Repeated blocks in indecomposable twofold extended triple systems

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Abstract

An extended triple system (a twofold extended triple system) with no idempotent element (ETS, TETS respectively) is a pair (V, B) where V is a v-set and B is a collection of unordered triples, called blocks, of type $\{x, y, z\}$ or $\{x, x, y\}$, such that each pair (whether distinct or not) is contained in exactly one (respectively, exactly two) blocks. For example, in the block $\{x, x, y\}$, the occurrence of the pair $\{x, y\}$ is counted once. It is well-known that an ETS(v) of order v (ETS(v)) exists if and only if $v \equiv 0 \pmod 3$, and it is trivial to see that a TETS of order v (TETS(v)) exists if and only if $v \equiv 0 \pmod 3$.

If a TETS(v) contains two blocks b_1, b_2 that are identical as subsets of V, then $b_1 = b_2$ is said to be a repeated block.

We are interested in the following question: Given $v \equiv 0 \pmod 3$ and a nonnegative integer k, does there exist a TETS(v) with exactly k repeated blocks? This question is related to the intersection problem for ETSs, solved by Lo Faro in 1995. The same question with the additional condition that the TETS be indecomposable (that is, cannot have its blocks partitioned into two ETS) is also of interest.

The purpose of this paper is to completely settle these questions.

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1 Introduction

The concept of an extended triple system was introduced by Johnson and N.S. Mendelsohn [8]. An extended triple system (a twofold extended triple system) is a pair (V, B) where V is a finite set and B is a collection of unordered triples from V, where each triple may have repeated elements, and such that every pair of elements of V, not necessarily distinct, is contained in exactly one (exactly two) triples of B. The triples of B are of three types: (1) $\{x, x, x\}$, (2) $\{y, y, z\}$, and (3) $\{a, b, c\}$. If there is a block $\{x, x, x\}$ then x is called idempotent.

From now on we restrict our attention to extended triple systems (twofold extended triple systems) with no idempotent elements (i.e., without the triples of type $\{x, x, x\}$). We shall denote such a design, based on a v-set, by ETS(v) (TETS(v)).

An ETS(v) has $s_v = v(v+3)/6$ blocks, and a TETS(v) has $v(v+3)/3$ blocks. A necessary and sufficient condition for existence is that $v \equiv 0 \pmod 3$ (shown in [1] for ETS(v)). Therefore when we state that a certain property concerning ETS(v) (or TETS(v)) is true, it is understood that $v \equiv 0 \pmod 3$.

Designs with repeated blocks have interesting applications in statistics and have been studied extensively. See for example [7].

Let $J(v) = \{k \mid$ there exist two ETS(v) having exactly k triples in common$\}$. Let $I(v) = \{0, 1, \ldots, s_v - 3, s_v\}$. It is shown in [11] that $J(9) = I(9) - \{13\}$ and $J(v) = I(v)$ for $v \neq 9$.

Let $E(v) = \{k \mid$ there exists a TETS(v) having exactly k repeated blocks$\}$; clearly $J(v) \subseteq E(v)$, since a pair of ETS(v)s having exactly k triples in common yield a (decomposable) TETS(v) with k repeated blocks.

By analogy, we now let $E_I(v) = \{k \mid$ there exists an indecomposable TETS(v) with exactly k repeated blocks$\}$. It is easy to see that $E_I(v) \subseteq E(v)$ with $E_I(v) \neq E(v)$ for all v. (If a TETS(v) has s_v repeated blocks, it is necessarily decomposable, i.e. $s_v \in E(v)$ but $s_v \notin E_I(v)$.)

Finally, let $T(v) = I(v) \cup \{s_v - 2\}$ and $T^*(v) = T(v) - \{s_v\}$.

