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New criteria for oscillation of nonlinear neutral differential equations

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Abstract

The aim of this work is to offer sufficient conditions for the oscillation of neutral differential equation second order

$$(r(t)[(y(t)+p(t)y(\tau(t)))']^{\gamma})'+f(t,y(\sigma(t)))=0,$$

where $\int^{\infty} r^{-1/\gamma}(s)\,\mathrm{d}s=\infty$. Based on the comparison with first order delay equations and by employ the Riccati substitution technique, we improve and complement a number of well-known results. Some examples are provided to show the importance of these results.

MSC: 34C10; 34K11

Keywords: Neutral delay differential equations; Oscillation criteria

1 Introduction

In this paper, we are interested in finding the sufficient conditions which ensure that the solutions of the equation

$$\bigl(r(t)\bigl[\bigl(y(t)+p(t)y(\tau(t))\bigr)'\bigr]^{\gamma}\bigr)'+f\bigl(t,y(\sigma(t))\bigr)=0 \tag{1.1}$$

are oscillatory, where $t\geq t_0$. Throughout this work, we suppose that:

($\boldsymbol{\Pi}_1$) $\gamma=a/b$ and a, b are odd positive integers;

($\boldsymbol{\Pi}_2$) $r,p\in C([t_0,\infty),\mathbb{R})$, $0\leq p(t)<1$, $r(t)>0$,

$$\eta(t,s):=\int_s^t r^{-1/\gamma}(u)\,\mathrm{d}u,\quad s\leq t,$$

and $\eta(\infty,t_0)=\infty$;

($\boldsymbol{\Pi}_3$) $\tau,\sigma\in C([t_0,\infty),\mathbb{R})$, $\tau(t)\leq t$, $0<\sigma(t)\leq t$, and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$;

($\boldsymbol{\Pi}_4$) $f(t,y)\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, $yf(t,y)>0$ for all $y\neq 0$, there exists $q\in C([t_0,\infty),(0,\infty))$ such that $|f(t,y)|\geq q(t)|y|^{\beta}$ and β is a quotient of odd positive integers.

Here, we define

$$u(t):=\bigl(y+py(\tau)\bigr)(t). \tag{1.2}$$

For a solution of (1.1), we propose a nontrivial function $y \in C([t_y,\infty),\mathbb{R})$, $t_y \geq t_0$, which satisfies (1.1) on $[t_y,\infty)$, and has the property $u(t)$ and $r(t)(u'(t))^{\gamma}$ are continuously differentiable for $t \in [t_y,\infty)$. We only focus on solutions of (1.1) which exist on $[t_0,\infty)$ and satisfy $\sup\{|y(t)| : t_y \leq t\} > 0$ for every $t \geq t_y$. If y is neither eventually positive nor eventually negative, then $y(t)$ is called oscillatory, otherwise it is called non-oscillatory.

Differential equations with neutral argument have interesting applications in problems of real world life. In the networks containing lossless transmission lines, the neutral differential equations appear in the modeling of these phenomena as is the case in high-speed computers. In addition, second order neutral equations appear in the theory of automatic control and in aeromechanical systems, in which inertia plays an important role. Moreover, second order delay equations play an important role in studying vibrating masses attached to an elastic bar, as the Euler equation, see [1, 2, 7]. One area of active research in recent times is to study the sufficient criterion for oscillation of delay differential equations, see [1–30].

Over this decade, a great amount of work has been done on development the oscillation theory of second order delay and advanced equations, see [3, 4, 9, 11–14, 16, 17, 22], and the oscillation theory of higher order delay equations, see [8, 10, 18, 19, 21, 24–26, 29].

