $\pi \in S_n$ is the poset $P(\pi)$ whose elements are the non-empty intervals of π; the order is defined by set inclusion. The minimal elements are the intervals of size 1.

If π is a simple permutation, the interval poset of π comprises the entire interval $[1,\ldots,n]$ with minimal elements $\{1\},\ldots,\{n\}$ as its only descendants. Hence, all simple permutations of a given order n share the same interval poset.

Definition 6. Given a poset P, the permutations, whose interval poset is P are called in [4] the *generators* of P. Formally, a set of such permutations is defined as

$$I(P) := \{w \in S_n : P(w) = P\}.$$

Following [4], we define a *dual-claw poset* as a poset that has a unique maximal element and $k \geq 4$ minimal elements, which are all covered by the maximal element. For our convenience, we omit the adjective 'dual' and call it here just *claw poset* .

A claw poset with k minimal elements is an interval poset of a permutation σ if and only if σ has no proper intervals; that is, σ is simple. Thus, its generators are the simple permutations of order k.

Remark 7. *A tree poset is a poset whose Hasse diagram is a tree. In [4] (Theorem 6.1), the author claimed that $P(\sigma)$ is a tree interval poset if and only if σ contains no interval of the form $p_1 \oplus p_2 \oplus p_3$ or $p_1 \ominus p_2 \ominus p_3$. Based on Theorem 3, it is evident that the interval poset of a block-wise simple permutation is a tree. In fact, by Definition 1, the interval poset of a block-wise simple permutation is a claw of claws. This means that the root is a claw and every node is either a claw or a leaf*

The first problem we tackle in this section is the enumeration of interval posets that represent block-wise simple permutations of order n. As described above, this problem reduces to counting claws of claws.

Let $\mathcal{C}$ be the combinatorial class of all claws of claws, so that for each n, $\mathcal{C}_n$ is the set of all claws of claws having n leaves. We apply the approach of symbolic combinatorics (see [3]) to count claw of claws.

The class $\mathcal{C}$ is described by the following symbolic combinatorial equation:

$$\mathcal{C} = \{\bullet\} \biguplus Seq_{\geq 4}(\mathcal{C})$$

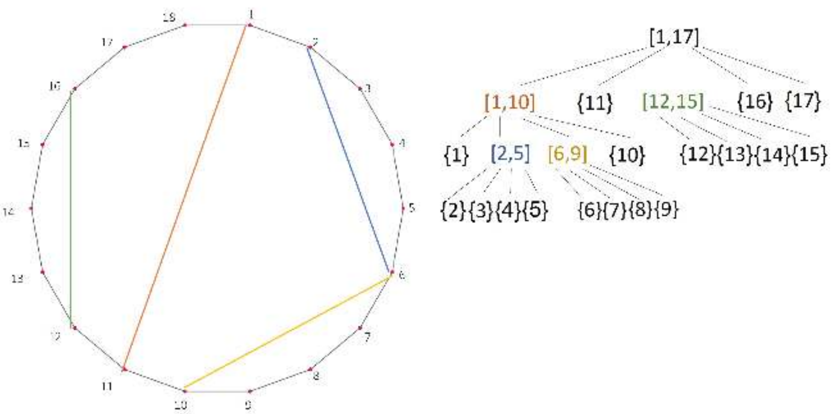
where $\{\bullet\}$ represents a leaf and $Seq_{\geq 4}(\mathcal{C})$ represents a sequence of elements of $\mathcal{C}$ with at least four components. The generating function is

$$C(z) = z + \frac{C(z)^4}{1 - C(z)}.$$

Theorem 8. *The number of interval posets corresponding to block-wise simple permutations of order $n \geq 4$ is*

$$\frac{1}{n} \sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} \binom{n+i-1}{i} \binom{n-2i-2}{i-1}.$$

The first few values of the sequence of these numbers are $1, 1, 1, 5, 10, 16, 45, 109, 222, 540$. This is sequence A054514 from OEIS [5] which also counts the number of ways to place non-crossing diagonals in a convex $(n+4)$-gon such that there are no triangles or quadrilaterals.
In our paper [1], we provide a combinatorial proof presenting a bijection from the set of interval posets corresponding to block-wise simple permutations to the above-mentioned set. Here we present only the framework and a quick graphic illustration.



Theorem 9. *The number of interval posets that represent a block-wise simple permutation of order n is equal to the number of ways to place non-crossing diagonals in a convex $(n+1)$-gon such that no triangles or quadrilaterals are present.*

We end this section with a simple consequence of Theorem 2.8 of [2], concerning Möbius function of the interval poset of a block-wise simple permutation.

Proposition 10. *Let $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$ be a block-wise simple permutation and let $P(\sigma)$ be its closed interval poset. For each interval I of $P(\sigma)$ we have*

$$\mu(I, [1, \ldots, n]) = \begin{cases} 1 & I = [1, \ldots, n] \\ -1 & I \text{ is a coatom} \\ k-1 & I = \emptyset \\ 0 & \textit{Otherwise} \end{cases}$$

Asymptotics — an open question and a conjecture

One might use Equation (1) to obtain an upper bound for the proportion of the number of blockwise simple permutations to the magnitude of the entire set of permutations. Thus, we write for each $n \in \mathbb{N}$, $A_n = W_n - Simp_n$ and we are interested in the asymptotic behavior of $R_n = \frac{|A_n|}{|S_n|}$. Experimental checks show that this ratio tends to zero when n tends to infinity as can be seen in the following table. Actually, we have corroborating data up to $n = 23$.

n	1	2	3	4	5	6	7	8	9	10	11	12
R_n	0	0	0	0	0	0	0.0031746	0.00267857	0.00303131	0.0029343	0.00273389	0.00247482

so we have the following conjecture:

Conjecture 11. *The proportion R_n tends to* 0 *when n tends to infinity.*

References

[1] E. Bagno, E. Eisenberg, S. Reches and S. Sigron, *Block-wise simple permutations*, arXiv:2303.13115 .

[2] M. Bouvel, L. Cionl and B. Izart ,*The interval posets of permutations seen from the decomposition tree perspective*, arXiv:2110.10000 .

[3] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*. Cambridge University Press, 2008.

[4] B. E. Tenner, *Interval Posets of Permutations*, Order 39, 523–536 (2022). https://doi.org/10.1007/s11083-021-09576-1

[5] Sloane, N. J. A., Sequence A054514.

Insertion encodings and bounded families

Eric Redmon Marquette University

This talk is based on joint work with Jay Pantone

A permutation of length n is called *k-bounded* if $|\pi_i - \pi_{i+1}| \leq k, 1 \leq i < n$. Avgustinovich and Kitaev in [3] proved that another family of permutations, the $(k+1)$-*determined* permutations, are equivalent to the inverses of the k-bounded permutations and developed a method of finding a finite state automaton by hand to enumerate them. They gave an example of the $k = 4$ case, and using the transfer matrix method they were able to find a rational generating function.

Now, we say a permutation π of length n is *anchored* if $\pi_1 = 1$ and $\pi_n = n$. Gillespie, Monks, and Monks in [2] sought similar information for *k-bounded anchored* permutations. Adapting methods from Avgustinovich and Kitaev in [3], they were able to show that the generating function for k-bounded anchored permutations is always rational. Instead of building a DFA to count these permutations in [2], the authors decided to give a recursion and combination proof for $k = 2, 3$ commenting that even at $k = 3$ the transfer matrix method becomes "computationally intractable due to the significantly larger size of the adjacency matrix."

We show that it is possible to enumerate k-bounded permutations with smaller finite state automata then those found in [3] and that using an automaton allows us to enumerate k-bounded anchored permutations for larger values of k. We can do so by working with the *insertion encoding* introduced by Albert, Linton, and Ruškuc in [1] and further studied by Vatter in [4]. With this method, we generate far more efficient machines, and have been able to enumerate up to the $k = 7$ case for both k-bounded and k-bounded anchored permutations in a reasonable amount of time. Figure 1 is one such machine.

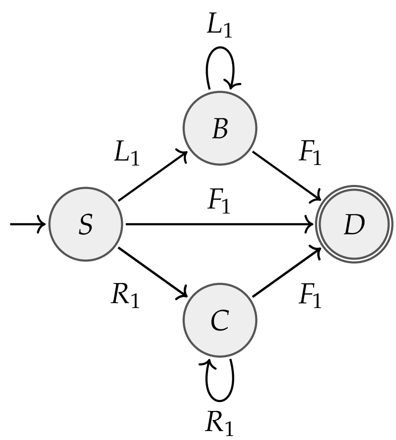


Figure 1: A DFA for 1-bounded permutations which happen to be the monotone permutations.

References

[1] M. Albert, S. Linton, and N. Ruškuc. The insertion encoding of permutations. *Electronic Journal of Combinatorics*, 12, 2005

[2] M. M. Gillespie, K. G. Monks, and K. M. Monks. Enumerating anchored permutations with bounded gaps. *Discrete Mathematics*, 343(9):111957, 2020.

[3] S. Avgustinovich and S. Kitaev. On uniquely k-determined permutations. *Discrete Mathematics*, 308(9):1500–1507, 2008.

[4] V. Vatter. Finding regular insertion encodings for permutation classes. *Journal of Symbolic Computation*, 47(3):259–265, 2012.

Bruce Sagan Michigan State University

This talk is based on joint work with Jinting Liang, Yan Zhuang

This talk concerns the relatively new concept of shuffle-compatibility for cyclic permutations. In particular, we take an algebraic approach using cyclic shuffle algebras.

Linear shuffle-compatibility

A *(linear) permutation* of a finite set P of positive integers is a sequence $\pi = \pi_1 \ldots \pi_n$ obtained by listing all the elements of P exactly once. Note, we do not require that $P = \{1, \ldots, n\} := [n]$. For example, one permutation of $P = \{2, 5, 7, 8\}$ is $\pi = 7285$. A *permutation statistic* is any function st whose domain is all permutations. Many such statistics which arise in practice can be stated in terms of permutation patterns. Of particular interest for us will be the following well-known statistics. The *descent set* of π is

$$\operatorname{Des} \pi = \{i \ : \ \pi_i > \pi_{i+1}\} \subseteq [n-1]$$

which records the positions of the occurrences of a consecutive 21 pattern in π. Two related statistics are the *descent number* and *major index*

$$\operatorname{des} \pi = |\operatorname{Des} \pi| \qquad \text{and} \qquad \operatorname{maj} \pi = \sum_{i \in \operatorname{Des} \pi} i.$$

If π and σ are permutations of disjoint sets then their *shuffle set* is

$$\pi \sqcup\!\sqcup \sigma = \{\tau \ : \ |\tau| = |\pi| + |\sigma| \text{ and } \pi, \sigma \text{ are subwords of } \tau\}.$$

For example

$$71 \sqcup\!\sqcup 25 = \{7125, 7215, 7251, 2715, 2751, 2571\}.$$

If st is a permutation statistic then its *distribution* on $\pi \sqcup\!\sqcup \sigma$ is the multiset

$$\operatorname{st}(\pi \sqcup\!\sqcup \sigma) = \{\{\operatorname{st} \tau \ : \ \tau \in \pi \sqcup\!\sqcup \sigma\}\}.$$

Note that this is a multiset and so keeps track of how many times each statistic value occurs. In our previous example, $\operatorname{des}(71 \sqcup\!\sqcup 25) = \{\{1, 1, 1, 2, 2, 2\}\}$. Call st *shuffle-compatible* if $\operatorname{st}(\pi \sqcup\!\sqcup \sigma)$ only depends on $\operatorname{st} \pi$, $\operatorname{st} \sigma$, $|\pi|$, and $|\sigma|$. All of the statistics we have described so far are shuffle-compatible.

Shuffle-compatibility is implicit in the work of Stanley [4] on P-partitions where he proved the generating function identity

$$\sum_{\tau \in \pi \sqcup\!\sqcup \sigma} q^{\operatorname{maj} \tau} = q^{\operatorname{maj} \pi + \operatorname{maj} \sigma} \begin{bmatrix} m+n \\ m \end{bmatrix}_q .$$

Here, $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is a q-binomial coefficient. Note that this equality immediately implies the shuffle-compatibility of maj. Another place where shuffle-compatibility arises is in the theory of quasisymmetric functions. Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. A monmial $x_{j_1}^{n_1} \cdots x_{j_k}^{n_k}$ where $i_1 < \ldots < i_k$ has *exponent sequence* $(n_1, \ldots, n_k)$ and *degree* $n = \sum_i n_i$. Gessel [5] defined a formal power series $f(\mathbf{x})$to be *quasisymmetric* if any two monomials with the same exponent sequence have the same coefficient. For example

$$f(\mathbf{x}) = 6x_1^2x_2 + 6x_1^2x_3 + 6x_2^2x_3 + \cdots - x_1^4 - x_2^4 - x_3^4 - \cdots$$

is quasisymmetric. The *algebra of quasisymmetric functions*, QSym, consists of all quasisymmetric functions of bounded degree. Of particular importance are the *fundamental quasisymmetric functions* associated with $S \subseteq [n-1]$ defined by

$$F_S(\mathbf{x}) = \sum_{\substack{i_1 \le \cdots \le i_n \\ j \in S \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

The way that these functions multiply is given by the rule that given any permutations π, σ we have

$$F_{\operatorname{Des}\pi}(\mathbf{x}) F_{\operatorname{Des}\sigma}(\mathbf{x}) = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} F_{\operatorname{Des}\tau}(\mathbf{x})$$

which is well defined since Des is shuffle-compatible.

Shuffle-compatibility was first explicitly defined by Gessel and Zhuang [6]. They used an algebraic approach to study this concept as follows. Given a permutation statistic st, say that permutations π, σ are st-*equivalent* if $\operatorname{st}\pi = \operatorname{st}\sigma$ and $|\pi| = |\sigma|$. Let π_{st} denote the st-equivalence class of π. If st is shuffle-compatible then there is an associated *shuffle algebra*, $\mathcal{A}_{\mathrm{st}}$, whose elements are linear combinations of the π_{st} with multiplication given by

$$\pi_{\mathrm{st}} \sigma_{\mathrm{st}} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} \tau_{\mathrm{st}}.$$

Conversely, if such a multiplication is well defined, then st is shuffle-compatible. Gessel and Zhuang use shuffle algebras to prove that a host of permutation statistics are shuffle-compatible.

