The strong edge colorings of a sparse random graph

Zbigniew Palka
Department of Discrete Mathematics
Adam Mickiewicz University
Matejki 48/49 Poznań, Poland

Abstract

The strong chromatic index of a graph G is the smallest integer k such that the edge set $E(G)$ can be partitioned into k induced subgraphs of G which form matchings. In this paper we consider the behavior of the strong chromatic index of a sparse random graph $K(n,p)$, where $p = p(n) = o(1)$.

1. Introduction

Let $G = (V, E)$ be a finite graph. The *chromatic index* $\gamma = \gamma(G)$ of a graph G is the least number of colors required in order to color each edge of G so that no two edges with a common vertex have the same color. In other words, $\gamma(G)$ is the smallest integer k such that the edge set $E(G)$ can be partitioned into k matchings. Vizing's theorem says that for every graph G,

$$\gamma(G) \in \{\Delta, \Delta + 1\}$$

where $\Delta = \Delta(G)$ is the maximum vertex degree of G. Futhermore, if $\gamma = \Delta + 1$ then G has two adjacent vertices of maximum degree Δ.

A *strong matching* in a graph G is an *induced* subgraph of G that forms a matching (i.e. a set of pairwise disjoint edges of G, no two of them being adjacent to the same edge).

The *strong chromatic index* $\gamma^* = \gamma^*(G)$ is the smallest integer k such that the edge set $E(G)$ can be partitioned into k strong matchings. Equivalently, γ^* is the smallest k such that the edge set $E(G)$ can be k-colored with the property that each color class is a strong matching of the graph G. Such a coloring is called *a strong edge coloring.*

A Vizing's-type problem is to give an upper bound for $\gamma^*(G)$ in terms of $\Delta = \Delta(G)$. A trivial bound is the following

$$\gamma^*(G) \leq 2\Delta^2 - 2\Delta + 1.$$

As a matter of fact, the color of an edge $\{v, w\}$ can be affected by the colors of at most $2(\Delta - 1)$ edges incident to $\{v, w\}$ and by the colors of at most $2(\Delta - 1)^2$ "second neighbors" of $\{v, w\}$. Also, a good strong edge coloring with at most $2\Delta^2 - 2\Delta + 1$ colors can be found by the greedy algorithm.

The open problem in this area is the conjecture (see e.g. [3]) that

$$\gamma^*(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if} \quad \Delta \text{ is even} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if} \quad \Delta \text{ is odd.} \end{cases}$$

The lower bounds for $\gamma^*(G)$ are provided by the following obvious inequalities

$$\gamma^*(G) \geq \max_{\{v,w\} \in E(G)} \{\deg(v) + \deg(w) - 1\} \tag{1}$$

where $\deg(v)$ denotes the degree of vertex v, and

$$\gamma^*(G) \geq \frac{|E(G)|}{\beta} \tag{2}$$

where $\beta = \beta(G)$ stands for the maximum number of edges in a strong matching of G.

The aim of this paper is to consider the strong edge colorings of a random graph model. Let $K(n,p)$ be a random graph on vertex set $\{1, 2, \ldots, n\}$, where each edge appears with the same probability p independently of all other edges.

It is well-known that for every $0 < p = p(n) < 1$, the chromatic index of a random graph, $\gamma_{n,p} = \gamma(K(n,p))$, satisfies a nice property, namely

$$\lim_{n \to \infty} P(\gamma_{n,p} = \Delta_{n,p}) = 1$$

where $\Delta_{n,p} = \Delta(K(n,p))$ is the maximum degree of a random graph $K(n,p)$.

In this paper we examine behavior of the strong chromatic index

$$\gamma^*_{n,p} = \gamma^*(K(n,p))$$

of a sparse random graph $K(n,p)$, i.e. when $p = p(n) \to 0$ as $n \to \infty$.

As usual, we say that *almost every* graph $K(n,p)$ has a given property $\mathcal{P}$ (or that $K(n,p)$ has *almost surely* property $\mathcal{P}$) if

$$\lim_{n \to \infty} P(K(n,p) \text{ has } \mathcal{P}) = 1.$$

2. Results

We begin with a simple observation.

