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Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives

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Abstract

In this paper, we obtain Hermite–Hadamard-type inequalities of convex functions by applying the notion of q^b-integral. We prove some new inequalities related with right-hand sides of q^b-Hermite–Hadamard inequalities for differentiable functions with convex absolute values of second derivatives. The results presented in this paper are a unification and generalization of the comparable results in the literature on Hermite–Hadamard inequalities.

Keywords: Hermite–Hadamard inequality; q-integral; Quantum calculus; Convex function

1 Introduction

The Hermite–Hadamard inequality introduced by Hermite and Hadamard (see also [1] and [2, p. 137]) is one of the most well-known inequalities in the theory of convex functional analysis. It has an interesting geometrical interpretation with several applications.

These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed manner if f is a concave function. Note that the Hermite–Hadamard inequalities may be viewed as a refinement of the concept of convexity and follows from Jensen's inequality. Hermite–Hadamard inequalities for convex functions have received much attention in the recent years, and, consequently, a remarkable variety of refinements and generalizations have been obtained.

Many well-known integral inequalities such as the Hölder, Hermite–Hadamard, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Gruss, Gruss-Chebyshev, and other integral inequalities have been studied in the setup of q-calculus using the concept of classical convexity. For more results in this direction, we refer to [3–18].

The purpose of this paper is to study Hermite–Hadamard-like inequalities for convex functions by applying the new concept of q^b-integral. We also discuss the relation of our results with comparable results existing in the literature.

The organization of this paper is as follows. In Sect. 2, we give a brief description of the concepts of q-calculus and some related works in this direction. In Sect. 3, we present the Hermite-Hadamard-type inequalities for the q^b-integrals. We also study the relation between the results presented herein and comparable results in the literature. Section 4 contains some conclusions and further directions for the future research. We believe that the study initiated in this paper may inspire new research in this area.

2 Preliminaries of q-calculus and some inequalities

In this section, we first present some known definitions and related inequalities in q-calculus. Set the following notation (see [19]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad q \in (0,1).$$

Jackson [20] defined the q-Jackson integral of a given function f from 0 to b as follows:

$$\int_0^b f(x)\, d_q x = (1-q)b \sum_{n=0}^{\infty} q^n f\bigl(bq^n\bigr), \quad \text{where } 0 < q < 1, \tag{2.1}$$

provided that the sum converges absolutely.

Jackson [20] defined the q-Jackson integral of a given function over the interval $[a,b]$ as follows:

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x.$$

Definition 1 ([21]) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the q_a-derivative of f at $x \in [a,b]$ is identified as

$$_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{2.2}$$

Since $f : [a,b] \to \mathbb{R}$ is a continuous function, we can define

$$_aD_q f(a) = \lim_{x\to a} {_aD_q f(x)}.$$

The function f is said to be q_a-differentiable on $[a,b]$ if $_aD_q f(x)$ exists for all $x \in [a,b]$. If we take $a = 0$ in (2.2), then we have $_0D_q f(x) = D_q f(x)$, where $D_q f(x)$ is the q-derivative of f at $x \in [0,b]$ (see [19]) given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.$$

Definition 2 ([22]) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the q^b-derivative of f at $x \in [a,b]$ is given by

$$^bD_q f(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}, \quad x \neq b.$$

Definition 3 Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then the second q^b-derivative of f at $x\in[a,b]$ is given by

$$\begin{aligned}{}^bD_q^2f(x)&={}^bD_q\big({}^bD_qf(x)\big)\\&=\frac{f(q^2ta+(1-tq^2)b)-(1+q)f(qta+(1-qt)b)+qf(ta+(1-t)b)}{(1-q)^2q(b-a)^2t^2}.\end{aligned}$$

Definition 4 ([21]) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then the q_a-definite integral on $[a,b]$ is defined by

$$\begin{aligned}\int_a^b f(x)\,{}_ad_qx&=(1-q)(b-a)\sum_{n=0}^{\infty}q^nf\big(q^nb+\big(1-q^n\big)a\big)\\&=(b-a)\int_0^1 f\big((1-t)a+tb\big)\,d_qt.\end{aligned}$$

Alp et al. [3] proved the following q_a-Hermite–Hadamard inequalities for convex functions in the setting of quantum calculus.