The cyclic case

There has been recent interest in a notion of shuffle-compatibility for cyclic permutations. If $\pi = \pi_1 \pi_2 \ldots \pi_n$ is a linear permutation then the corresponding *cyclic permiutation* is the set of all rotations of π which is denoted

$$[\pi] = \{\pi_1 \pi_2 \ldots \pi_n,\ \pi_2 \ldots \pi_n \pi_1,\ \ldots,\ \pi_n \pi_1 \ldots \pi_{n-1}\}.$$

As an example

$$[47362] = \{47362,\ 73624,\ 36247,\ 62473,\ 24736\}.$$

A *cyclic permutation statistic* is a function cst whose domain is cyclic permutations. We can construct an important example as follows. If $\pi = \pi_1 \ldots \pi_n$ is a linear permutation then define its *cyclic descent set* to be

$$\operatorname{cDes}\pi = \{i \ :\ \pi_i > \pi_{i+1} \text{ where } i \text{ is taken modulo } n\}.$$

To illustrate

$$\mathrm{cDes}\,429768 = \{1,3,4,6\}.$$

Now for a circular permutation $[\pi]$, define its *cyclic descent set* and *cyclic descent number*

$$\mathrm{cDes}[\pi] = \{\{\mathrm{cDes}\,\pi' \;:\; \pi' \in [\pi]\}\} \qquad \text{and} \qquad \mathrm{cdes}[\pi] = |\,\mathrm{cDes}\,\pi|.$$

Note that the latter is well defined since $|\,\mathrm{cDes}\,\pi'| = |\,\mathrm{cDes}\,\pi|$ for all $\pi' \in [\pi]$.

Disjoint cyclic permutations $[\pi]$ and $[\sigma]$ have *cyclic shuffle set*

$$[\pi] \sqcup [\sigma] = \{[\tau] \;:\; \tau \in \pi' \sqcup \sigma' \text{ where } \pi' \in [\pi] \text{ and } \sigma' \in [\sigma]\}.$$

The definition of the *cyclic distribution* $\mathrm{cst}([\pi] \sqcup [\sigma])$ is as expected. And we say that cst is *cyclic shuffle-compatible* if $\mathrm{cst}([\pi] \sqcup [\sigma])$ depends only on $\mathrm{cst}[\pi]$, $\mathrm{cst}[\sigma]$, $|\pi|$, and $|\sigma|$. Adin et al. [1] defined cyclic quasisymmetric functions and a cyclic analogue of the fundamental quasisymmetric functions. Multiplication of the latter being well-defined is equivalent to the cyclic shuffle-compatibility of cDes, although they did not define this notion. Domagalski et al. [2] were the first to explicitly define cyclic shuffle-compatibility. They studied its properties using combinatorial methods and gave a way to lift linear shuffle-compatibility results to cyclic ones.

In this talk we will show how to adapt the algebraic approach of [6] to the cyclic case. In particular we will define a cyclic analogue of the shuffle algebra and use it to prove many results. For example, we have the following where cPk and cpk are cyclic analogues of the linear peak set and peak number.

Theorem 1. *The cyclic statistics* cDes, cdes, cPk, cpk, *and* (cdes, cpk) *are shuffle-compatible.*

We will end with some open problems, including that of finding a cyclic version of maj which is shuffle-compatible. Further details will be found in the preprint [3].

References

[1] R.M. Adin, I.M. Gessel, V. Reiner, and Y. Roichman, Cyclic quasi-symmetric functions, *Israel J. Math.*, vol.243 (2021), n.1, p.437–500.

[2] R. Domagalski, J. Liang, Q. Minnich, B.E. Sagan, J. Schmidt, and A. Sietsema, Cyclic shuffle compatibility. *Sém. Lothar. Combin.*, vol.85 (2020–2021), art.B85d, 11 p.

[3] J. Liang, B.E. Sagan, Y. Zhuang, Cyclic shuffle-compatibility via cyclic shuffle algebras, preprint, arXiv:2212.14522, 34 p.

[4] R.P. Stanley, *Ordered structures and partitions*, Memoirs of the American Mathematical Society, No.119 (1972), American Mathematical Society.

[5] I.M. Gessel, Multipartite P-partitions and inner products of skew Schur functions, *Contemp. Math.*, vol.34 (1984), p.289–317.

[6] I.M. Gessel and Y. Zhuang, Shuffle-compatible permutation statistics, *Adv. Math.*, vol.332 (2018), p.85–141.

Patterns in trivalent maps and normalisation of linear λ-terms

Alexandros Singh LIPN, Université Sorbonne Paris Nord

This talk is based on joint work with Olivier Bodini, Bernhard Gittenberger, Michael Wallner, and Noam Zeilberger.

Motivated by the study of normalisation for the linear λ-calculus, we present in this talk our recent work on the enumeration of some patterns in trivalent maps and in linear λ-terms. As a result of this work, we obtain an asymptotic lower bound on the average number of steps required to reduce a linear λ-term to its normal form, as well as a sampler for normal linear λ-terms (terms avoiding a specific pattern called a "redex").

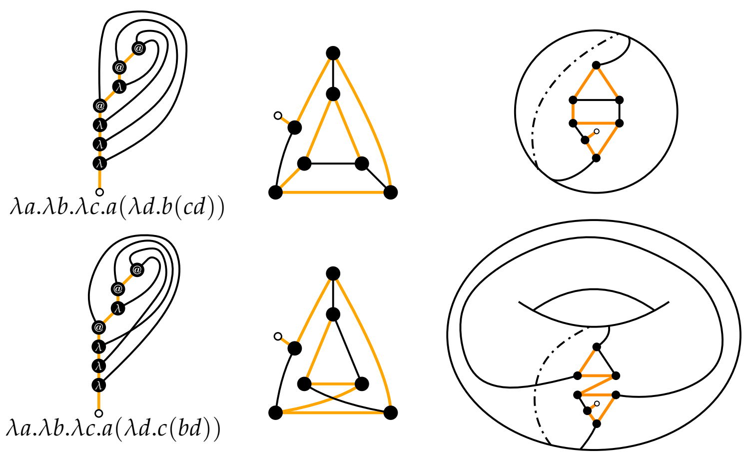


Figure 1: Two rooted trivalent maps of genus 0 and 1 respectively, together with their corresponding λ-terms.

Combinatorics of the λ-calculus and maps

The λ-calculus is a formal system of computation whose terms are formed inductively, using the following three ingredients:

- Variables, denoted by lower-case letters: $x, y, z, \ldots$.
- Applications, denoted by $(s\ t)$, where s, t are themselves terms.
- Abstractions, denoted by $\lambda x.t$, where t is a term and x is a variable that may appear zero or more times inside t.

In this talk, we focus on the *linear* λ-calculus, which is obtained by imposing the following restriction on the general calculus: variables must appear *exactly once* inside a term. As an example, the term $(\lambda x.\lambda y.(x\ y))(\lambda z.z)$ is linear but $\lambda x.(x\ x)$ is not. Linear λ-terms are especially interesting from a combinatorial point of view: they correspond to cubic maps of arbitrary genus as shown in [2] - see Figure 1 for a visual depiction of this correspondence. Furthermore, this bijection is "robust": natural fragments of this calculus correspond to interesting families of cubic maps (for example planar and bridgeless cubic maps) while logically-significant statistics on such terms come paired with combinatorially-significant statistics defined on maps (see, for example, [3]). This correspondence enables the study of terms and maps via the use of tools drawn from both logic and combinatorics.

Computing with the λ-calculus: β-reduction

The *λ-calculus* comes equipped with the notion of *β-reduction*, a rule which, informally, transforms a (sub)term of the form $(\lambda x.t)u$ (called a *redex*) to $t[x := u]$, i.e. an instance of t where all free occurrences of x are replaced with u (in a *capture-avoiding* manner that respects the structure of terms involved). This notion of reduction endows the λ-calculus with computational dynamics, turning it into a powerful system of computation: the general calculus is Turing complete, while computing the β-normal form of a linear λ-term is PTIME-complete, see [1]. A term with no redices is called a *normal form*. Normal forms have natural combinatorial and logical interpretations: they can be viewed as terms and maps avoiding the redex pattern or as proofs of tautologies in implicational linear logic.

In the context of the linear λ-calculus, β-reduction is strongly normalising and has a so-called *strong diamond property* from which we can deduce the following informal statement: the number of steps required to fully reduce a term to its normal form is uniquely determined. Our main result is the following lower bound on the average of this number:

Theorem 1. *Let $W_n(t)$ be the random variable equal to the number of steps required to reduce a closed linear λ-term of size n to its normal form. Then, for $n \in 3\mathbb{N} + 2$ large enough, we have*

$$\mathbb{E}(W_n) \geq \frac{11n}{240}. \tag{1}$$

where the size of a term t is set to be the number of its subterms, or equivalently the number of edges in its corresponding map.

Patterns in λ-terms and maps

To derive Theorem 1, we study the following three patterns in random linear λ-terms, whose occurences provide a great deal of information about the dynamics of normalisation as shown in [4]:

- $(\lambda x.C[(x\ u)])(\lambda y.t_1)t_2; \qquad (p_1)$

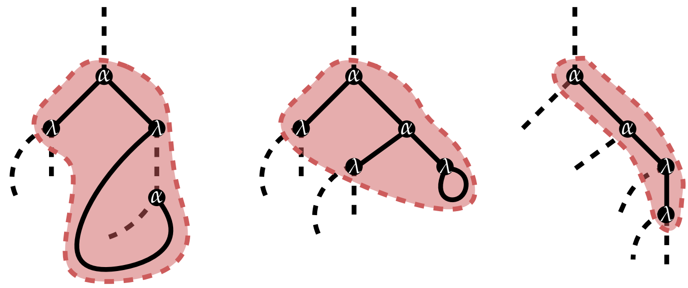


Figure 2: The three patterns in maps induced by the corresponding patterns p_1, p_2, and p_3 in closed linear terms.

- $(\lambda x.x)(\lambda y.t_1)t_2$; $\quad (p_2)$
- $(\lambda x.\lambda y.t_1)t_2t_3$; $\quad (p_3)$

where C stands for some context.

These three patterns correspond to the three patterns of Figure 2, a fact which we use to derive the following result:

Theorem 2. *Let P_1, P_2, P_3 be the random variables given by the number of occurences of the patterns p_1, p_2, p_3, respectively, in a closed linear term of size $n \in 3\mathbb{N}+2$. Then, for $n \in 3\mathbb{N}+2$ large enough,*

$$\mathbb{E}(P_1) = \frac{1}{6},\ \mathbb{E}(P_2) = \frac{1}{48},\ \mathbb{E}(P_3) \geq \frac{n}{240}.$$

Conclusion

Apart from the aforementioned results, this work has also allowed us to derive an algorithm for sampling closed linear normal forms (i.e. terms avoiding the redex pattern) which is of independent interest since these correspond to proofs of tautologies in implicational linear logic.

As for future directions, a primary one would be the derivation of the exact asymptotics for the expectated number of steps required to normalise a random linear term. More generally, it would be of interest to further explore the possible behaviour of various patterns in maps and terms for example by deriving full limit laws and studying correlations in the numbers of occurences of such patterns.

References

[1] H.G Mairson. Linear lambda calculus and PTIME-completeness. *Journal of Functional Programming*, vol.14(2004), n.6,p.623-633.

[2] O.Bodini, D. Gardy, and A. Jacquot. Asymptotics and random sampling for BCI and BCK lambda terms. *Theoretical Computer Science*, vol.502(2013), p.227-238.

[3] O. Bodini, A. Singh, and N. Zeilberger. Asymptotic Distribution of Parameters in Trivalent Maps and Linear Lambda Terms. *Submitted work*. Preprint available under: arXiv:2106.08291.

[4] J.J. Lévy. Réductions correctes et optimales dans le lambda-calcul. *Thesis, Univ. Paris VII, France* (1978).

Grassmannian permutations avoiding identity

Anurag Singh Indian Institute of Technology Bhilai

This talk is based on joint work with Krishna Menon

A permutation is called *Grassmannian* if it has at most one descent. The study of pattern avoidance in such permutations was initiated by Gil and Tomasko [1]. We continue this work by studying Grassmannian permutations that avoid the identity permutation of a given size. We prove most of our results by relating Grassmannian permutations to binary words and Dyck paths.

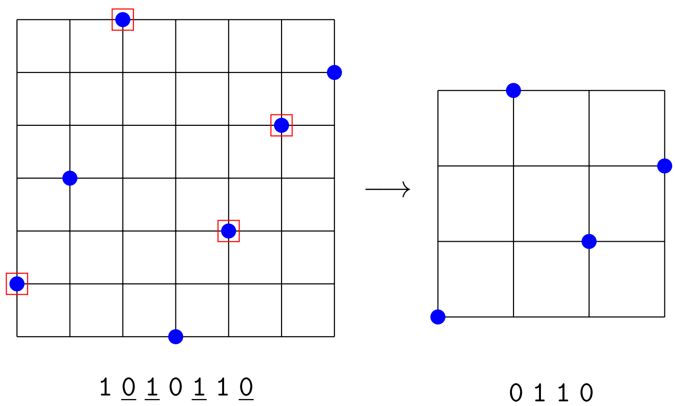


Figure 1: Pattern in a Grassmannian permutation and corresponding binary word.

Preliminaries

We denote the set of Grassmannian permutations of $[n]$ by $\mathcal{G}_n$. We use binary words to denote these permutations. Let $w = w_1 w_2 \cdots w_n$ be a binary word of length n, and we construct the Grassmannian permutation $G(w)$ as follows: If $A = \{i \in [n] \mid w_i = 0\}$ is a set of size k, then we set the first k terms of $G(w)$ to be those of A listed in increasing order. The remaining $n-k$ terms are those of $[n] \setminus A$ listed in increasing order.

Each permutation in $\mathcal{G}_n$ is of the form $G(w)$ for some binary word w of length n. This representation is unique for any non-identity permutation, and the binary words that correspond to identity are those of the form $0^j 1^{n-j}$ for $j \in [0, n]$.

We say a binary word w' *contains* a binary word w if it contains w as a subsequence. We say a binary word w' *avoids* w if it does not contain w. If $G(w)$ is not the identity permutation, $G(w')$ contains $G(w)$ if and only if w' contains w. Similarly, $G(w')$ contains the identity permutation of size k if and only if w' contains $0^j 1^{k-j}$ for some $j \in [0, k]$. For any binary word w, denote by $\mathcal{G}_n(w)$ the set of Grassmannian permutations of length n that avoid $G(w)$.

Example 1. The binary word $w = 1010110$ has corresponding Grassmannian permutation $G(w) = 2471356$ and contains the permutation $G(v) = 1423$ where $v = 0110$

since w contains v (see Figure 1).

As we make use of Dyck paths in the sequel, we assume familiarity with terms related to them (for example, see [3]). The number of Dyck paths of semilength n is the n^{th} Catalan number given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

We now give an overview of our results and refer to [2] for the details.