The purpose of this paper is to determine $E(v)$ and $E_I(v)$ for all v; in particular we prove the following result:

MAIN THEOREM $E(3) = \{0, 3\}$, $E_I(3) = \emptyset$, $E(6) = I(6)$, $E_I(6) = I(6) - \{s_6\}$, $E(v) = T(v)$ *and* $E_I(v) = T^*(v)$, *for all* $v \geq 9$.

The corresponding problems for twofold triple systems (TTS(v)) were settled in [12] and [13]; for balanced ternary designs with block size 3 and index 2 in [4] (when $\rho_2 = 1$) and [5] (when $\rho_2 = 2$); and a related problem for multiset designs in [6]. (See for example [2] and [3] for more on balanced ternary designs and multiset designs.)

2 Preliminaries and recursive constructions

Trivially there is no TETS(v) having exactly two nonrepeated blocks, so in order to prove the Main Theorem we only have to show that:

$$E(3) = \{0, 3\}; E_I(3) = \emptyset; E(6) = \{0, 1, 2, 3, 4, 5, 6, 9\} \text{ and}$$

$$E_I(v) = \{0, 1, 2, \ldots, s_v - 2\} \text{ for } v \geq 9 \text{ where } s_v = v(v + 3)/6.$$

Obviously $s_v \in E(v)$ and $s_v \notin E_I(v)$, for all v.

Constructions for ETS(v) were stated explicitly in [11]; here we state them in the form for TETS(v).

Let (V, B) be a TETS(v), where $V = \{a_i \mid i = 1, 2, \ldots, v\}$.

(1a) v to $2v$
Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v-1\}$ be a 2-factorization of $2K_v$ on $X = \{x_1, x_2, \ldots, x_v\}$, where $V \cap X = \emptyset$. Put $S = V \cup X$ and $T = B \cup C \cup 2D$ where

$$C = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \ldots, v - 1\},\quad D = \{a_v xx, \text{ for each } x \in X\}$$

and $2D$ consists of blocks of D, each block occurring twice. Then (S, T) is a TETS($2v$).

(1b) v to $2v$
Put $X = \{x_1, x_2, \ldots, x_v, x_{v+1}\}$ with $V \cap X = \emptyset$. Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v\}$ be a 2-factorization of $2K_{v+1}$ on X. Put $S = V \cup (X - \{x_{v+1}\})$ and $T = B \cup C \cup H$ where $C = \{a_i xy \mid xy \in F_i \text{ and } x_{v+1} \notin \{xy\}; i = 1, 2, \ldots, v\}$, and $H = \{a_i x_j x_j \mid x_j x_{v+1} \in F_i;\ i = 1, 2, \ldots, v\}$.

Then (S, T) is a TETS($2v$).

Let F_a and F_b be two 2-factors of $2K_n$ on the vertex set $N = \{1, 2, \ldots, n\}$. The notation $F_{a,b}$ will denote a set of blocks $\{iij_{i_1}, iij_{i_2} \mid i = 1, 2, \ldots, n\}$ where $F_a \cup F_b = \{1j_{1_1}, 1j_{1_2}, 2j_{2_1}, 2j_{2_2}, \ldots, nj_{n_1}, nj_{n_2}\}$.

(2) v to $2v + 3$
Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v + 2\}$ be a 2-factorization of $2K_{v+3}$ on the vertex set $X = \{x_1, x_2, \ldots, x_{v+3}\}$ with $V \cap X = \emptyset$. Put $S = V \cup X$ and $T = B \cup C \cup F_{v+1,v+2}$ where $C = \{a_i xy \mid xy \in F_i, i = 1, 2, \ldots, v\}$. Then (S, T) is an ETS($2v + 3$).