Proposition 1. *Let* $p = p(n) = o(n^{-3/2})$. *Then*

$$\lim_{n\to\infty} P(\gamma^*_{n,p} = 1) = 1. \quad \square$$

Indeed, for the edge probability $p = o(n^{-3/2})$ almost every graph $K(n,p)$ consists of independent edges only.

When $p = p(n)$ is of the order at least $n^{-3/2}$ but $p = o(n)$ then the following result holds.

Theorem 1. *Let* $k \geq 2$ *be fixed,* $0 < c < \infty$ *and*

$$p = cn^{-\frac{k+1}{k}}. \tag{3}$$

Then

$$\lim_{n\to\infty} P(\gamma^*_{n,p} = t) = \begin{cases} e^{-\lambda} & \text{if} \quad t = k-1 \\ 1 - e^{-\lambda} & \text{if} \quad t = k \end{cases}$$

where

$$\lambda = \frac{c^k}{k!}(k2^{k-2} - k + 1).$$

Proof. It is well-known that if $np \to 0$ as $n \to \infty$, then $K(n,p)$ is almost surely a forest. Moreover, for the edge probability p given by (3), every isolated tree in $K(n,p)$ has almost surely at most $k+1$ vertices. Also (see [3]) if a graph G is a tree then

$$\gamma^*(G) = \max_{\{v,w\}\in E(G)} \{\deg(v) + \deg(w) - 1\} := \phi(G), \tag{4}$$

(compare with (1)).

What are the structures of trees on $k+1$ vertices that give maximum value in (4)? There are only two such structures: a star, i.e. a tree with exactly one non-pendant vertex and a tree with exactly two non-pendant vertices (see Fig. 1, where i can take any integer value from 1 to $k-2$).

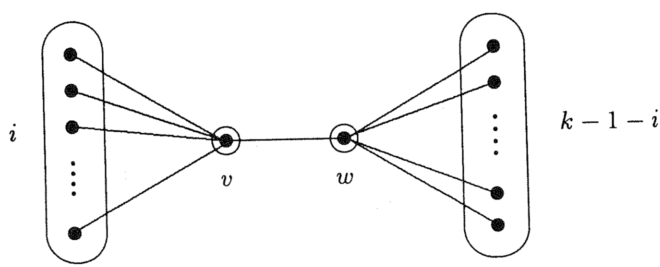

Fig. 1

Indeed, let v be adjacent with l vertices different from w and w with s vertices different from v. Then $l, s \geq 0$ and $l + s \leq k - 1$. Now

$$\phi(G) = \max_{\{v,w\} \in E(G)} (l + s + 1) = k$$

and all extremal graphs are easily characterized.

Let the random variables X_1 and X_2 count the number of configurations in $K(n,p)$ which are isomorphic to a star on $k+1$ vertices and to a tree on $k+1$ vertices with two non-pendant vertices, respectively. Then

$$P(\gamma^*_{n,p} = k) = P(X_1 + X_2 \geq 1). \tag{5}$$

For a fixed $k \geq 2$, we have

$$Exp(X_1) = \binom{n}{k+1}(k+1)p^k \sim c^k/k! = \lambda_1.$$

Moreover, if $k \geq 3$ then

$$\begin{aligned} Exp(X_2) &= \sum_{i=1}^{k-2} \binom{n}{k+1}\binom{k+1}{2}\binom{k-1}{i} p^k \\ &= \binom{n}{k+1}\binom{k+1}{2} p^k \left(\sum_{i=0}^{k-1} \binom{k-1}{i} - 2 \right) \\ &\sim \frac{c^k}{(k-1)!}(2^{k-2} - 1) = \lambda_2. \end{aligned}$$

A standard method (see e.g. [1]) shows that both random variables X_1 and X_2 have asymptotically Poisson distribution with parameter λ_1 and λ_2, respectively. Therefore, by (5),

$$P(\gamma^*_{n,p} = k) = 1 - P(X_1 = 0, X_2 = 0) \sim 1 - e^{-\lambda},$$

where

$$\lambda = \lambda_1 + \lambda_2 = \frac{c^k}{k!}(k2^{k-2} - k + 1).$$

On the other hand, if $X_1 = 0$ and $X_2 = 0$ (which, for $p = p(n)$ given by (3), holds with probability tending to $e^{-\lambda}$ as $n \to \infty$) then using Chebyshev's inequality one can show that almost every graph $K(n,p)$ contains a star on k vertices and a structure of a tree that gives the maximum in (4) is such a star. Obviously, one needs $k-1$ colors in order to color its edges. Consequently,

