On the Cycle Structure of In-Tournaments

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Abstract

An in-tournament is an oriented graph such that the in-neighborhood of every vertex induces a tournament. Therefore, in-tournaments are a generalization of local tournaments where, for every vertex, the set of in-neighbors as well as the set of out-neighbors induce a tournament. While local tournaments have been intensively studied very little is known about in-tournaments. It is the purpose of this paper to give more information about in-tournaments where we will focus mainly on the cycle structure of these digraphs. We will investigate the extendability of cycles and the influence of the minimum indegree on the cycle structure. In particular, we show that every strong in-tournament of order n with minimum indegree at least $\frac{n}{3}$ is pancyclic.

1 Terminology and Introduction

Throughout this paper we will consider digraphs that contain no multiple arcs, no loops and no cycles of length 2. We call these digraphs *oriented graphs*. An *in-tournament* is an oriented graph such that the set of negative neighbors of every vertex induces a tournament, i.e. every pair of distinct vertices that have a common positive neighbor are connected by exactly one arc.

A digraph D is determined by its set of vertices and its set of arcs, denoted by $V(D)$ and $E(D)$, respectively. We call D a connected digraph if the underlying graph is connected. For $xy \in E(D)$ where $x, y \in V(D)$, we write $x \to y$ and we say that x *dominates* y or y *is dominated by* x. Furthermore, y is a *positive neighbor* or *out-neighbor* of x and x is a *negative neighbor* or *in-neighbor* of y. Let S_1 and S_2 be disjoint subsets of $V(D)$. If $s_1 \to s_2$ for every $s_1 \in S_1$ and $s_2 \in S_2$ we denote this by $S_1 \to S_2$. If $S_2 = \{y\}$ then we use $S_1 \to y$ instead of $S_1 \to \{y\}$.

For $S \subseteq V(D)$, the digraph which is induced by the vertices of S is denoted by $D[S]$. The *outdegree* $d^+(x, S)$ and *indegree* $d^-(x, S)$ *with respect to* S of a vertex $x \in V(D)$ are defined to be the number of positive and negative neighbors of x in S, respectively. In the case when $S = V(D)$, we also write $d^+(x)$ and $d^-(x)$.

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The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\{d^+(x) \mid x \in V(D)\}$ and $\min\{d^-(x) \mid x \in V(D)\}$, respectively. Furthermore, $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ is the *minimum degree* of D. Analogously, the *maximum outdegree* of D is defined as $\Delta^+(D) = \max\{d^+(x) \mid x \in V(D)\}$. If $d^+(x) = d^-(x) = p$ for every $x \in V(D)$, then D is called *p-regular*.

All cycles and paths mentioned here are oriented cycles and oriented paths. A *Hamiltonian path* of a digraph D is a path that consists of all the vertices of D. Analogously, a *Hamiltonian cycle* is a cycle containing all the vertices of D. The length of a shortest cycle in a digraph D is called the *girth of* D, denoted by $g(D)$. A cycle C in D is *extendable* if D contains a cycle C' such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. We call C to be a k-cycle if C consists of k vertices. A digraph D of order n is called *pancyclic* if D contains a k-cycle for every $3 \leq k \leq n$. If every vertex of D belongs to a cycle of length k for every $3 \leq k \leq n$, then D is called *vertex pancyclic*.

The study of tournaments and their different generalizations is one of the most attractive subjects in the work on digraphs. One type of generalization transfers the adjacency between every pair of distinct vertices in tournaments to only those pairs where both vertices belong either to the positive or to the negative neighborhood of some vertex of the digraph. This leads to the class of local tournaments, or more generally, to the class of locally semicomplete digraphs where adjacent vertices may be connected by two mutually opposite arcs. The research about the structure of these digraphs evolved into a very productive area. In particular, the Ph. D. theses of Y. Guo [5] and J. Huang [8] have been devoted to this subject.