(3) v to $2v + 9$, v odd
Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v + 2\}$ be a set of 1-factors of K_{v+9} on $X = \{x_i \mid i = 1, 2, \ldots, v+9\}$ such that its leave (that is, its complement) is the graph consisting of the $v+9$ triangles $L = \{\{i, i+1, i+3\} \mid i = 1, 2, \ldots, v+9\}$ (cf. [14]). Put $S = V \cup X$ and $T = B \cup 2C \cup 2L \cup 2F_{v+1,v+2}$ where $C = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \ldots, v\}$. (Note that here F_{v+1}, F_{v+2} are 1-factors, and so the definition of $F_{v+1,v+2}$ is modified accordingly; see [11] p. 212.) It is then straightforward to see that (S, T) is an ETS($2v + 9$).

3 Small cases

In this section we shall omit braces when listing blocks, and also the comma separators between the elements; this will save space and increase clarity. Subsequently we shall denote the ETS(3) aab, bbc, cca by $E(abc)$. Also in the following B_N will denote single occurrences of blocks, while B_R will denote blocks to be repeated.

LEMMA 1 $E(3) = \{0, 3\}$; $E_I(3) = \emptyset$.

Proof There are precisely two TETS(3), and both are decomposable:

NO repeated blocks:	112	113	221	223	331	332
3 repeated blocks:	112	223	331	112	223	331

□

LEMMA 2 $E_I(6) = E(6) \setminus \{9\} = \{0, 1, 2, 3, 4, 5, 6\}$.

Proof

$0 \in E_I(6)$ $B_N =$ 115, 116, 224, 226, 332, 336, 445, 446, 551, 552, 661, 664, 123, 124, 134, 256, 356, 345. (Indecomposable.)

$1 \in E_I(6)$ $B_R =$ 446;
$B_N =$ 115, 116, 224, 225, 332, 336, 551, 554, 661, 662, 123, 124, 134, 256, 356, 345. (Indecomposable.)

$2 \in E_I(6)$ $B_R =$ 226, 356;
$B_N =$ 115, 116, 332, 334, 445, 446, 551, 552, 661, 664, 123, 124, 134, 245. (Indecomposable.)

$3 \in E_I(6)$ $B_R =$ 116, 551, 356;
$B_N =$ 225, 226, 332, 334, 445, 446, 662, 664, 123, 124, 134, 245. (Indecomposable.)

$4 \in E_I(6)$ $B_R =$ 115, 446, 554, 661.
$B_N =$ 224, 226, 334, 336, 123, 124, 134, 235, 256, 356. (Indecomposable.)

$5 \in E_I(6)$ $B_R =$ 115, 226, 446, 661, 356.
$B_N =$ 332, 334, 552, 554, 123, 124, 134, 245. (Indecomposable.)

$6 \in E_I(6)$ $B_R =$ 115, 226, 336, 446, 556, 661.
$B_N =$ 123, 124, 134, 235, 245, 345. (Indecomposable.)

Finally, if a TETS(6) has 7 repeated blocks, then without loss of generality we can suppose that the four non-repeated triples are 123, 124, 134, 234. It is easy to see that this is impossible. □

LEMMA 3 $E_I(9) = I(9) \setminus \{s_9\}$.

Proof

$\underline{0 \in E_I(9)}$:

$$B_N = \left\{ \begin{array}{cccccccccccc} 114 & 115 & 225 & 226 & 336 & 337 & 447 & 448 & 558 & 559 & 669 & 661 \\ 771 & 772 & 883 & 882 & 993 & 994 & 123 & 136 & 234 & 247 & 345 & 358 \\ 456 & 469 & 571 & 567 & 682 & 678 & 789 & 793 & 891 & 814 & 912 & 925 \end{array} \right\}.$$

(Indecomposable.)