$$\lim_{n \to \infty} P(\gamma^*_{n,p} = k - 1) = e^{-\lambda},$$

and the proof is completed. □

Theorem 1 shows that for the edge probability p given by (3), the strong chromatic index of a random graph $K(n,p)$ behaves similarly as its maximum vertex degree, since (see e.g. [5]):

$$\lim_{n\to\infty} P(\Delta_{n,p}=t) = \begin{cases} e^{-\theta} & \text{if } t=k-1 \\ 1-e^{-\theta} & \text{if } t=k \end{cases}$$

where $\theta = \frac{c^k}{k!}$.

Now let $np \to c$ as $n \to \infty$, where $0 < c < \infty$. In such a case the structure of $K(n,p)$ changes dramatically when c passes 1 (see e.g. [1]). However during this period of the evolution, the number of cycles contained in $K(n,p)$ is fixed, i.e. does not depend on n. This ensures that the strong chromatic index is still of the order of magnitude $O(\Delta_{n,p})$. As a matter of fact the following results hold.

Theorem 2. *Let* $p = \frac{c}{n}$.

(i) If $0 < c < 1$, *then for arbitrary small* $\epsilon > 0$

$$P\left(\gamma^*_{n,p} < \left(\frac{5}{2}+\epsilon\right)\Delta_{n,p}\right) = 1-o(1).$$

(ii) If $1 \le c < \infty$, *then there is a constant* C *such that*

$$P\left(\gamma^*_{n,p} \le C\cdot\Delta_{n,p}\right) = 1-o(1).$$

Before we prove this theorem, let us comment on a case when $0 < c < 1$. It is known that during this period of the evolution the random graph $K(n,p)$ is almost surely planar (see e.g. [1]). A simple consequence of Vizing's theorem and the four color theorem shows (see e.g. [3]), that if G is a planar graph, then

$$\gamma^*(G) \le 4\Delta(G)+4.$$

The first part of Theorem 2 claims, that in a case of a random graph, this upper bound can be, asymptotically, improved by a small factor.

Proof of Theorem 2. Let T be a tree with maximum vertex degree $\Delta(T)$. It is not hard to see, that adding to the edge set of T a new edge (of course a cycle is formed) may increase the strong chromatic index of a new graph G, by $\Delta(T)$. Therefore

$$\gamma^*(G) \le \phi(G)+\Delta(T) \tag{6}$$

where $\phi(G)$ is defined by (4). A little more effort shows that if one adds to a tree T some l new edges, in such a way that a longest cycle of a new graph G has k

vertices, then the strong chromatic index can be increased by $l+(k-2)(\Delta(T)-1)$. To see this consider a tree T which has k non-pendant vertices, each of degree $\Delta = \Delta(T)$. By (4) one needs $2\Delta - 1$ colors in a strong edge coloring of such a tree. Now add $\binom{k-1}{2}$ missing edges between those k non-pendant vertices. Each new edge receives a new color and each of $(k-2)(\Delta - 1)$ already colored edges must turn into a different color. Finally add $l - \binom{k-1}{2}$ edges in such a way that each must be colored by a new color (it is easy to see that this is always possible). Thus

$$\gamma^*(G) \leq \phi(G) + l + (k-2)(\Delta(T) - 1). \tag{7}$$

Now we show that it is unlikely for $K(n,p)$ where $p = \frac{c}{n}$, $0 < c < \infty$, to contain a pair of adjacent vertices, both of large degrees. Let $X = X(r,s)$ be the number of ordered pairs of adjacent vertices (v,w) such that

$$r \leq deg(v) \leq deg(w) \leq deg(v) + s - 1.$$

Let $\Delta = \Delta_{n,p}$. It is known (see e.g. [1, p.72]) that for almost every graph $K(n, c/n)$

$$\Delta = \frac{\log n}{\log \log n}(1 + o(1)). \tag{8}$$

We will show that for $r = c_1\Delta$ and $s = c_2\Delta$, where $c_1 > \frac{1}{2}$ and $c_2 = 1 - c_1$ are constants,