As a generalization of local tournaments, J. Bang-Jensen, J. Huang, and E. Prisner [1] studied the class of in-tournaments, where only the set of in-neighbors of every vertex induces a tournament. But very little work has been done concerning in-tournaments and it is the purpose of this paper to give more information about the properties of this family of digraphs. We focus on the cycle structure of in-tournaments where we consider the extendability of cycles and the influence of the minimum indegree on the cycle structure. In particular, we show that every strong in-tournament of order n with minimum indegree at least $\frac{n}{3}$ is pancyclic.

2 Preliminary results

In this section we will state some known results which either will be useful in our investigations or will be generalized later on. The first two are due to Rédei [10] and Moon [9], respectively, and deal with the structure of tournaments.

Theorem 2.1 *Every tournament contains a Hamiltonian path.*

Theorem 2.2 *Every strong tournament is vertex pancyclic.*

The next two results on long cycles in strong in-tournaments were found by Bang-Jensen, Huang, and Prisner [1].

Theorem 2.3 *An in-tournament is Hamiltonian if and only if it is strong.*

Theorem 2.4 *Let D be a strong in-tournament of order n that contains a k-cycle for some $k \geq \frac{n}{2}$. Then D has cycles of length $k, k+1, ..., n$.*

In the following, we continue the investigation of long cycles and we present some sufficient conditions for the existence of short cycles in in-tournaments. In particular, a generalization of Theorem 2.4 is given in Section 4.

3 Extending cycles

Given a cycle C of length k in an in-tournament D, we consider conditions for the existence of cycle lengths in D that are related to k. In two cases we show the extendability of C, i.e. there is a $(k+1)$-cycle in D containing all the vertices of C.

The first result was already mentioned in [1]. Since the authors of [1] made a minor slip in the statement of the hypothesis, we restate it here with a slightly different proof.

Theorem 3.1 *Let D be a strong in-tournament of order n containing a k-cycle, $k < n$, that is not extendable. Then D has cycles of length $l+1, l+2, ..., l+k$ for some $2 \leq l \leq n-k$.*

Proof. Let $C = x_1 x_2 ... x_k x_1$ be a cycle of length $k < n$ which is not extendable. Since D is strong, there exists a vertex $u \in V(D-C)$ such that u has a positive neighbor on C. Without loss of generality, $u \to x_k$. Obviously, u and x_{k-1} are adjacent, and since C is not extendable, we have $u \to x_{k-1}$. Gradually, this implies $u \to C$. Hence, there is a path P of length l, $2 \leq l \leq n-k$, in D leading from C to u. Since $u \to C$, this implies the existence of the desired cycles. □

The argumentation in the proof of Theorem 3.1 leads to the following corollary.

Corollary 3.2 *Let D be strong in-tournament of order n such that $\Delta^+(D) \leq k-1$. If D has a k-cycle C_k then D contains cycles $C_{k+1}, C_{k+2}, ..., C_n$ of length $k+1, k+2, ..., n$, respectively, such that $V(C_k) \subset V(C_{k+1}) \subset ... \subset V(C_n)$.*

By considering the diameter, it is possible to give more precise information about the cycle lengths in Theorem 3.1.

Theorem 3.3 *A strong in-tournament D of order n with diameter* $\mathrm{diam}(D) = p$ *contains cycles of length $p+1, p+2, ..., n$.*

Proof. We proceed by induction on the cycle length. Let uv be an arbitrary arc in D. Since D contains a path from v to u of length $k \leq p$, there exists a $(k+1)$-cycle C in D. If $k < p$ and C is not extendable then, by following the proof of Theorem 3.1, we deduce that there are cycles of length $l+1, l+2, ..., l+k$ for some $2 \leq l \leq p$

in D. Hence D contains a cycle of length k' where $k+1 \leq k' \leq p+1$, and inductively we obtain a $(p+1)$-cycle in D. Clearly, the same holds if C is extendable or if $k = p$.