Theorem 1 *If $f:[a,b]\to\mathbb{R}$ is a convex differentiable function on $[a,b]$ and $0<q<1$, then we have*

$$f\left(\frac{qa+b}{1+q}\right)\le\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\le\frac{qf(a)+f(b)}{1+q}.\tag{2.3}$$

In [3] and [23] authors established some bounds for the left- and right-hand sides of inequality (2.3).

On the other hand, Bermudo et al. [22] gave the following definition and obtained the related Hermite–Hadamard-type inequalities.

Definition 5 ([22]) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then the q^b-definite integral on $[a,b]$ is given by

$$\begin{aligned}\int_a^b f(x)\,{}^bd_qx&=(1-q)(b-a)\sum_{n=0}^{\infty}q^nf\big(q^na+\big(1-q^n\big)b\big)\\&=(b-a)\int_0^1 f\big(ta+(1-t)b\big)\,d_qt.\end{aligned}$$

Theorem 2 ([22]) *If $f:[a,b]\to\mathbb{R}$ is a convex differentiable function on $[a,b]$ and $0<q<1$, then we have the following q-Hermite-Hadamard inequalities*:

$$f\left(\frac{a+qb}{1+q}\right)\le\frac{1}{b-a}\int_a^b f(x)^bd_qx\le\frac{f(a)+qf(b)}{1+q}.\tag{2.4}$$

From Theorems 1 and 2 we obtain the following inequalities.

Corollary 1 [22] *For any convex function $f:[a,b]\to\mathbb{R}$ and $0<q<1$, we have*

$$f\left(\frac{qa+b}{1+q}\right)+f\left(\frac{a+qb}{1+q}\right)\le\frac{1}{b-a}\left\{\int_a^b f(x)\,{}_ad_qx+\int_a^b f(x)\,{}^bd_qx\right\}\le f(a)+f(b)\tag{2.5}$$

and

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2(b-a)}\left\{\int_a^b f(x)\,{}_a d_q x + \int_a^b f(x)\,{}^b d_q x\right\} \le \frac{f(a)+f(b)}{2}. \tag{2.6}$$

Theorem 3 (Hölder's inequality, [24, p. 604]) *Suppose that* $x > 0$, $0 < q < 1$, $p_1 > 1$. *If* $\frac{1}{p_1} + \frac{1}{r_1} = 1$, *then*

$$\int_0^x \left|f(x)g(x)\right| d_q x \le \left(\int_0^x \left|f(x)\right|^{p_1} d_q x\right)^{\frac{1}{p_1}} \left(\int_0^x \left|g(x)\right|^{r_1} d_q x\right)^{\frac{1}{r_1}}.$$

In this paper, we will also find some bounds for right-hand side of inequality (2.4).

3 New Hermite–Hadamard-type inequalities for quantum integrals

We now give some new Hermite–Hadamard-type inequalities for functions whose second q^b-derivatives in absolute value are convex.

We start with the following useful lemma.

Lemma 1 *If* $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *is a twice* q^b*-differentiable function on* (a,b) *such that* ${}^bD_q^2 f$ *is continuous and integrable on* $[a,b]$, *then we have*:

$$\begin{aligned} &\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}^b d_q x \\ &\quad = \frac{q^2(b-a)^2}{1+q}\int_0^1 t(1-qt)\,{}^bD_q^2 f\big(ta+(1-t)b\big)\,d_q t, \end{aligned} \tag{3.1}$$

where $0 < q < 1$.