Step 1. $\{1,4,7,10\} \subseteq E_I(9)$.
Let (V,B) be the following indecomposable twofold triple system:
$B_R = 123,\ 456,\ 789,\ 169;$
$B_N = 147,\ 258,\ 267,\ 348,\ 357,\ 249,\ 157,\ 268,\ 349,\ 247,\ 367,\ 148,\ 259,$
$158,\ 359,\ 368.$
Let $C_1 = (B - 2.\{123,456,789\}) \cup E(123) \cup E(132) \cup E(456) \cup E(465) \cup E(789) \cup E(798)$;
let C_2 be obtained by replacing the blocks $E(132)$ of C_1 with the blocks $E(123)$;
let C_3 be obtained by replacing the blocks $E(465)$ of C_2 with the blocks $E(456)$;
let C_4 be obtained by replacing the blocks $E(798)$ of C_3 with the blocks $E(789)$.
Then (V,C_1), (V,C_2), (V,C_3) and (V,C_4) are indecomposable TETS(9) with exactly 1, 4, 7 and 10 repeated blocks, respectively.

Step 2. $\{2,3,5,6,8\} \subseteq E_I(9)$.
Let (V,B) be the following indecomposable twofold triple system:
$B_R = 123,\ 145,\ 167,\ 189;$
$B_N = 247,\ 349,\ 356,\ 378,\ 248,\ 256,\ 259,\ 268,\ 279,\ 347,\ 358,\ 369,\ 468,\ 579,\ 469,\ 578.$
Let $D_1 = (B - \{2.\{123\},468,579,469,578\}) \cup E(123) \cup E(132) \cup E(496) \cup E(468) \cup E(587) \cup E(579)$;
let D_2 be obtained by replacing the blocks $E(132)$ of D_1 with the blocks $E(123)$;
let D_3 be obtained by replacing the blocks $E(496)$ and $E(587)$ of D_2 with the blocks $E(469)$ and $E(578)$;
let D_4 be obtained by replacing the blocks $E(496)$ and $E(587)$ of D_1 with the blocks $E(469)$ and $E(578)$.
Then (V,D_1), (V,D_2), (V,D_3) and (V,D_4) are indecomposable TETS(9) with exactly 3, 6, 8 and 5 repeated blocks, respectively.

Let $D_5 = (B - \{123,189,468,579,347,256\}) \cup E(123) \cup E(468) \cup E(579) \cup E(189) \cup E(347) \cup E(256)$; then (V,D_5) is an indecomposable TETS(9) with exactly 2 repeated blocks.

$9 \in E_I(9)$:
$B_R = 115,\ 331,\ 448,\ 559,\ 883,\ 994,\ 147,\ 258,\ 369;$
$B_N = 221,\ 229,\ 661,\ 662,\ 772,\ 779,\ 168,\ 357,\ 246,\ 237,\ 234,\ 345,$
$456,\ 567,\ 678,\ 789,\ 891,\ 912.$ (Indecomposable.)

Step 3. $\{11,12,13,15\} \subseteq E_I(9)$.
Let (V,B) be the following indecomposable TETS(9):
$B_R = 115,\ 226,\ 337,\ 445,\ 559,\ 661,\ 775,\ 883,\ 994,\ 235,\ 346,\ 568,\ 679,\ 289,\ 139;$
$B_N = 124,\ 127,\ 478,\ 247,\ 178,\ 148.$
Let $B_1 = (B - \{445,775,478\}) \cup \{448,778,457\}$; let $B_2 = (B - E(459)) \cup E(495)$; let $B_3 = (B_1 - E(378)) \cup E(387)$. Then (V,B), (V,B_1), (V,B_2) and (V,B_3) are indecomposable TETS(9) with exactly 15, 13, 12 and 11 repeated blocks, respectively.

$14 \in E_I(9)$:
$B_R = 119,\ 226,\ 448,\ 664,\ 772,\ 887,\ 993,\ 158,\ 167,\ 289,\ 357,\ 368,\ 479,\ 569;$
$B_N = 332,\ 334,\ 552,\ 554,\ 123,\ 124,\ 134,\ 245.$ (Indecomposable.)