Now let D contain cycles of length $p+1, p+2, ..., p+s$ for some $1 \leq s \leq n-p-1$. Either the $(p+s)$-cycle is extendable or analogously to the case above, D contains cycles of length $l+1, l+2, ..., l+p+s$ for some $2 \leq l \leq p$. Since $l+1 \leq p+1$ and $l+p+s \geq p+s+2$, the second alternative leads to a $(p+s+1)$-cycle, too. □

The following example shows that Theorem 3.3 is best possible. Consider the in-tournament D with the vertex set $V(D) = \{x_1, x_2, ..., x_{3k+1}\}$, where $k \geq 3$. For every $1 \leq i \leq 3k+1$, let $x_i \to \{x_{i+1}, x_{i+2}, x_{i+3}\}$ (all indices modulo $3k+1$). Then $\mathrm{diam}(D) = k$ and there is no cycle of length k in D.

Next we present a sufficient condition for cycle extendability in connected in-tournaments that is based on the minimum degree.

Theorem 3.4 *Let D be a connected in-tournament of order n such that $\delta(D) \geq p > 0$. Then every k-cycle with $n - \lfloor \frac{4p+1}{3} \rfloor \leq k < n$ is extendable.*

Proof. Let C be a cycle of length k and suppose to the contrary that C is not extendable. Let $R \subseteq V(D-C)$ such that $v \in R$ if and only if $d^-(v,C) > 0$ and assume, without loss of generality, that $R \neq \emptyset$. Furthermore, define $Q = V(D-C) \backslash R$. It follows that $k+|R|+|Q| = n \leq k + \lfloor \frac{4p+1}{3} \rfloor$ and therefore, $|R|+|Q| \leq \lfloor \frac{4p+1}{3} \rfloor$. By definition, $d^-(u,C) = 0$ for every $u \in Q$ and since C is not extendable, we have $d^+(v,C) = 0$ for every $v \in R$. Let $v_0 \in R$ such that $d^+(v_0,R) \leq \frac{|R|-1}{2}$.

If $Q = \emptyset$, then $d^+(v_0,D) = d^+(v_0,R)$. Since $|R| \leq \lfloor \frac{4p+1}{3} \rfloor$, it follows that $p \leq d^+(v_0,D) \leq \frac{|R|-1}{2} \leq \frac{1}{2}\lfloor \frac{4p+1}{3} \rfloor - \frac{1}{2} \leq \frac{2p-1}{3}$ which is clearly a contradiction.

For $Q \neq \emptyset$, let $u_0 \in Q$ such that $d^-(u_0,Q) \leq \frac{|Q|-1}{2}$. The contradiction $2p \leq d^+(v_0,D) + d^-(u_0,D) \leq \frac{|R|-1}{2} + |Q| + \frac{|Q|-1}{2} + |R| \leq \frac{3}{2}(|R|+|Q|) - 1 \leq \frac{3}{2}\lfloor \frac{4p+1}{3} \rfloor - 1 \leq 2p - \frac{1}{2}$ completes the proof. □

The lower bound $n - \lfloor \frac{4p+1}{3} \rfloor$ for the cycle lengths in Theorem 3.4 is best possible. To see this consider for example the case when $p = 3t+1$ for some $t \geq 0$. Let D be a tournament of odd order $n \geq 6t+3 \geq 5$ with the vertex set $V(D) = V_1 \cup V_2 \cup V_3$ where $|V_2| = |V_3| = 2t+1$. Let $D[V_1], D[V_2]$ and $D[V_3]$ be regular tournaments and let $V_1 \to V_2 \to V_3 \to V_1$. It is easy to see that $\delta(D) = 3t+1 = p$. Furthermore, the Hamiltonian cycle of $D[V_1]$ has length $k = n - (4t+2) = n - \lceil \frac{4p+1}{3} \rceil$ and is not extendable.

Analogously, examples can be constructed for the cases $p = 3t$ and $p = 3t+2$.