Proof From Definition 2 it follows that

$$\begin{aligned} &{}^bD_q^2 f\big(ta+(1-t)b\big) \\ &\quad = {}^bD_q\big({}^bD_q\big(f\big(ta+(1-t)b\big)\big)\big) \\ &\quad = {}^bD_q\left(\frac{f(qta+(1-qt)b)-f(ta+(1-t)b)}{(1-q)(b-a)t}\right) \\ &\quad = \frac{1}{(1-q)(b-a)t}\left[\frac{f(q^2ta+(1-tq^2)b)-f(qta+(1-qt)b)}{(1-q)q(b-a)t}\right. \\ &\qquad \left. - \frac{f(qta+(1-qt)b)-f(ta+(1-t)b)}{(1-q)(b-a)t}\right] \\ &\quad = \frac{f(q^2ta+(1-tq^2)b)-f(qta+(1-qt)b)}{(1-q)^2q(b-a)^2t^2} \\ &\qquad - \frac{f(qta+(1-qt)b)-f(ta+(1-t)b)}{(1-q)^2(b-a)^2t^2} \\ &\quad = \frac{f(q^2ta+(1-tq^2)b)-(1+q)f(qta+(1-qt)b)+qf(ta+(1-t)b)}{(1-q)^2q(b-a)^2t^2}. \end{aligned} \tag{3.2}$$

Also,

$$
\begin{aligned}
&\int_0^1 t(1-qt)\,{}^bD_q^2 f\big(ta+(1-t)b\big)\,d_qt \\
&= \int_0^1 \frac{f(q^2ta+(1-tq^2)b)-(1+q)f(qta+(1-qt)b)+qf(ta+(1-t)b)}{(1-q)^2q(b-a)^2t}\,d_qt \\
&\quad - \int_0^1 q\left[\frac{f(q^2ta+(1-tq^2)b)-(1+q)f(qta+(1-qt)b)+qf(ta+(1-t)b)}{(1-q)^2q(b-a)^2}\right]d_qt.
\end{aligned} \tag{3.3}
$$

By equality (2.1) we obtain that

$$
\begin{aligned}
&\int_0^1 \frac{f(q^2ta+(1-tq^2)b)-(1+q)f(qta+(1-qt)b)+qf(ta+(1-t)b)}{(1-q)^2q(b-a)^2t}\,d_qt \\
&= (1-q)\sum_{n=0}^{\infty}\frac{f(q^{n+2}a+(1-q^{n+2})b)}{(1-q)^2q(b-a)^2} - (1-q)(1+q)\sum_{n=0}^{\infty}\frac{f(q^{n+1}a+(1-q^{n+1})b)}{(1-q)^2q(b-a)^2} \\
&\quad + q(1-q)\sum_{n=0}^{\infty}\frac{f(q^{n}a+(1-q^{n})b)}{(1-q)^2q(b-a)^2} \\
&= \sum_{n=0}^{\infty}\frac{f(q^{n+2}a+(1-q^{n+2})b)}{(1-q)q(b-a)^2} - \sum_{n=0}^{\infty}\frac{f(q^{n+1}a+(1-q^{n+1})b)}{(1-q)q(b-a)^2} \\
&\quad - q\left[\sum_{n=0}^{\infty}\frac{f(q^{n+1}a+(1-q^{n+1})b)}{(1-q)q(b-a)^2} - \sum_{n=0}^{\infty}\frac{f(q^{n}a+(1-q^{n})b)}{(1-q)q(b-a)^2}\right] \\
&= \frac{f(b)-f(qa+(1-q)b)}{(1-q)q(b-a)^2} - q\left[\frac{f(b)-f(a)}{(1-q)q(b-a)^2}\right].
\end{aligned} \tag{3.4}
$$