Counting Grassmannian permutations avoiding identity

Our main result is the proof of an expression conjectured by Michael Weiner, stated in [1, Page 4], for the number of Grassmannian permutations of size m avoiding the identity permutation of size k, that is, for $|\mathcal{G}_m(0^k)|$. For any $k \geq 1$ and $m \geq 0$, we set $B(k,m)$ to be the number of binary words of length m that avoid $0^j 1^{k-j}$ for all $j \in [0,k]$. If $m \geq k$, for any $i \in [0,m]$, $0^i 1^{m-i}$ contains $0^j 1^{k-j}$ for some $j \in [0,k]$. The results in the previous section imply that $B(k,m) = |\mathcal{G}_m(0^k)|$ for $m \geq k \geq 2$.

Theorem 2 ([1, Conjecture])**.** *For any $k \geq 2$ and $m \in [k, 2k-2]$, we have*

$$B(k,m) = \sum_{j=1}^{2k-m} (-1)^{j-1} j \cdot \binom{2k-m-j}{j} \cdot C_{k-j}. \tag{1}$$

The above result can be proved using recursions on these binary words. However, we also obtain a different expression for the same numbers by representing the binary words as Dyck paths. In particular, we show that $B(k,m)$ is the number of Dyck paths of semilength $(k+1)$ where the sum of the heights of the first and last peak is $(2k-m)$. Combining this with Theorem 2, we get the following result.

Proposition 3. *The number of Dyck paths of semilength n where $s \leq 2n-2$ is the sum of the heights of the first and last peaks is*

$$\sum_{j=1}^{\lfloor s/2 \rfloor} (-1)^{j-1} j \binom{s-j}{j} C_{n-1-j}.$$

It would be interesting to see if the above result can be proved directly, possibly by the Principle of Inclusion-Exclusion, instead of using recursions. Breaking up the count according to the height of the first peak gives us the following expression for $B(k,m)$.

Theorem 4. *For any $k, m \geq 1$, we have*

$$B(k,m) = \sum_{a=1}^{2k-m-1} \left[\binom{m}{k-a} - \binom{m}{k} \right].$$

Restricted Grassmannian permutations avoiding identity

We now turn to special classes of Grassmannian permutations. A permutation is said to be *odd* if it has an odd number of inversions (occurrences of the pattern 21). We say that a binary w is odd if the corresponding permutation $G(w)$ is odd. We use $O(k,m)$ to denote the number of odd binary words of length m that avoid 0^j1^{k-j} for all $j \in [0,k]$. Hence, $O(k,m)$ is the number of odd permutations in $\mathcal{G}_m(0^k)$.

In particular, for $m = 2k-2$, we have the following result, where we set C_n to be 0 if n is not an integer.

Proposition 5. *For any $k \geq 2$, we have*

$$O(k, 2k-2) = \frac{C_{k-1} + C_{(k-2)/2}}{2}.$$

We prove this by representing binary words as Dyck paths and defining a sign-reversing involution on them. Another result used is that the number of Dyck paths of semilength n that have all peaks and valleys at odd height is $C_{(n-1)/2}$. We use similar ideas to obtain a general expression for $O(k,m)$ in terms of $B(k,m)$ [2, Theorem 4.7].

We also study avoidance of identity in special classes such as Grassmannian involutions and biGrassmannian permutations (Grassmannian permutations whose inverse is Grassmannian as well). For example, we have the following results.

Theorem 6. *The number of Grassmannian involutions of size m that avoid the identity permutation of size k where $k \leq m < 2k$ is*

$$\left\lfloor \frac{(2k-m)^2}{4} \right\rfloor.$$

Theorem 7. *The number of biGrassmannian permutations of size m that avoid the identity permutation of size k where $k \leq m$ is*

$$\binom{2k-m+1}{3}.$$

References

[1] J. B. Gil and J. A. Tomasko. Restricted grassmannian permutations. *Enumer. Combin. Appl.*, 2(S4PP6), 2022.

[2] K. Menon and A. Singh. Grassmannian permutations avoiding identity. *arXiv preprint arXiv:2212.13794*, 2022.

[3] R. P. Stanley. *Catalan numbers*. Cambridge University Press, 2015.

Permuton limit of 1342-avoiding permutations

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This talk is based on joint work with Kaitlyn Hohmeier

A permuton [4] is a probability measure on the unit square with uniform marginals. For a permutation π, there is a corresponding permuton μ_π defined as follows:

$$\mu_\pi = n \sum_{i=1}^{n} \delta_{((i-1)/n, i/n) \times ((\pi(i)-1)/n, \pi(i)/n)}$$

where δ is Lebesgue measure. The permuton serves as an useful object when describing certain large scale behavior of permutations. In particular, we can describe the limit of a sequence of permutations by considering weak convergence of the corresponding sequence of permutons. We show the permuton limit of 1342-avoiding permutations converge to the unique permuton supported on line $x + y = 1$.

In [1], Bóna provides a bijection between 1342-avoiding indecomposable permutations and certain labeled rooted plane trees called $\beta(0,1)$ trees [2]. This bijection has the useful property that if two permutations π and σ have the same set of left-to-right minima, their corresponding $\beta(0,1)$ trees are the same after removing the labels. These unlabeled trees are in bijection with 132-avoiding, which are determined by the set of left-to-right minima. The maximum distance of these left-to-right minima from the line $x + y = n + 1$ can be determined by the maximum distance of a leaf to the root in the corresponding tree.

For fixed $\epsilon > 0$, there exists $c > 0$ such that the probability that the maximum distance from a leaf to the root in a uniformly random rooted plane tree exceeds $n^{1/2+\epsilon}$ is bounded by $c^{-1}e^{-n^c}$ for some positive constant c (see for example [3]).

Using these ingredients we show that most 1342-avoiding permutations have left-to-right minima that are not too far from the line $x + y = n + 1$ which allows us to conclude that the permuton limit for uniformly random 1342-avoiding permutations converges weakly to the unique permuton supported on the anti-diagonal $x + y = 1$.

References

[1] M. Bóna. Exact enumeration of 1342-avoiding permutations: a close link with labeled trees and planar maps. *Journal of Combinatorial Theory Series A*, vol. 80 (1997), n. 2, p. 257-272.

[2] R. Cori and G.Schaeffer. Description trees and Tutte formulas. *Theoretical Computer Science*, vol.292, 2003, n. 1, p 165-183.

[3] Le Gall J. Random trees and applications *Probability Surveys* vo.2, 2005, p. 245-311

[4] R. Kenyon, D. Král, C. Radin, and P. Winkler. Permutations with fixed pattern densities. *Random Structures & Algorithms*, vo. 56, 2020, no.1, p. 220–250.

Pattern avoidance in alternating sign matrices

Rebecca Smith SUNY Brockport

This talk is based on joint work with Mathilde Bouvel and Jessica Striker

We consider the notion of pattern avoidance in alternating sign matrices with respect to the key of the alternating sign matrix defined by Lascoux [4] and further studied by Aval [1]. We then refine the case of such pattern avoidance studied by Ayyer, Cori, and Gouyou-Beauchamps [2] to similarly characterize the alternating sign matrices avoiding a pair of patterns. We show the class of alternating sign matrices whose key avoids these patterns is enumerated by the Catalan numbers.

Introduction

Alternating sign matrices were introduced and further investigated by Mills, Robbins, and Rumsey [6, 7]. These matrices are in bijection with the possible vertex states of square ice of the same dimensions.

Definition 1. An *alternating sign matrix* is a square matrix where

1. Each entry is one of $0, 1, -1$.
2. Every row and column sum to 1.
3. The non-zero entries of each row and column alternate in sign.

Alternating sign matrices can also be thought of a generalization of permutation matrices. Permutation matrices are exactly the alternating sign matrices with no -1 entries. When extending the notion of pattern avoidance to alternating sign matrices, there are two natural considerations. One is to simply treat -1s like 0s in the alternating sign matrix and then determine whether or not the permutation matrix appears as a submatrix of this modified version of the alternating sign matrix. This notion was studied by Johansson and Linusson [3]. We employ a different definition of avoidance by utilizing the key of an alternating sign matrix defined by Lascoux [4] which systematically removes the -1s of an alternating sign matrix such that the result is a permutation matrix. Then pattern avoidance in alternating sign matrices is the traditional permutation avoidance in the key.

In this vein, the counting of Gog words avoiding 312 which are associated with a certain class of monotone triangles, which can then be associated with alternating sign matrices was studied by Ayyer, Cori, and Gouyou-Beauchamps [2]. We note that our calculation of the key of an alternating sign matrix defined below is a symmetry of the one defined by Lascoux [4] and the key that would be related to the work of Ayyer, Cori, and Gouyou-Beauchamps [2]. Specifically, the left version of the key we employ (consistent with Aval [1]) would relate the triangles Ayyer, Cori, and Gouyou-Beauchamps [2] study to alternating sign matrices whose key avoids 213 instead of Gog words avoiding 312.

Definition 2. The (left) *key* of an alternating sign matrix A is computed by removing the -1s one by one as follows:

1. A -1 entry is removable if there are no other -1 entries that are weakly Northwest of it.
2. For each removable -1,
   (a) Consider the Ferrer's shape that results using the -1 as the Southeast corner, the next 1 entry to the North of the removable -1 as the Northeast corner, the next 1 to the West of the removable -1 as the Southwest corner and any 1s that appear in this rectangle as corners in the Ferrers diagram.
   (b) Replace the North-most 1 from the Ferrer's diagram with a 0. For each subsequent 1, place a new 1 in the row of the previously replaced 1 and column of the current 1, then replace the old 1 with a 0. Finally replace the -1 with a 0.
3. The resulting permutation matrix determines the key.

Example 3. The process of determining the key of an alternating sign matrix is shown.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 0 & \mathbf{0} & \mathbf{1} & 0 & 0 \\ 1 & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & 0 & 1 \\ 0 & \mathbf{1} & \mathbf{-1} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} & 0 & 0 \\ 1 & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & 0 & 1 \\ 0 & \mathbf{0} & \mathbf{0} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \to \quad 21543$$

In a similar manner of previous work on monotone triangles shown in Mills, Robbins, and Rumsey [6, 7] and Ayyer, Cori, and Gouyou-Beauchamps [2], we associate monotone triangles with alternating sign matrices as follows:

- Create a 0-1 matrix by summing rows from the bottom to the current row.
- Record the columns where the 1s sit from bottom to top in the matrix (columns numbered from left to right) to create the triangle from top to bottom.

Example 4. The process of obtaining a monotone triangle from an alternating sign matrix is shown below.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \to \quad \begin{matrix} 2 & & & & \\ 1 & 3 & & & \\ 1 & 3 & 4 & & \\ 1 & 2 & 3 & 5 & \\ 1 & 2 & 3 & 4 & 5 \end{matrix}$$

Definition 5. The monotone triangles created in this manner will have properties:

1. The columns will be weakly increasing from bottom to top.
2. The rows will be strictly increasing from left to right, with bottom row $1\ 2\ 3 \cdots n$.
3. The diagonals will be weakly increasing from top left to bottom right.

Specifically, Ayyer, Cori, and Gouyou-Beauchamps [2] considered *gapless* monotone triangles where in addition to the rules given above, there are no gaps between column entries. Example 4 shows a gapless monotone triangle. As described here, gapless monotone triangles are in bijection with alternating sign matrices avoiding 213.

Left Key avoidance of 213 and 123

In our work, we classify the alternating sign matrices whose key avoids both 213 and 123 with a further refinement to the gapless monotone triangles.

Theorem 6. *The class of $n \times n$ alternating sign matrices whose key avoids both* 123 *and* 213 *is in bijection with the set of gapless monotone triangles of size n with at most two distinct values in any column.*

There is then a nice bijection between these triangles and a class of inversion sequences known to be counted by the Catalan numbers from work of Martinez and Savage [5].

Proposition 7. *Gapless monotone triangles with n columns with at most two distinct values in each column are in bijection with inversion sequences avoiding* 10*.*

References

[1] Jean-Christophe Aval. Keys and alternating sign matrices. *Sém. Lothar. Combin.*, 59:Art. B59f, 13, 2007/10.

[2] Arvind Ayyer, Robert Cori, and Dominique Gouyou-Beauchamps. Monotone triangles and 312 pattern avoidance. *Electron. J. Comb.*, 18(2), 2011.

[3] Robert Johansson and Svante Linusson. Pattern avoidance in alternating sign matrices. *Ann. Comb.*, 11(3-4):471–480, 2007.

[4] Alain Lascoux. Chern and Yang through ice. preprint.

[5] Megan Martinez and Carla Savage. Patterns in inversion sequences II: inversion sequences avoiding triples of relations. *J. Integer Seq.*, 21(2):Art. 18.2.2, 44, 2018.

[6] W. H. Mills, David P. Robbins, and Howard Rumsey, Jr. Proof of the Macdonald conjecture. *Invent. Math.*, 66(1):73–87, 1982.

[7] W. H. Mills, David P. Robbins, and Howard Rumsey, Jr. Alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 34(3):340–359, 1983.

Bijective proofs of shuffle-compatibility for colored descent statistics

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This talk is based on joint work with Yan Zhuang

In this talk, we present a proof of a standardization lemma and equivalency theorem, from which we derive a bijective method to verify shuffle-compatibility of colored descent statistics. This method is an analogue of an uncolored method used by Baker-Jarvis and Sagan. Afterwards, we apply the method to a subset of colored descent statistics studied by Moustakas. These include the colored peak set sPk, color sum csum, and flag major index fmaj. We conclude the talk by providing possible avenues of research from our results.

Introduction

For this talk, we allow permutations to be arrangements of any finite number of positive integers, not only elements in the set $[n]$. We thus have the *ground set* of a permutation π, written $\mathrm{gs}(\pi)$, to be the set of its entries. Denote the set of all permutations with ground set U by $L_1(U)$.

Definition 1. Let $\pi \in L_1(U)$ and $\sigma \in L_1(V)$ be disjoint permutations, i.e. their ground sets have empty intersection. Then a permutation $\tau \in L_1(U \cup V)$ is a *shuffle* of π and σ if both π and σ are subsequences of τ. Denote the set of shuffles of π and σ by $\pi \sqcup\!\sqcup \sigma$.

Definition 2. A permutation statistic St is *shuffle-compatible* provided the distribution of the statistic over the shuffle set of any two disjoint permutations π and σ, written $\mathrm{St}(\pi \sqcup\!\sqcup \sigma)$, depends only on the values $\mathrm{St}(\pi)$, $\mathrm{St}(\sigma)$, $|\pi|$, and $|\sigma|$.