4 Pancyclic in-tournaments

In this section we show that every strong in-tournament of order n with minimum indegree $\frac{n}{3}$ is pancyclic. In the first two steps we consider long and short cycles in strong in-tournaments with arbitrary minimum indegree p. The following result is a generalization of Theorem 2.4 which represents the case $p = 1$.

Theorem 4.1 *Let D be a strong in-tournament of order n such that $\delta^-(D) \geq p$. If D contains a k-cycle for some $n > k > \min\{\frac{1}{4}(2n-3p+1), \frac{2}{3}(n-2p)+\frac{1}{2}\}$ then D has a cycle of length $k+1$.*

Proof. Let C be a cycle of length k, where k satisfies the desired inequality. Suppose to the contrary that D does not contain a cycle of length $k+1$. Since D is strong, there is a vertex $x \in V(D-C)$ such that x has a positive neighbor in C. By the assumption, C is not extendable and hence this implies analogously to the proof of Theorem 3.1 that $x \to C$. Let N denote the set of negative neighbors of x. It follows that $N \cap V(C) = \emptyset$, $|N| \geq p$ and $D[N]$ is a tournament. Furthermore, let P be a shortest path leading from C to x. Clearly, $|V(P) \cap N| = 1$. For $|V(P)| \leq k+1$, the path P and $k+1-|V(P)|$ further vertices from C form a $(k+1)$-cycle, a contradiction. Therefore, let $|V(P)| \geq k+2 \geq 5$. This, together with the minimality of P, implies that $d^-(y,C) = 0$ and $d^-(y, P-N) \leq 1$ for every $y \in N$.

Let $y_1 \in N$ such that $d^-(y_1, N) \leq \frac{|N|-1}{2}$. Then y_1 has at least $p - (\frac{|N|-1}{2}+1)$ negative neighbors in $D-(C \cup \{x\} \cup N \cup P)$. Therefore, we obtain

$$\begin{aligned} n &\geq |V(C)| + |\{x\}| + |N| + (|V(P)|-3) + p - \left(\frac{|N|-1}{2}+1\right) \\ &\geq 2k + p + \frac{|N|-1}{2} \geq 2k + \frac{3p-1}{2}. \end{aligned}$$

which implies $k \leq \frac{1}{4}(2n-3p+1)$.

To obtain the second upper bound for k consider the strong component of $D[N]$, say D_1, such that there is no arc leading from $D[N]-D_1$ to D_1. By Theorem 2.2, we derive a contradiction to the assumption if D_1 contains more than k vertices. Hence let $|D_1| \leq k$, and let $y_2 \in D_1$ such that $d^-(y_2, N) = d^-(y_2, D_1) \leq \frac{|D_1|-1}{2} \leq \frac{k-1}{2}$. This implies that $d^-(y_2, D-(C \cup \{x\} \cup N \cup P)) \geq p - (\frac{k-1}{2}+1)$, which yields

$$\begin{aligned} n &\geq |V(C)| + |\{x\}| + |N| + (|V(P)|-3) + p - \left(\frac{k-1}{2}+1\right) \\ &\geq 2k + 2p - \left(\frac{k-1}{2}+1\right). \end{aligned}$$

It follows that $k \leq \frac{2}{3}(n-2p)+\frac{1}{2}$. Since $k \leq \frac{1}{4}(2n-3p+1)$, we derive a contradiction.
$\square$

For every p there exists an in-tournament with minimum indegree p that is not strong and that contains a cycle of length k, where k satisfies the condition of Theorem 4.1, but no cycle of length $k+1$. Therefore, we cannot drop the condition on the strong connectivity in Theorem 4.1. To see this, let $p \geq 1$ be an arbitrary integer and consider the digraph D of order n where $2p+1 < n < \frac{11p+3}{2}$. Let D consist of the vertex set $V(D) = V_1 \cup V_2 \cup ... \cup V_r$, $r \geq 2$, where $|V_1| = ... = |V_{r-1}| = 2p+1$ and $|V_r| \leq 2p+1$. For $1 \leq i \leq r-1$, let $D[V_i]$ induce a p-regular tournament, and let $D[V_r]$ be a set of independent vertices. Furthermore, let $V_i \to V_j$ for every $1 \leq i < j \leq r$.