$$\lim_{n\to\infty} Exp(X(r,s)) = 0. \tag{9}$$

We have, with $q = 1 - p$,

$$\begin{aligned} Exp(X) &\leq n^2 p \sum_{k=r-1}^{\Delta-1} \binom{n-2}{k} p^k q^{n-2-k} \sum_{i=k}^{k+s-1} \binom{n-2}{i} p^i q^{n-2-i} \\ &< n \cdot c \cdot s \sum_{k=r-1}^{\Delta-1} \left(\frac{ce}{k}\right)^{2k} \\ &= O\left(n\Delta^2 \left(\frac{ce}{c_1\Delta}\right)^{2c_1\Delta}\right). \end{aligned}$$

In the above estimation the second inequality is implied, among other things, by the inequality

$$e^k > \frac{k^k}{k!}$$

(which follows, for example, from Stirling's formula). Therefore, by (8),

$$\log Exp(X) \leq (1 - 2c_1)\log n$$

and, since $c_1 > \frac{1}{2}$, we obtain (9). This implies that for all adjacent pairs of vertices (v, w)

$$P(deg(v) < c_1\Delta) = 1 - o(1).$$

Consequently, for arbitrarily small but fixed $\epsilon > 0$,

$$\phi(K(n, c/n)) < \left(\frac{3}{2} + \epsilon\right)\Delta. \tag{10}$$

Now if $0 < c < 1$ then almost every graph $K(n,p)$, $p = \frac{c}{n}$, consists of components with at most one cycle. Consequently by (6) and (10) we obtain the first part of theorem.

The second part follows similarly by (7) and (10), since for $1 \le c < \infty$, a random graph $K(n, c/n)$ contains almost surely a fixed number of cycles (see [1, p.79]). □

Presented results show that as long as $p = p(n) \le \frac{c}{n}$, where $0 < c < \infty$, the strong chromatic index of a random graph $K(n,p)$ is at most of the order of magnitude $O(\Delta)$, where $\Delta = \Delta_{n,p}$.

Our next result shows that if $p = p(n) \ge \frac{c(n)}{n}$, where $c(n)$ is a function tending to infinity as $n \to \infty$, the strong chromatic index of $K(n,p)$ is of the order of magnitude greater than the maximum vertex degree $\Delta = \Delta_{n,p}$.

Theorem 3. *Let $p \ge \frac{c}{n} = o(1)$, where $c = c(n) \to \infty$ as $n \to \infty$. Then*

$$\gamma^*_{n,p} \ge O\left(\frac{\Delta^2}{\log \Delta}\right). \tag{11}$$

Proof. Let $\alpha_{n,p}$ denote the size of the largest independent set in $K(n,p)$. Then (see [4]), for $np = c(n) = o(n)$, almost surely

$$\alpha_{n,p} \le \frac{2n}{c(n)} \log c(n)(1 + o(1)).$$

Clearly, $\beta_{n,p}$, the maximum number of edges in a strong matching satisfies

$$\beta_{n,p} \le \alpha_{n,p}.$$

Thus, by (2),

$$\gamma^*_{n,p} \ge C(1)\frac{c(n)^2}{\log c(n)}$$

where $C(1)$ is a positive constant. On the other hand, in the case when $np = c(n) = o(n)$ almost every graph $K(n,p)$ is such that (see e.g. [5])

$$\Delta_{n,p} = O(c(n))$$

and we arrive at (11). □

3. Acknowledgements

This research was supported in part by grant *KBN - 2 P03A 023 09*.
The author is indebted to a referee for constructive suggestions.

References

[1] B. Bollobás, Random Graphs, Academic Press Inc. (London) Ltd., 1985.

[2] A. El Maftouhi, L. Marquez Gordones, The maximum order of a strong matching in a random graph, Australasian J. of Comb. 10 (1994), 97-104.

[3] R. Faudree, R. Schelp, A. Gyáfrás, Zs. Tuza, The strong chromatic index of graphs, Ars Comb. 29 (1990) B, 205-211.

[4] A. Frieze, On the independence number of random graphs, Discrete Math. 81 (1990), 171-175.

[5] Z. Palka, Asymptotic properties of random graphs, Dissertationes Mathematicae CCLXXV, PWN, Warszawa 1988.

(Received 9/10/97; revised 6/1/98)