Inspired by Richard Stanley's theory of P-partitions [4], Ira Gessel and Yan Zhuang [2] formalized and explored shuffle-compatible permutation statistics. They defined the *shuffle algebra* of a descent statistic and used it to prove shuffle-compatibility. These results are complemented by Duff Baker-Jarvis and Bruce E. Sagan (abbreviated BJS) [1], who developed a method using a series of bijections to uniformly prove the shuffle-compatibility of many descent statistics.

Later, Vasileios Moustakas [3] proved results of shuffle-compatibility on colored permutations. Specifically, he showed that several colored descent statistics, including the colored peak set, color sum, and flag major index, are shuffle-compatible. The ideas used to prove these results are similar to those of Gessel & Zhuang. We synthesize the theory of shuffle-compatibility of colored permutation statistics with the bijective methods of BJS below.

Definition 3. Let $\pi = \pi_1\pi_2 \cdots \pi_n \in L_1(U)$ be a permutation. If we choose an integer $r \geq 1$ and an n-list

$$\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in [0, r-1]^n$$

then we consider the pair (π, ϵ) an *r-colored permutation* of U, and write (π, ϵ) as follows:

$$\pi^{\epsilon} = \pi_1^{\epsilon_1} \pi_2^{\epsilon_2} \cdots \pi_n^{\epsilon_n}.$$

We can assign a total color order on $\mathrm{gs}(\pi^{\epsilon})$ so that $a^b < c^d$ when one of the 2 following conditions hold:

- $b > d$, or
- $b = d$ and $a < c$.

This allows us to analyze r-colored permutations with colored and uncolored Eulerian and Mahonian descent statistics ($r = 1$ for the uncolored case).

Definition 4. The permutations π^{ϵ} and ρ^{α} have the same *standardization*, written $\mathrm{std}(\pi^{\epsilon}) = \mathrm{std}(\rho^{\alpha})$, when π^{ϵ} and ρ^{α} share the same relative order.

Definition 5. A *colored permutation statistic* St is a map from r-colored permutations to a set of values such that the value $\mathrm{St}(\pi^{\epsilon})$ depends only on $\mathrm{std}(\pi^{\epsilon})$ *and* ϵ.

Note that we can realize uncolored permutation statistics as colored ones by extending the total ordering on the integers to the total color ordering mentioned earlier. We now introduce our first colored descent statistic.

Definition 6. The *colored descent set* of π^{ϵ}, written $\mathrm{sDes}(\pi^{\epsilon})$, is the pair $(\widehat{S}, \epsilon)$ where

$$\widehat{S} := \mathrm{Des}^*(\pi^{\epsilon}) \cup \left\{ i \in [n-1] : \epsilon_i \neq \epsilon_{i+1} \text{ or } \pi_i^{\epsilon_i} > \pi_{i+1}^{\epsilon_{i+1}} \right\} \cup \{n\}.$$

Observe that the condition $\pi_i^{\epsilon_i} > \pi_{i+1}^{\epsilon_{i+1}}$ is equivalent to π^{ϵ} having a descent at index i, hence the name of the statistic. The definitions for shuffles and shuffle-compatibility are defined on colored permutations and their statistics similarly to their uncolored counterparts, so we avoid stating their definitions for brevity.

Main Results

Lemma 7 (Standardization). *For $r \geq 1$, let* St *be a colored descent statistic, and consider the permutations $\pi^{\epsilon}, \rho^{\epsilon}\ \sigma^{\delta}, \tau^{\delta}$ such that $|\pi| = |\rho|$, $|\sigma| = |\tau|$, and $\mathrm{gs}(\pi) \cap \mathrm{gs}(\sigma) = \mathrm{gs}(\rho) \cap \mathrm{gs}(\tau) = \emptyset$. If $\mathrm{std}(\pi) = \mathrm{std}(\rho)$ and $\mathrm{std}(\sigma) = \mathrm{std}(\tau)$ then*

$$\mathrm{St}(\pi^{\epsilon} \sqcup \sigma^{\delta}) = \mathrm{St}(\rho^{\epsilon} \sqcup \tau^{\delta}).$$

□

The Standardization Lemma is necessary to prove the following result, which is core to our project. Denote the set of all r-colored permutation statistics of U by $L_r(U)$.

Corollary 8 (Equivalency)**.** *Suppose* St *is a colored descent statistic. The following are equivalent.*

1. St *is shuffle-compatible.*

2. *If* $\mathrm{St}(\pi^\epsilon) = \mathrm{St}(\rho^\alpha)$ *where* $\pi^\epsilon, \rho^\alpha \in L_r([m])$ *and* $\sigma^\delta \in L_r([n]+m)$ *for some* $m, n \geq 0$, $r \geq 1$, *then*
$$\mathrm{St}(\pi^\epsilon \sqcup\!\sqcup\, \sigma^\delta) = \mathrm{St}(\rho^\alpha \sqcup\!\sqcup\, \sigma^\delta).$$

3. *If* $\mathrm{St}(\sigma^\delta) = \mathrm{St}(\tau^\gamma)$ *where* $\pi^\epsilon \in L_r([m])$ *and* $\sigma^\delta, \tau^\gamma \in L_r([n]+m)$ *for some* $m, n \geq 0$, $r \geq 1$, *then*
$$\mathrm{St}(\pi^\epsilon \sqcup\!\sqcup\, \sigma^\delta) = \mathrm{St}(\pi^\epsilon \sqcup\!\sqcup\, \tau^\gamma).$$

□

References

[1] Duff Baker-Jarvis and Bruce E Sagan. Bijective proofs of shuffle compatibility results. *Advances in Applied Mathematics*, vol. 113 (2020).

[2] Ira M Gessel and Yan Zhuang. Shuffle-compatible permutation statistics. *Advances in Mathematics*, vol. 332 (2018), p. 85–141.

[3] Vasileios Dionysios Moustakas. *Enumerative combinatorics, representations and quasisymmetric functions* (2021). PhD thesis, Εθνικό και Καποδιστ ριακό Πανεπιστήμιο Αθηνών (ΕΚΠΑ). Σχολή Θετικών Επιστημών...

[4] Richard P Stanley. *Ordered structures and partitions*, vol. 119 (1972). American Mathematical Society.

Inversion sequences avoiding pairs of patterns

Benjamin Testart Université de Lorraine

In this talk, we will present new enumerative results about inversion sequences avoiding pairs of patterns of length 3.

Context and summary of results

Inversion sequences are integer sequences $(\sigma_1, \ldots, \sigma_n)$ such that $0 \leqslant \sigma_i < i$ for all $i \in \{1, \ldots, n\}$. The study of pattern-avoiding inversion sequences began in two independent articles [1] and [2] in 2015 and 2016. These two initial articles solved the enumeration of inversion sequences avoiding a single pattern for every pattern of length 3 except for two. Subsequent works [4, 5, 3], among others, left the enumeration open for only one single pattern, and 23 pairs of patterns of length 3. In our work, we obtain recurrence formulas for all of these cases (as well as a few closed formulas), using four different decompositions of pattern-avoiding inversion sequences. These formulas allow us to compute between 100 and 350 terms of each enumeration sequence in one minute, using C++ programs running on a personal computer.

Definitions and notation

We denote by $\mathbf{I}_n$ the set of *inversion sequences* of length n, that is integer sequences $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that $0 \leqslant \sigma_i < i$ for all $i \in \{1, \ldots, n\}$. There is a natural bijection between $\mathbf{I}_n$ and the set of permutations of n elements, called the *Lehmer code*, which explains the name "inversion sequence". If $\pi = (\pi_1, \ldots, \pi_n)$ is a permutation, the inversion sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ associated with π by the Lehmer code is defined by $\sigma_i = \#\{j \mid j < i \text{ and } \pi_j > \pi_i\}$ for all $i \in \{1, \ldots, n\}$, i.e. σ_i counts the number of inversions of π whose second entry is at position i.

We call *pattern* any integer sequence whose set of values is $\{0, \ldots, k\}$ for some $k \geqslant 0$. Our definition for pattern containment and avoidance is the same as the one for permutations (note that unlike permutations, our patterns may contain repeated values). Given a set of patterns Θ, we denote by $\mathbf{I}_n(\Theta)$ the set of Θ-avoiding inversion sequences of length n. If two sets of patterns Θ and Θ' are such that $|\mathbf{I}_n(\Theta)| = |\mathbf{I}_n(\Theta')|$ for all n, we say that Θ and Θ' are Wilf-equivalent (for inversion sequences), which we denote by $\Theta \sim \Theta'$.

Generating trees

Generating trees have often been used to enumerate pattern-avoiding inversion sequences, see [3] for example. In our work, we make use of two different ways of growing inversion sequences which result in generating trees.

The first way is the most common and simply consists in inserting a new entry at the end of an inversion sequence. We use this approach only for inversion sequences avoiding the pair of patterns {000,100}, since all other patterns it could be applied to were already solved (many of them in [3]).

The second way consists in inserting a maximal entry somewhere in an inversion sequence. The inserted value can be either a repetition of the current maximum of the sequence, or a larger value which will become its new maximum. We use this approach to solve the enumeration of inversion sequences avoiding the pairs of patterns {000, 102}, {000, 201} $\sim$ {000, 210}, {100, 101}, {100, 110}, {101, 210}, and {110, 201}.

Splitting at the leftmost maximum

Given an inversion sequence $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbf{I}_n$, let $m = \max(\sigma)$ and p be the leftmost position of m in σ. We then split σ into $\alpha \cdot \beta$, where $\alpha = (\sigma_1, \ldots, \sigma_{p-1})$ is an inversion sequence of length $p-1$, and $\beta = (\sigma_p, \ldots, \sigma_n)$ is a word of length $n-p+1$ on the alphabet $\{0, \ldots, m\}$ such that $\beta_1 = m$. Clearly, if σ avoids a pattern θ, then α and β must also avoid θ. However, requiring α and β to avoid θ is not enough to guarantee that σ avoids θ in general: an occurrence of θ in σ could take some entries in α and some in β.

We express conditions on β, which depend on α, that are necessary and sufficient for σ to avoid θ. For some patterns θ, the number of words β that can follow a prefix α depends only on statistics of α. For example if $\theta = 010$, then α and β cannot share any common values, hence we only need to know the number of distinct values in α in order to count how many words β are such that $\alpha \cdot \beta$ is a 010-avoiding inversion sequence.

In most cases, this decomposition requires to also enumerate words avoiding θ (which can be done by applying this same decomposition to words). However, the family of words which can play the role of β is sometimes simpler. For example if $\theta = 201$, then once the positions of the maximum m are set, the remaining entries in β are nonincreasing.

Using this method, we obtain recurrence formulas for inversion sequences avoiding the pattern 010, and those avoiding the pairs of patterns {000, 120}, {010, 000}, {010, 110}, {010, 120}, {010, 201} $\sim$ {010, 210}, {011, 120}, {100, 120}, {101, 120}, {110, 120}, {102, 201}, and {120, 201}.

Shifted inversion sequences and splitting at the leftmost zero

Let $n, h \geqslant 0$ be two integers, and $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an integer sequence of length n. We call σ a *h-shifted inversion sequence* if $0 \leqslant \sigma_i < i + h$ for all $i \in \{1, \ldots, n\}$, and we denote by $\mathbf{I}_n^h$ the set of h-shifted inversion sequences of length n. The elements of $\mathbf{I}_n^h$ can be seen as inversion sequences of length $n+h$ whose first h entries were deleted.

When splitting an inversion sequence into two parts, shifted inversion sequences naturally appear on the right part. In the previous section, shifted inversion sequences did not appear because we split inversion sequences after an occurrence of their maximum, removing the condition of "being an inversion sequence" on this right part and turning it into a word instead. However, shifted inversion sequences are useful when studying pattern-avoiding inversion sequences which are more naturally split around a minimum than a maximum.

Given a shifted inversion sequence $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbf{I}_n^h(\theta)$, let p be the position of the leftmost zero (sequences with no zeros are counted by $\mathbf{I}_n^{h-1}(\theta)$, so they may be ignored). We proceed to a decomposition similar to that of the previous section. This time, we split σ into $\alpha \cdot 0 \cdot \gamma$, where $\alpha = (\sigma_1, \ldots, \sigma_{p-1})$ is a θ-avoiding h-shifted inversion sequence which contains no zeros, and $\gamma = (\sigma_{p+1}, \ldots, \sigma_n)$ is a θ-avoiding $(h+p)$-shifted inversion sequence.

Using this method, we obtain recurrence formulas for inversion sequences avoiding the pairs of patterns {010, 102}, {100, 102}, and {102, 210}.

References

[1] Sylvie Corteel, Megan A. Martinez, Carla D. Savage, and Michael Weselcouch. Patterns in inversion sequences I. *Discrete Mathematics and Theoretical Computer Science*, 18(2), 2016.

[2] Toufik Mansour and Mark Shattuck. Pattern avoidance in inversion sequences. *Pure Mathematics and Applications*, 25(2):157–176, 2015.

[3] Toufik Mansour and Gökhan Yildirim. An algorithmic approach based on generating trees for enumerating pattern-avoiding inversion sequences. 2022. arXiv:2210.04489.

[4] Megan A. Martinez and Carla D. Savage. Patterns in inversion sequences II: Inversion sequences avoiding triples of relations. *Journal of Integer Sequences*, 21(18.2.2), 2018.

[5] Chunyan Yan and Zhicong Lin. Inversion sequences avoiding pairs of patterns. *Discrete Mathematics and Theoretical Computer Science*, 22(1), 2020.

Thresholds for patterns in random permutations

Daniel Threlfall University of Strathclyde

This talk is based on joint work with David Bevan.

In this work we investigate thresholds for the appearance and disappearance of consecutive patterns occurring within large random permutations as the number of inversions increases. Let $\sigma_{n,m}$ denote a permutation chosen uniformly from the set of n-permutations with exactly m inversions. We call $\sigma_{n,m}$ the *uniform random permutation.*

As the number of inversions increases, consecutive patterns appear before eventually disappearing in the following way: if π is any non-monotonic consecutive pattern, then there exist functions $f_\pi^-(n)$ and $f_\pi^+(n)$ such that

$$\lim_{n\to\infty} \mathbb{P}\left[\sigma_{n,m} \text{ contains } \pi\right] = \begin{cases} 0 & \text{if } m \ll f_\pi^-(n) \\ 1 & \text{if } f_\pi^-(n) \ll m \ll f_\pi^+(n) \\ 0 & \text{if } f_\pi^+(n) \ll m \end{cases}$$

where $f(n) \ll g(n)$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. We establish these lower and upper thresholds for any fixed consecutive pattern π. We also consider thresholds for classical patterns.