Then D is an in-tournament with minimum indegree p, and since $D[V_1]$ is Hamiltonian, D contains a cycle of length $k = 2p+1$. Since $2p+1 < n < \frac{11p+3}{2}$ it follows that $n > k > \frac{1}{4}(2n - 3p + 1)$, but clearly D has no $(k+1)$-cycle.

In 1970, Behzad, Chartrand, and Wall [2] conjectured that a d-regular oriented graph of order n has girth at most $\lceil \frac{n}{d} \rceil$. This conjecture, and even a stronger one of Caccetta and Häggkvist [4] that the same is valid when the minimum indegree is at least d, has been verified in particular cases: $d = 2$ by Caccetta and Häggkvist [4], $d = 3$ by Hamidoune [6], and $d = 4, 5$ by Hòang and Reed [7]. For further information, we refer the reader to a recent article of Bondy [3]. Next we shall show that the conjecture of Behzad, Chartrand, and Wall is valid for the special family of in-tournaments.

Theorem 4.2 *Let D be an in-tournament of order n with $\delta^-(D) \geq p > 0$. Then $g(D) \leq \lceil \frac{n}{p} \rceil$.*

Proof. Let $x \in V(D)$ be an arbitrary vertex of D and let N_1 denote the set of negative neighbors of x. By the hypothesis, $D[N_1]$ is a tournament and $|N_1| \geq p$. We may assume that $D[N_1]$ is transitive, since if it contains any cycle, it follows by Theorem 2.2 that $g(D) = 3$. Note that $p \leq \frac{n-1}{2}$ and hence, $\lceil \frac{n}{p} \rceil \geq 3$. Let $y_1 \in N_1$ such that $d^-(y_1, N_1) = 0$ and define N_2 to be the set of negative neighbors of y_1. Clearly, $D[N_2]$ is a tournament, $(N_1 \cup \{x\}) \cap N_2 = \emptyset$ and $|N_2| \geq p$. Again assume that $D[N_2]$ is transitive and let $y_2 \in N_2$ such that $d^-(y_2, N_2) = 0$. Let $c = \lfloor \frac{n}{p} \rfloor$. Analogously, for $2 \leq j < c$, we define the vertex sets N_{j+1} to be the set of negative neighbors of the unique vertex $y_j \in N_j$, where $d^-(y_j, N_j) = 0$. Such a vertex exists for every j since $D[N_j]$ is a tournament which we suppose to be transitive. Clearly, $|N_{j+1}| \geq p$ for every $2 \leq j < c$. If y_j has a negative neighbor $z_i \in N_i$ for some $1 \leq i < j$ (possibly, $z_i = y_i$), then the cycle $z_i y_j y_{j-1} \ldots y_i z_i$ has length t for some $3 \leq t \leq c$, and the same holds if $x \to y_j$. Therefore, we may assume that $(N_1 \cup \ldots \cup N_j \cup \{x\}) \cap N_{j+1} = \emptyset$ for every $2 \leq j < c$. If n is divisible by p, then, for $j = c - 1$, we obtain the contradiction $n \geq |\{x\}| + \sum_{i=1}^{c} |N_i| \geq 1 + p \cdot \frac{n}{p} = n + 1$. Otherwise let $y_c \in N_c$ such that $d^-(y_c, N_c) = 0$. It follows that $n - |\{x\}| - \sum_{i=1}^{c} |N_i| \leq n - 1 - p \cdot \lfloor \frac{n}{p} \rfloor \leq p - 2$, which implies that y_c has at least one negative neighbor in $\bigcup_{i=1}^{c-1} N_i$. Clearly, this leads to a t-cycle in D for some $3 \leq t \leq c + 1 = \lceil \frac{n}{p} \rceil$. □