In particular, by using the comparison technique, the equation

$$\left(r\left[u'\right]^{\gamma}\right)'(t) + \left(qy^{\beta}(\sigma)\right)(t) = 0 \tag{1.3}$$

have been studied by Baculikova and Dzurina in [6] when $\gamma \geq \beta$, $\sigma(t)$ and $\tau(t)$ are nondecreasing, $\tau(\sigma(t)) = \sigma(\tau(t))$. By using the Riccati transformation technique, in [23, 28, 30], the oscillatory properties of solutions of the equation

$$\left(r\left|u'\right|^{\gamma-1}u'\right)'(t) + \left(q\left|y(\sigma)\right|^{\beta-1}y(\sigma)\right)(t) = 0 \tag{1.4}$$

have been considered. Liu et al. [23] studied the oscillation properties for (1.4) under the conditions $\gamma \geq \beta$, $r'(t) > 0$, and $\sigma'(t) > 0$. Zeng et al. [30] used the technique of Riccati transformation to obtain oscillation conditions for (1.4), which improves the results in [23]. Under a more general case, namely for all $\gamma > 0$ and $\beta > 0$, Wu et al. [28] studied the oscillation criteria of equation (1.4).

The purpose of this work is to contribute to the development of the oscillation theory of second order nonlinear equations with delay argument. Firstly, by using comparison theorems that compare the second order equation with first order delay equation, we obtain two different conditions to ensure oscillation of (1.1) when $\gamma < \beta$ and $\gamma > \beta$. The results of this part improve and complement the results in [6].

Secondly, we present a new result for oscillation of (1.1) by using the technique of Riccati transformation, which improves the related results reported in [23, 28]. In order to show the importance of our results, we introduce two examples and compare the results in this paper with the previous results.

We will need the following two lemmas in the next parts.

Lemma 1.1 ([5, Lemma 3]) *If the function w satisfies $w > 0$, $w' > 0$, and $w'' \leq 0$ for $t \geq t_0$, then there exists $t_\mu \geq t_0$ such that*

$$w(\sigma)(t) \geq \mu \frac{1}{t}\sigma(t)w(t)$$

for all $\mu \in (0,1)$.

Lemma 1.2 *Assume that* $\Psi(s) := ks - ls^{1+1/\gamma}$, *where* k *and* l *are real constants*, $l > 0$ *and* γ *is defined as* (Π_1). *Then we have*

$$\Psi(s) \leq \max_{s\in\mathbb{R}} \Psi(s) = \frac{1}{\gamma}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} k^{\gamma+1} l^{-\gamma}.$$

2 Main results

Throughout this paper, we will be employing the following notation:

$$\begin{aligned}
G(t) &:= q(t)\bigl(1 - p\bigl(\sigma(t)\bigr)\bigr)^{\beta}, \\
\Theta(t) &:= \eta(t,t_0) + \frac{\mu^{\beta}}{\gamma}\int_{t_0}^{t} \frac{\sigma^{\beta}(\nu)}{\nu^{\beta}} \rho(\nu)\eta^{1+\gamma}(\nu,t_0)G(\nu)\,\mathrm{d}\nu, \\
\widehat{\Theta}(t) &:= \eta(t,t_1) + \frac{C^{\beta-\gamma}}{\gamma}\int_{t_1}^{t} \eta(\nu,t_1)\eta^{\gamma}\bigl(\sigma(\nu),t_1\bigr)G(\nu)\,\mathrm{d}\nu,
\end{aligned}$$

and

$$\rho(t) := \begin{cases} \lambda_1 & \text{if } \gamma < \beta; \\ 1 & \text{if } \gamma = \beta; \\ \lambda_2 \eta^{\beta-\gamma}(t,t_0) & \text{if } \gamma > \beta, \end{cases}$$

where $\mu \in (0,1)$ and C, λ_1, and λ_2 are positive real constants.