To do so, we work with inversion sequences, which we consider to be weak integer compositions. As a result, we introduce the following model of random integer compositions. Let $\mathcal{C}_{n,m}$ denote the set of all compositions of length n such that all terms sum to m. Let $p \in [0,1)$ and $q = 1-p$, then for each $m \geq 0$, we assign to each composition $C \in \mathcal{C}_{n,m}$ the probability $p^m q^n$. Each term is sampled independently from the geometric distribution with parameter q; that is, $\mathbb{P}\left[C(i) = k\right] = p^k q$ for each $k \geq 0$ and $i \in [n]$. We call such a random composition a *geometric random composition.* We establish that, asymptotically with probability tending to 1, a geometric random composition is an inversion sequence if and only if $p \ll 1$.

In this work we focus on how we transfer thresholds for patterns in the geometric random composition to get thresholds for patterns in the uniform random permutation.

Generalized characters of the generalized symmetric group

Omar Tout Sultan Qaboos University

We will define and study the generalized characters of the generalized symmetric group. The generalized characters of the symmetric group were first studied by Strahov in [1].

Let $\mathbb{Z}_k := \{0, 1, \cdots, k-1\}$ be the cyclic group of order k. The *generalized symmetric group* is the wreath product $\mathbb{Z}_k \wr \mathcal{S}_n$, that is the group with underlying set $\mathbb{Z}_k^n \times \mathcal{S}_n$ and product defined as follows:

$$((\sigma_1, \ldots, \sigma_n); p) \cdot ((\epsilon_1, \ldots, \epsilon_n); q) = ((\sigma_{q^{-1}(1)}\epsilon_1, \ldots, \sigma_{q^{-1}(1)}\epsilon_n); pq),$$

for any $((\sigma_1, \ldots, \sigma_n); p), ((\epsilon_1, \ldots, \epsilon_n); q) \in \mathbb{Z}_k^n \times \mathcal{S}_n$. Since there are exactly k conjugacy classes of $\mathbb{Z}_k$ consisting of $\{i\}$ for each $0 \leq i \leq k-1$, the *type* of $x = (g; p) \in \mathbb{Z}_k \wr \mathcal{S}_n$ is the k-partite partition $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ of n, where each partition λ_i is formed out of cycles c of p whose cycle sum equals i. We define the *marked-type* of x to be the marked k-partite partition of n obtained from the type of x by marking the part corresponding to the cycle containing n in p. For example, consider the element $x = (g, p) \in \mathbb{Z}_3 \wr \mathcal{S}_{10}$ where $g = (1, 1, 2, 0, 1, 1, 1, 2, 1, 0)$ and $p = (1, 4)(2, 5)(3)(6)(7, 8, 9, 10)$. The cycle sum of $(1, 4)$ is $1 + 0 = 1$, of $(2, 5)$ is $1 + 1 = 2$, of (3) is 2, of (6) is 1 and of $(7, 8, 9, 10)$ is $1 + 2 + 1 + 0 = 1$ in $\mathbb{Z}_3$. Thus, $\text{type}(x) = (\emptyset, (4, 2, 1), (2, 1))$ and $\text{marked-type}(x) = (\emptyset, (4^*, 2, 1), (2, 1))$. Two elements $x, y \in \mathbb{Z}_k \wr \mathcal{S}_n$ are said to be $\mathbb{Z}_k \wr \mathcal{S}_{n-1}$-conjugate if there exists $k \in \mathbb{Z}_k \wr \mathcal{S}_{n-1}$ such that $x = kyk^{-1}$.

The details of the following results can be found in [2].

Proposition 1. *Two elements of $\mathbb{Z}_k \wr \mathcal{S}_n$ are in the same $\mathbb{Z}_k \wr \mathcal{S}_{n-1}$-conjugacy class if and only if they have the same marked-type.*

Let $\text{diag}(\mathbb{Z}_k \wr \mathcal{S}_{n-1})$ denote the diagonal subgroup of $\mathbb{Z}_k \wr \mathcal{S}_n \times \mathbb{Z}_k \wr \mathcal{S}_{n-1}$, that is the subgroup of $\mathbb{Z}_k \wr \mathcal{S}_n \times \mathbb{Z}_k \wr \mathcal{S}_{n-1}$ formed of all the elements (k, k) with $k \in \mathbb{Z}_k \wr \mathcal{S}_{n-1}$. As a consequence of Proposition 1, we have the following result.

Corollary 2. *The pair $(\mathbb{Z}_k \wr \mathcal{S}_n \times \mathbb{Z}_k \wr \mathcal{S}_{n-1}, \text{diag}(\mathbb{Z}_k \wr \mathcal{S}_{n-1}))$ is a symmetric Gelfand pair.*

If λ is a partition of n that can be obtained from the partition μ of $n-1$ by adding only one square in an exterior corner of the Young diagram of μ then we will write $\mu \nearrow \lambda$. We extend this notation to k-partite partitions of n and we write $\boldsymbol{\mu} \nearrow \boldsymbol{\lambda}$ if $\mu_i \nearrow \lambda_i$ for some i and $\mu_j = \lambda_j$ for $j \neq i$. If $\mu \nearrow \lambda$, where λ is a partition of n, then λ can be seen as a marked partition of n where the marked part corresponds to the row in which the exterior corner belongs. For example, $(\emptyset, (4^*, 2, 1), (2, 1))$ corresponds to $(\emptyset, (3, 2, 1), (2, 1)) \nearrow (\emptyset, (4, 2, 1), (2, 1))$.

Since $(\mathbb{Z}_k \wr \mathcal{S}_n \times \mathbb{Z}_k \wr \mathcal{S}_{n-1}, \text{diag}(\mathbb{Z}_k \wr \mathcal{S}_{n-1}))$ is a Gelfand pair, it follows that the induced representation $1_{\text{diag}(\mathbb{Z}_k \wr \mathcal{S}_{n-1})}^{\mathbb{Z}_k \wr \mathcal{S}_n \times \mathbb{Z}_k \wr \mathcal{S}_{n-1}}$ is multiplicity free and its character is given in the following proposition

Proposition 3.

$$\mathrm{char}\left(1_{\mathrm{diag}(\mathbb{Z}_k\wr\mathcal{S}_{n-1})}^{\mathbb{Z}_k\wr\mathcal{S}_n\times\mathbb{Z}_k\wr\mathcal{S}_{n-1}}\right) = \sum_{\substack{\rho \text{ } k\text{-partite partition of } n,\\ \sigma\nearrow\rho}} \chi^{\rho}\times\chi^{\sigma}. \tag{1}$$

Corollary 4. *The zonal spherical functions of the pair* $(\mathbb{Z}_k\wr\mathcal{S}_n\times\mathbb{Z}_k\wr\mathcal{S}_{n-1}, \mathrm{diag}(\mathbb{Z}_k\wr\mathcal{S}_{n-1}))$ *are*

$$\omega^{\sigma\nearrow\rho}(x,y) = \frac{1}{k^{n-1}(n-1)!}\sum_{h\in\mathbb{Z}_k\wr\mathcal{S}_{n-1}} \chi^{\rho}(xh)\chi^{\sigma}(yh),$$

where σ *is a k-partite partition of* $n-1$ *and* ρ *is a k-partite partition of n with* $\sigma\nearrow\rho$.

The $\mathbb{Z}_k\wr\mathcal{S}_{n-1}$-generalized character of $\mathbb{Z}_k\wr\mathcal{S}_n$ associated to $\sigma\nearrow\rho$, where ρ is a k-partite partition of n, is defined by

$$\chi^{\sigma\nearrow\rho}(x) = \chi^{\sigma}(1)\omega^{\sigma\nearrow\rho}(x,1) = \frac{\chi^{\sigma}(1)}{k^{n-1}(n-1)!}\sum_{h\in\mathbb{Z}_k\wr\mathcal{S}_{n-1}} \chi^{\rho}(xh)\chi^{\sigma}(h) \tag{2}$$

for any $x\in\mathbb{Z}_k\wr\mathcal{S}_n$. We list below some of the important properties of the generalized characters:

- They form an orthogonal basis for the algebra $C(\mathbb{Z}_k\wr\mathcal{S}_n\times\mathbb{Z}_k\wr\mathcal{S}_{n-1}, \mathrm{diag}(\mathbb{Z}_k\wr\mathcal{S}_{n-1}))$ of all complex-valued functions on $\mathbb{Z}_k\wr\mathcal{S}_n\times\mathbb{Z}_k\wr\mathcal{S}_{n-1}$ that are constant on the $\mathrm{diag}(\mathbb{Z}_k\wr\mathcal{S}_{n-1})$-conjugacy classes of $\mathbb{Z}_k\wr\mathcal{S}_n\times\mathbb{Z}_k\wr\mathcal{S}_{n-1}$.
- $\chi^{\sigma\nearrow\rho}(1) = \chi^{\sigma}(1)$.
- $\chi^{\sigma\nearrow\rho}(yxy^{-1}) = \chi^{\sigma\nearrow\rho}(x)$ for all $x\in\mathbb{Z}_k\wr\mathcal{S}_n$, $y\in\mathbb{Z}_k\wr\mathcal{S}_{n-1}$.
- $\langle\chi^{\mu\nearrow\lambda},\chi^{\nu\nearrow\rho}\rangle_{\mathbb{Z}_k\wr\mathcal{S}_n} = \frac{\chi^{\mu}(1)}{\chi^{\lambda}(1)}\delta^{\lambda\rho}\delta^{\mu\nu}$.

When $k=1$, $\mathbb{Z}_k\wr\mathcal{S}_n$ is isomorphic to $\mathcal{S}_n$ and tables of the generalized characters of $\mathcal{S}_3$ and $\mathcal{S}_4$ can be found in [1, page 118]. We reproduce below the table of the $\mathcal{S}_2$-generalized characters of $\mathcal{S}_3$.

	$C_{(3^*)}$	$C_{(2,1^*)}$	$C_{(2^*,1)}$	$C_{(1,1,1^*)}$
Order	2	1	2	1
Elements	$(123),(132)$	$(12)(3)$	$(13)(2),(23)(1)$	$(1)(2)(3)$
$\chi^{(3^*)}$	1	1	1	1
$\chi^{(2,1^*)}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	1
$\chi^{(2^*,1)}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1
$\chi^{(1,1,1^*)}$	1	-1	-1	1

Table 1: $\mathcal{S}_2$-generalized characters of $\mathcal{S}_3$

When $k=2$, $\mathbb{Z}_k\wr\mathcal{S}_n$ is isomorphic to the Hyperoctahedral group $\mathcal{H}_n$, and the generalized characters in this case were first defined in [3]. We refer to [4, Pages 51, 52 and 53] for the character tables of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. For example, by definition, $\chi^{((1^*),(1))}((13)(24))$ equals

$$\frac{\chi^{(\emptyset,(1))}(1)}{2^{2-1}(2-1)!}\big(\chi^{((1),(1))}((13)(24))\chi^{(\emptyset,(1))}((1)(2)) + \chi^{((1),(1))}((13)(24)(12))\chi^{(\emptyset,(1))}((12))\big)$$

$$= \frac{1}{2}\big(0.1 + 0. - 1\big) = 0.$$

We use the character tables of $\mathcal{H}_1$ and $\mathcal{H}_2$ along with the above properties to produce, in Table 2, the $\mathcal{H}_1$-generalized characters of $\mathcal{H}_2$.

	$C_{((1,1^*),\emptyset)}$	$C_{((2^*),\emptyset)}$	$C_{((1^*),(1))}$	$C_{((1),(1^*))}$	$C_{(\emptyset,(1,1^*))}$	$C_{(\emptyset,(2^*))}$
Order	1	2	1	1	1	2
Elements	$(1)(2)(3)(4)$	$(13)(24)$, $(14)(23)$	$(12)(3)(4)$	$(1)(2)(34)$	$(12)(34)$	(1324), (1423)
$\chi^{((2^*),\emptyset)}$	1	1	1	1	1	1
$\chi^{((1,1^*),\emptyset)}$	1	-1	1	1	1	-1
$\chi^{((1^*),(1))}$	1	0	-1	1	-1	0
$\chi^{((1),(1^*))}$	1	0	1	-1	-1	0
$\chi^{(\emptyset,(2^*))}$	1	1	-1	-1	1	-1
$\chi^{(\emptyset,(1,1^*))}$	1	-1	-1	-1	1	1

Table 2: $\mathcal{H}_1$-generalized characters of $\mathcal{H}_2$

References

[1] E. Strahov. Generalized characters of the symmetric group. *Advances in Mathematic*, vol.212(1) (2007), p.109-142.

[2] O. Tout. Generalized characters of the generalized symmetric group. *arXiv preprint arXiv:2212.13241*, (2022).

[3] O. Tout. On the symmetric Gelfand pair $(\mathcal{H}_n \times \mathcal{H}_{n-1}, \mathrm{diag}(\mathcal{H}_{n-1}))$. *Journal of Algebra and Its Applications*, vol.20(05) (2021), p.2150085.

[4] A. Islami. Symmetric Functions as Characters of Hyperoctahedral Group. *Ph.D. thesis, York University*, (2020).

Growth rates of permutations with a given descent set

Justin M. Troyka California State University, Los Angeles

This talk is based on joint work with Mohamed Omar

Let $\mathbb{P}$ denote the set of positive integers. Significant work has been done on the peak polynomial for a finite set $I \subseteq \mathbb{P}$ [2, 3, 5]. More recent work has found analogous results for the descent polynomial for a finite set $I \subseteq \mathbb{P}$ [6, 9]. In this research we study the asymptotic growth of $d_n(I)$, the number of permutations of size n with descent set $I \cap [n-1]$, when $I \subseteq \mathbb{P}$ is an infinite set. If I is neither finite nor cofinite, the sequence $d_n(I)$ has greater than polynomial growth. ("Cofinite" means its complement is finite).

Some prior work has been done on the asymptotic growth of $d_n(I)$ for infinite I. Bender, Helton, and Richmond [1] prove an asymptotic formula of the form $cL^n n!$ for the number of permutations with a given "nearly periodic" descent set. Elizalde and Troyka [7, Sec. 5] show that $d_n(I) \gg (n/2)\lfloor n/2 \rfloor!$ for "almost all" $I \subseteq \mathbb{P}$ (see Conjecture 3 below). Other prior work on this topic includes [4] and [11]; the latter is from the point of view of statistical physics.

Preliminaries

For each n, $d_n(I)$ is maximized by $I = \{2, 4, 6, \ldots\}$ and $I = \{1, 3, 5, \ldots\}$, a fact first proved by Viennot [13]; in this case, $d_n(I)$ counts the alternating permutations and equals the Euler number E_n. Asymptotically, this number is

$$E_n \sim \frac{4}{\pi}\left(\frac{2}{\pi}\right)^n n! \qquad (1)$$

(see [8, p. 268–269]).