The digraph that illustrates the sharpness of Theorem 3.3 can be varied to show that the conjecture of Behzad, Chartrand, and Wall [2] is best possible even for in-tournaments. For the integers $p \geq 2$ and $k \geq 3$, let D be the digraph with the vertex set $V(D) = \{x_1, x_2, \ldots, x_{p \cdot k+1}\}$. For every $1 \leq i \leq p \cdot k + 1$, let $x_i \to \{x_{i+1}, x_{i+2}, \ldots, x_{i+p}\}$. Then D is a p-regular in-tournament that contains no t-cycle for every $3 \leq t \leq k = \lfloor \frac{p \cdot k+1}{p} \rfloor$.

Since we will deal with the case $p = \frac{n}{3}$ from now on, we state the following corollaries from Theorem 4.1 and Theorem 4.2, respectively.

Corollary 4.3 *Let D be a strong in-tournament of order n such that $\delta^-(D) \geq \frac{n}{3}$. If D contains a k-cycle for some $n > k > \frac{2n}{9} + \frac{1}{2}$ then D has a cycle of length $k + 1$.*

Corollary 4.4 *Every in-tournament of order n with $\delta^-(D) \geq \frac{n}{3}$ contains a 3-cycle.*

Now we consider short cycles in in-tournaments with the desired minimum in-degree of $\frac{n}{3}$. In this situation we do not need to restrict ourselves to strong in-tournaments. The following result fills the gap of cycle lengths between 3 and $\frac{2n}{9}+\frac{1}{2}$.

Theorem 4.5 *Let D be an in-tournament of order n such that $\delta^-(D) \geq \frac{n}{3}$. If D contains a k-cycle for some $3 \leq k \leq \frac{2n}{9}+1$ then D has a cycle of length $k+1$.*

Proof. Let C be a cycle of length k for some $3 \leq k \leq \frac{2n}{9}+1$ and suppose to the contrary that D does not contain a $(k+1)$-cycle. Let $x_0 \in V(C)$ such that $d^-(x_0, C) \leq \frac{|V(C)|-1}{2} = \frac{k-1}{2}$, and let N_1 denote the set of negative neighbors of x_0 in $D - C$. By the hypothesis, $D[N_1]$ is a tournament and $|N_1| \geq \frac{n}{3} - \frac{k-1}{2}$. Note that $k \leq \frac{2n}{9}+1$ implies that $\frac{n}{3} - \frac{k-1}{2} \geq k-1 \geq 2$. By the assumption, C is not extendable and therefore $N_1 \to x_0$ leads to $N_1 \to C$. Let D_1 be the strong component of $D[N_1]$ such that there is no arc leading from $D[N_1] - D_1$ to D_1. Since D_1 is a strong tournament, we derive a contradiction to the assumption if D_1 contains more than k vertices. Hence let $|D_1| \leq k$, and let $x_1 \in V(D_1)$ such that $d^-(x_1, N_1) = d^-(x_1, D_1) \leq \frac{|D_1|-1}{2} \leq \frac{k-1}{2}$. Since $d^-(x_1, C) = 0$, we obtain $d^-(x_1, D - (V(C) \cup N_1)) \geq \frac{n}{3} - \frac{k-1}{2}$.