From (2.1) and Definition 5 it follows that

$$
\begin{aligned}
&\int_0^1 q\left[\frac{f(q^2ta+(1-tq^2)b)-(1+q)f(qta+(1-qt)b)+qf(ta+(1-t)b)}{(1-q)^2q(b-a)^2}\right]d_qt \\
&= q\Bigg[(1-q)(b-a)\sum_{n=0}^{\infty}\frac{q^{n+2}f(q^{n+2}a+(1-q^{n+2})b)}{(1-q)^2q^3(b-a)^3} \\
&\quad - (1-q)(1+q)(b-a)\sum_{n=0}^{\infty}\frac{q^{n+1}f(q^{n+1}a+(1-q^{n+1})b)}{(1-q)^2q^2(b-a)^3} \\
&\quad + q(1-q)(b-a)\sum_{n=0}^{\infty}\frac{q^{n}f(q^{n}a+(1-q^{n})b)}{(1-q)^2q(b-a)^3}\Bigg] \\
&= q\Bigg[\frac{1}{(1-q)^2q^3(b-a)^3} \\
&\quad \times\left(\int_a^b f(x)\,{}^bd_qx-(1-q)(b-a)f(a)-(1-q)(b-a)qf\big(qa+(1-q)b\big)\right) \\
&\quad - \frac{1+q}{(1-q)^2q^2(b-a)^3}\left(\int_a^b f(x)\,{}^bd_qx-(1-q)(1+q)(b-a)f(a)\right)
\end{aligned}
$$

$$+\frac{1}{(1-q)^2(b-a)^3}\int_a^b f(x)\,{}^b d_q x\Bigg]$$

$$=\frac{1+q}{(b-a)^2q^2}\int_a^b f(x)\,{}^b d_q x+\frac{q^2+q-1}{(1-q)q^2(b-a)^2}f(a)-\frac{f(qa+(1-q)b)}{(1-q)q(b-a)^2} \tag{3.5}$$

Using (3.4) and (3.5) in (3.3), we have

$$\int_0^1 t(1-qt)\,{}^bD_q^2 f\big(ta+(1-t)b\big)\,d_q t$$
$$=\frac{f(b)-f(qa+(1-q)b)}{(1-q)q(b-a)^2}-q\bigg[\frac{f(b)-f(a)}{(1-q)q(b-a)^2}\bigg]$$
$$-\frac{1+q}{(b-a)^2q^2}\int_a^b f(x)\,{}^b d_q x-\frac{q^2+q-1}{(1-q)q^2(b-a)^2}f(a)+\frac{f(qa+(1-q)b)}{(1-q)q(b-a)^2}$$
$$=\frac{f(a)+qf(b)}{(b-a)^2q^2}-\frac{1+q}{(b-a)^2q^2}\int_a^b f(x)\,{}^b d_q x. \tag{3.6}$$

Multiplying both sides of (3.6) by $\frac{(b-a)^2q^2}{1+q}$, we obtain the required identity (3.1) and hence we complete the proof of Lemma 1. □

Remark 1 If we take the limit as $q\to 1^-$ in Lemma 1, then we have

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''\big(ta+(1-t)b\big)\,dt,$$

as given in [25].

Theorem 4 *If $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is a twice q^b-differentiable function on (a,b) such that ${}^bD_q^2 f$ is continuous and integrable on $[a,b]$, then we have the following inequality, provided that $|{}^bD_q^2 f|$ is convex on $[a,b]$:*

$$\bigg|\frac{f(a)+qf(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)\,{}^b d_q x\bigg|$$
$$\le\frac{q^2(b-a)^2}{(1+q)(q^2+q+1)(q^3+q^2+q+1)}\big[\big|{}^bD_q^2 f(a)\big|+q^2\big|{}^bD_q^2 f(b)\big|\big],$$

where $0<q<1$.

Proof Taking the modulus in Lemma 1 and applying the convexity of $|{}^bD_q^2 f|$, we obtain

$$\bigg|\frac{f(a)+qf(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)\,{}^b d_q x\bigg|$$
$$\le\frac{q^2(b-a)^2}{1+q}\int_0^1\big(t(1-qt)\big)\big|{}^bD_q^2 f\big(ta+(1-t)b\big)\big|\,d_q t$$
$$\le\frac{q^2(b-a)^2}{1+q}\int_0^1\big(t(1-qt)\big)\big[t\big|{}^bD_q^2 f(a)\big|+(1-t)\big|{}^bD_q^2 f(b)\big|\big]\,d_q t$$