Suppose we randomly select $I \subseteq \mathbb{P}$ by deciding, independently for each $i \in \mathbb{P}$, whether to put $i \in I$ or to put $i \notin I$ with probability $1/2$ each. For each n, the expected value of $d_n(I)$ is

$$\mathbb{E}[d_n(I)] = 2\left(\frac{1}{2}\right)^n n!. \qquad (2)$$

The asymptotic enumerations in (1) and (2) suggest the following definitions as a coarse asymptotic statistic: we define the *upper growth rate* and *lower growth rate* of $I \subseteq \mathbb{P}$ to be

$$\overline{\mathrm{gr}}(I) = \limsup_{n\to\infty} \left(\frac{d_n(I)}{n!}\right)^{1/n} \quad \text{and} \quad \underline{\mathrm{gr}}(I) = \liminf_{n\to\infty} \left(\frac{d_n(I)}{n!}\right)^{1/n}$$

If $\overline{\mathrm{gr}}(I) = \underline{\mathrm{gr}}(I)$, then this shared value is denoted by $\mathrm{gr}(I)$ and called the *proper growth rate* of I, and we say that $\mathrm{gr}(I)$ exists.

We know that $\mathrm{gr}(I) = 0$ if I is finite or cofinite, since this case is the previously studied descent polynomial. We also know that $\mathrm{gr}(I) = 2/\pi$ if $I = \{2, 4, 6, \ldots\}$ or $I = \{1, 3, 5, \ldots\}$, as seen in (1). Thus, for every $I \subseteq \mathbb{P}$, we have

$$0 \leq \underline{\mathrm{gr}}(I) \leq \overline{\mathrm{gr}}(I) \leq 2/\pi$$

(and note that $2/\pi \approx 0.6366$). If $d_n(I) \sim cn^p L^n n!$ for constants c, p, L, then $\mathrm{gr}(I) = L$.

We have thus seen that the number of permutations with descent set $I \cap [n-1]$ is at most an exponentially decaying fraction of the total number of permutations, and the base of the exponential decay is at most $2/\pi$.

Results

In fact, every upper and lower growth rate between 0 and $2/\pi$ is possible:

Theorem 1. *For every $L, L' \in [0, 2/\pi]$ such that $L \leq L'$, there exists $I \subseteq \mathbb{P}$ such that $\underline{\mathrm{gr}}(I) = L$ and $\overline{\mathrm{gr}}(I) = L'$. In particular, if $L = L'$, then there exists I such that $\mathrm{gr}(I)$ exists and equals L.*

Roughly speaking, this is proved by recursively constructing the set I with (i) a run of consecutive $i \notin I$ until the growth rate is low enough, followed by (ii) a run of alternating $i \notin I$ and $i \in I$ until the growth rate is high enough again, continuing to alternate between step (i) and step (ii) as $i \to \infty$.

In the process of proving that theorem, we also prove the following:

Theorem 2. *For $I \subseteq \mathbb{P}$ and $m \geq 0$, define $\Sigma^m(I) = \{i - m \colon i \in I \text{ and } i > m\}$; that is, $\Sigma^m(I)$ is the m-fold left shift of I. If $I, I' \subseteq \mathbb{P}$ and there exist $m, m' \geq 0$ such that $\Sigma^m(I) = \Sigma^{m'}(I')$, then $\overline{\mathrm{gr}}(I) = \overline{\mathrm{gr}}(I')$ and $\underline{\mathrm{gr}}(I) = \underline{\mathrm{gr}}(I')$. In particular, descent sets that are eventually equal have the same upper (resp. lower) growth rate.*

Work on this project is ongoing. We hope to prove the following:

Conjecture 3. *Randomly select $I \subseteq \mathbb{P}$ by deciding, independently for each $i \in \mathbb{P}$, whether to put $i \in I$ or to put $i \notin I$ with probability $1/2$ each. Then $\underline{\mathrm{gr}}(I) > 0$ with probability 1.*

On the other hand, the stronger statement "if I is neither finite nor cofinite then $\underline{\mathrm{gr}}(I) > 0$" is false. We can construct I to be sparse enough that $\mathrm{gr}(I) = 0$, with the resulting sequence $d_n(I)$ being super-polynomial and sub-exponential. Conjecture 3 is stronger than the result from Elizalde and Troyka [7, Sec. 5] that, with I randomly selected as above, $\frac{d_n(I)}{(n/2)\lfloor n/2 \rfloor!} = \infty$ with probability 1.

Remarks on peak sets

We have some similar results for $p_n(I)$, the number of permutations of size n with peak set $I \cap [n-1]$. We can define the upper and lower growth rates of I in the same way as before, using $p_n(I)$ instead of $d_n(I)$. Knowing that $\max_{I \subseteq \mathbb{P}} p_n(I) = \Theta\big(3^{-n/3} n!\big)$ (see [10, 3]), it follows that the upper and lower growth rates lie in $\left[0, 1/\sqrt[3]{3}\right]$. Note that $1/\sqrt[3]{3} \approx 0.6934$, just a bit higher than the maximum growth rate $2/\pi \approx 0.6366$ that we saw in the $d_n(I)$ case.

In the case where I is periodic, we found an explicit formula for the growth rate in terms of Euler numbers. Work on growth rates of permutations with a given peak set is still in progress.

References

[1] E. A. Bender, W. J. Helton, and L. B. Richmond. Asymptotics of permutations with nearly periodic patterns of rises and falls. *Electron. J. Combin.*, vol. 10 (2003), #R40.

[2] S. Billey, K. Burdzy, and B. E. Sagan. Permutations with given peak set. *J. Integer Seq.*, vol. 16 (2013), Article 13.6.1.

[3] S. Billey, M. Fahrbach, and A. Talmage. Coefficients and roots of peak polynomials. *Exp. Math.*, vol. 25 (2016), n. 2, p. 165–175.

[4] F. C. S. Brown, T. M. A. Fink, and K. Willbrand. On arithmetic and asymptotic properties of up–down numbers. *Discrete Math.*, vol. 307 (2007), p. 1722–1736.

[5] A. Diaz-Lopez, P. E. Harris, E. Insko, and M. Omar. A proof of the peak polynomial positivity conjecture. *J. Combin. Theory Ser. A*, vol. 149 (2017), p. 21–29.

[6] A. Diaz-Lopez, P. E. Harris, E. Insko, M. Omar, and B. E. Sagan. Descent polynomials. *Discrete Math.*, vol. 342 (2019), n. 6, p. 1674–1686.

[7] S. Elizalde and J. M. Troyka. Exact and asymptotic enumeration of cyclic permutations according to descent set. *J. Combin. Theory Ser. A*, vol. 165 (2019), p. 360–391.

[8] P. Flajolet and R. Sedgewick. *Analytic Combinatorics* (2009). Cambridge Univ. Press.

[9] E. Kantarci Oğuz. Connecting descent and peak polynomials. Preprint (2018), `arXiv:1806.05353`.

[10] A. Kasraoui. The most frequent peak set of a random permutation. Preprint (2012), `arXiv:1210.5869`.

[11] J. M. Luck. On the frequencies of patterns of rises and falls. *Phys. A*, vol. 407 (2014), p. 252–275.

[12] R. P. Stanley. *Enumerative Combinatorics Vol. I*, 2nd ed. (2012). Cambridge Stud. Adv. Math. 49, Cambridge Univ. Press.

[13] G. Viennot. Permutations ayant une forme donnée [Permutations having a given up–down sequence]. *Discrete Math.*, vol. 26 (1979), p. 279–284.

Counting permutations by inversions

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This talk is based on joint work with Anders Claesson, Christian Bean, Atli Fannar Franklin and Jay Pantone

A permutation is (*sum*) *indecomposable* if it can not be written as a direct sum of two non-empty permutations. Every permutation can be written as a direct sum of indecomposable permutations, called its *components*. The following result appears in Claesson, Jelínek, and Steingrímsson [1].

Lemma. *For a permutation of length n with c components and k inversions, we have*

$$c + k \geq n.$$

Let I_k be the set of indecomposable permutations with k inversions. The lemma implies that I_k is finite. In this talk, we will study subsets of I_k that avoid classical patterns. We use $I_k(B)$ to denote the subset of permutations in I_k that avoid all of the patterns in B. Some of our results are summarized in the table below.

B	Counting sequence of $I_k(B)$	OEIS
$\{12\}$	Characteristic function of triangular numbers	A010054
$\{123\}$	Recurrence relation	
$\{132\}$	Number of partitions of k	A000041
$\{231\}$	Number of fountains on k coins	A005169
$\{321\}$	Number of parallelogram polyominoes with k cells	A006958
$\{123, 132\}$	Number of almost triangular partitions of k	A135278
$\{123, 231\}$	Recurrence relation	
$\{132, 213\}$	Number of Gorenstein partitions of k	A117629
$\{132, 231\}$	Number of partitions of k into distinct parts	A000009
$\{132, 321\}$	Number of divisors of k	A000005

We will also mention sets defined by the avoidance of longer patterns, as well as how these counting sequences can be used to bound the usual counting sequences of permutation classes by length.

One can also consider the problem of counting sets of permutations by other statistics, besides inversions, as well as using a different restriction than indecomposability.

References

[1] Anders Claesson, and Vít Jelínek and Einar Steingrímsson. Upper bounds for the Stanley-Wilf limit of 1324 and other layered patterns, *Journal of Combinatorial Theory, Series A*, vol.119 (2012), n.8, p.1680-1691.

Sorting with restricted containers

Vincent Vatter University of Florida

This talk is based on joint work with Michael Albert and Jay Pantone

In [1], we investigated a generalization of stacks, queues, and numerous other sorting machines that we called $\mathcal{C}$-machines. Our initial work focused on showing how this viewpoint rapidly leads to functional equations (that can sometimes even be solved) for the classes of permutations that $\mathcal{C}$-machines generate. In addition, the viewpoint of $\mathcal{C}$-machines identifies several fairly well-behaved and simply defined permutation classes that do not fit any D-finite generating function of reasonable size.

In this talk, I will discuss how the viewpoint of $\mathcal{C}$-machines leads to new shape-Wilf-equivalence results, and further problems and conjectures regarding these machines.

References

[1] M. Albert, C. Homberger, J. Pantone, N. Shar, and V. Vatter. Generating permutations with restricted containers. *J. Combin. Theory Ser. A*, 157:205–232, 2018.

Dyck paths and inversion tables

Michael Wallner TU Wien

In a permutation $\pi = \pi_1\pi_2\dots\pi_n$, a pair (π_j, π_k) is called an inversion if $j < k$ and $\pi_j > \pi_k$. We define its *right-inversion table* $(r_1, r_2, \dots, r_n)$ by

$$r_i := |\{\pi_j : (i, \pi_j) \text{ is an inversion}\}|.$$

In other words, r_i is the number of elements right of i that are smaller than i. Note that an inversion table uniquely characterizes a permutation; see [4, Section 5.1.1]. From this definition, it is clear that for all $i \in [n]$ we have

$$0 \leq r_i \leq i - 1.$$

A key property is that all r_i's are mutually independent, which, e.g., directly proves that there are $n!$ permutations of size n. This observation motivates the illustration of inversion tables as *boxed staircase paths* shown in Figure 1. It consists of a *staircase path* $(EN)^n$ and unit boxes between the path and the line $y = -1$, where below each E step exactly one unit box is marked. We label the lowest row of boxes by 0, the one above by 1, etc. If box k is marked in column i then $r_i = k$. Now, the lattice path acts as an upper bound for the r_i's: the staircase path gives $r_i \leq i$. Figure 1 also illustrates two different ways to bijectively map it to a permutation.

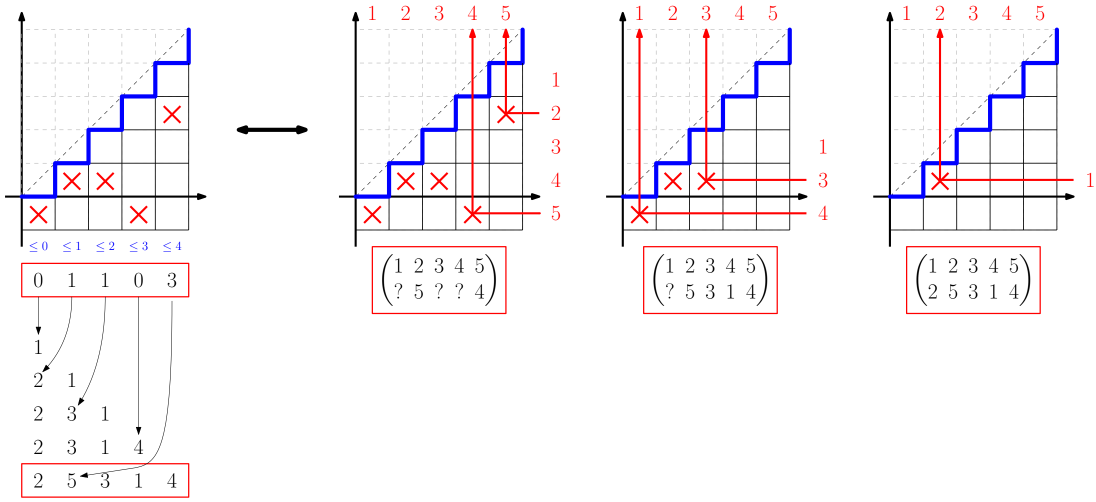

Figure 1: Top left: A visual representation of the inversion table $(0, 1, 1, 0, 3)$; top right and bottom left: two ways to bijectively map it to the permutation $(2, 5, 3, 1, 4)$.

We generalize this construction now by allowing *other* Dyck paths than the staircase path. Recall that a *Dyck path* D of length $2n$ is a path from $(0, 0)$ to (n, n) with steps $E = (1, 0)$ and $N = (0, 1)$ staying always weakly below the diagonal $y = x$. Let $y_i(D)$ be the ordinate of the ith E step in D. This brings us to the following definition.

Definition 1 (Boxed Dyck paths). A *boxed Dyck path* B is a Dyck path D in which the ith E step is decorated by a number from $\{0, \dots, y_i(D)\}$.

As for staircase paths we may associate a (restricted) right-inversion table to each boxed Dyck path that we call a *Dyck inversion table*. The upper bounds of this table are defined by the underlying Dyck path D: $r_i := y_i(D)$. For example, in the path *EENEEENNNNEN* shown in Figure 2 we have:

$$r_1 \leq 0, \qquad r_2 \leq 0, \qquad r_3 \leq 1, \qquad r_4 \leq 1, \qquad r_5 \leq 4.$$

As before, we may associate to each boxed Dyck path a permutation. The total number of permutations for a given Dyck path is $\prod_{i=1}^{n}(r_i + 1)$. For example, the Dyck path above can be associated with 20 permutations; three possible choices are shown in Figure 2. Hence, the associated Dyck path naturally imposes certain restrictions on the associated permutations (which depend on the chosen bijection).