$\underline{16 \in E_I(9)}$:
$B_R = 117,\ 229,\ 335,\ 446,\ 559,\ 668,\ 779,\ 881,\ 991,\ 258,\ 457,\ 369,\ 156,\ 267,\ 378,\ 489$;
$B_N = 123,\ 124,\ 134,\ 234$. (Indecomposable.) $\square$

LEMMA 4 $E_I(15) = E(15) \setminus \{s_{15}\} = \{0, 1, \ldots, s_{15} - 2\}$.

Proof

Step 1. $\{0, 1, \ldots, 15\} \subseteq E_I(15)$.
Let $V = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $X = V \cup \mathbb{Z}_9$. Let h_1, h_2, h_3, h_4 be the following 2-factors of $2K_9$ on the vertex set $\mathbb{Z}_9$:
$h_1 = \{i(i+1) \mid i \in \mathbb{Z}_9\}$; $h_2 = \{i(i+2) \mid i \in \mathbb{Z}_9\}$; $h_3 = \{i(i+3) \mid i \in \mathbb{Z}_9\}$; $h_4 = \{i(i+4) \mid i \in \mathbb{Z}_9\}$.

For an element a and a set of pairs P, let $a * P$ denote the set $\{axy \mid \{x, y\} \in P\}$.

Let (V, B) be an indecomposable TETS(6) with exactly k repeated blocks. (So, from Lemma 2, $0 \leq k \leq 6$.) Let $\mathcal{F}^{(t)} = \{F_1^{(t)}, F_2^{(t)}, F_3^{(t)}, F_4^{(t)}, F_5^{(t)}, F_6^{(t)}\}$, $t = 1, 2$, be the following sets of six 2-factors:

$$\mathcal{F}^{(1)} = \{h_1, h_2, h_3, h_1, h_2, h_3\}; \quad \mathcal{F}^{(2)} = \{h_1, h_2, h_3, h_4, h_1, h_2\}.$$

Furthermore, let $C^1 = \{ii(i+4) \mid i \in \mathbb{Z}_9\}$ and $C^2 = \{ii(i+3), ii(i+4) \mid i \in \mathbb{Z}_9\}$.
Put $H^t = B \cup a_i * F_i^{(t)} \cup Y^{(t)}$, where $Y^{(t)} = \begin{cases} 2 \cdot C^1 & \text{if } t = 1; \\ C^2 & \text{if } t = 2. \end{cases}$
It is straightforward to verify that $(X, H^{(1)})$ and $(X, H^{(2)})$ are indecomposable TETS(15) with exactly $9 + k$ and k repeated blocks, respectively.
By Lemma 2, $k \in \{0, 1, 2, 3, 4, 5, 6\}$ and this gives $\{0, 1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15\} \subseteq E_I(15)$.
Now in $H^{(2)}$ replace $\{114, 115, 225, 994, 19a_1, 12a_1, 336, 337, 226, 447, 23a_1, 34a_1\}$ with $\{2 \cdot \{11a_1, 33a_1, 22a_1\}, 99a_1, 149, 125, 44a_1, 236, 347\}$. The result is an indecomposable TETS(15) with exactly $k + 3$ repeated blocks. This implies that $\{7, 8\} \subseteq E_I(15)$, which completes the verification of Step 1.

Step 2. $\{16, 17, \ldots, 29\} \cup \{31, 32, 35\} \subseteq E_I(15)$.
Let $X = V \cup \mathbb{Z}_8$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$. Let h_1, h_2, h_3, h_4 be the following 1-factors of K_8 on the vertex set $\mathbb{Z}_8$, and let S_6, S_7, S_8 be the following 2-factors of $2K_8$ on the vertex set $\mathbb{Z}_8$.

$$h_1 = \{04, 16, 23, 57\},\ h_2 = \{05, 13, 26, 47\},\ h_3 = \{06, 17, 24, 35\},\ h_4 = \{07, 14, 25, 36\};$$

$$S_5 = \{02, 27, 67, 46, 34, 03, 15, 15\};\ S_6 = \{01, 12, 02, 45, 56, 46, 37, 37\};$$

$$S_7 = \{01, 12, 27, 67, 56, 45, 34, 03\}.$$

If Q_i is the 2-factor on $\mathbb{Z}_8$ obtained by doubling each edge of h_i, $i = 1, 2, 3, 4$, then $\mathcal{F} = \{Q_1, Q_2, Q_3, Q_4, S_5, S_6, S_7\}$ is a 2-factorization of $2K_8$ on $\mathbb{Z}_8$.