$$= \frac{q^2(b-a)^2}{1+q}\left[\left|{}^bD_q^2 f(a)\right|\int_0^1 t\big(t(1-qt)\big)\,d_qt + \left|{}^bD_q^2 f(b)\right|\int_0^1 (1-t)\big(t(1-qt)\big)\,d_qt\right]$$
$$= \frac{q^2(b-a)^2}{1+q}\left[\frac{|{}^bD_q^2 f(a)|}{(q^2+q+1)(q^3+q^2+q+1)} + \frac{q^2|{}^bD_q^2 f(b)|}{(q^2+q+1)(q^3+q^2+q+1)}\right],$$

which completes the proof. □

Remark 2 Under the assumptions of Theorem 4 with the limit as $q \to 1^-$, we have the following trapezoidal inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)^2}{12}\left[\frac{|f''(a)|+|f''(b)|}{2}\right],$$

as given by Sarikaya and Aktan [26, Proposition 2].

Theorem 5 *Suppose that* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *is a twice* q^b*-differentiable function on* (a,b) *and* ${}^bD_q^2 f$ *is continuous and integrable on* $[a,b]$. *If* $|{}^bD_q^2 f|^{p_1}, p_1>1$, *is convex on* $[a,b]$, *then we have the following inequality*:

$$\left|\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}^bd_qx\right|$$
$$\leq \frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{p_1}}(1+q+q^2)}\left(\frac{1}{q^3+q^2+q+1}\right)^{\frac{1}{p_1}}\left(\left|{}^bD_q^2 f(a)\right|^{p_1} + q^2\left|{}^bD_q^2 f(b)\right|^{p_1}\right)^{\frac{1}{p_1}},$$

where $0<q<1$.

Proof Taking the modulus in Lemma 1 and applying the well-known power mean inequality, we have

$$\left|\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}^bd_qx\right|$$
$$\leq \frac{q^2(b-a)^2}{1+q}\int_0^1 \big(t(1-qt)\big)\left|{}^bD_q^2 f\big(ta+(1-t)b\big)\right|\,d_qt$$
$$\leq \frac{q^2(b-a)^2}{1+q}\left(\int_0^1 \big(t(1-qt)\big)\,d_qt\right)^{1-\frac{1}{p_1}}$$
$$\times\left(\int_0^1 \big(t(1-qt)\big)\left|{}^bD_q^2 f\big(ta+(1-t)b\big)\right|^{p_1}\,d_qt\right)^{\frac{1}{p_1}}.$$

By the convexity of $|{}^bD_q^2 f|^{p_1}$ we have

$$\left|\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}^bd_qx\right|$$
$$\leq \frac{q^2(b-a)^2}{1+q}\left(\int_0^1 \big(t(1-qt)\big)\,d_qt\right)^{1-\frac{1}{p_1}}$$
$$\times\left(\int_0^1 \big(t(1-qt)\big)\left[t\left|{}^bD_q^2 f(a)\right|^{p_1} + (1-t)\left|{}^bD_q^2 f(b)\right|^{p_1}\right]\,d_qt\right)^{\frac{1}{p_1}}$$

$$= \frac{q^2(b-a)^2}{1+q}\left(\int_0^1 \big(t(1-qt)\big)\,d_qt\right)^{1-\frac{1}{p_1}}$$
$$\times\left(\big|{}^bD_q^2f(a)\big|^{p_1}\int_0^1 t\big(t(1-qt)\big)\,d_qt + \big|{}^bD_q^2f(b)\big|^{p_1}\int_0^1 (1-t)\big(t(1-qt)\big)\,d_qt\right)^{\frac{1}{p_1}}$$
$$= \frac{q^2(b-a)^2}{1+q}\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{p_1}}$$
$$\times\left(\frac{|{}^bD_q^2f(a)|^{p_1}}{(q^2+q+1)(q^3+q^2+q+1)} + \frac{q^2|{}^bD_q^2f(b)|^{p_1}}{(q^2+q+1)(q^3+q^2+q+1)}\right)^{\frac{1}{p_1}},$$

which completes the proof. □

Remark 3 If we take the limit as $q \to 1^-$ in Theorem 5, then we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{12.2^{\frac{1}{p_1}}}\big(\big|f''(a)\big|^{p_1} + \big|f''(b)\big|^{p_1}\big)^{\frac{1}{p_1}}.$$