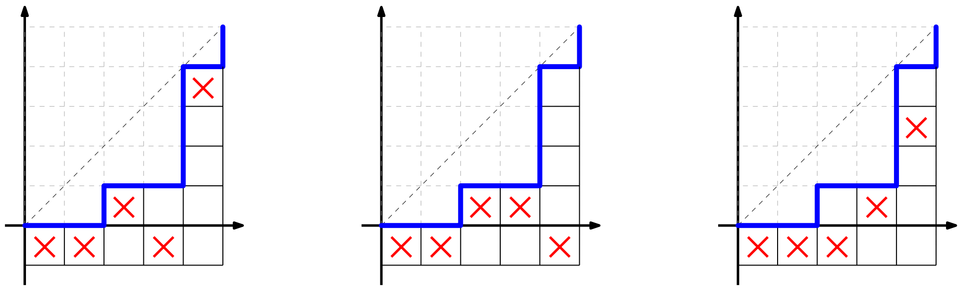

Figure 2: Three different boxed Dyck paths associated with the Dyck path *EENEEENNNNEN*.

Bijections

Inversion tables are in bijection with permutations, but also with many other combinatorial objects [3], such as regressive mappings, increasing Cayley trees, increasing plane binary trees, and many more; see Figure 3. Dyck inversion tables now allow us to define restricted classes of all these. Thereby, each Dyck path may be associated to a subclass of the respective combinatorial objects. Moreover, the full class of boxed Dyck paths, i.e., all Dyck inversion tables of fixed size, are in bijection with a class of directed acyclic graphs called relaxed binary trees [1].

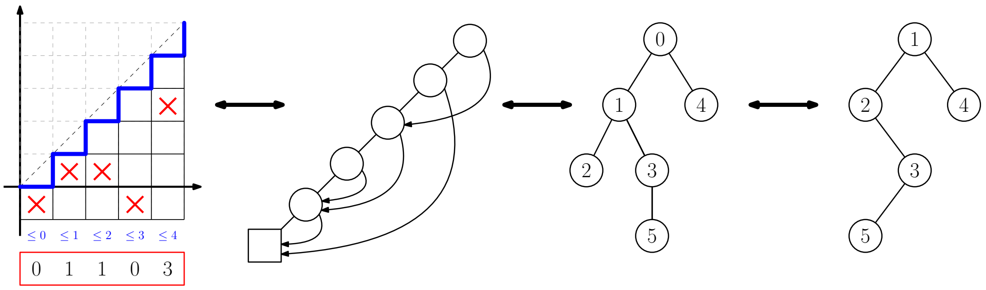

Figure 3: Four combinatorial objects that are bijectively related (from left to right): (1) Boxed staircase paths, (2) relaxed (plane) binary chains, (3) increasing (non-plane) Cayley trees, (4) increasing (plane) binary trees.

Enumeration

Let b_n be the number of boxed Dyck paths of size $2n$. Previously, we showed that

$$b_n = \Theta\left(n!\,4^n e^{3a_1 n^{1/3}} n\right),$$

where $a_1 \approx -2.338$ is the largest root of the Airy function $\mathrm{Ai}(x)$ defined as the unique function satisfying $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and $\lim_{n\to\infty} \mathrm{Ai}(x) = 0$; see [1, Theorem 1.1].

Now, boxed Dyck paths allow to combinatorially explain the involved terms. First, the term $n!$ arises due to the decorations by permutations. Second, recall that Dyck paths of length $2n$ (and, thus, binary trees of size n) are enumerated by the Catalan numbers $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n/\sqrt{\pi n^3}$. Thus, we get the following result. Note that the base of the stretched exponential is quite small: $e^{3a_1 n^{1/3}} \approx 0.0008989^{n^{1/3}}$.

Theorem 2. *The probability that a random Dyck path of length $2n$ may be decorated by an independent random permutation of size n, both drawn uniformly at random, behaves like*

$$\frac{b_n}{n!\,\mathrm{Cat}_n} = \Theta\left(e^{3a_1 n^{1/3}} n^{5/2}\right).$$

We can also consider other subclasses defined by certain patterns. For example, using the Gessel–Lindström–Viennot formula for counting non-intersecting lattice paths, we can show the following result.

Theorem 3. *The number of boxed Dyck paths with weakly increasing markers is equal to*

$$\mathrm{Cat}_n\mathrm{Cat}_{n+2} - \mathrm{Cat}_{n+1}^2 = \frac{24}{\pi}\frac{16^n}{n^5}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

which is given by OEIS A005700. This sequence is D-finite but not algebraic.

Next to restrictions on the markers, we may also restrict the allowed Dyck paths or combine both. This gives even more interesting subclasses, where some have been shown to be in bijection with different classes of directed acyclic graph classes: phylogenetic trees, compacted trees [1], and deterministic finite automata [2].

References

[1] Andrew Elvey Price, Wenjie Fang, and Michael Wallner. Compacted binary trees admit a stretched exponential. *J. Combin. Theory Ser. A*, 177:105306, 40, 2021.

[2] Andrew Elvey Price, Wenjie Fang, and Michael Wallner. Asymptotics of Minimal Deterministic Finite Automata Recognizing a Finite Binary Language. In *AofA 2020*, volume 159 of *LIPIcs*, pages 11:1–11:13, 2020.

[3] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[4] Donald E. Knuth. *The Art of Computer Programming. Vol. 3*. Addison-Wesley, Reading, MA, 1998. Sorting and Searching, Second edition.

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This talk is based on joint work with Juan B. Gil and Oscar A. Lopez

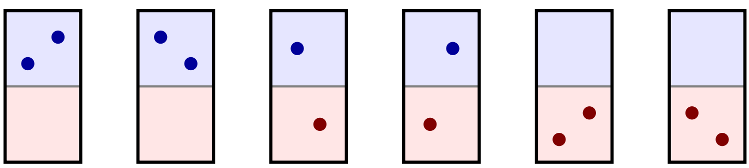
In this talk, we consider the class $S_n(1324)$ of permutations of size n that avoid the pattern 1324, and we examine the subset $S_n^{1\prec n}(1324)$ of such permutations for which 1 appears to the left of n (in one-line notation). For $n \geq 2$, we establish a connection between the set of permutations in $S_n^{1\prec n}(1324)$ having the 1 adjacent to the n, and the set of 1324-avoiding dominoes with $n-2$ points (counted by A000139 in [3]). We then introduce a constructive algorithm to enumerate the elements of $S_n^{1\prec n}(1324)$ by the position of the 1 relative to the position of the n. Time permitting, we will discuss generating functions for the enumeration of $S_n^{1\prec n}(1324, \sigma)$ for certain patterns σ of size 3 or 4, and will share some conjectures related to other positional statistics.

Bijection to dominoes

As studied in [1], a 1324-avoiding vertical *domino* is a two-cell gridded permutation in $\mathrm{Grid}^{\#}\binom{\mathrm{Av}(213)}{\mathrm{Av}(132)}$ whose underlying permutation avoids 1324.

Theorem 1. *There is a one-to-one correspondence between* 1324*-avoiding dominoes with* $n-2$ *points and permutations in* $S_n^{1\prec n}(1324)$ *having the* 1 *adjacent to the* n*.*

For example, the six distinct 1324-avoiding dominoes with two points



correspond to the permutations 1423, 1432, 3142, 2143, 2314, and 3214.

Enumeration by relative position

Let $S_{n,k}^{1\prec n}(1324)$ be the set of $\sigma \in S_n^{1\prec n}(1324)$ such that $\sigma^{-1}(n) - \sigma^{-1}(1) = k$. Then, by the above theorem, we have

$$\left|S_{n,1}^{1\prec n}(1324)\right| = \frac{2\cdot(3n-3)!}{(2n-1)!n!} \text{ for } n \geq 2.$$

Let $f(x)$ be the generating function for this sequence, and let

$$g(x,t) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} \left|S_{n,k}^{1\prec n}(1324)\right| t^k x^n.$$

Theorem 2. *We have*

$$g(x,t) = \frac{xtf(x)}{x - tf(x)}.$$

In order to prove this theorem, we introduce an operation $\odot$ that allows us to uniquely factor every element in $S^{1\prec n}_{n,k}(1324)$ as a 'product' of elements in $S^{1\prec n}_{n,1}(1324)$.

References

[1] D. Bevan, R. Brignall, A. Elvey Price, J. Pantone, A structural characterisation of Av(1324) and new bounds on its growth rate, *European J. Combin.* **88** (2020), 103115, 29 pp.

[2] S. Kitaev, *Patterns in permutations and words*, Monographs in Theoretical Computer Science, an EATCS Series, Springer, Heidelberg, 2011.

[3] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

Pattern Avoidance for k-Catalan Sequences

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The k-Catalan sequences count k-ary trees and k-ary Dyck words (e.g. OEIS A001764 for $k = 3$). We prove that they are enumerated by $(k-1)$-regular words that avoid *consistent* patterns (α, β) with $\alpha \in \{123, 132, 213, 231, 312, 321\}$ and $\beta \in \{112, 121, 122, 211, 212, 221\}$. For example, the 2-regular words with $n = 3$ distinct symbols that avoid 231 and 221,

112233 112323 113223 121233 121323 131223 132123 211233 211323 311223 312123 321123

are equinumerous to ternary trees with three nodes. Since 1-regular words are permutations that avoid all β, our result generalizes the foundational result by MacMahon and Knuth that the Catalan numbers $\mathcal{C}(n)$ count $\mathsf{Av}(\mathbf{123})$ and $\mathsf{Av}(\mathbf{231})$. In our setting there are three distinct *r-Wilf classes*, $\mathsf{Av}_r(\mathbf{123}, \mathbf{112})$, $\mathsf{Av}_r(\mathbf{231}, \mathbf{121})$, and $\mathsf{Av}_r(\mathbf{231}, \mathbf{221})$, where r indicates r-regular words. Our result is most elegantly understood in terms of pattern avoidance with both strong and weak inequalities. For example, $\mathsf{Av}(\mathbf{123})$ forbids $p_i < p_j < p_k$ for $i < j < k$, while $\mathsf{Av}_r(\mathbf{123}, \mathbf{112})$ forbids $p_i \leq p_j < p_k$. Defant and Kravitz previously established that the k-Catalan numbers $\mathcal{C}_k(n)$ count $\mathsf{Av}_r(\mathbf{231}, \mathbf{221})$ [1].

Let $\mathfrak{S}_n^r$ be the set of *r-regular words of length* $r \cdot n$, which are strings with r copies of each symbol in $[n] = \{1, 2, \ldots, n\}$. Permutations are the special case of $r = 1$ (i.e., $\mathfrak{S}_n = \mathfrak{S}_n^1$). When $r > 1$, the words have duplicated symbols, and it makes sense to consider the avoidance of patterns with duplicated symbols. For example, 2-regular words and those avoiding 112 are shown below, where Av_r generalizes Av to r-regular words,

$$\cup_{n=0}^{\infty} \mathfrak{S}_n^2 = \{\epsilon, 11, 1122, 1212, 1221, 2112, 2121, 2211, 112233, 112323, 112332, 113223, \ldots\}$$
$$\mathsf{Av}_2(\mathbf{112}) = \{\epsilon, 11, \cancel{1122}, \cancel{1212}, 1221, \cancel{2112}, 2121, 2211, \cancel{112233}, \cancel{112323}, \cancel{112332}, \cancel{113223}, \ldots\}$$

The same generalization results in *r-Wilf equivalence* where the standard symmetries of the square apply. In particular, there are two r-Wilf equivalence classes when avoiding a single pattern over the multiset $\{1, 1, 2\}$ or $\{1, 2, 2\}$,

$$\mathsf{Av}_r(\mathbf{112}) \equiv \mathsf{Av}_r(\mathbf{211}) \equiv \mathsf{Av}_r(\mathbf{221}) \equiv \mathsf{Av}_r(\mathbf{122}) \qquad \mathsf{Av}_r(\mathbf{121}) \equiv \mathsf{Av}_r(\mathbf{212}). \tag{1}$$

Classic pattern avoidance places restrictions on the relative values of symbols using strict equalities. For example, if $\sigma = s_1 s_2 \ldots s_n$, then σ avoids 123 if

$$\nexists\, i < j < k \text{ where } s_i < s_j < s_k. \tag{2}$$

By pairing a classic pattern with a *multi-pattern* (i.e., a pattern over a multiset) we can enforce weak inequalities. For example, σ avoids 123 and 112 if

$$\nexists\, i < j < k \text{ where } s_i \leq s_j < s_k. \tag{3}$$

In this paper we focus on forbidding pairs of patterns (α, β) selected from below. Among the $6 \cdot 6 = 36$ pairs, there are 12 that enforce a strong and a weak inequality as in (3) (see the proof of Corollary 4). We name these pairs *consistent*.

$$\alpha \in \{123, 132, 213, 231, 312, 321\} \text{ and } \beta \in \{112, 121, 122, 211, 212, 221\}. \tag{4}$$

Now we prove that $(k-1)$-regular words that avoid a consistent pair of patterns from 4 are counted by the k-Catalan sequence. In each case, we provide an explicit mapping from $\mathfrak{S}_n^{k-1}$ to $\mathfrak{D}_n^k$, where the latter is the set of k-ary Dyck words with n copies of 1. Note that the strings in $\mathfrak{S}_n^{k-1}$ have length $(k-1)\cdot n$, while those in $\mathfrak{D}_n^k$ have length $k \cdot n$. As a result, each map finds a unique way of adding n symbols.

Theorem 3 was previously established in [1]. The map f_3 introduced here is particularly tricky, and is accompanied by Figure 1.