Let (V, B) be a twofold triple system of order 7 (TTS(7)) with exactly k repeated blocks, where $k \in \{1, 3, 7\}$ (cf. [9]), and suppose that the triple $a_3a_4a_7$ is a repeated block. Then $C = B \cup a_i * Q_i \cup a_j * S_j$, $i = 1, 2, 3, 4$ and $j = 5, 6, 7$ is a TTS(15) containing the repeated blocks of $R = \{15a_5, 37a_6, 04a_1, 26a_2, a_3a_4a_7\}$. Put

$$H = E(15a_5) \cup E(37a_6) \cup E(04a_1) \cup E(26a_2) \cup E(a_3a_4a_7)$$

and let $D = (C - 2 \cdot R) \cup 2 \cdot H$. It is straightforward to verify that (V, D) is an indecomposable TETS(15) with exactly $15 + 14 + (k - 1) = 28 + k$ repeated blocks. Let D_1 be obtained by replacing the blocks $E(15a_5)$ of D with the blocks $E(1a_55)$; let D_2 be obtained by replacing the blocks $E(37a_6)$ of D_1 with the blocks $E(3a_67)$; let D_3 be obtained by replacing the blocks $E(04a_1)$ of D_2 with the blocks $E(0a_14)$; let D_4 be obtained by replacing the blocks $E(26a_2)$ of D_3 with the blocks $E(2a_26)$; let D_5 be obtained by replacing the blocks $E(a_3a_4a_7)$ of D_4 with the blocks $E(a_3a_7a_4)$; let M_i be obtained by replacing the blocks $a_3 * h_3$ and $a_4 * h_4$ of D_i with the blocks $a_3 * h_4$ and $a_4 * h_3$, $i = 1, 2, 3, 4, 5$.

Then (X, D_i) is an indecomposable TETS(15) with exactly $28 + k - 3i$ repeated blocks, and (X, M_i) is an indecomposable TETS(15) with exactly $20 + k - 3i$ repeated blocks, $i = 1, 2, 3, 4, 5$.

This completes Step 2.

Step 3. $\{30, 33, 37\} \subseteq E_I(15)$.
Let $X = \{0, 1, 2, \ldots, 9, a, b, c, d, e\}$ and let (X, B) be the following indecomposable TETS(15) with exactly 37 repeated blocks:

$$B_R = \left\{ \begin{array}{cccccccccccccc} 07a & 09d & 0be & 18d & 19a & 1ce & 27c & 28a & 2bd & 37d & 38e & 39b & 3ac & 47e \\ 48b & 49c & 77b & bb1 & 117 & 88c & cc0 & 008 & 99e & ee2 & 229 & aad & dd4 & 44a \\ 335 & 556 & 663 & 016 & 025 & 034 & 123 & 145 & 246 & & & & & \end{array} \right\};$$

$$B_N = \{578, 678, 589, 689, 579, 679, 5ab, 6ab, 5bc, 6bc, 5cd, 6cd, 5de, 6de, 5ae, 6ae\}.$$

Let B_1 be obtained by replacing the blocks $016, 025, 145, 246$ of B with the blocks $015, 026, 146, 245$.
Let B_2 be obtained by replacing the blocks $E(17b)$ of B_1 with the blocks $E(1b7)$.
Then (X, B_1) and (X, B_2) are two indecomposable TETS(15) with exactly 33 and 30 repeated blocks, respectively.