Theorem 6 *Suppose that $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a twice q^b-differentiable function on (a,b) and ${}^bD_q^2f$ is continuous and integrable on $[a,b]$. If $|{}^bD_q^2f|^{p_1}$ is convex on $[a,b]$ for some $p_1 > 1$ and $\frac{1}{r_1} + \frac{1}{p_1} = 1$, then we have*

$$\left|\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}^bd_qx\right|$$
$$\le \frac{q^2(b-a)^2}{1+q}(u_1)^{\frac{1}{r_1}}\left(\frac{|{}^bD_q^2f(a)|^{p_1} + q|{}^bD_q^2f(b)|^{p_1}}{q+1}\right)^{\frac{1}{p_1}}, \tag{3.7}$$

where $u_1 = (1-q)\sum_{n=0}^{\infty}(q^n)^{r_1+1}(1-q^{n+1})^{r_1}$ and $0 < q < 1$.

Proof Taking the modulus in Lemma 1 and applying well-known Hölder's inequality, we obtain

$$\left|\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}^bd_qx\right|$$
$$\le \frac{q^2(b-a)^2}{1+q}\int_0^1 \big(t(1-qt)\big)\big|{}^bD_q^2f\big(ta+(1-t)b\big)\big|\,d_qt$$
$$\le \frac{q^2(b-a)^2}{1+q}\left(\int_0^1 \big(t(1-qt)\big)^{r_1}\,d_qt\right)^{\frac{1}{r_1}}\left(\int_0^1 \big|{}^bD_q^2f\big(ta+(1-t)b\big)\big|^{p_1}\,d_qt\right)^{\frac{1}{p_1}}.$$

Since $|{}^bD_q^2f|^{p_1}$ is convex, we have

$$\left|\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}^bd_qx\right|$$
$$\le \frac{q^2(b-a)^2}{1+q}\left(\int_0^1 \big(t(1-qt)\big)^{r_1}\,d_qt\right)^{\frac{1}{r_1}}$$

$$\times \left(\left|{}^bD_q^2 f(a)\right|^{p_1} \int_0^1 t \, d_q t + \left|{}^bD_q^2 f(b)\right|^{p_1} \int_0^1 (1-t) \, d_q t \right)^{\frac{1}{p_1}}$$
$$= \frac{q^2(b-a)^2}{1+q} (u_1)^{\frac{1}{r_1}} \left(\frac{|{}^bD_q^2 f(a)|^{p_1} + q|{}^bD_q^2 f(b)|^{p_1}}{q+1} \right)^{\frac{1}{p_1}}.$$

Thus

$$u_1 = \int_0^1 \left(t(1-qt)\right)^{r_1} d_q t = (1-q) \sum_{n=0}^{\infty} \left(q^n\right)^{r_1+1} \left(1-q^{n+1}\right)^{r_1},$$

which completes the proof. □

Remark 4 If we take the limit as $q \to 1^-$ in Theorem 6, then we have

$$u_1 = \int_0^1 \left(t(1-t)\right)^{r_1} dt = B(r_1+1, r_1+1),$$

where $B(x,y)$ is the Euler beta function. Moreover, inequality (3.7) reduces to

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right|$$
$$\leq \frac{(b-a)^2}{2} \left(B(r_1+1, r_1+1)\right)^{\frac{1}{r_1}} \left(\frac{|f''(a)|^{p_1} + |f''(b)|^{p_1}}{2} \right)^{\frac{1}{p_1}}.$$

We obtain another Hermite–Hadamard-type inequality for powers in terms of the second quantum derivatives.