Theorem 2.1 *Assume that* $r'(t) \geq 0$. *If every solution of*

$$v'(t) + G(t)\Theta^{\beta}\bigl(\sigma(t)\bigr)v^{\beta/\gamma}\bigl(\sigma(t)\bigr) = 0 \tag{2.1}$$

oscillates for any $\lambda_1, \lambda_2 > 0$, *then every solution of* (1.1) *oscillates.*

Proof Assume the contrary and suppose that equation (1.1) has a nonoscillatory solution y on $[t_0,\infty)$. Now, we suppose, without loss of generality, that $y > 0$, $y(\tau) > 0$, and $y(\sigma) > 0$ for $t \geq t_1 \geq t_0$. Hence, we find $u(t) > 0$ and $(r(t)(u'(t))^{\gamma})' \leq 0$ for $t \geq t_1$. From [6, Lemma 3], we have that $u'(t) > 0$ for $t \geq t_1$. From definition (1.2), we get

$$\begin{aligned}
y(t) &= u(t) - p(t)y\bigl(\tau(t)\bigr) \\
&\geq u(t) - p(t)u\bigl(\tau(t)\bigr) \\
&\geq \bigl(1 - p(t)\bigr)u(t).
\end{aligned} \tag{2.2}$$

From (1.1), (Π_4), and (2.2), we obtain

$$\begin{aligned}
\bigl(r(t)\bigl(u'(t)\bigr)^{\gamma}\bigr)' &= -f\bigl(t,y\bigl(\sigma(t)\bigr)\bigr) \leq -q(t)y^{\beta}\bigl(\sigma(t)\bigr) \\
&\leq -G(t)u^{\beta}\bigl(\sigma(t)\bigr).
\end{aligned} \tag{2.3}$$

Since $(r(t)(u'(t))^{\gamma})' \leq 0$, we see that the function $U(t) := r^{1/\gamma}(t)u'(t)$ is nonincreasing, and hence

$$\begin{aligned} u(t) &= u(t_1) + \int_{t_1}^{t} u'(\nu)\,\mathrm{d}\nu = u(t_1) + \int_{t_1}^{t} \frac{1}{r^{1/\gamma}(\nu)} U(\nu)\,\mathrm{d}\nu \\ &\geq U(t) \int_{t_1}^{t} \frac{1}{r^{1/\gamma}(\nu)}\,\mathrm{d}\nu = \eta(t,t_1)U(t). \end{aligned} \tag{2.4}$$

By a simple computation, we note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigl(u(t) - \eta(t,t_1)U(t)\bigr) &= u'(t) - \eta'(t,t_1)r^{1/\gamma}(t)u'(t) - \eta(t,t_1)\bigl(r^{1/\gamma}(t)u'(t)\bigr)' \\ &= -\eta(t,t_1)U'(t) \end{aligned} \tag{2.5}$$

and

$$\bigl(r(t)\bigl(u'(t)\bigr)^{\gamma}\bigr)' = \frac{\mathrm{d}}{\mathrm{d}t} U^{\gamma}(t) = \gamma U^{\gamma-1}(t)U'(t),$$

which with (2.3) yields

$$\begin{aligned} -\eta(t,t_1)U'(t) &= -\frac{1}{\gamma}\eta(t,t_1)U^{1-\gamma}(t)\bigl(r(t)\bigl(u'(t)\bigr)^{\gamma}\bigr)' \\ &\geq \frac{1}{\gamma}\eta(t,t_1)G(t)U^{1-\gamma}(t)u^{\beta}\bigl(\sigma(t)\bigr). \end{aligned} \tag{2.6}$$

Combining (2.5) with (2.6), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(u(t) - \eta(t,t_1)U(t)\bigr) \geq \frac{1}{\gamma}\eta(t,t_1)G(t)U^{1-\gamma}(t)u^{\beta}\bigl(\sigma(t)\bigr). \tag{2.7}$$