Let N_2 be the set of negative neighbors of x_1 in $D - (V(C) \cup N_1)$. As above, $n_2 = |N_2| \geq k-1$. Analogously, let D_2 with $|D_2| \leq k$ denote the strong component of $D[N_2]$ that has no negative neighbors in $D[N_2] - D_2$, and let $x_2 \in D_2$ such that $d^-(x_2, N_2) \leq \frac{k-1}{2}$. By Theorem 2.1, there exists a Hamiltonian path $u_1u_2...u_{n_2}$ of $D[N_2]$ such that $u_1 = x_2$. Assume now that $y \to x_2$ for some $y \in V(C) \cup N_1$. Since $n_2 \geq k-1$, D contains the $(k+1)$-cycle $u_1u_2...u_{k-1}x_1yu_1$ if $y \in V(C) \cup N_1 \backslash V(D_1)$, a contradiction. For $y \in V(D_1)$, let P denote a path from x_1 to y in D_1. Obviously, $2 \leq |V(P)| \leq |D_1| \leq k$ and we obtain the cycle $u_1u_2...u_{k+1-|V(P)|}Pu_1$ of length $k+1$ in D. Hence, altogether we deduce that $d^-(x_2, V(C) \cup N_1 \cup N_2) \leq \frac{k-1}{2}$ which leads to $|N_3| \geq \frac{n}{3} - \frac{k-1}{2} \geq k-1 \geq 2$, where N_3 denotes the negative neighborhood of x_2 in $D - (V(C) \cup N_1 \cup N_2)$.

Again, let D_3 be the strong component of $D[N_3]$ such that there is no arc leading from $D[N_3] - D_3$ to D_3, and let $x_3 \in D_3$ such that $d^-(x_3, N_3) \leq \frac{k-1}{2}$. Since $x_3 \to x_2 \to x_1 \to C$, the vertex x_3 has no negative neighbor in $V(C)$. Analogously to the argumentation for x_2 and N_1, we see that $d^-(x_3, N_2) = 0$. Furthermore, x_3 has no negative neighbor $y \in N_1 \backslash V(D_1)$ since otherwise $k-2$ vertices of N_3 and the vertices x_2, x_1 and y form a cycle of length $k+1$. Assume that $y \to x_3$ for some $y \in V(D_1)$, and let P be a shortest path from x_1 to y in D_1. If $|V(P)| \leq k-1$ then again P, x_2 and $k - |V(P)|$ vertices of N_3 build up a $(k+1)$-cycle. The remaining case $|V(P)| = k$ can occur at most once because D_1 is Hamiltonian. Hence, $d^-(x_3, D_1) \leq 1$ and we summarize that $d^-(x_3, V(C) \cup N_1 \cup N_2 \cup N_3) \leq \frac{k-1}{2}+1$. This implies $n_4 = d^-(x_3, D - (V(C) \cup N_1 \cup N_2 \cup N_3)) \geq \frac{n}{3} - \frac{k-1}{2} - 1$.

Finally, it follows that

$$n \geq |V(C)| + |N_1| + |N_2| + |N_3| + n_4 \geq k + 4\left(\frac{n}{3} - \frac{k-1}{2}\right) - 1$$

which leads to $k \geq \frac{n}{3} + 1$, a contradiction. $\square$

Corollary 4.4, Theorem 4.5 and Corollary 4.3 can be summarized to obtain the desired result.

Corollary 4.6 *Let D be a strong in-tournament of order n with $\delta^-(D) \geq \frac{n}{3}$. Then D is pancyclic.*

Now we vary the example showing the sharpness of Theorem 3.3 such that $x_i \rightarrow \{x_{i+1}, x_{i+2}, ..., x_{i+k}\}$ for every $1 \leq i \leq 3k+1$, to underline that Corollary 4.6 is best possible. Having order $3k+1$ and minimum indegree k, the digraph contains no cycle of length 3.

Moreover, the following example shows that Corollary 4.6 cannot be extended to vertex pancyclicity. Let D be the digraph of order $3k$, $k \geq 3$, with the vertex set $V(D) = \{x\} \cup N \cup P \cup Z$. Let $D[N]$ be a strong tournament on k vertices, and let $D[P]$ and $D[Z]$ be arbitrary tournaments on k and $k-1$ vertices, respectively. Furthermore, let $Z \rightarrow N \rightarrow x \rightarrow P \rightarrow Z$ and $N \rightarrow P$. Then D is a strong in-tournament with $\delta^-(D) \geq k$ and the vertex x is not contained in a cycle of length 3.

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