$34 \in E_I(15)$.

$$B_R = \left\{ \begin{array}{ccccccccccccc} 07a & 0be & 18d & 19a & 27c & 28a & 2bd & 37d & 38e & 39b & 3ac & 47e & 48b \\ 49c & 17b & 08c & 29e & 4ad & 009 & 99d & dd0 & 11c & cce & ee1 & 223 & 334 \\ 442 & 778 & 013 & 025 & 046 & 126 & 145 & 356 & & & & & \end{array} \right\};$$

$$B_N = \left\{ \begin{array}{ccccccccccccc} 885 & 557 & 886 & 667 & 66b & bba & aa6 & aab & bb5 & 55a & 589 & 689 & 579 \\ 679 & 5bc & 6bc & 5cd & 6cd & 5de & 6de & 5ae & 6ae & & & & \end{array} \right\}.$$

$36 \in E_I(15)$.

$$B_R = \left\{ \begin{array}{ccccccccccccc} 07b & 08d & 0ce & 18a & 19d & 1be & 27d & 28e & 29b & 2ac & 37e & 39c & 3ad \\ 123 & 145 & 016 & 025 & 034 & 356 & 49e & 6ae & 009 & 117 & 224 & 338 & 446 \\ 55d & 662 & 77c & 88b & 99a & aa0 & bb3 & cc1 & dde & ee5 & & & \end{array} \right\};$$

$$B_N = \left\{ \begin{array}{ccccccccccccc} 478 & 48c & 4cd & 4bd & 4ab & 47a & 579 & 589 & 58c & 5bc & 5ab & 57a & 678 \\ 689 & 679 & 6bc & 6cd & 6bd & & & & & & & & \end{array} \right\}.$$

Step 4. $\{38, 39, 40, 42\} \subseteq E_I(15)$.
Let $X = \{0, 1, 2, \ldots, 9, a, b, c, d, e\}$ and let (X, B) be the following indecomposable TETS(15) with exactly 42 repeated blocks:

$$B_R = \left\{ \begin{array}{cccccccccccccc} 078 & 09a & 0bc & 0de & 147 & 168 & 19b & 1ad & 1ce & 239 & 246 & 25a & 27e & 28c \\ 2bd & 37c & 38d & 3ae & 45d & 489 & 4be & 57b & 59c & 67a & 69e & 6cd & 8ab & 004 \\ 112 & 220 & 334 & 44a & 558 & 66b & 779 & 88e & 99d & aac & bb3 & cc4 & dd7 & ee5 \end{array} \right\};$$

$$B_N = \{013 \;\; 015 \;\; 036 \;\; 056 \;\; 135 \;\; 356\}.$$

Let B_1 be obtained by replacing the blocks $004, 334, 036$ of B with the blocks $006, 336, 034$.
Let B_2 be obtained by replacing the blocks $334, cc4, 37c$ of B_1 with the blocks $337, cc7, 34c$.
Let B_3 be obtained by replacing the blocks $E(ac4)$ of B with the blocks $E(a4c)$.
Then (X, B_1), (X, B_2) and (X, B_3) are three indecomposable TETS(15) with exactly 40, 38 and 39 repeated blocks, respectively.

Step 5. $\{41, 43\} \subseteq E_I(15)$.
Let $X = \{0, 1, 2, \ldots, 9, a, b, c, d, e\}$ and let (X, B) be the following indecomposable TETS(15) with exactly 43 repeated blocks:

$$B_R = \left\{ \begin{array}{ccccccccccccccc} 056 & 078 & 09a & 0bc & 0de & 146 & 179 & 18a & 1bd & 1ce & 24e & 25b & 26a & 28c & 29d \\ 36d & 38e & 3ac & 39b & 45c & 47b & 489 & 4ad & 57a & 59e & 67e & 68b & 7cd & 004 & 115 \\ 227 & 335 & 443 & 55d & 669 & 773 & 885 & 99c & aab & bbe & cc6 & dd8 & eea & & \end{array} \right\};$$

$$B_N = \{012 \;\; 013 \;\; 023 \;\; 123\}.$$

Let B_1 be obtained by replacing the blocks $115, 335, 013$ of B with the blocks $110, 330, 135$. Then (X, B_1) is an indecomposable TETS(15) with exactly 41 repeated blocks.