Since $r'(t) \geq 0$ and $(r(t)(u'(t))^{\gamma})' \leq 0$, we have that $u''(t) \leq 0$. Thus, from Lemma 1.1, we obtain

$$u\bigl(\sigma(t)\bigr) \geq \mu \frac{\sigma(t)}{t} u(t) \tag{2.8}$$

for all $t \geq t_{\mu}$. From (2.4), (2.7), and (2.8), we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigl(u(t) - \eta(t,t_1)U(t)\bigr) &\geq \frac{\mu^{\beta}}{\gamma}\frac{\sigma^{\beta}(t)}{t^{\beta}}\eta(t,t_1)G(t)U^{1-\gamma}(t)u^{\beta}(t) \\ &= \frac{\mu^{\beta}}{\gamma}\frac{\sigma^{\beta}(t)}{t^{\beta}}\eta(t,t_1)G(t)U^{1-\gamma}(t)u^{\beta-\gamma}(t)u^{\gamma}(t) \\ &\geq \frac{\mu^{\beta}}{\gamma}\frac{\sigma^{\beta}(t)}{t^{\beta}}\eta^{1+\gamma}(t,t_1)G(t)U(t)u^{\beta-\gamma}(t) \end{aligned} \tag{2.9}$$

for $t \geq \max\{t_1, t_{\mu}\}$. Now, since $u(t)$ is positive and increasing, we have that $u(t) \geq u(t_2) \geq m > 0$ for $t \geq t_2 \geq t_1$. Moreover, since $r(t)(u'(t))^{\gamma}$ is positive and decreasing, we see that $r(t)(u'(t))^{\gamma} \leq r(t_2)(u'(t_2))^{\gamma} = M$ for $t \geq t_2$, and hence

$$u(t) \leq u(t_2) + M^{1/\gamma}\eta(t,t_2). \tag{2.10}$$

Since $\eta(\infty, t_1) = \infty$, there exist constant $N > 0$ and $t_N > t_\mu$ such that $\eta(t, t_1) > N$ for all $t \geq t_N$. Hence, from (2.10), we find

$$u(t) \leq K\eta(t, t_2),$$

where $K := (\frac{1}{N}u(t_2) + M^{1/\gamma})$. Then we can pick $t_2 \geq t_N$ sufficiently large such that

$$u^{\beta-\gamma}(t) \geq \begin{cases} \lambda_1 & \text{if } \gamma < \beta; \\ 1 & \text{if } \gamma = \beta; \\ \lambda_2 \eta^{\beta-\gamma}(t, t_2) & \text{if } \gamma > \beta, \end{cases} \tag{2.11}$$

for $t \geq t_2$, where $\lambda_1 = m^{\beta-\gamma}$ and $\lambda_2 = K^{\beta-\gamma}$. Combining (2.11) with (2.9), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(u(t) - \eta(t, t_1)U(t)\big) \geq \frac{\mu^\beta}{\gamma}\frac{\sigma^\beta(t)}{t^\beta}\rho(t)\eta^{1+\gamma}(t, t_1)G(t)U(t).$$

By integrating this inequality from t_2 to t, we obtain

$$u(t) \geq \eta(t, t_1)U(t) + \frac{\mu^\beta}{\gamma}\int_{t_2}^{t}\frac{\sigma^\beta(\nu)}{\nu^\beta}\rho(\nu)\eta^{1+\gamma}(\nu, t_1)G(\nu)U(\nu)\,\mathrm{d}\nu.$$

From the monotonicity of $U(t)$, we get

$$u(t) \geq U(t)\left(\eta(t, t_1) + \frac{\mu^\beta}{\gamma}\int_{t_2}^{t}\frac{\sigma^\beta(\nu)}{\nu^\beta}\rho(\nu)\eta^{1+\gamma}(\nu, t_1)G(\nu)\,\mathrm{d}\nu\right). \tag{2.12}$$

This means that

$$u\big(\sigma(t)\big) \geq U\big(\sigma(t)\big)\Theta\big(\sigma(t)\big),$$

which together with (2.3) implies that

$$\big(U^\gamma(t)\big)' + G(t)\Theta^\beta\big(\sigma(t)\big)\big(U^\gamma\big(\sigma(t)\big)\big)^{\beta/\gamma} \leq 0. \tag{2.13}$$