Theorem 7 *With assumptions of Theorem 6, we have the inequality*

$$\left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) \, {}^b d_q x \right|$$
$$\leq \frac{q^2(b-a)^2}{1+q} \left(\frac{1}{[r_1+1]_q} \right)^{\frac{1}{r_1}} \left(u_2 \left|{}^bD_q^2 f(a)\right|^{p_1} + u_3 \left|{}^bD_q^2 f(b)\right|^{p_1} \right)^{\frac{1}{p_1}}, \tag{3.8}$$

where

$$u_2 = (1-q) \sum_{n=0}^{\infty} q^{2n} \left(1-q^{n+1}\right)^{p_1} \quad \textit{and} \quad u_3 = (1-q) \sum_{n=0}^{\infty} q^n \left(1-q^n\right)\left(1-q^{n+1}\right)^{p_1}.$$

Proof Taking the modulus of the right-hand side of the equality in Lemma 1 and applying well-known Hölder's inequality, we obtain

$$\left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) \, {}^b d_q x \right|$$
$$\leq \frac{q^2(b-a)^2}{1+q} \int_0^1 \left(t(1-qt)\right) \left|{}^bD_q^2 f\left(ta+(1-t)b\right)\right| d_q t$$
$$\leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t^{r_1} \, d_q t \right)^{\frac{1}{r_1}} \left(\int_0^1 (1-qt)^{p_1} \left|{}^bD_q^2 f\left(ta+(1-t)b\right)\right|^{p_1} d_q t \right)^{\frac{1}{p_1}}.$$

Since $|{}^bD_q^2 f|^{p_1}$ is convex, we have

$$\begin{aligned}
&\left|\frac{f(a)+qf(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)\,{}^b d_q x\right| \\
&\quad\le \frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t^{r_1}\,d_q t\right)^{\frac{1}{r_1}} \\
&\qquad\times\left(\left|{}^bD_q^2 f(a)\right|^{p_1}\int_0^1 (1-qt)^{p_1} t\,d_q t+\left|{}^bD_q^2 f(b)\right|^{p_1}\int_0^1 (1-qt)^{p_1}(1-t)\,d_q t\right)^{\frac{1}{p_1}} \\
&\quad= \frac{q^2(b-a)^2}{1+q}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(u_2\left|{}^bD_q^2 f(a)\right|^{p_1}+u_3\left|{}^bD_q^2 f(b)\right|^{p_1}\right)^{\frac{1}{p_1}}.
\end{aligned}$$

We can easily see that

$$u_2=\int_0^1 (1-qt)^{p_1} t\,d_q t=(1-q)\sum_{n=0}^{\infty} q^{2n}\left(1-q^{n+1}\right)^{p_1}$$

and

$$u_3=\int_0^1 (1-qt)^{p_1}(1-t)\,d_q t=(1-q)\sum_{n=0}^{\infty} q^{n}\left(1-q^{n}\right)\left(1-q^{n+1}\right)^{p_1}.$$

This completes the proof. □

Remark 5 If we take the limit as $q\to 1^-$ in Theorem 7, then we have

$$u_2=\int_0^1 (1-t)^{p_1} t\,d_q t=\frac{1}{(p_1+1)(p_1+2)}$$

and

$$u_3=\int_0^1 (1-t)^{p_1}(1-t)\,dt=\frac{1}{p_1+2}.$$

Moreover, inequality (3.8) reduces to

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{(b-a)^2}{2}\left(\frac{1}{r_1+1}\right)^{\frac{1}{r_1}}\left(\frac{1}{(p_1+1)(p_1+2)}\right)^{\frac{1}{p_1}}\left((p_1+2)\left|f''(a)\right|^{p_1}+\left|f''(b)\right|^{p_1}\right)^{\frac{1}{p_1}}.
\end{aligned}$$

4 Conclusions

In this paper, we obtained Hermite–Hadamard-type inequalities for convex functions by applying the newly defined q^b-integral. The results proved in this paper are a potential generalization of the existing comparable results in the literature. As future directions, we can find similar inequalities through different types of convexities.

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