This completes the proof of Lemma 4. $\square$

4 Conclusion

Let $U(2n)$ be the set of integers k such that a pair of 1-factorizations of order $2n$ on $N = \{1, 2, \ldots, 2n\}$, having k edges in common, exist. In [10], Lindner and Wallis gave a complete solution to this problem by showing that:

$$U(2) = \{1\}; \quad U(4) = \{0, 2, 6\}; \quad U(6) = \{0, 1, 2, 3, 5, 6, 7, 9, 15\} \quad \text{and}$$

$$U(2n) = \{0, 1, \ldots, u = n(2n-1)\} \setminus \{u-5, u-3, u-2, u-1\}, \quad \text{for all } n \geq 4.$$

The basic ingredients used in [11] to study the intersection problem for ETS(v) are included in the following lemma:

LEMMA 5
(a) For v even, if $(k, h) \in J(v) \times U(v)$, then $v + k + h \in J(2v)$.

(b) *For v even, $J(v) \subseteq J(2v)$.*
(c) *For $v \geq 9$, $J(v) = I(v)$ implies $J(2v) = I(2v)$.*
(d) *For v odd, $v \geq 9$, $J(v) = I(v)$ implies $J(2v+3) = I(2v+3)$.*
(e) *For v odd, $v \geq 15$, $J(v) = I(v)$ implies $J(2v+9) = I(2v+9)$.*

Note that a 2-factorization of $2K_v$ is formed when two 1-factorizations of K_v are taken and paired together. So if $h \in U(v)$, then certainly we can find two 2-factorizations of $2K_v$ having h common edges. So using ideas similar to those in the lemma above, but using indecomposable TETS(v) with exactly k repeated blocks instead of a pair of ETS(v) intersecting in exactly k blocks, the following result is obtained in exactly the same way:

LEMMA 6
(a) *For v even, if $(k,h) \in E_I(v) \times U(v)$, then $v + k + h \in E_I(2v)$.*
(b) *For v even, $E_I(v) \subseteq E_I(2v)$.*
(c) *For $v \geq 9$, $E_I(v) = T^*(v)$ implies $E_I(2v) = T^*(2v)$.*
(d) *For v odd, $v \geq 9$, $E_I(v) = T^*(v)$ implies $E_I(2v+3) = T^*(2v+3)$.*
(e) *For v odd, $v \geq 15$, $E_I(v) = T^*(v)$ implies $E_I(2v+9) = T^*(2v+9)$.*

LEMMA 7 $E_I(12) = T^*(12)$.

Proof By *(a)* and *(b)* of Lemma 6, we only need to prove that $s_{12} - 2 = 28 \in E_I(12)$. Let $X = \{0,1,2,\ldots,9,a,b\}$ and let (X,B) be the following indecomposable TETS(12) with exactly 28 repeated blocks:

$$B_R = \{045, 069, 07b, 156, 179, 18b, 267, 25b, 28a, 346, 378, 39b, 489, 47a, 59a,$$

$$6ab, 00a, 11a, 224, 335, 441, 558, 668, 775, 880, 992, aa3, bb4\};$$

$$B_N = \{012, 013, 023, 123\}.$$

Hence $28 \in E_I(12)$. □

We now have our required result:

MAIN THEOREM *$E(3) = \{0,3\}$, $E_I(3) = \emptyset$, $E(6) = I(6)$, $E_I(6) = I(6) - \{s_6\}$, $E(v) = T(v)$ and $E_I(v) = T^*(v)$, for all $v \geq 9$.*

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