We can see that $\nu(t) = U^\gamma(t)$ is a positive solution of the first order delay differential inequality (2.13). In view of [27, Lemma 1], the associated delay differential equation (2.1) also has a positive solution. This contradiction completes the proof. □

Theorem 2.2 *If every solution of*

$$\nu'(t) + G(t)\widehat{\Theta}^\beta\big(\sigma(t)\big)\nu^{\beta/\gamma}\big(\sigma(t)\big) = 0 \tag{2.14}$$

oscillates for all $C > 0$, *then every solution of* (1.1) *oscillates.*

Proof As in the proof of Theorem 2.1, we get that (2.2)–(2.7) hold. Now, we consider the following cases: In the case where $\gamma > \beta$, combining (2.4) with (2.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(u(t) - \eta(t, t_1)U(t)\big) \geq \frac{1}{\gamma}\eta(t, t_1)\eta^\beta\big(\sigma(t), t_1\big)G(t)U^{1-\gamma}(t)U^\beta\big(\sigma(t)\big).$$

Since $\sigma(t) \leq t$ and $U'(t) \leq 0$, we have $U(\sigma(t)) \geq U(t)$, and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(u(t) - \eta(t,t_1)U(t)\bigr) \geq \frac{1}{\gamma}\eta(t,t_1)\eta^{\beta}\bigl(\sigma(t),t_1\bigr)G(t)U(t)U^{\beta-\gamma}(t).$$

From the fact that $U(t)$ is positive and nonincreasing, we get that $U(t) \leq A$, where $A > 0$ and $t \geq t_2 \geq t_1$. Hence, $U^{\beta-\gamma}(t) \geq A^{\beta-\gamma}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(u(t) - \eta(t,t_1)U(t)\bigr) \geq \frac{A^{\beta-\gamma}}{\gamma}\eta(t,t_1)\eta^{\beta}\bigl(\sigma(t),t_1\bigr)G(t)U(t). \tag{2.15}$$

In the case where $\gamma \leq \beta$, using the facts $u(t) > 0$ and $u'(t) > 0$, we have $u(t) \geq B > 0$ for t sufficiently large. It follows from (2.4) and (2.7) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigl(u(t) - \eta(t,t_1)U(t)\bigr) &\geq \frac{1}{\gamma}\eta(t,t_1)G(t)U^{1-\gamma}(t)u^{\gamma}\bigl(\sigma(t)\bigr)u^{\beta-\gamma}\bigl(\sigma(t)\bigr) \\
&\geq \frac{B^{\beta-\gamma}}{\gamma}\eta(t,t_1)G(t)U^{1-\gamma}(t)u^{\gamma}\bigl(\sigma(t)\bigr) \\
&\geq \frac{B^{\beta-\gamma}}{\gamma}\eta(t,t_1)\eta^{\gamma}\bigl(\sigma(t),t_1\bigr)G(t)U^{1-\gamma}(t)U^{\gamma}\bigl(\sigma(t)\bigr) \\
&\geq \frac{B^{\beta-\gamma}}{\gamma}\eta(t,t_1)\eta^{\gamma}\bigl(\sigma(t),t_1\bigr)G(t)U(t).
\end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(u(t) - \eta(t,t_1)U(t)\bigr) \geq \frac{C^{\beta-\gamma}}{\gamma}\eta(t,t_1)\eta^{\gamma}\bigl(\sigma(t),t_1\bigr)G(t)U(t), \tag{2.17}$$

where $C = \min\{A,B\}$. The rest of the proof is similar to that of Theorem 2.1 and so we omit it. □

For the oscillatory behavior of

$$v'(t) + Q(t)v^{\alpha}\bigl(\sigma(t)\bigr) = 0, \tag{2.18}$$

where Q is a positive continuous function, Erbe et al. [15] and Ladde et al. [20] showed that every solution of (2.18) is oscillatory if and only if

$$\int_{t_0}^{\infty} Q(\nu)\,\mathrm{d}\nu = \infty \quad \text{for all } \alpha \in (0,1). \tag{2.19}$$