Theorem 1. *The number of $(k-1)$-regular words over $[n]$ that avoid 123 and 112 is the k-ary Catalan number $C_k(n)$.*

Proof. The map $f_1 : \mathfrak{S}_n^{k-1} \to \mathfrak{D}_n^k$ scans through the word from left-to-right and creates the k-ary Dyck word by inserting 1s whenever a new smallest symbol is encountered, and 0 for every occurrence of each symbol. More specifically, if i is the new smallest symbol, and $i, i+1, \ldots, i+c$ have not yet been encountered, then $c+1$ copies of 1 are inserted. For example,

$$f_1(344431221) = 110000110000 \text{ and } f_1(332132211) = 100101000000. \tag{5}$$

The map f_1 is a bijection between the words in $\mathfrak{S}_n^{k-1}$ that avoid 123 and 112 and $\mathfrak{D}_n^k$. □

Theorem 2. *The number of $(k-1)$-regular words over $[n]$ that avoid 231 and 121 is the k-ary Catalan number $C_k(n)$.*

Proof. The map $f_2 : \mathfrak{S}_n^{k-1} \to \mathfrak{D}_n^k$ scans through the word from left-to-right and creates the k-ary Dyck word by inserting 1 for the first occurrence of each symbol, and 0 for every occurrence of each symbol. The order in which the insertions is done is important. If there is a run of c consecutive first occurrences, then all c copies of 1 are inserted before the c copies of 0. For example,

$$f_2(211244433) = 110000101000 \text{ and } f_2(111322233) = 100011000000. \tag{6}$$

The map f_2 is a bijection between the words in $\mathfrak{S}_n^{k-1}$ that avoid 231 and 121 and $\mathfrak{D}_n^k$. □

Theorem 3 ([1])**.** *The number of $(k-1)$-regular words over $[n]$ that avoid 231 and 221 is the k-ary Catalan number $C_k(n)$.*

Proof. Given a $(k-1)$-regular word σ, let $s(i)$ be the number of smaller symbols to the right of the leftmost copy of each $i \in [n]$. Define the following quantity for each $i \in [n]$,

$$r(i) = s(i) + (n-i+1) \cdot (k-1). \tag{7}$$

Let $f_3(\sigma)$ be the k-ary Dyck word of length kn with the following property: For each $1 \le i \le n$, there is a unique copy of 1 with $r(i)$ copies of 0 to its right. For example,

$$f_3(12123434) = 101000101000 \text{ and } f_3(321112233) = 111000000000. \tag{8}$$

The map f_3 is a bijection between the words in $\mathfrak{S}_n^{k-1}$ that avoid 231 and 221 and $\mathfrak{D}_n^k$. □

0	1	2
116121253223334444555666		
18	10	0

(a) The $r(i)$ quantities for $i \in [6]$. For example, the leftmost copy of 6 has $r(i) = 18$ smaller values to its right.

0+24	1+20	2+16
100101000110000000100000000000		
18+4	10+8	0+12

(b) Add $r(i)$ to the baseline quantity to determine the number of 0s to the right of each 1. For example, there are $18 + 4 = 22$ copies of 0 to the right of 1.

Figure 1: Example of f_3 for $k = 5$ and $n = 6$. The Dyck word is created by adding the $r(i)$ values to a baseline quantity $(n - i + 1) \cdot (k - 1)$. The result gives the number of 0s to the right of each 1 (and the baseline is relative to 100001000010000100001000010000).

Now we consolidate the three theorems into our main result.

Corollary 4. *The $(k-1)$-regular words that avoid α and β are counted by the k-ary Catalan numbers $C_k(n)$, whenever α and β are consistent pairs selected from* (4).

Proof. The consistent pairs partition into three cases based on standard symmetries of the square (i.e., reverse and/or inverse). The cases are summarized below along with references to their proofs and the specific pair of patterns they considered in bold.

patterns:	**(123,112)**	(132,121)	(132,122)
reverse:	(321,211)	**(231,121)**	**(231,221)**
inverse:	(321,221)	(312,212)	(213,112)
both:	(123,122)	(213,212)	(312,211)
	Theorem 1	Theorem 2	[1], Theorem 3

□

Note that the classic result by MacMahon [3] and Knuth [2] is the special case of $k = 2$ in Corollary 4. This is due to the fact that 1-regular words are permutations, and permutations avoids all $\beta \in \{112, 121, 122, 211, 212, 221\}$.

In the full version of this paper, we'll consider k-regular words avoiding all pairs of patterns in (4) (regardless of consistency). Specific results include generalizations of Fibonacci (132 and 122) and Pellian (132 and 221) numbers.

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References

[1] Colin Defant and Noah Kravitz. Stack-sorting for words. *Australasian Journal of Combinatorics*, 77(1):51–68, 2020.

[2] Donald E Knuth. The Art of Computer Programming, Vol 1: Fundamental Algorithms. *Algorithms. Reading, MA: Addison-Wesley*, 1968.

[3] Percy Alexander MacMahon. *Combinatory analysis, volumes I and II*, volume 1. American Mathematical Soc., 2001.

Inversion sequences avoiding 021 and a five-letter pattern

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This talk is based on joint work with Toufik Mansour-University of Haifa

An integer sequence $e = e_1 \cdots e_n$ of length n is called an *inversion sequence* if $0 \leq e_i < i$ for each $1 \leq i \leq n$. We use I_n to denote the set of length n inversion sequences. A pattern τ is any word of length k over the alphabet $[k] := \{0, 1, \cdots, k-1\}$. For a set of patterns B, we use $I_n(B)$ to denote the set of all inversion sequences of length n that avoids every pattern in B. The first systematic study of pattern-avoidance for inversion sequences was initiated by Mansour and Shattuck [1] and Corteel et al. [2] for the patterns of length three in 2015. We study the generating functions for inversion sequences avoiding 021 and another pattern of length five. There are 106 pattern pairs $(021, \tau)$ where τ avoids 021; for each of them, we characterize the corresponding generating trees and obtain analytic expressions for the generating functions

$$F_{(021,\tau)}(x) = \sum_{n \geq 1} |I_n(021, \tau)| x^n.$$

We use the algorithmic approach developed in [3] to determine the succession rules of the generating trees corresponding to the given pattern class. We then translate these rules into equations of generating functions and solve them using the kernel method. In the following sections, we present two theorems for the patterns 00011 and 00012 to exemplify the general results; for the complete list of results, see [5].

An enumeration procedure based on generating trees and kernel method

Let $\mathcal{I}_B = \cup_{n=1}^{\infty} I_n(B)$. We use a tree representation, denoted by $T(B)$, for the class $\mathcal{I}_B$. The tree $T(B)$ will be empty if no inversion sequence of arbitrary length avoids the patterns in B. Otherwise, the root's label will be 0, that is, $0 \in T(B)$. From the root, we construct the remainder of the tree $T(B)$ in a recursive way where the n^{th} level of the tree consists of exactly the elements of $I_n(B)$ arranged in such a way that the parent of an inversion sequence $e_1 \cdots e_n \in I_n(B)$ is the unique inversion sequence $e_1 \cdots e_{n-1} \in I_{n-1}(B)$. We obtain the children of $e_1 \cdots e_{n-1} \in I_{n-1}(B)$ from the set $\{e_1 \cdots e_{n-1} e_n \mid e_n = 0, 1, \ldots, n\}$ by applying the pattern restrictions from B. We arrange the nodes from the left to the right so that if $e_1 \cdots e_{n-1} i$ and $e' = e_1 \cdots e_{n-1} j$ are children of the same parent $e_1 \cdots e_{n-1}$, then e appears on the left of e' whenever $i < j$. We define an equivalence relation on the set of the nodes of the tree $T(B)$ and obtain a second tree representation of $\mathcal{I}_B$, denoted by $T[B]$, which is more convenient for enumerating purposes; see [3, 4] for the details. Let $T(B; e)$ denote the subtree of $T(B)$ where the inversion sequence e is its root. We say that e is *equivalent* to e', denoted by $e \sim e'$, if and only if the corresponding subtrees are isomorphic, that is, $T(B; e) \cong T(B; e')$ as plane trees. Lemma 2.1 of [3] provides a criterion to check the condition $T(B; e) \cong T(B; e')$ in finite time. The algorithm developed in [3], called KMY algorithm, provides the

[1]G. Yıldırım was partially supported by Tübitak-Ardeb project no: 120F352

rules to generate the tree $T[B]$ whose labeling uses the equivalence classes of the above equivalence relation. $T[B]$ is called the generating tree of $\mathcal{I}_B$. It is again a rooted, labeled, plane tree whose vertices are the equivalence class representations of the objects in $\mathcal{I}_B$ with the following properties: (i) each element of $\mathcal{I}_B$ appears exactly once in the tree; (ii) element of size n appears at level n in the tree; (iii) there is a set of succession rules that determine the number of children and their labels for each vertex. As the details outlined in [3, 4] for several cases, the enumeration of $I_n(B)$ is obtained by the following steps: (i) an educated guess for the rules of the generating tree based on the KMY algorithm's output, (ii) verification of the succession rules, (iii) translating rules of the generating tree into a one-parameter infinite system of equations involving related generating functions, (iv) using bivariate generating functions and obtaining a finite system of equations, (v) using kernel method to obtain an expression for the generating function of the pattern class. This note extends the results of [4] to the pairs of patterns 021 and another pattern of length five, whereas the earlier paper studied the pattern pairs 021 and another pattern of length four for which there are 33 cases. In the next sections, we present two representative results from the complete list of [5], where we studied all 106 cases.

Enumeration of $I_n(021, 00011)$

Based on the KMY algorithm [3], we first determine the succession rules for the generating tree $T[021, 00011]$. We introduce the vertex labeling symbols as $a_m = 0^m$, $b_m = 01^m$, $c_m = 001^m$, $d_m = 0^2 1^2 \dots m^2$, $e_m = 0^2 1^2 \dots (m-1)^2 m$, $f_m = 01^2 2^2 \dots m^2$, and $g_m = 01^2 2^2 ..(m-1)^2 m$, for all $m \geq 1$; and $a_e = e$ for an inversion sequence e. The generating tree $T[021, 00011]$ with the root labeled as $a_0 = 0$ has the following succession rules:

$$
\begin{array}{ll}
a_0 \rightsquigarrow a_{00} g_1, & a_{00} \rightsquigarrow a_3 e_1 a_{002}, \\
a_{002} \rightsquigarrow a_3 a_{0022} a_{002}, & a_{0022} \rightsquigarrow a_4 a_{00222} e_1 a_{002}, \\
a_{00222} \rightsquigarrow a_5 c_3 a_3 a_{0002} a_{0003}, & a_m \rightsquigarrow a_{m+1} \cdots a_3 a_{0002} a_{0003}, \\
b_m \rightsquigarrow b_{m+1} c_m a_m \cdots a_3 a_{0002} a_{0003}, \quad m \geq 3 & d_m \rightsquigarrow a_{m+4} c_{m+2} e_{m+1} \cdots e_1 a_{002}, \quad m \geq 1 \\
c_m \rightsquigarrow a_{m+3} c_{m+1} a_{m+1} \cdots a_3 a_{0002} a_{0003}, & f_m \rightsquigarrow d_m b_{m+2} g_{m+1} \cdots g_1, \quad m \geq 1 \\
e_m \rightsquigarrow a_{m+3} d_m e_m \cdots e_1 a_{002}, & g_m \rightsquigarrow e_m f_m g_m \cdots g_1, \quad m \geq 1
\end{array}
$$

As detailed in [3, 4] for similar results, these rules translate into a system of equations for the generating functions. We then use the kernel method to solve equations and obtain the generating function for the given class.

Theorem 1. *We have*

$$
\begin{aligned}
F_{(021,00011)}(x) &= \frac{(1-x-3x^2)\sqrt{1-4x} - 9x^3 - 3x^2 + 4x - 1}{2x^3\sqrt{(1+x)(1-3x)}} \\
&\quad - \frac{(2-2x+3x^2)\sqrt{1-4x} + 2x^3 - 8x^2 + 7x - 2}{2x^3}.
\end{aligned}
$$

Enumeration of $I_n(021,00012)$

We use the KMY algorithm to determine the succession rules for the generating tree $T[021,00012]$. Let $a_m = 0^m$, $b_m = 01^m$, $c_m = 001^m$, $d_m = 0^21^2\dots m^2$, $e_m = 0^21^2\dots(m-1)^2m$, $f_m = 01^22^2\dots m^2$, and $g_m = 01^22^2\dots(m-1)^2m$, for all $m \geq 1$; and $a_e = e$ for an inversion sequence e. The generating tree $T[021,00012]$ has the root $a_0 = 0$ and satisfies the following rules:

$$
\begin{aligned}
a_0 &\rightsquigarrow a_{00}g_1, & a_{00} &\rightsquigarrow a_3e_1a_{002},\\
a_{002} &\rightsquigarrow a_3a_{0022}a_{002}, & a_{0022} &\rightsquigarrow a_4e_1a_{002}a_{00222},\\
a_{00222} &\rightsquigarrow a_5c_3a_{0001}^3, & a_{0001} &\rightsquigarrow a_{0001}^2,\\
a_m &\rightsquigarrow a_{m+1}a_{0001}^m, & b_m &\rightsquigarrow b_{m+1}c_ma_{0001}^m,\\
c_m &\rightsquigarrow a_{m+3}c_{m+1}a_{0001}^{m+1}, & d_m &\rightsquigarrow a_{m+4}c_{m+2}e_{m+1}\cdots e_1a_{002},\\
e_m &\rightsquigarrow a_{m+3}d_me_m\cdots e_1a_{002}, & f_m &\rightsquigarrow d_mb_{m+2}g_{m+1}\cdots g_1,\\
g_m &\rightsquigarrow e_mf_mg_m\cdots g_1.
\end{aligned}
$$

We obtain the following result by translating these rules into the equations for generating functions and solving them by the kernel method.

Theorem 2. *We have*

$$
\begin{aligned}
F_{(021,00012)}(x) &= \frac{4x^9 - 8x^8 - 8x^7 + 10x^6 - 18x^5 + 6x^4 + 8x^3 - 9x^2 + 4x - 1}{2x^2(1+x)^2(1-x)^4(1-2x)}\\
&\quad + \frac{18x^7 - 24x^6 + 24x^5 + 8x^4 - 26x^3 + 17x^2 - 6x + 1}{2x^2(1+x)(1-x)^4(1-2x)\sqrt{(1+x)(1-3x)}}.
\end{aligned}
$$

References

[1] T. Mansour, M. Shattuck, *Pattern avoidance in inversion sequences*, Pure Math. Appl. 25 (2), 157–176, 2015.

[2] S. Corteel, M.A. Martinez, C.D. Savage, M. Weselcouch, *Patterns in inversion sequences I*, Discrete Math. Theor. Comput. Sci. 18 (2), 2016.

[3] I. Kotsireas, T. Mansour, and G. Yıldırım, *An algorithmic approach based on generating trees for enumerating pattern-avoiding inversion sequences*, preprint.

[4] T. Mansour, G. Yıldırım, *Inversion sequences avoiding* 021 *and another pattern of length four* preprint.

[5] T. Mansour and G. Yıldırım, Counting $I_n(\{021,\tau\})$ whenever τ is a 5-letter pattern that avoids 021.
http://math.haifa.ac.il/toufik/enumerative/paper021x5letters.pdf

View of the spire of Dijon Cathedral Saint-Bénigne, showing roofs with polychrome tiles. Photo by Jochen Jahnke at German Wikipedia.

The cover photo of glazed roof tiles of Hôtel de Vogüé à Dijon is provided by François de Dijon at Wikipedia.