Moreover, Baculikova and Dzurina [6] proved that equation (2.18) is oscillatory if $\alpha \in (0,1]$ and

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} Q(\nu)\,\mathrm{d}\nu > \frac{1}{\mathbf{e}}.$$

For the case $\alpha > 1$, Tang [27] studied the oscillation behavior of (2.18). In the following, by using the results of [6, 15], and [27], we obtain new criteria for oscillation of solutions of (1.1).

Corollary 2.1 *Assume that $\gamma > \beta$. If*

$$\int_{t_0}^{\infty} G(\nu)\widehat{\Theta}^{\beta}\bigl(\sigma(\nu)\bigr)\,\mathrm{d}\nu = \infty, \tag{2.20}$$

then every solution of (1.1) *oscillates.*

Proof From [20], the associated delay differential equation (2.14) is oscillatory if and only if (2.20) holds. □

Corollary 2.2 *Assume that $\gamma \geq \beta$. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} G(\nu)\widehat{\Theta}^{\beta}\bigl(\sigma(\nu)\bigr)\,\mathrm{d}\nu > \frac{1}{\mathbf{e}}, \tag{2.21}$$

then every solution of (1.1) *oscillates.*

Proof In view of [6, Lemma 4], the first order delay equation (2.14) is oscillatory if (2.21) holds. □

Corollary 2.3 *Assume that $\gamma < \beta$. Let there exist a function $\psi \in C^1([t_0,\infty),\mathbb{R})$ such that* $\psi' > 0$, $\lim_{t\to\infty}\psi(t) = \infty$,

$$\limsup_{t\to\infty} \frac{\beta\psi'(\sigma(t))\sigma'(t)}{\gamma\psi'(t)} < 1,$$

and

$$\liminf_{t\to\infty} \left(\frac{1}{\psi'(t)} e^{-\psi(t)} G(t)\widehat{\Theta}^{\beta}\bigl(\sigma(t)\bigr)\right) > 0 \tag{2.22}$$

for any $C > 0$, then every solution of (1.1) *oscillates.*

Proof By using the results of [27] in Theorem 1, we get that equation (2.14) is oscillatory if (2.22) holds. □

In the next theorem, we use the technique of Riccati to get a new oscillation condition for equation (1.1).

Theorem 2.3 *Assume that there exists a function $\varphi \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\int_{t_1}^{\infty} \left(\varphi(\nu)\widehat{G}(\nu) - \frac{r(\nu)(\varphi'_{+}(\nu))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\varphi^{\gamma}(\nu)}\right)\mathrm{d}\nu = \infty \tag{2.23}$$

for some sufficiently large $t_1 \geq t_0$, where

$$\widehat{G}(t) := G(t)\rho(t)\exp\left(-\gamma\int_{\sigma(t)}^{t} \frac{1}{r^{1/\gamma}(\nu)\Theta(\nu)}\,\mathrm{d}\nu\right)$$

and $\varphi'_{+}(t) = \max\{\varphi'(t),0\}$, then every solution of (1.1) *oscillates.*

Proof As in the proof of Theorem 2.1, we get (2.2)–(2.12) hold. From (2.12), we deduce that

$$\frac{u'(t)}{u(t)} \leq \frac{1}{r^{1/\gamma}(t)\Theta(t)}.$$

By integrating this inequality from $\sigma(t)$ to t, we get

$$u\bigl(\sigma(t)\bigr) \geq u(t)\exp\biggl(-\int_{\sigma(t)}^{t} \frac{1}{r^{1/\gamma}(\nu)\Theta(\nu)}\,\mathrm{d}\nu\biggr). \tag{2.24}$$

Now, we define the following function:

$$R(t) := \varphi(t)r(t)\biggl(\frac{u'(t)}{u(t)}\biggr)^{\gamma}, \quad t \geq t_1. \tag{2.25}$$

Then we have

$$R'(t) = \frac{\varphi'(t)}{\varphi(t)}R(t) + \varphi(t)\biggl(\frac{(r(t)(u'(t))^{\gamma})'}{u^{\gamma}(t)} - \gamma r(t)\biggl(\frac{u'(t)}{u(t)}\biggr)^{\gamma+1}\biggr). \tag{2.26}$$

By (2.3) and (2.11), we deduce that

$$\begin{aligned} \frac{(r(t)(u'(t))^{\gamma})'}{u^{\gamma}(t)} &\leq -G(t)\frac{u^{\beta}(\sigma(t))}{u^{\gamma}(t)} \\ &= -G(t)\frac{u^{\gamma}(\sigma(t))}{u^{\gamma}(t)}u^{\beta-\gamma}\bigl(\sigma(t)\bigr) \\ &\leq -G(t)\rho(t)\exp\biggl(-\gamma\int_{\sigma(t)}^{t} \frac{1}{r^{1/\gamma}(\nu)\Theta(\nu)}\,\mathrm{d}\nu\biggr) \\ &= -\widehat{G}(t). \end{aligned} \tag{2.27}$$

By the definition (2.25), we obtain

$$\biggl(\frac{u'(t)}{u(t)}\biggr)^{\gamma+1} = \frac{1}{r^{1+1/\gamma}(t)\varphi^{1+1/\gamma}(t)}R^{1+1/\gamma}(t). \tag{2.28}$$

From (2.26)–(2.28), we find

$$R'(t) \leq -\varphi(t)\widehat{G}(t) + \frac{\varphi'(t)}{\varphi(t)}R(t) - \frac{\gamma}{r^{1/\gamma}(t)\varphi^{1/\gamma}(t)}R^{1+1/\gamma}(t).$$

Next, by using Lemma 1.2 with $k = \varphi'_{+}/\varphi$, $l = \gamma r^{-1/\gamma}\varphi^{-1/\gamma}$ and $s = R$, we have

$$R'(t) \leq -\varphi(t)\widehat{G}(t) + \frac{r(t)(\varphi'_{+}(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\varphi^{\gamma}(t)}.$$

Integrating the latter inequality from t_1 to t, we get

$$\int_{t_1}^{t}\biggl(\varphi(\nu)\widehat{G}(\nu) - \frac{r(\nu)(\varphi'_{+}(\nu))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\varphi^{\gamma}(\nu)}\biggr)\,\mathrm{d}\nu \leq R(t_1) - R(t) \leq R(t_1),$$

which contradicts (2.23). Therefore, equation (1.1) is oscillatory. □

Example 2.1 Consider the equation

$$\left(\left[\left(y(t)+p_0 y(\delta t)\right)'\right]^{\gamma}\right)' + \frac{q_0}{t^{(\gamma+3)/2}} y(\lambda t) = 0, \tag{2.29}$$

where $t > 0$, $\gamma \geq 1$ is a quotient of odd positive integers, $\beta = 1$, $p_0 \in [0,1)$, $\delta, \lambda \in (0,1]$, and $q_0 > 0$. By using Theorem 2.2, we get that equation (2.29) is oscillatory if every solution of

$$v'(t) + \phi(t) v^{1/\gamma}(\lambda t) = 0$$

oscillates, where

$$\phi(t) := q_0(1-p_0)\left(\frac{\lambda}{t^{(\gamma+1)/2}} + q_0(1-p_0)\lambda^{(3\gamma+1)/2} \frac{2C^{1-\gamma}}{\gamma(\gamma+1)} \frac{1}{t}\right)$$

for all $C > 0$.

By Corollary 2.1, if $\gamma > 1$, then

$$\int_{t_0}^{\infty} \phi(v)\, \mathrm{d}v = \infty,$$

and hence equation (2.29) is oscillatory. Obviously, the results of Baculikova and Dzurina in [6] fail to apply on this equation.

Let $\gamma = 1$. From Corollary 2.2, we obtain that equation (2.29) is oscillatory if

$$q_0(1-p_0)\left(\lambda + q_0(1-p_0)\lambda^2\right) \ln \frac{1}{\lambda} > \frac{1}{\mathbf{e}}.$$

As a special case of equation (2.29), the equation

$$\left(y(t) + \frac{1}{2} y\left(\frac{1}{2} t\right)\right)'' + \frac{q_0}{t^2} y\left(\frac{1}{3} t\right) = 0 \tag{2.30}$$

is oscillatory if $q_0 > 1.58856$. By applying Corollary 2 in [6], the known related criterion for (2.30) is $q_0 > 5.44381$. On the other hand, equation (2.30) has nonoscillatory solution $y(t) := \sqrt{t}$ when

$$q = \frac{\sqrt{3}}{16}(4+\sqrt{2}) < 1.58856.$$

Example 2.2 Consider the equation

$$\left(y(t) + \frac{1}{2} y(\delta t)\right)'' + \frac{q_0}{t^2} y(\lambda t) = 0, \tag{2.31}$$

where $t > 0$ and $\delta, \lambda \in (0,1]$, the known related criteria for oscillation of this equation are as follows:

1. By applying Corollary 2 in [6],we get

$$q_0 \lambda \ln \frac{1}{2\lambda} > \frac{2}{\mathrm{e}}; \tag{C1}$$

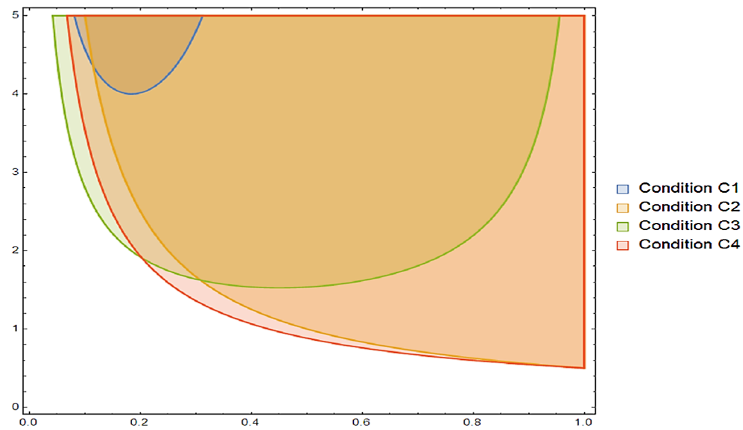

Figure 1 Regions for which Conditions (C1)–(C4) are satisfied

2. By applying Theorem 1 in [28] or Theorem 2.1 in [23],we obtain

$$q_0\lambda > \frac{1}{2}; \tag{C2}$$

3. By applying our results in Corollary 2.2, we have

$$\frac{1}{2}q_0\lambda\left(1+\frac{1}{2}\lambda q_0\right)\ln\frac{1}{\lambda} > \frac{1}{e}; \tag{C3}$$

4. By applying our results in Theorem 2.3, we find

$$q_0\lambda^{1/(1+\lambda q_0/2)} > \frac{1}{2}. \tag{C4}$$

In Fig. 1, we test the strength of oscillation criteria (C1)–(C4).

Remark 2.1 From the previous examples, we note that:

– By using the technique of comparison with first order delay equations, Corollary 2.1 improves Corollary 2 in [6].
– Based on the technique of Riccati transformation, Theorem 2.3 improves Theorem 1 in [28] and Theorem 2.1 in [23].
– Condition (C3) supports the most efficient condition for values of $\lambda \in (0, 0.2)$, and Condition (C4) supports the most efficient condition for values of $\lambda \in (0.2, 1)$.

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